The M-Homomorphism and M-Anti Homomorphism of an M-Fuzzy Subgroup and its Level M-Subgroups

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Abstract
In this paper, we introduce the concept of an M-fuzzy subgroup of an M-group and discussed some of its properties.
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M-group, fuzzy set, fuzzy subgroup, M-fuzzy subgroup of an M-group , level subset , level M-subgroups , M-homomorphism , M-anti homomorphism.

Introduction
The concept of fuzzy sets was initiated by Zadeh. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld gave the idea of fuzzy subgroups. Author N. Jacobson introduced the concept of M-group, M-subgroup.

1. Preliminaries
This section contains some definitions and results to be used in the sequel.

1.1 Definition
Let S be a set. A fuzzy subset A of S is a function A: $S \rightarrow [0,1]$.

1.2 Definition
Let G be a group. A fuzzy subset A of G is called a fuzzy subgroup if for $x, y \in G$,

(i) $A(xy) \geq \min \{ A(x), A(y)\}$,

(ii) $A(x^{-1}) = A(x)$.

1.3 Definition
A group with operators is an algebraic system consisting of a group G , a set M and a function defined in the product set $M \times G$ and having values in G such that, if ma denotes the element in G determined by the element a of G and the element m of M, then $m(ab) = (ma)(mb)$ holds for all $a, b \in G$ and $m \in M$. We shall use the phrases "G is an M-group" to a group with operators.

A subgroup H of an M-group G is said to be an M-subgroup if $mx \in H$ for all $m \in M$ and $x \in H$.

1.4 Definition
Let G be an M-group and A be a fuzzy subgroup of G . Then A is called an M-fuzzy subgroup of G if for all $x \in G$ and $m \in M$, then $A(mx) \geq A(x)$.

1.5 Definition
Let A be a fuzzy subset of S. For $t \in [0, 1]$, the level subset of A is the set, $A_t = \{ x \in S : A(x) \geq t \}$.

1.6 Definition
Let G be a finite group of order n and A be a fuzzy subgroup of G. Let $Im(A) = \{ t_i : A(x) = t_i \text{ for some } x \in G \}$. Then $\{ A_{t_i} \}$ are the only level subgroups of A.

1.1 Example
Let A be a fuzzy subset of an M-group G, then A is defined by

$$A(x) = \begin{cases} 0.7 & \text{if } x \in G \\ 0.2 & \text{otherwise.} \end{cases}$$

Then it is easy to verify that A is an M-fuzzy subgroup of G.

1.7 Definition
Let G and G′ be any two M-groups. Then the function $f: G \rightarrow G'$ is said to be an M-homomorphism if

(i) $f(xy) = f(x) f(y)$ for all x, y in G.

(ii) $f(mx) = mf(x)$ for all m in M and x in G.

1.8 Definition

Let G and G′ be any two M-groups (not necessarily commutative). Then the function $f: G \to G'$ is said to be an M-anti homomorphism if

(i) $f(xy) = f(y) f(x)$ for all $x, y \in G$.

(ii) $f(mx) = m f(x)$ for all m in M and x in G.

2. M-fuzzy subgroups of an M-group G under M-homomorphism and M-anti homomorphism

2.1 Theorem

Let f be a M-homomorphism from an M-group G onto an M-group G′. If A is an M- fuzzy subgroup of G and A is f-invariant, then f(A) , the image of A under f, is an M- fuzzy subgroup of G′.

Proof

Let $\alpha \in$ Image f(A).

Then for some $y \in G'$, $(f(A))(y) = \sup_{x \in f^{-1}(y)} A(x) = \alpha$, where $\alpha \leq A(e)$.

Clearly A_α is an M-subgroup of G.

If $\alpha = 1$, then $(f(A))_\alpha = G'$.

