J. DIFFERENTIAL GEOMETRY
**54** (2000) 99-138

# THE MAGID-RYAN CONJECTURE FOR EQUIAFFINE HYPERSPHERES WITH CONSTANT SECTIONAL CURVATURE

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## Abstract

We study affine hyperspheres with constant sectional curvature. More precisely we obtain a classification of the affine hyperspheres with constant sectional curvature $c$, provided $c \neq H$, where $H$ denotes the affine mean curvature of the immersion. Our classification gives a complete and positive answer to a conjecture of M. Magid and P. Ryan about these hyperspheres.

## 1. Introduction

In this paper, we study nondegenerate affine hypersurfaces $M^n$ in $\mathbb{R}^{n+1}$. It is well known that on such hypersurfaces there exists a canonical transversal vector field $\xi$ called the affine normal vector field. If for all $p \in M$, $\xi(p)$ passes through a fixed point (resp. is parallel), $M^n$ is called a proper affine sphere (resp. improper affine sphere).

The standard models of affine spheres are the quadrics. Unlike in Euclidean geometry, where the only umbilical submanifolds are the spheres and the linear subspaces, the class of all equiaffine spheres is simply too large to classify. Therefore, in order to better understand the geometry of affine spheres, it is necessary to impose an extra condition. This

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*Subject class.* 53A15.

*Key words and phrases.* Affine differential geometry, affine hyperspheres, constant sectional curvature.

can either be a completeness assumption, which so far only works in the positive definite case, as studied by Blaschke, Calabi, Pogorelov, Cheng, Yau, Sasaki, Li, Ferrer, Martinez, Milan and others (see [9] and the references contained therein) or an additional assumption about the curvature.

At the conference on Affine Differential Geometry at Oberwolfach in 1986, it was proposed to study in a systematic way those affine hyperspheres with constant sectional curvature (with respect to the affine metric). In the case where the dimension is two, the first result in this direction was already obtained by Radon [12] at the beginning of this century and the classification was completed by Simon [13]. Also in higher dimensions, several results were obtained:

**Theorem 1** ([8]). *Let $M^n$ be a positive definite affine hypersphere in $R^{n+1}$ with constant sectional curvature $c$ with respect to the affine metric. Then either $M^n$ is an open part of a positive definite quadric or $M^n$ is affine equivalent with an open part of $x_1 \dots x_{n+1} = 1$.*

**Theorem 2** ([10]). *Let $M^3$ be an affine hypersphere with constant sectional curvature $c$. Assume that the affine metric $h$ is Lorentzian and that $c \neq H$, where $H$ is the affine mean curvature. Then $M^3$ is affine congruent with an open part of either $(x_1^2 + x_2^2)x_3x_4 = 1$ or $(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1$.*

The above theorem motivated M. Magid and P. Ryan to formulate the following conjecture in 1989:

**Conjecture 1.** *Let $M^n$ be an affine hypersphere in $\mathbb{R}^{n+1}$ with constant sectional curvature $c$ and with nonzero Pick invariant $J$. Then $c = 0$ and $M^n$ is equivalent to*

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \dots (x_{2m-1}^2 \pm x_{2m}^2) = 1,$$

*if $n = 2m - 1$ or with*

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \dots (x_{2m-1}^2 \pm x_{2m}^2)x_{2m+1} = 1,$$

*if $n = 2m$.*

In case that $c = H$, or equivalently $J = 0$, the above conjecture is no longer true. In that case, many nontrivial examples can be constructed. See among others [4] or [2]. Also, if either of the conditions that $M$ is an affine hypersphere or that $M$ has constant sectional curvature is weakened, see [3] and [14], several new examples will occur. The previously mentioned theorem of [8] shows that the conjecture is true if the affine metric is positive definite. Similar partial results, in case that the metric is Lorentzian or in case that the dimension is 4 were obtained respectively in [7] and [1].

