The M/G/1 retrial queue: An information theoretic approach

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Abstract

In this paper, we give a survey of the use of information theoretic techniques for the estimation of the main performance characteristics of the $M/G/1$ retrial queue. We focus on the limiting distribution of the system state, the length of a busy period and the waiting time. Numerical examples are given to illustrate the accuracy of the maximum entropy estimations when they are compared versus the classical solutions.

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1 Introduction

In classical queueing theory it is usually assumed that any customer who cannot get service automatically upon arrival either joins a waiting line or leaves the system forever. However, the consideration of loss queueing models is just a first approximation to a more sophisticated situation. Usually the real behaviour of a blocked customer consists of leaving the service area temporarily but he returns to repeat his demand after some random time. This queueing behaviour is studied in the so-called retrial queues (see Falin and Templeton (1987) and Artalejo (1999a, 1999b) for a survey and bibliographical information).

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This paper gives a survey on the application of information theoretic techniques for the estimation of the performance descriptors of the $M/G/1$ queue with retrials. More concretely, we use the principle of maximum entropy (PME) which provides an elegant methodology for computing a unique estimate for an unknown probability distribution based on information expressed in terms of some given mean value constraints. Although we focus on the $M/G/1$ queue, the methodology can be easily extended to other queueing models with retrials (see Artalejo and Martin (1994), Artalejo and Gomez-Corral (1995) and Aissani and Smail (2003)) and without retrials (see Kouvatsos (1994), Tadj and Hamdi (2001) and Wang *et al.* (2002)). For a review of other (non-queueing) applications of the information theoretic methods, we refer the reader to the book by Kapur (1989). As examples of more recent applications we mention the papers by Smolders and Urbach (2002) in the context of x-ray fluorescence spectroscopy and the paper by Oganian and Domingo-Ferrer (2003) published recently in this journal where the reciprocal of Shannon's entropy is used to measure disclosure risk in tabular data.

The paper will be organized as follows. In Section 2 we describe the mathematical model and summarize its main characteristics in terms of generating functions (discrete characteristics) or Laplace transforms (continuous characteristics). A brief overview of the maximum entropy formalism is given in Section 3. Then, the general theory is used to get maximum entropy estimations of the limiting distribution of the system state (Section 4), the length of a busy period (Section 5) and the waiting time (Section 6) (see Falin et al (1994), Lopez-Herrero (2004) and Artalejo et al. (2002)). The numerical experiments show the goodness of the maximum entropy solutions based on the first two moments and the value of the generating function (respectively the Laplace transform) at a given point.

2 The mathematical model

Primary customers arrive to a single server queueing system following a Poisson process of rate λ. Any customer finding the server busy is blocked and leaves temporarily the service area. Such customers join a group of unsatisfied customers called orbit. We assume that the access from the retrial group to the service facility is governed by the classical linear policy, i.e., the probability of a repeated attempt during the interval $(t, t + \Delta t)$, given that j customers were in orbit at time t, is $j\mu\Delta t + o\Delta(t)$. The service times follow a common distribution function $B(x)$ ($B(0) = 0$), with kth moment β_k and Laplace transform $\beta(s)$. The input flow of primary arrivals, intervals between repeated attempts and service times are mutually independent.

The system state at time t can be described by means of the process $X = \{(C(t), N(t), \xi(t)); t \geq 0\}$, where $C(t)$ denotes the state of the server, 0 or 1 according to whether the server is free or busy, $N(t)$ is the number of customers in orbit at time

t and, if $C(t) = 1$, then $\xi(t)$ represents the elapsed time of the customer being served. In what follows, we neglect the component $\xi(t)$ and consider only the pair $(C(t), N(t))$ which takes values on the state space $S = \{0, 1\} \times \mathbb{N}$. We assume that $\rho = \lambda\beta_1 < 1$ so our queueing model is stable and the limiting probabilities $P_{ij} = \lim_{t\to\infty} P\{C(t) = i, N(t) = j\}$, $(i, j) \in S$, exist and are positive. Then, their corresponding partial generating functions $P_i(z) = \sum_{j=0}^{\infty} z^j P_{ij}$, $i \in \{0, 1\}$, are given by (see Falin and Templeton (1997)):

$$P_0(z) = (1 - \rho) \exp\left\{-\frac{\lambda}{\mu} \int_z^1 \frac{1 - \beta(\lambda - \lambda u)}{\beta(\lambda - \lambda u) - u} du\right\}, \tag{1}$$

$$P_1(z) = \frac{1 - \beta(\lambda - \lambda z)}{\beta(\lambda - \lambda z) - z} P_0(z). \tag{2}$$

By routine differentiation of formulas (1)-(2) we get after some algebra the following expressions for the partial moments $m_i^k = \sum_{j=0}^{\infty} j^k P_{ij}$, for $i \in \{0, 1\}$ and $k = 0, 1, 2$:

$$m_0^0 = 1 - \rho, \; m_1^0 = \rho \tag{3}$$

$$m_0^1 = \frac{\lambda\rho}{\mu}, \; m_1^1 = \frac{\lambda^2\beta_2}{2(1 - \rho)} + \frac{\lambda\rho^2}{\mu(1 - \rho)}, \tag{4}$$

$$m_0^2 = m_0^1 + \frac{\lambda}{\mu} m_1^1, \tag{5}$$

$$m_1^2 = \frac{\lambda^3\beta_3}{3(1 - \rho)} + \frac{\lambda^4\beta_2^2}{4(1 - \rho)^2} + \frac{\lambda^2\beta_2}{2(1 - \rho)} + \frac{\lambda^3\beta_2}{2\mu(1 - \rho)^2} + \frac{\lambda\rho}{\mu(1 - \rho)}$$
$$+ \frac{\lambda^4}{(1 - \rho)^2}\left(\frac{\beta_1}{\mu} + \frac{\beta_2}{2}\right)^2 - m_0^2. \tag{6}$$

The busy period of the $M/G/1$ retrial queue, L, starts with the arrival of a primary customer who finds the system empty and ends at the first departure epoch in which the system becomes empty again. The analysis of L in terms of Laplace transforms leads to the following expression

$$L^*(s) = \frac{\int_0^{L_\infty^*(s)} \frac{\beta(s + \lambda - \lambda u)}{e(s, u)(\beta(s + \lambda - \lambda u) - u)} du}{\int_0^{L_\infty^*(s)} \frac{du}{e(s, u)(\beta(s + \lambda - \lambda u) - u)}}, \; s > 0, \tag{7}$$

where $L_\infty^*(s)$ represents the Laplace transform for the busy period in the standard retrial queue without retrials given by $L_\infty^*(s) = \beta(s + \lambda - \lambda L_\infty^*(s))$ and $e(s, u)$ is

$$e(s, u) = \exp\left\{\frac{1}{\mu} \int_0^u \frac{s + \lambda - \lambda\beta(s + \lambda - \lambda v)}{\beta(s + \lambda - \lambda v) - v} dv\right\}, \; 0 \le u < L_\infty^*(s).$$