If $0 < \alpha < 1$, then $(f(A))_\alpha = f(A_\alpha)$, because ,

$$z \in (f(A))_\alpha \Leftrightarrow (f(A))(z) \geq \alpha.$$
$$\Leftrightarrow \sup_{x \in f^{-1}(z)} A(x) \geq \alpha. \text{ (since } 0 < \alpha < 1)$$
$$\Leftrightarrow \text{there exists x in G such that } f(x) = z \text{ and } A(x) \geq \alpha.$$
$$\Leftrightarrow z \in (f(A_\alpha)).$$

Hence, $(f(A))_\alpha = (f(A_\alpha))$.

Since f is an M-homomorphism , $(f(A_\alpha))$ is an M-subgroup of G′.

Hence $(f(A))_\alpha$ is an M-subgroup of G′.

Hence f(A) is an M- fuzzy subgroup of G′.

2.2 Theorem

The M-homomorphic pre-image of an M-fuzzy subgroup of an M-group G′ is an M-fuzzy subgroup of an M-group G.

Proof

Let $f: G \to G'$ be an M-homomorphism. Let the fuzzy set V on G′ be an M-fuzzy subgroup.

We have to prove that any fuzzy set A on G is an M-fuzzy subgroup, where V = f(A).

Now,
$$\begin{aligned} A(xy) &= V(f(xy)). \\ &= V(f(x)f(y)) \quad \text{as f is an M-homomorphism.} \\ &\geq \min\{ V(f(x)), V(f(y))\} \\ &\qquad \text{as V is an M-fuzzy subgroup of G′.} \\ &= \min\{ A(x), A(y)\}. \end{aligned}$$

That is, $A(xy) \geq \min\{ A(x), A(y)\}$.

For $x \in G$,
$$\begin{aligned} A(x^{-1}) &= V(f(x^{-1})) \\ &= V((f(x)^{-1})) \quad \text{as f is an M-homomorphism} \\ &= V(f(x)) \quad \text{as V is an M-fuzzy subgroup of G′} \\ &= A(x). \end{aligned}$$

That is, $A(x^{-1}) = A(x)$.

Clearly,
$$\begin{aligned} A(mx) &= V(f(mx)) \\ &= V(mf(x)), \quad \text{as f is an M-homomorphism} \\ &\geq V(f(x)) \quad \text{as V is an M-fuzzy subgroup of G′} \\ &= A(x). \end{aligned}$$

That is, $A(mx) \geq A(x)$.

Hence A is an M-fuzzy subgroup of G.

2.3 Theorem

Let f be an M-anti homomorphism from an M-group G onto an M-group G′. If A is an M- fuzzy subgroup of G and A is f-invariant, then f(A) , the image of A under f, is an M-fuzzy subgroup of G′.

Proof

Let $\alpha \in$ Image f(A).

Then for some $y \in G'$, $(f(A))(y) = \sup_{x \in f^{-1}(y)} A(x) = \alpha$, where $\alpha \leq A(e)$.

Clearly A_α is an M-subgroup of G.

If $\alpha = 1$, then $(f(A))_\alpha = G'$.

If $0 < \alpha < 1$, then $(f(A))_\alpha = f(A_\alpha)$, because ,

$$z \in (f(A))_\alpha \Leftrightarrow (f(A))(z) \geq \alpha.$$
$$\Leftrightarrow \sup_{x \in f^{-1}(z)} A(x) \geq \alpha. \text{ (since } 0 < \alpha < 1)$$
$$\Leftrightarrow \text{there exists x in G such that } f(x) = z$$

and $A(x) \geq \alpha$.

$\Leftrightarrow z \in (f(A_\alpha))$.

Hence, $(f(A))_\alpha = (f(A_\alpha))$.

Since f is an M- anti homomorphism , $(f(A_\alpha))$ is an M-subgroup of G′.

Hence $(f(A))_\alpha$ is an M-subgroup of G′.

Hence f(A) is an M- fuzzy subgroup of G′.