In this paper we further develop the ideas of [7], to show that the conjecture is **true** in general. The paper is organized as follows. In Section 2 we shortly recall some basic formulas of affine differential geometry and we derive the equations that an affine hypersphere with constant sectional curvature has to satisfy. In particular, we show that in each tangent space, we have that

(i) $h(K(x,y),z)$ is symmetric in $x,y,z$

(ii) the trace of the linear operator $y \mapsto K(x,y)$ vanishes for every vector x,

(iii) Denote by $K_x y = K(x,y)$. Then

$$[K_x, K_y]z = a(h(y,z)x - h(x,z)y),$$

where $a = -J$ is a non-zero number and $K$ is the difference tensor.

The problem, given $a$ and the index of the metric, to classify all such metrics $h$ and tensors $K$ is a highly non trivial problem from linear algebra. In Section 3, we will investigate this problem and show that upto a natural equivalence there is at most one solution.

Using this solution, we then show in Section 4 that we can construct a special frame in a neighborhood of each point. The non-algebraic equations stating that $M$ is an affine hypersphere with constant sectional curvature then imply that the connection coefficients of this frame vanishes identically, implying already that $M$ is flat. An explicit integration then completes the proof of the conjecture.

## 2. Preliminaries

We briefly recall the basic formulas for affine differential geometry. For more details, we refer to [11]. Let $M^n$ be a connected differentiable $n$-dimensional hypersurface of the equiaffine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection $D$ and a parallel volume element $\omega$, given by the determinant. We allow $M$ to be immersed by an immersion $f$, but we will not denote the immersion if there is no confusion possible. Let

$\xi$ be an arbitrary local transversal vector field to $f(M)$. For any vector fields $X, Y, X_1, \dots, X_n$, we write

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \tag{1}$$

$$\theta(X_1, \dots, X_n) = \omega(X_1, \dots, X_n, \xi), \tag{2}$$

thus defining an affine connection $\nabla$, a symmetric (0,2)-type tensor $h$, called the second fundamental form, and a volume element $\theta$. $M$ is said to be non-degenerate if $h$ is non-degenerate (and this condition is independent of the choice of transversal vector field $\xi$). If $M$ is non-degenerate it is known that there is a unique choice (up to sign) of transversal vector field such that the induced connection $\nabla$, the induced second fundamental form $h$ and the induced volume element $\theta$ satisfy the following conditions:

(i) $\nabla\theta = 0$

(ii) $\theta = \omega_h$,

where $\omega_h$ is the metric volume element induced by $h$. $\nabla$ is called the induced affine connection, $\xi$ the affine normal and $h$ the affine metric. By replacing $\xi$ by $-\xi$ if necessary, we may assume that the signature $s$ of the affine metric $h$ satisfies $2s \leq n$. Condition (i) implies that $D_X\xi$ is tangent to $M$ for any tangent vector $X$ to $M$. Hence we can define a (1,1)-tensor field $S$, called the affine shape operator, by $D_X\xi = -SX$. If $S = HI$, then $M$ is called an affine sphere with affine mean curvature $H$. If $M$ is an affine sphere and $n \geq 2$ then $H$ is constant. $M$ is called a proper affine sphere if $H \neq 0$ and an improper affine sphere if $H = 0$.

Let $\widehat{\nabla}$ denote the Levi Civita connection of the affine metric $h$. The difference tensor $K$ is defined by $K(X, Y) = K_X Y = \nabla_X Y - \widehat{\nabla}_X Y$ for tangent vector fields $X$ and $Y$. Notice that $K$ is symmetric in $X$ and $Y$. From (i) and (ii), we deduce

$$\operatorname{trace} K_X = 0 \text{ (apolarity condition)}.$$

If we define the cubic form $C$ by $C(X, Y, Z) = (\nabla h)(X, Y, Z)$, then the Codazzi equation says that $C$ is totally symmetric. Moreover, we have the following relation