The above expression provides a theoretical solution but it has serious limitations in practice. In particular, the moments of L cannot be obtained by direct differentiation. From the theory of regenerative processes, it is easy to get the following formula for the expectation:

$$E[L] = \frac{1}{\lambda}\left(\frac{1}{P_0(0)} - 1\right). \tag{8}$$

A direct method of calculation (see Artalejo and Lopez-Herrero (2000)) for the second moment yields

$$E[L^2] = \frac{1}{P_0(0)}\left(\frac{1}{(1-\rho)^2}\left(\beta_2 + \frac{2\rho\beta_1}{\mu}\right) - \int_0^1 \frac{2}{\lambda\mu(\beta(\lambda - \lambda t) - t)}\right. \tag{1}$$

$$\times\left(1 - \frac{\lambda(1-t)\beta'(\lambda - \lambda t)}{\beta(\lambda - \lambda t) - t} - \frac{1}{1-\rho}\exp\left\{\frac{\lambda}{\mu}\int_t^1 \frac{1 - \beta(\lambda - \lambda u)}{\beta(\lambda - \lambda u) - u}du\right\}\right)dt\right). \tag{9}$$

Unfortunately it does not seem possible to numerically invert the density function of L by applying well-known algorithms (see Press *et al.* (1992)) because the above solution (7) is derived when s is a real value, and such algorithmic methods require to evaluate the Laplace transform at any desired complex s. In this sense, the maximum entropy estimation developed in Section 5 provides an elegant alternative to solve this drawback.

Let us assume that a primary customer arrives to the system at time t. His virtual waiting time $W(t)$ is defined as the time that the customer spends in the orbit waiting for service. According to the definition, $W(t)$ excludes the service time. We consider the system at steady state so, in what follows, we simply denote $W(t)$ by W. The analysis of W is intricate because customers in the orbit operate under a random order discipline. Following the book by Falin and Templeton (1997) we observe that the Laplace transform of W is given by

$$W^*(s) = 1 - \rho + \frac{\lambda(1-\rho)}{s}\int_{L_\infty^*(s)}^1 \frac{(1-u)(\beta(\lambda - \lambda u) - \beta(s + \lambda - \lambda u))}{(\beta(\lambda - \lambda u) - u)(u - \beta(s + \lambda - \lambda u))} \tag{2}$$

$$\times\exp\left\{\int_u^1 \frac{s + \mu + \lambda - \lambda v}{\mu(\beta(s + \lambda - \lambda v) - v)}dv\right\}\exp\left\{\int_1^u \frac{\lambda - \lambda v}{\mu(\beta(\lambda - \lambda v) - v)}dv\right\}du. \tag{10}$$

An appeal to Little's formula gives

$$E[W] = \frac{\lambda\beta_2}{2(1-\rho)} + \frac{\rho}{\mu(1-\rho)}. \tag{11}$$

Recently, Artalejo *et al.* (2002) obtained the following explicit expression for the second moment:

$$E[W^2] = \frac{2\lambda\beta_3}{3(1-\rho)(2-\rho)} + \frac{\lambda^2\beta_2^2}{(1-\rho)^2(2-\rho)}$$
$$+ \frac{\lambda\beta_2}{\mu}\left(\frac{2}{(1-\rho)^2(2-\rho)} + \frac{\rho}{(1-\rho)^2}\right) + \frac{2\rho}{\mu^2(1-\rho)^2}. \tag{12}$$

Since *s* is real, we notice again that the most typical techniques for the numerical inversion of *W* do not apply. However, the above formulas (11)-(12) will be helpful in the sequel. In fact, they constitute the basis for the main value constraints needed to construct maximum entropy estimations.

3 The maximum entropy formalism

Some "classical" queueing techniques include the general framework of birth-and-death processes and methods of solution for non-Markovian stochastic processes (such as embedded Markov chains, supplementary variables, matrix-analytic techniques, etc.). One elegant alternative for this is given by information theoretic methods that use the principles of maximum entropy and minimum cross-entropy (if a prior distribution is available) to estimate probability distributions given information in the form of known mean values. We refer the reader to the survey paper by Kouvatsos (1994) and the references therein.

A novel reader having a first approach to the literature could feed the idea that maximum entropy solutions only provide a reasonable approximation to the true (but complex) queueing system modelled by "classical" techniques. Such interpretation of the information theoretic techniques is poor and trivial. The aim of the PME is to provide a self-contained method of inference for estimating uniquely an unknown probability distribution. The maximum entropy distribution gives the most random solution; i.e., it introduces the minimum additional information beyond what is implied in the original available mean constraints. It should be pointed out that information theoretic analysis neither pretends to replace the "classical" queueing solutions not to be an approximation to that "classical" results. The idea is just to apply the maximum entropy formalism in order to get the widest probability distribution subject to the known constraints. Hence, when in what follows we present "classical" queueing results versus maximum entropy solutions, we only pretend to display two alternative tools for analyzing an unique real underlying queueing phenomenon. It is so far of our intention to suggest a possible (philosophical or numerical) superiority of the "classical" methodology over the maximum entropy approach or vice versa.

We next summarize the maximum entropy formalism (see Shore and Johnson (1981) and Kouvatsos (1994)). The general theory is common for both the discrete and

continuous cases. Thus, we simply denote by $f(x)$ the corresponding mass probability function or density function associated with the queueing performance measure under study. We assume that $f(x)$ takes values in a state space χ, so we have the normalization condition

$$\int_\chi f(x)dx = 1. \tag{13}$$

The known information about $f(x)$ can be expressed in terms of mean value constraints of the form

$$\int_\chi F_k(x)f(x)dx = F_k, \qquad 1 \le k \le m, \tag{14}$$

for known functions $F_k(x)$ and known values F_k. We note that the structural form of the constraints (14) covers important special cases such as:

(a) $F_k(x) = x^k$ (central moments of order k).

(b) $F_k(x) = I_{(-\infty, x_k]}(x)$ (value of the distribution function at the point x_k).

(c) $F_k(x) = e^{-s_k x}$ (value of the Laplace transform or the moment generating function at the point s_k).