2.4 Theorem

The M-anti homomorphic pre-image of an M-fuzzy subgroup of an M-group G′ is an M-fuzzy subgroup of an M-group G.

Proof

Let $f: G \to G'$ be an M-anti homomorphism. Let the fuzzy set V on G′ be an M-fuzzy subgroup.

We have to prove that any fuzzy set A on G is an M-fuzzy subgroup, where V = f(A).

Now, $A(xy) = V(f(xy))$.

$= V(f(x)f(y))$

as f is an M-anti homomorphism.

$\geq \min \{ V(f(x)) , V(f(y))\}$

as V is an M- fuzzy subgroup of G′.

$= \min \{ A(x) , A(y)\}$.

That is, $A(xy) \geq \min\{ A(x) , A(y)\}$.

For $x \in G$,

$A(x^{-1}) = V(f(x^{-1}))$

$= V((f(x)^{-1}))$ as f is an M-anti homomorphism

$= V(f(x))$ as V is an M-fuzzy subgroup of G′

$= A(x)$.

That is, $A(x^{-1}) = A(x)$.

Clearly, $A(mx) = V(f(mx))$

$= V(mf(x))$, as f is an M-anti homomorphism

$\geq V(f(x))$ as V is an M-fuzzy subgroup of G′

$= A(x)$.

That is, $A(mx) \geq A(x)$.

Hence A is an M-fuzzy subgroup of G.

3. Properties of level subsets of an M-fuzzy subgroup of an M-group:

3.1 Theorem

Let A be a fuzzy subset of an M-group G. If A is an M-fuzzy subgroup of G, then the level subsets A_t , $t \in Im(A)$ are M-subgroups of G.

Proof

Let $t \in Im(A)$ and $x , y \in A_t$.

Then A(x) = t and A(y) = t.

Given that A is an M-fuzzy subgroup of G.

Therefore, A is a fuzzy subgroup of G.

Hence $A(xy) \geq \min \{ A(x) , A(y) \} = t$.

That is, $A(xy) \geq t$.

That is, $xy \in A_t$.

Moreover, if $x \in A_t$,then $A(x^{-1}) = A(x) \geq t$.

Hence $x^{-1} \in A_t$.

Hence A_t is a subgroup of G.

Now, for any $x \in A_t$ and $m \in M$, then

$A(mx) \geq A(x) \geq t$.

Hence $mx \in A_t$.

Hence A_t is an M-subgroup of G.

3.2 Theorem

Let A be a fuzzy subset of an M-group G. If the level subsets A_t , $t \in Im(A)$ are M-subgroups of G, then A is an M-fuzzy subgroup of G.

Proof

Let the level subsets A_t , $t \in Im(A)$ are M-subgroups of G.

If there exist $x_0 , y_0 \in G$ such that $A(x_0y_0) < \min\{A(x_0) ,A(y_0)\}$.

Let $t_0 = (A(x_0y_0) + \min\{A(x_0) ,A(y_0)\}) /2$, we have $A(x_0y_0) < t_0 < \min\{A(x_0) ,A(y_0)\}$.

It follows that $x_0 , y_0 \in A_{t0}$,but $x_0y_0 \notin A_{t0}$.

Which is a contradiction.

Hence $A(xy) \geq \min \{ A(x) , A(y)\}$.

Similarly, we have $A(x^{-1}) \geq A(x)$.

Hence A is a fuzzy subgroup of G.

Now, suppose, for $m \in M$ and $x \in G$, $A(mx) < A(x)$.

Let $t_0 = (A(mx) + A(x))/2$.

Then, $A(mx) < t_0 < A(x)$.

That is , for $m \in M$ and $x \in G$, then $x \in A_{t0}$, but $mx \notin A_{t0}$.

Which is a contradiction to A_{t0} is a M-subgroup of G.

Hence $A(mx) \geq A(x)$.

Hence A is an M-fuzzy subgroup of G.