$$h(K_X Y, Z) = -\tfrac{1}{2} C(X, Y, Z),$$

such that $K_X$ is a symmetric operator w.r.t. $h$. The Pick invariant $J$ is defined by $J = \frac{1}{n(n-1)} h(K, K)$. It is then well known that the basic equations for an affine sphere with constant sectional curvature $c$ are given by

$$h(K(X,Y),Z) \quad \text{is symmetric in } X,\ Y \text{ and } Z \tag{3}$$

$$\operatorname{trace} K_X = 0 \quad \text{for every X} \tag{4}$$

$$[K_X, K_Y]Z = a(h(Y,Z)X - h(X,Z)Y), \tag{5}$$

$$(\widehat{\nabla}_X K)(Y,Z) = (\widehat{\nabla}_Y K)(X,Z), \tag{6}$$

$$(M,h) \text{ is a space of constant sectional curvature } c \tag{7}$$

where $a = -J = H - c$.

## 3. An algebraic classification

In this section, we will work at a point $p$ and we will assume that we have on the tangent space, which is an $n$-dimensional vector space $\mathbb{R}^n$ a metric $h$ with signature $s$, $2s \leq n$ and a tensor $K$ satisfying (3), (4) and (5) with $a \neq 0$. Our purpose in this section will be to classify all such metrics and tensors in terms of the non-zero number $a$ and the signature $s$ of the metric. The purpose of this section is to show the following result:

**Theorem 3.** *Let $a$ be a nonzero number, let $n$ be the dimension and let $s$ be a nonnegative integer number with $n - 2s \geq 0$. Then*

*(i) If $n - 2s > 1$ and $a$ is negative, then there does not exist a metric $h$ (with signature $s$) and a tensor $K$ on $\mathbb{R}^n$ which satisfy (3), (4) and (5),*

*(ii) Otherwise, solutions $h$ (with index $s$) and $K$ do exist. Moreover any two solutions $(h,K)$ and $(\tilde{h},\tilde{K})$ are related by*

$$\tilde{h}(x,y) = h(Ax, Ay), \tag{8}$$

$$\tilde{K}(x,y) = A^{-1}K(Ax, Ay), \tag{9}$$

*where $A$ is an arbitrary change of basis of our vector space.*

It is straightforward to show that if we have a solution, a new solution can be constructed by the procedure described in the above theorem. The proof of the existence and uniqueness is much more difficult and will be divided into several lemmas throughout this section. The idea of the proof will be to show that, if $n - 2s < 1$ or $a < 0$, we can construct a special basis $\{e_1, \dots, e_n\}$ such that

$$h(e_i, e_j) = h_{ij} \tag{10}$$

$$h(K(e_i, e_j), e_k) = K_{ij}^k \tag{11}$$

where $h_{ij}$ and $K_{ij}^k$ are numbers determined by $a$ and the signature of the metric.

In case that the metric is positive definite, i.e., $s = 0$, this result was obtained in [6], see also [8] and [7]:

**Lemma 1.** *Assume that $s = 0$, i.e., the metric $h$ is positive definite. Then $a$ has to be negative. Moreover there exists a basis $\{e_1, \dots, e_n\}$ such that*

$$h(e_i, e_j) = \delta_{ij}, \tag{12}$$

*and such that*

$$h(K(e_i, e_i), e_i) = (n - i)\sqrt{-\tfrac{a_i}{n-i+1}} \tag{13}$$

$$h(K(e_i, e_i), e_j) = \begin{cases} 0 & j > i \\ -\sqrt{-\tfrac{a_j}{n-i+1}} & i > j \end{cases} \tag{14}$$

$$h(K(e_i, e_j), e_k) = 0, \tag{15}$$

*for mutually different indices $i$, $j$, $k$ and the numbers $a_i$ are defined by*

$$a_i = \tfrac{(n+1)}{(n-i+2)} a.$$

One of the tools used to prove the above lemma in [8] is the following reduction lemma, of which the proof is straightforward:

**Lemma 2.** *Let $V$ be a $k$-dimensional vector subspace of $\mathbb{R}^n$ and assume that*

*(i) the metric $h$ restricted to the vector space $V$ is non-degenerate,*

*(ii) there exist a one-form $\mu$ defined on $V$, such that*

$$K(v, w) = \mu(v)w,$$

*where $v \in V$ and $w \in V^\perp$,*

*(iii)* $K(V,V) \subset V$, *i.e., for each* $v_1, v_2 \in V$, *we have that* $K(v_1, v_2) \in V$,

*(iv)* $K$ *restricted to vectors belonging to* $V$ *satisfies* (3) *and* (5).