The PME states that, of all the distributions satisfying the mean value constraints (13) and (14), the minimal prejudiced is the one maximizing the Shannon's entropy functional

$$H(f) = -\int_\chi f(x)\ln f(x)dx. \tag{15}$$

Suppose that a prior distribution $g(x)$ is given as current estimate, then the principle of minimum cross-entropy generalizes the PME by stating that, of all the distributions satisfying the mean constraints, the minimum cross-entropy solution is chosen by minimizing the functional

$$H(f, g) = \int_\chi f(x)\ln \frac{f(x)}{g(x)}dx. \tag{16}$$

In fact, the PME corresponds to the particular case when the prior distribution $g(x)$ in (16) is uniformly distributed on the state space χ.

The maximization of $H(f)$ can be carried out with the help of the method of Lagrange's multipliers. If there exists a distribution that maximizes the entropy (15) and satisfies the constraints (13) and (14), then it has the following form

$$\hat{f}(x) = \exp\left\{-\hat{\alpha}_0 - \sum_{k=1}^m \hat{\alpha}_k F_k(x)\right\}, \qquad x \in \chi, \tag{17}$$

where $\hat{\alpha}_k$, for $0 \le k \le m$, are the Lagrangian multipliers. $\hat{\alpha}_0$ is determined from the normalization condition (13), so we obtain

$$\exp\{\hat{\alpha}_0\} = \int_{\chi} \exp\left\{-\sum_{k=1}^{m} \hat{\alpha}_k F_k(x)\right\} dx. \tag{18}$$

The rest of Lagrangian multipliers satisfy the relations

$$-\frac{\partial \hat{\alpha}_0}{\partial \hat{\alpha}_k} = F_k, \qquad 1 \le k \le m. \tag{19}$$

In general, it is impossible to solve (19) for $\hat{\alpha}_k$ explicitly. As an exception, we mention the special case where $m = 1$ and $F_1(x) = x$, which yields the explicit distribution

$$\hat{f}_1(x) = \frac{1}{F_1} e^{-x/F_1}, \qquad x \in \chi. \tag{20}$$

Suppose that we add the second moment as an additional constraint, then the pair $(\hat{\alpha}_1, \hat{\alpha}_2)$ must be computed numerically. By combining (14) and (17) we observe that a standard method for finding the optimal α_k is to solve the system

$$\int_{\chi} (F_i(x) - F_i) \exp\left\{-\sum_{k=1}^{m} \alpha_k (F_k(x) - F_k)\right\} dx = 0, \qquad 1 \le i \le m. \tag{21}$$

The above equations (21) for the Lagrangian multipliers are implicit and non-linear. It can be proved then that the problem of solving (21) is equivalent to minimizing the potential function

$$F(\alpha_1, ..., \alpha_m) = \ln \int_{\chi} \exp\left\{-\sum_{k=1}^{m} \alpha_k (F_k(x) - F_k)\right\} dx, \tag{22}$$

or, alternatively, the balanced function

$$G(\alpha_1, ..., \alpha_m) = \sum_{i=1}^{m} p_i \left(\int_{\chi} (F_i(x) - F_i) \exp\left\{-\sum_{k=1}^{m} \alpha_k (F_k(x) - F_k)\right\} dx\right)^2, \tag{23}$$

where $0 < p_i < 1$ and $\sum_{i=1}^{m} p_i = 1$.

The balanced function $G(\alpha_1, ...\alpha_m)$ in (23) takes the value 0 at the optimal solution $(\widehat{\alpha}_1, ..., \widehat{\alpha}_m)$ which provides a computational advantage over the potential function F in (22). For computing the minimum in (23) we will employ a method of direct search (see Nelder and Mead (1964)) which does not involve derivatives, avoiding problems arising when the Hessian of G is algorithmically almost singular. A complete discussion of this technical problem can be found in Agmon *et al.* (1979).

4 Maximum entropy estimation of the system state

After the preceding preliminaries we are ready to apply the maximum entropy methodology to the distribution of the system state in the $M/G/1$ retrial queue. Firstly, we assume that the available information consists of the marginal distribution of the server state and the partial expectations of the number of customers in orbit, so we know expressions (3) and (4).

Distinguishing the server state is important in order to provide a more detailed information. Then, the constraints (3) play the role of the normalization condition (13). According to (20), we expect to find a first order maximum entropy solution $\{\hat{P}_{ij}^1; i \in \{0, 1\}, j \geq 0\}$ of geometric type. This is formalized in the following result.

Proposition 1 *If the available information is given by m_i^k, for $i \in \{0, 1\}$ and $k \in \{0, 1\}$, then according to the PME the estimation of the probability distribution of the system state is*

$$\hat{P}_{0j}^1 = \frac{(m_0^0)^2}{m_0^0 + m_0^1} \left(\frac{m_0^1}{m_0^0 + m_0^1} \right)^j , \ j \geq 0, \tag{24}$$

$$\hat{P}_{1j}^1 = \frac{(m_1^0)^2}{m_1^0 + m_1^1} \left(\frac{m_1^1}{m_1^0 + m_1^1} \right)^j , \ j \geq 0. \tag{25}$$

Proof. It is sufficient to consider the case $i = 0$. Applying the method of Lagrangian multipliers we get a solution $\hat{P}_{0j}^1$ of the form

$$\hat{P}_{0j}^1 = uv^j, \ j \geq 0.$$

Since $\{\hat{P}_{0j}^1; j \geq 0\}$ satisfies the constraints m_0^0 and m_0^1, we find that

$$u = \frac{(m_0^0)^2}{m_0^0 + m_0^1}, \ v = \frac{m_0^1}{m_0^0 + m_0^1}.$$

This proves the desired expression (24).   □

According to the geometric structural form, the first order estimations (24) and (25) are decreasing sequences. Nevertheless, the limiting probabilities $\{P_{ij}; j \geq 0\}$ may have a mode at any arbitrary level of the orbit, we say j_i^*, for $i \in \{0, 1\}$. In particular, in the case of the $M/M/1$ retrial queue, the distribution is unimodal and the modes are given by

$$
j_0^* = \begin{cases} 0, & \text{if } \lambda\rho < \mu, \\[2ex] \left[\dfrac{(\lambda - \mu)\rho}{\mu(1 - \rho)}\right], & \text{if } \lambda\rho \geq \mu, \end{cases}
$$

$$
j_1^* = \left[\dfrac{\lambda\rho}{\mu(1 - \rho)}\right],
$$

where $[x]$ is the integer part of x. We observe that if $(\lambda - \mu)\rho/\mu(1 - \rho)$ (respectively $\lambda\rho/\mu(1 - \rho)$) is integer, then $j_0^* - 1$ (respectively $j_1^* - 1$) is also a mode.