3.1 Definition

Let A be an M- fuzzy subgroup of an M-group G. Then the M-subgroups A_t, for $t \in [0,1]$ and $t \geq A(e)$, are called level M-subgroups of A.

4. Level M-subgroups of M-fuzzy subgroups of an M-group G under M-homomorphism and M-anti homomorphism

4.1 Theorem

The M-homomorphic image of a level M-subgroup of an M-fuzzy subgroup A of an M-group G is a level M-subgroup of an M-fuzzy subgroup f(A) of an M-group G′ where A is f-invariant.

Proof

Let G and G′ be any two M-groups.

Let f: G → G′ be an M-homomorphism.

Let A be an M-fuzzy subgroup of G.

Clearly, f(A) is an M-fuzzy subgroup of G′.

Let A_α be a level M-subgroup of an M-fuzzy subgroup A of G.

Since f is an M-homomorphism , f (A_α) is an M-subgroup f(A) of G′ and f $(A_\alpha) = (f(A))_\alpha$.

Hence $(f(A))_\alpha$ is a level M-subgroup f(A) of G′ .

4.2 Theorem

The M-homomorphic pre-image of a level M-subgroup of an M-fuzzy subgroup V of an M-group G′ is a level M-subgroup of an M-fuzzy subgroup $f^{-1}(V)$ of an M-group G.

Proof

Let G and G′ be any two M-groups.

Let f: G → G′ be an M-homomorphism.

Let V be an M-fuzzy subgroup of G′.

Clearly $f^{-1}(V)$ is an M-fuzzy subgroup of G .

Let V_t be a level M-subgroup of an M-fuzzy subgroup V of G′.

Since , f is an M-homomorphism , $f^{-1}(V_t)$ is an M-subgroup of $f^{-1}(V)$ of G

and $f^{-1}(V_t) = (f^{-1}(V))_t$, is an M-subgroup of an M-fuzzy subgroup $f^{-1}(V)$ of G.

That is , $(f^{-1}(V))_t$ is a level M-subgroup of an M-fuzzy subgroup $f^{-1}(V)$ of G.

4.3 Theorem

The M-anti homomorphic image of a level M-subgroup of an M-fuzzy subgroup A of an M-group G is a level M-subgroup of an M-fuzzy subgroup f(A) of an M-group G′ where A is f-invariant.

Proof

Let G and G′ be any two M-groups.

Let f: G → G′ be an M-anti homomorphism.

Let A be an M-fuzzy subgroup of G.

Clearly, f(A) is an M-fuzzy subgroup of G′.

Let A_α be a level M-subgroup of an M-fuzzy subgroup A of G.

Since f is an M-anti homomorphism , f (A_α) is an M-subgroup f(A) of G′ and f $(A_\alpha) = (f(A))_\alpha$.

Hence $(f(A))_\alpha$ is a level M-subgroup f(A) of G′ .

4.4 Theorem

The M-anti homomorphic pre-image of a level M-subgroup of an M-fuzzy subgroup V of an M-group G′ is a level M-subgroup of an M-fuzzy subgroup $f^{-1}(V)$ of an M-group G.

Proof

Let G and G′ be any two M-groups.

Let f: G → G′ be an M-anti homomorphism.

Let V be an M-fuzzy subgroup of G′.

Clearly $f^{-1}(V)$ is an M-fuzzy subgroup of G .

Let V_t be a level M-subgroup of an M-fuzzy subgroup V of G′.

Since , f is an M-anti homomorphism , $f^{-1}(V_t)$ is an M-subgroup of $f^{-1}(V)$ of G

and $f^{-1}(V_t) = (f^{-1}(V))_t$, is an M-subgroup of an M-fuzzy subgroup $f^{-1}(V)$ of G.

That is , $(f^{-1}(V))_t$ is a level M-subgroup of an M-fuzzy subgroup $f^{-1}(V)$ of G.

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