*Then if* $K$ *satisfies* (3), (4), (5), *then the bilinear mapping* $K^*$, *defined on* $V^\perp$ *by*

$K^*(w_1, w_2)$ *is the* $V^\perp$ *component of* $K(w_1, w_2)$, *for* $w_1, w_2 \in V^\perp$

*satisfies*

$h(K^*(x,y),z)$ *is symmetric in* $x, y, z \in V^\perp$
*the trace of the linear operator* $y \mapsto K^*(x,y)$ *vanishes for every vector* $x$ *in* $V^\perp$,
$[K^*_x, K^*_y]z = a^*(h(y,z)x - h(x,z)y)$,

*where* $a^*$ *is related to* $a$ *by*

$$a^* = a - h^V(\mu, \mu), \tag{16}$$

*where* $h^V(\mu,\mu)$ *denotes the length of the* 1*-form* $\mu$ *restricted to the space* $V$.

The structure of the proof will now be that we first construct a $2s$ dimensional vector subspace with index $s$ which satisfies the conditions of Lemma 2, after which applying Lemma 1 to the orthogonal complement will complete the proof. In order to construct our vector space, we will from now on assume that $s > 0$. The main tool in our proof will be the study of null vectors, i.e., non-zero vectors $u$ with $h(u,u) = 0$ and null directions. We say that 2 null vectors $u$ and $w$ determine the same null direction if there exists a positive number $\lambda$ such that $u = \lambda w$. It is clear that the set of all nulldirections (equipped with the quotient topology) is a compact set.

First, we need the following technical lemma which can be seen as an extension of Lemma 3.1 of [7] and which will allow us to construct differentiable submanifolds (without singularities) in each tangent space. Before formulating the lemma, we introduce some notation first. For a fixed vector $v$, we define the linear operators $K_v^0 w = w$ and $K_v^{i+1} w = K(v, K_v^i w)$.

**Lemma 3.** *There does not exist a (nonzero) vector* $v$ *such that*

$$h(K_v^i(v), v) = 0,$$

*for all $i \leq k$ and such that*

$$K_v^j(v) \text{ is linearly dependent of } v, K(v,v), \dots, K_v^{j-1}(v),$$

*where $j$ is any index between 1 and $k+1$.*

*Proof.* Instead of proving this theorem over the real numbers, we will prove it over the complex numbers. It is easily seen that this implies the result over the reals too. First we prove the result for $k=0$, i.e we want to show that there does not exist a null vector $v \in T_pM$ such that $K(v,v)$ is a multiple of $v$. Assuming that there exists such a vector $v$, we proceed as in Lemma 3.1 of [7] to show that then $K(v,v)=0$. We now take a complementary vector $u$ to $v$, i.e., $h(u,u)=0$ and $h(u,v)=1$. Then, we have that

$$K_vK_vu = K_vK_uv = [K_v, K_u]v = ah(u,v)v,$$

We now write $w = K_vu$ and we have $h(v,v)=0$, $h(v,w)=h(v,K_vu)=h(K_vv,u)=0$, and

$$h(w,w) = h(K_vu, K_vu) = h(K_vK_vu, u) = ah(u,v)^2 = a.$$

Since $a \neq 0$, the above formulas imply that the space $W$ spanned by $u$, $v$ and $w$ is a nondegenerate, 3-dimensional space which is invariant under $K_v$. Since $K_v$ is a symmetric operator, also $W^\perp$ is invariant under $K_v$. Notice that we can rechoose $u$ such that moreover $h(u, K_vu)=0$.