In the light of the information about the modes, it is clear that the first order estimation will be accurate only when $j_0^* = j_1^* = 0$, which is equivalent to the inequality $\lambda\rho < \mu(1 - \rho)$.

To illustrate the above comments, in Table 1 we consider an $M/M/1$ retrial queue with a small retrial rate $\mu = 0.05$, so that the distribution is sparse and $j_0^* = j_1^* = 6$. Hence, the maximum entropy solution $\hat{P}_{ij}^1$ gives a bad estimation of the probabilities P_{ij}. Hence, the necessity of deriving new estimations of the system state is clear.

Two initial reasons justify the use of two moment estimations. Firstly, in Falin and Templeton (1997) is mentioned that the number of customers in orbit is asymptotically Gaussian, as $\mu \to 0$. This fact agrees with the structural form (17)-(18) of the maximum entropy distribution. Furthermore, by treating j as a continuous variable, we easily see that the k-moment estimation has at most $k - 1$ relative extremes.

By adapting the maximum entropy formalism to the case under consideration, we see that the Lagrangian multipliers can be obtained by minimizing the potential functions

$$
F_i(\alpha_1^i, \alpha_2^i) = \ln \sum_{j=0}^{\infty} \exp\left\{ -\sum_{k=1}^{2} \alpha_k^i\left(j^k - \frac{m_i^k}{m_i^0}\right)\right\}, \quad i \in \{0, 1\}. \tag{26}
$$

The computation of the infinite series on the right-hand side of (26) implies the consideration of a truncation threshold K which can be determined with the help of Tchebychev's inequality.

The second order estimations $\hat{P}_{ij}^2$ in Table 1 have modes at the seventh level of the orbit. The last row of the table gives the value of the Shannon entropy (15) for the classical distribution and the maximum entropy estimations. As expected, we observe that the entropy decreases when we increase the number of known moments.

It should be noted that the moments m_i^k are obtained by taking derivatives of the partial generating function $P_i(z)$, for $i \in \{0, 1\}$, at the point $z = 1$. Hence, it should be interesting to improve the estimation by considering any other constraint providing information related to another different point z_0. To this end, we consider $P_i(z_0)$ which satisfies the structural form described in (14). Now the maximum entropy solution has

Table 1: $M/M/1$ *retrial queue with* $(\lambda, \mu) = (1.0, 0.05)$ *and* $\rho = 0.25$

j	P_{0j}	P_{1j}	$\widehat{P}^1_{0j}$	$\widehat{P}^1_{1j}$	$\widehat{P}^2_{0j}$	$\widehat{P}^2_{1j}$	$\widehat{P}^{2,1}_{0j}$	$\widehat{P}^{2,1}_{1j}$
0	0.00237	0.00059	0.09782	0.03125	0.01010	0.00285	0.00186	0.00043
1	0.01189	0.00312	0.08506	0.02734	0.01912	0.00545	0.01141	0.00294
2	0.03121	0.00858	0.07397	0.02392	0.03260	0.00941	0.03265	0.00899
3	0.05723	0.01645	0.06432	0.02093	0.05002	0.01471	0.05984	0.01734
4	0.08226	0.02468	0.05593	0.01831	0.06911	0.02077	0.08341	0.02520
5	0.09872	0.03085	0.04863	0.01602	0.08597	0.02652	0.09719	0.03048
6	0.10283	0.03342	0.04229	0.01402	0.09628	0.03061	0.09975	0.03240
7	0.09549	0.03222	0.03677	0.01227	0.09707	0.03194	0.09277	0.03119
8	0.08057	0.02819	0.03197	0.01073	0.08812	0.03013	0.07944	0.02763
9	0.06266	0.02271	0.02780	0.00939	0.07203	0.02570	0.06317	0.02274
10	0.04543	0.01703	0.02418	0.00822	0.05300	0.01982	0.04687	0.01747
11	0.03097	0.01200	0.02102	0.00719	0.03511	0.01382	0.03253	0.01257
12	0.02000	0.00800	0.01828	0.00629	0.02094	0.00871	0.02116	0.00847
13	0.01231	0.00507	0.01589	0.00550	0.01124	0.00496	0.01290	0.00536
14	0.00725	0.00308	0.01382	0.00481	0.00543	0.00255	0.00738	0.00318
15	0.00411	0.00179	0.01202	0.00421	0.00236	0.00119	0.00396	0.00177
SE	3.05175		3.51236		3.07099		3.05262	

the form

$$\hat{P}^{2,1}_{ij} = \exp\left\{-\left(\widehat{\alpha}^i_0 + j\widehat{\alpha}^i_1 + j^2\widehat{\alpha}^i_2 + z_0^j\widehat{\alpha}^i_{2,1}\right)\right\}, \ i \in \{0, 1\}, \ j \geq 0.$$

The Lagrangian coefficients $(\widehat{\alpha}^i_0, \widehat{\alpha}^i_1, \widehat{\alpha}^i_2, \widehat{\alpha}^i_{2,1})$ can be computed after a new appeal to the use of Nelder and Mead's algorithm.

The entries in Table 1 show that the estimation improves when we employ $\hat{P}^{2,1}_{ij}$, with $z_0 = 0.55$, instead of $\hat{P}^2_{ij}$. In particular, the estimations $\hat{P}^{2,1}_{ij}$ fit the modes of the classical probabilities P_{ij}.

Another different possibility is to employ as constraints the relationships $\lambda P_{1j} = (j + 1)\mu P_{0,j+1}$, for $j \geq 0$, which express the conservation of flow across the level j of the orbit. The details and some numerical examples can be found in Falin *et al.* (1994).

In Table 2 we consider a second numerical example in which the system parameters are chosen to fix the traffic intensity $\rho = 0.9$. As a consequence of increasing the value of ρ, the limiting probabilities P_{ij} become sparse. Thus, a good estimation typically demands the use of higher truncation thresholds. For example, to calculate $\{\hat{P}^{2,1}_{1j}; j \geq 0\}$ we take $K = 80$. The maximum entropy solution $\hat{P}^{2,1}_{ij}$ based on the values $P_i(0.55)$, for $i \in \{0, 1\}$, seems to be an accurate estimation. In particular, it fits the modes $j^*_0 = 0$ and $j^*_1 = 1$ of the classical distribution P_{ij}.

The preceding numerical examples deal with a model with exponential service times. However, this assumption is not restrictive and similar conclusions can be obtained for other service time distributions. In fact, once the mean value constraints are fixed, the formalism is independent of the service time distribution.

Table 2: $M/M/1$ *retrial queue with* $(\lambda, \mu) = (0.9, 5.0)$ *and* $\rho = 0.9$

j	P_{0j}	P_{1j}	$\widehat{P}_{0j}^1$	$\widehat{P}_{1j}^1$	$\widehat{P}_{0j}^2$	$\widehat{P}_{1j}^2$	$\widehat{P}_{0j}^{2,1}$	$\widehat{P}_{1j}^{2,1}$
0	0.06606	0.05946	0.03816	0.07745	0.04492	0.07163	0.06453	0.05889
1	0.01070	0.06314	0.02360	0.07078	0.02383	0.06647	0.01386	0.06307
2	0.00568	0.06194	0.01459	0.06469	0.01294	0.06162	0.00556	0.06289
3	0.00371	0.05909	0.00902	0.05912	0.00719	0.05709	0.00315	0.06030
4	0.00265	0.05558	0.00557	0.05403	0.00408	0.05285	0.00215	0.05657
5	0.00200	0.05182	0.00344	0.04938	0.00237	0.04889	0.00163	0.05243
6	0.00155	0.04804	0.00213	0.04513	0.00141	0.04519	0.00131	0.04827
7	0.00123	0.04435	0.00131	0.04125	0.00086	0.04174	0.00109	0.04427
8	0.00099	0.04081	0.00081	0.03770	0.00053	0.03852	0.00092	0.04051
9	0.00081	0.03746	0.00050	0.03445	0.00034	0.03553	0.00078	0.03702
10	0.00067	0.03432	0.00031	0.03149	0.00022	0.03274	0.00067	0.03381
11	0.00056	0.03139	0.00019	0.02878	0.00014	0.03014	0.00058	0.03085
12	0.00047	0.02868	0.00011	0.02630	0.00010	0.02774	0.00050	0.02814
13	0.00039	0.02617	0.00007	0.02404	0.00007	0.02550	0.00043	0.02566
14	0.00033	0.02385	0.00004	0.02197	0.00005	0.02343	0.00037	0.02339
15	0.00028	0.02173	0.00002	0.02008	0.00003	0.02151	0.00032	0.02132
SE	3.52914		3.56020		3.55273		3.53329	

Table 3: *Sensitivity analysis of the function G on the Lagrangian multipliers*

	$(\lambda, \nu, \mu) = (1.0, 4.0, 0.05)$		$(\lambda, \nu, \mu) = (0.9, 1.0, 5.0)$	
	$i = 0$	$i = 1$	$i = 0$	$i = 1$
$\left(\widehat{\alpha}_1^i + \varepsilon/2, \widehat{\alpha}_2^i + \varepsilon/2\right)$	68.9365	94.6871	911.1997	1884403.5
$\left(\widehat{\alpha}_1^i - \varepsilon/2, \widehat{\alpha}_2^i + \varepsilon/2\right)$	52.7580	73.2812	857.8347	1745129.8
$\left(\widehat{\alpha}_1^i - \varepsilon/2, \widehat{\alpha}_2^i - \varepsilon/2\right)$	72.9542	100.4483	6224.688	14647723.4
$\left(\widehat{\alpha}_1^i + \varepsilon/2, \widehat{\alpha}_2^i - \varepsilon/2\right)$	55.6646	77.5092	5413.781	12915677.3
$\left(\widehat{\alpha}_1^i + \varepsilon, \widehat{\alpha}_2^i, \widehat{\alpha}_{2,1}^i\right)$	0.7506	0.9492	0.5796	4596.297
$\left(\widehat{\alpha}_1^i - \varepsilon, \widehat{\alpha}_2^i, \widehat{\alpha}_{2,1}^i\right)$	0.7545	0.9542	0.6063	4843.183
$\left(\widehat{\alpha}_1^i, \widehat{\alpha}_2^i + \varepsilon, \widehat{\alpha}_{2,1}^i\right)$	177.7164	242.7601	184.5680	2869660.8
$\left(\widehat{\alpha}_1^i, \widehat{\alpha}_2^i - \varepsilon, \widehat{\alpha}_{2,1}^i\right)$	207.7439	285.3387	1189.341	65279520.6
$\left(\widehat{\alpha}_1^i, \widehat{\alpha}_2^i, \widehat{\alpha}_{2,1}^i + \varepsilon\right)$	0.0002	0.0002	0.0010	0.2752
$\left(\widehat{\alpha}_1^i, \widehat{\alpha}_2^i, \widehat{\alpha}_{2,1}^i - \varepsilon\right)$	0.0002	0.0002	0.0010	0.2754

We conclude this section with some practical tips for the computation of Lagrangian multipliers. Some numerical results for sensitivity analysis are also presented in Table 3. Although the potential function F is a strictly convex function and therefore a Newton-Raphson method should converge for any initial guess $(\alpha_1, ..., \alpha_m)$, there are some practical problems. Due to the exponential structure of the maximum entropy solution (17), F becomes asymptotically linear along some directions. Thus, its Hessian eventually becomes algorithmically singular when the initial guess for the multiplier is

chosen close to the asymptotic region. It is also typical that F has a long valley in some direction, then the gradient of F is in the direction of the valley, but not necessarily in the direction of the optimal solution.

The discussion for the balanced function G is analogous but we recall that $G(\hat{\alpha}_1, ..., \hat{\alpha}_m) = 0$. Thus, in Table 3 we deal with G and discuss the sensitivity of the Lagrangian multipliers based on small changes in some determined directions. For the scenarios in Tables 1 and 2, we allow the optimal solutions $(\hat{\alpha}_1^i, \hat{\alpha}_2^i)$ and $(\hat{\alpha}_1^i, \hat{\alpha}_2^i, \hat{\alpha}_{2,1}^i)$ to change along ten directions described in the first column of the table. Then, the entries give the value of G for the choice $\varepsilon = 10^{-3}$. This sensitivity analysis shows the very strong incidence of the initial guess. The effect is stronger when we perturb the Lagrangian multiplier $\hat{\alpha}_2^i$ associated with the second order moment, and also when the distribution is very sparse (see the case $i = 1$ for the example with $\rho = 0.9$).