Since $K_v$ is symmetric, it is clear that we can divide $W^\perp$ up into say $i_1$ Jordan blocks of say length $r_{i_1}$. Say that $u_1, \dots, u_{r_i}$ are a Jordan basis for one of the blocks, i.e., $K(v, u_{i_2}) = u_{i_2+1}$, where we assume that $u_{r_i+1} = u_{r_i+2} = \dots = 0$ and we used the fact that $K_v$ has only the zero eigenvalue. Then, we have from (5) that

$$0 = [K_v, K_{u_1}]v = K_vu_2 = u_3.$$

This means that each Jordan block is either 1-dimensional or 2-dimensional. Elementary linear algebra now shows that we can choose the blocks in such a way that they are all mutually orthogonal and such that if $u_1$ spans a 1-dimensional block, then $h(u_1,u_1) \neq 0$ and if $u_2$ and $u_3$ span a 2-dimensional block with $K_vu_2 = u_3$, then $h(u_2,u_2) = h(u_3,u_3) = 0$ and $h(u_2,u_3) \neq 0$.

Then, we have, in case $u_1$ spans a 1-dimensional Jordan block that

$$-au_1 = [K_u, K_{u_1}]v = -K_{u_1}w,$$

and in case that $u_2$ and $u_3$ span a 2-dimensional block, with $h(u_2, u_2) = h(u_3, u_3) = 0$ and $h(u_2, u_3) \neq 0$, we get that

$$-au_3 = [K_u, K_{u_3}]v = -K_{u_3}w.$$

Hence $h(K_w u_3, u_2) = h(K_w u_2, u_3) = ah(u_2, u_3)$, implying that $a$ is the component of $K_w u_2$ in the direction of $u_2$ as well as the component of $K_w u_3$ in the direction of $u_3$.

Finally, we have

$$-aw = [K_u, K_w]v = K_u(av) - K_w(w) = aw - K_w w.$$

So, we have that

$a$ is the component of $K_w u_2$ in the direction of $u_2$, $\{u_2, u_3\}$ block of dimension 2

$a$ is the component of $K_w u_3$ in the direction of $u_3$, $\{u_2, u_3\}$ block of dimension 2

$a$ is the component of $K_w u_1$ in the direction of $u_1$, $\{u_1\}$ block of dimension 1

$2a$ is the component of $K_w w$ in the direction of $w$

$a$ is the component of $K_w v$ in the direction of $v$

$a$ is the component of $K_w u$ in the direction of $u$.

Hence the apolarity condition for $K_w$ yields $0 = (n + 1)a$. Hence $a = 0$, which is a contradiction.

We now proceed to the general case and we suppose that the result is not true. Call $\ell$ the smallest number such that $K_v^\ell v$ is linearly dependent from $v, K(v, v), \ldots K_v^{\ell-1}v$. Since we assumed the result to be false we must have that $\ell \leq k + 1$ and because of the previous part of the proof we must have that $\ell > 1$. So, we can write

$$K_v^\ell v = \sum_{i=0}^{\ell-1} a_i K_v^i v. \tag{17}$$

However this allows us to show that

$$h(v, K_v^{i_1} v) = 0, \tag{18}$$

for all numbers $i_1$. Hence the space spanned by $v, \ldots, K_v^{\ell-1}v$, which we denote by $V$ is a null-space, i.e., the inner product of any two elements in this space equals zero. It now follows from (5) that $K(K_v^{i_1}v, K_v^{i_2}v) = K_v^{i_1+i_2+1}v$ So, if we write $u = xv + K(v,v)$, where since $\ell > 1$ we have that $v$ and $K(v,v)$ are linearly independent, we get that

$$\begin{aligned} K(u,u) &= x^2K(v,v) + 2xK(v,K(v,v)) + K(K(v,v),K(v,v)) \\ &= x^2K_vv + 2xK_v^2v + K_v^3v \\ K_u^{i_1}u &= \sum_{i_2=0}^{i_1+1} x^{i_1-i_2+1}\binom{i_1+1}{i_2}K_v^{i_1+i_2}v \\ K_u^{\ell-1}u &= \sum_{i_2=0}^{\ell} x^{\ell-i_2}\binom{\ell}{i_2}K_v^{\ell+i_2-1}v. \end{aligned}$$