5 Maximum entropy estimation of the busy period

In this section we illustrate numerically the use of the PME to get an estimation for the density of L. Although the mathematical formalism and numerical techniques are common for both discrete and continuous distributions, the numerical effort to carry out the latter is considerably superior. More precisely, numerical implementation in a discrete case implies the estimation of a finite set of probabilities which can be done in a personal computer after a few minutes run. However, in a continuous situation is necessary to estimate a density function maybe defined over $(0,+\infty)$. It typically demands several hours of running time, and so often the program stops without converging to the optimal Lagrangian multipliers.

Initially we assume that the available information consists of the first and second moments of L, which are provided by formulas (8) and (9). After that, we add one more constraint by using the value of the Laplace transform $L^*(s)$ at a given real point $s = s_0$ (see equation (7)). The methodology described in Section 3 yields maximum entropy densities $\widehat{f_2}(x)$ and $\widehat{f_{2,1}}(x)$, respectively. Their corresponding functional forms look as follows

$$\widehat{f_2}(x) = \exp\left\{-\left(\widehat{\alpha_0} + x\widehat{\alpha_1} + x^2\widehat{\alpha_2}\right)\right\}, \; x \in (0, T),$$

$$\widehat{f_{2,1}}(x) = \exp\left\{-\left(\widehat{\alpha_0} + x\widehat{\alpha_1} + x^2\widehat{\alpha_2} + e^{-s_0 x}\widehat{\alpha_{2,1}}\right)\right\}, \; x \in (0, T).$$

As a practical remark, we observe that the potential function F and the balanced function G, given in formulas (22) and (23) respectively, involve integrals defined over $(0,+\infty)$. Thus, solving the minimization problem implies firstly the consideration of a truncated interval $(0, T)$. The upper bound T may be chosen with the help of Tchebychev's inequality, such as $P\{L > T\} \leq 10^{-2}$.

Note that the maximum entropy densities satisfy the given constraints, in particular the first two moments of $\widehat{f_2}(x)$ and $\widehat{f_{2,1}}(x)$ coincide with the ones of $f_L(x)$, and so does

the Laplace transform associated with $\widehat{f_{2,1}}(x)$, at $s = s_0$, with $L^*(s)$. Thus, we propose to check the accuracy of the maximum entropy estimation by measuring the relative errors associated with their estimates for the Laplace transforms; i.e., we consider

$$E_2(s) = \left| \frac{L_2^*(s)}{L^*(s)} - 1 \right| \text{ and } E_{2,1}(s) = \left| \frac{L_{2,1}^*(s)}{L^*(s)} - 1 \right|,$$

where $L_2^*(s) = \int_0^T e^{-sx}\widehat{f_2}(x)dx$ and $L_{2,1}^*(s) = \int_0^T e^{-sx}\widehat{f_{2,1}}(x)dx$.

Table 4: *Comparing Laplace transforms in an $M/E_3/1$ retrial queue*

s	$\rho = 0.25$		$\rho = 0.5$		$\rho = 0.75$	
	$L^*(s)$	$L_2^*(s)$	$L^*(s)$	$L_2^*(s)$	$L^*(s)$	$L_2^*(s)$
0.05	0.89166	0.90632	0.84622	0.85323	0.76778	0.70380
0.1	0.83598	0.83324	0.76951	0.75077	0.67777	0.54867
0.25	0.69107	0.67587	0.61400	0.55568	0.53178	0.33193
0.5	0.53792	0.51711	0.46800	0.38979	0.40388	0.20069
1.0	0.35856	0.35355	0.30994	0.24492	0.26869	0.11219
1.5	0.25426	0.26907	0.22107	0.17874	0.19324	0.07787
3.0	0.11078	0.15700	0.09882	0.09880	0.08854	0.04061
4.5	0.05802	0.11090	0.05282	0.06828	0.04822	0.02747
6.0	0.03409	0.08574	0.03150	0.05217	0.02916	0.02075
10.0	0.01159	0.05343	0.01095	0.03202	0.01038	0.01256

We next analyze the length of a busy period in an $M/G/1$ retrial queue with Erlang service times; i.e., we have

$$B(x) = \int_0^x \frac{v^m}{(m-1)!}e^{-vx}x^{m-1}dx, \ x \geq 0,$$

where $m \in \{1, 2, ...\}$ and $v > 0$. In particular, we take $m = 3$ phases and $\beta_1 = m/v = 1$. Then, the arrival rate is chosen as $\lambda = 0.25, 0.5$ and 0.75. For a given λ, we assume that the retrial rate is $\mu = 2\lambda$. Table 4 presents a comparison between the classical Laplace transform $L^*(s)$ and the maximum entropy solution based on two moments $L_2^*(s)$. For most fixed $s > 0$, we observe that the classical and maximum entropy solutions are closer when the traffic intensity decreases.

We next improve the estimation by considering the maximum entropy solution $\widehat{f_{2,1}}(x)$. For practical purposes, the choice of s_0 can be done by taking into account that the behaviour of $f_L(x)$ and $L^*(s)$ near the boundaries of their domains is determined by the Tauberian relations

$$\lim_{s \to 0} sL^*(s) = \lim_{x \to +\infty} f_L(x) \text{ and } \lim_{s \to +\infty} sL^*(s) = \lim_{x \to 0} f_L(x). \tag{27}$$

Consequently, small values of s_0 provide a better description of the tail behaviour of $f_L(x)$, while a large value of s_0 describes better the behaviour near the origin.
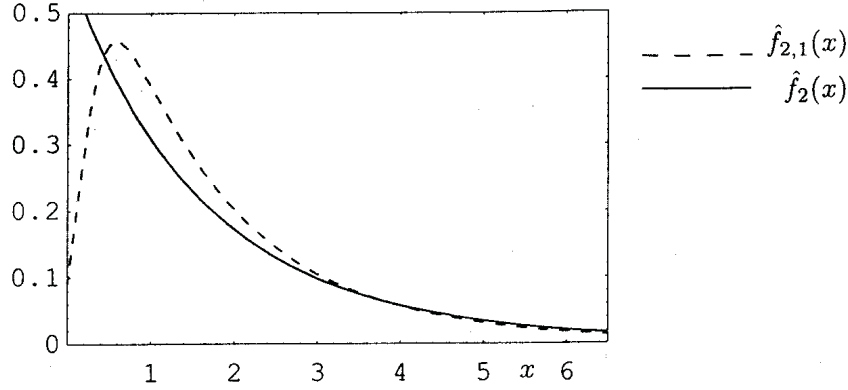

Figure 1: *ME estimations in an $M/E_3/1$ retrial queue with $\rho = 0.25$.*

In Table 5 and Figure 1 we consider and $M/E_3/1$ retrial queue with $\beta_1 = 1$ and $\rho = 0.25$, and we use as constraint the value of $L^*(s)$ at point $s_0 = 4.5$. We can observe in Table 5 that the relative errors are moderately small as s is close to zero. On the other hand, they are notably diminished when we compare large values of s and we employ $\widehat{f_{2,1}}(x)$ rather than $\widehat{f_2}(x)$. This fact and the boundary behaviour given in (27) indicate that, near the origin, the classical density function is much better described by $\widehat{f_{2,1}}(x)$ than by $\widehat{f_2}(x)$.

Table 5: *Relative errors in the $M/E_3/1$ retrial queue for $\rho = 0.25$*

s	$L^*(s)$	$L_2^*(s)$	$E_2(s)$	$L_{2,1}^*(s)$	$E_{2,1}(s)$
0.05	0.89166	0.90632	0.01644	0.90586	0.01593
0.1	0.83598	0.83324	0.00327	0.83101	0.00593
0.25	0.69107	0.67587	0.02200	0.66439	0.03861
0.5	0.53792	0.51711	0.03868	0.48998	0.08912
1.0	0.35856	0.35355	0.01395	0.30709	0.14352
1.5	0.25426	0.26907	0.05823	0.21404	0.15821
3.0	0.11078	0.15700	0.41719	0.09887	0.10750
4.5	0.05802	0.11090	0.91138	0.05802	0.00000
6.0	0.03409	0.08574	1.51504	0.03871	0.13564
10.0	0.01159	0.05343	3.60810	0.01841	0.58837

Figure 1 shows different shapes of the maximum entropy densities $\widehat{f_2}(x)$ and $\widehat{f_{2,1}}(x)$. Previous discussion permits to assert that the classical distribution near the origin should be better represented by $\widehat{f_{2,1}}(x)$; i.e., a bell-shaped function. Moreover, the figure

illustrates the importance of including information about the Laplace transform, because the mode of L is not reproduced unless a constraint on $L^*(s)$ is specified.
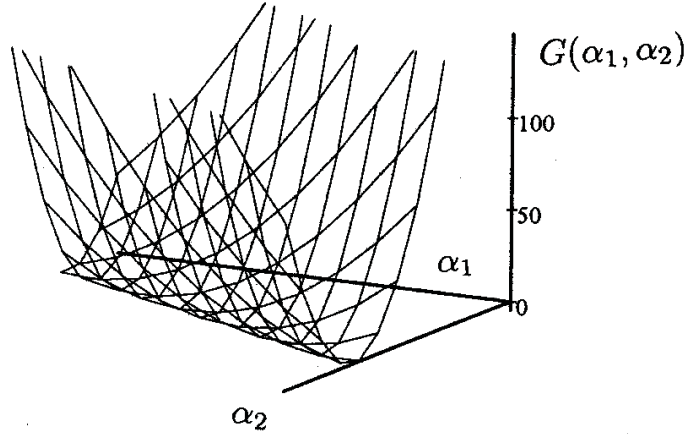

Figure 2: *The balanced function $G(\alpha_1, \alpha_2)$.*

In a second numerical example, see Table 6 and Figure 2, we consider an $M/H_2/1$ retrial queue so $B(x)$ is given by

$$B(x) = \sum_{i=1}^{2} p_i \left(1 - e^{-\nu_i x}\right), \; x \geq 0,$$

where $0 \leq p_1, p_2 \leq 1$, $p_1 + p_2 = 1$ and $\nu_1, \nu_2 > 0$. We consider that the mean service time is $\beta_1 = 1$ and the coefficient of variation $(\beta_2 - \beta_1^2)^{1/2}/\beta_1$ is 1.25. The parameters of the H_2 distribution cannot uniquely determined fitting the above values unless we add another additional condition. Thus, we assume that the distribution has balanced means; i.e., $p_1/\nu_1 = p_2/\nu_2$. The retrial rate is chosen as $\mu = \lambda/2$.

In Table 6 we compare the Laplace transforms associated with the maximum entropy densities based on two moments and two moments plus the value of $L^*(s)$ at the point $s_0 = 4.5$. The entries for the relative errors $E_{2,1}(s)$ are smaller than the errors $E_2(s)$, showing the superiority of the estimation $\widehat{f_{2,1}}(x)$.

In Figure 2, we plot the balanced function $G(\alpha_1, \alpha_2)$ in a neighbourhood of the Lagrangian multipliers $(\hat{\alpha}_1, \hat{\alpha}_2)$. In agreement with the comments expressed in Section 4, the surface shows a rapid growth of G along some directions. The existence of a valley is also observed.

For a numerical analysis of the number of customers served during a busy period, we refer to the paper by Lopez-Herrero (2002).

Table 6: *Relative errors in the M/H₂/1 retrial queue for ρ = 0.25*

s	$L^*(s)$	$L_2^*(s)$	$E_2(s)$	$L_{2,1}^*(s)$	$E_{2,1}(s)$
0.05	0.86545	0.82936	0.04170	0.83688	0.03301
0.1	0.80664	0.71491	0.11371	0.73749	0.08573
0.25	0.70620	0.50750	0.28135	0.57245	0.18939
0.5	0.60442	0.34316	0.43224	0.45084	0.25410
1.0	0.47820	0.20868	0.56361	0.35129	0.26539
1.5	0.39784	0.15000	0.62294	0.30388	0.23617
3.0	0.26601	0.08140	0.69398	0.23558	0.11438
4.5	0.20016	0.05586	0.72090	0.20016	0.00000
6.0	0.16049	0.04252	0.73503	0.17635	0.09887
10.0	0.10486	0.02598	0.75224	0.13675	0.30408

6 Maximum entropy estimation of the waiting time

We now present a maximum entropy analysis of the waiting time W based on the knowledge of $E[W]$, $E[W^2]$ and $W^*(s)$ at a given positive real point $s = s_0$ (see formulas (10)-(12)). Since the definition of W excludes the service time, we notice that the distribution function of W, $F_W(x)$, has a jump at $x = 0$ and is absolutely continuous in the interval $(0,+\infty)$. Thus, we have

$$\frac{dF_W(x)}{dx} = (1 - \rho)u_0(x) + f_W(x), \ x \geq 0,$$

where $u_0(x)$ is the unit impulse at the origin defined by

$$u_0(x) = \begin{cases} +\infty, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

Hence, the estimation of the distribution of W reduces to the density $f_W(x)$ of the continuous contribution. We first assume that the available information consists of the first two moments $E[W]$ and $E[W^2]$. In a second step, we add the knowledge of $W^*(s)$ at the point $s = 0.25$. We denote both estimations by $\widehat{f_2}(x)$ and $\widehat{f_{2,1}}(x)$ respectively.

In Table 7 we consider the $M/M/1$ retrial queue with $\lambda = 0.5$, $\nu = 1.0$ and $\mu = 2.0$, so the traffic intensity is $\rho = 0.5$. We evaluate the accuracy of the maximum entropy solutions by comparing the classical Laplace transform $W^*(s)$ versus the maximum entropy versions $W_2^*(s)$ and $W_{2,1}^*(s)$, which are given by $W_2^*(s) = 1 - \rho + \int_0^T e^{-sx}\widehat{f_2}(x)dx$ and $W_{2,1}^*(s) = 1 - \rho + \int_0^T e^{-sx}\widehat{f_{2,1}}(x)dx$.

The entries $E_2(s)$ and $E_{2,1}(s)$ correspond to the relative errors which are defined analogously to those given in Section 5 for the busy period. We observe that the relative errors decrease when we employ $\widehat{f_{2,1}}(x)$ rather than $\widehat{f_2}(x)$. The upper bound T is 29.5.

Table 7: *Relative errors in the M/M/1 retrial queue for ρ = 0.5*

| s	$W^*(s)$	$W_2^*(s)$	$E_2(s)$	$W_{2,1}^*(s)$	$E_{2,1}(s)$
0.01	0.98548	0.98548	1.27×10^{-6}	0.98548	7.58×10^{-8}
0.05	0.93549	0.93559	0.00010	0.93548	9.56×10^{-6}
0.1	0.88632	0.88682	0.00057	0.88627	5.98×10^{-5}
0.25	0.78842	0.79120	0.00325	0.78842	0.00000
0.3	0.76562	0.76928	0.00478	0.76579	0.00022
0.6	0.67751	0.68605	0.01261	0.67987	0.00407
0.9	0.63035	0.64237	0.05078	0.63532	0.00788
1.2	0.60109	0.61536	0.02374	0.60823	0.01187
1.5	0.58130	0.59699	0.02699	0.59007	0.01508
10.0	0.50633	0.51763	0.02232	0.51545	0.01801
20.0	0.50193	0.50898	0.01405	0.50781	0.01171
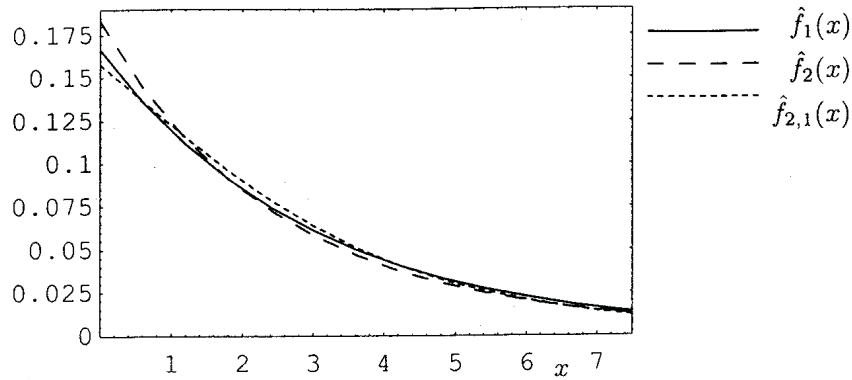
Figure 3: *ME estimations in an M/M/1 retrial queue with ρ = 0.5.*

For the same numerical example, in Figure 3 we display the maximum entropy densities $\widehat{f_1}(x)$, $\widehat{f_2}(x)$ and $\widehat{f_{2,1}}(x)$. In the light of the decreasing shape of the three densities, we conclude that all these solutions are enough close. However, at this point we remember the Tauberian relations (27), which give some light about the effect of the auxiliary point s_0. Accordingly, in Figure 4 we allow s_0 to take values 0.25, 0.5, 1.0 and 2.0. As far as s_0 increases we expect to get a better description of the behaviour of $f_W(x)$ near the origin $x = 0$. In fact, we observe that the densities associated with the values 0.25 and 0.5 are decreasing functions whereas the densities based on the values 1.0 and 2.0 exhibit a bell-shaped form.

Finally, in Figure 5 we plot the potential function $F(\alpha_1, \alpha_2)$. The resulting surface is complementary to Figure 2, where we consider the busy period and plot the balanced function G. Once more we observe the existence of a long valley and asymptotic linearity when we leave a neighbourhood of the Lagrangian multipliers.

The numerical results show that the use of the first two moments and the value of the Laplace transform in a given point is, in general, sufficient to obtain accurate estimations.
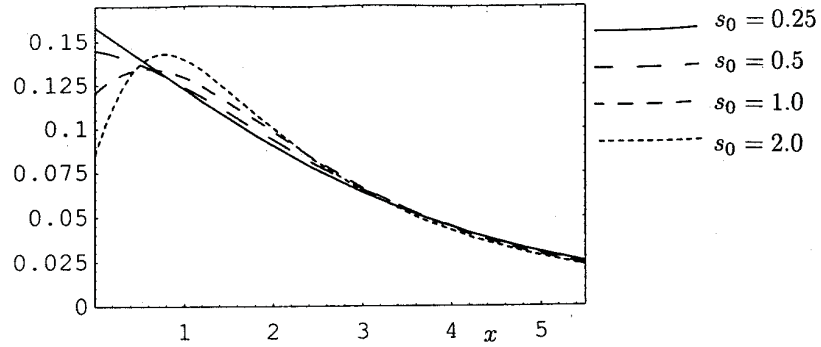

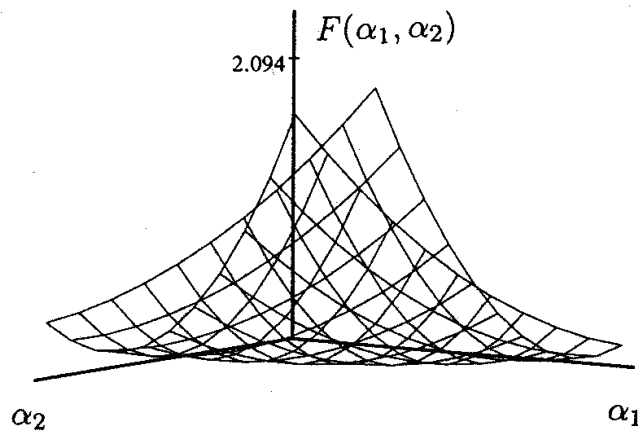**Figure 4**: *The effect of the point s_0.*

Figure 5: *The potential function $F(\alpha_1, \alpha_2)$.*

7 Acknowledgements

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