In the above equations, we can use (17) in order to express everything as a linear combination of the linearly independent vectors $v, K(v,v), \dots, K_v^{\ell-1}v$. Sorting with respect to powers of $x$, we get that the condition that $u, \dots, K_u^{\ell-1}u$ are linearly dependent reduces to a non-trivial polynomial equation in $x$ of degree $1 + 2 + \cdots + \ell$. Since every polynomial over the complex numbers has roots, we can find an $x$ such that $u, K_uu, \dots, K_u^{\ell-1}u$ are linearly independent. Since $v$ and $K(v,v)$ are linearly independent, $u$ is non-zero and since $u, K_uu, \dots, K_u^iu \in V$, which is a nullspace, $u$ satisfies the conditions of the lemma. This yields a contradiction. q.e.d.

We now introduce several sets $M_j$ by

$$M_j = \{v|0 \neq v \text{ and } h(v, K_v^i(v)) = 0, \forall i \leq j\}. \tag{19}$$

For example, we have that $M_0$ is the set of all nullvectors. Clearly, we also have that $M_{j+1} \subset M_j$. By $N_j$ we denote the corresponding set of null-directions.

Before showing now that the sets are actually differentiable manifolds, we need the following technical lemma.

**Lemma 4.** *Let $v$ be a nonzero vector such that $h(v, K_v^i(v)) = 0$ for all $i \leq j$. Then, for any vector $u$, we have that*

$$h(u, K_v^\ell(v)) = h(v, K_v^\ell(u)) = h(v, K_v^{\ell-k}K_uK_v^{k-1}(v)),$$

*for $\ell \leq j+2$ and $1 \leq k \leq \ell$.*

Then, we have:

**Theorem 4.** *If $M_j$ is not the empty set, then $M_j$ defines an $(n-j-1)$-dimensional manifold.*

*Proof.* Let $v \in M_j$. Denote by $k$ the smallest index such that $h(v, K_v^k v) = \alpha \neq 0$. From Lemma 3 such $k$ must exist and because $v \in M_j$, we also have that $k > j$. Lemma 3 implies that the vectors $v, K(v,v), \ldots, K_v^k v$ are linearly independent vectors and thus span a $k+1$-dimensional vector subspace of $V$. We now define vectors $e_i$ by

$$e_i = K_v^{i-1} v,$$

for $i = 1, \ldots, k+1$. Since $K_v$ is a symmetric operator it follows that

$$\begin{aligned} h(e_{i_1}, e_{i_2}) &= h(K_v^{i_1-1} v, K_v^{i_2-1} v) \\ &= h(v, K_v^{i_1+i_2-2} v) = 0, \quad i_1 + i_2 < k+2 \end{aligned} \tag{20}$$

$$\begin{aligned} h(e_{i_1}, e_{i_2}) &= h(K_v^{i_1-1} v, K_v^{i_2-1} v) \\ &= h(v, K_v^{i_1+i_2-2} v) = \alpha, \quad i_1 + i_2 = k+2, \end{aligned} \tag{21}$$

So we see that the vector space $V$ is a nondegenerate subspace of $\mathbb{R}^n$. This implies that the $\mathbb{R}^n$ can be written as the direct sum of $V$ and $V^{\perp}$. We take now for $e_{k+2}, \ldots, e_n$ an arbitrary orthonormal basis (i.e., $h(e_{i_2}, e_{i_3}) = \epsilon_i \delta_{i_2 i_3}$, $i_2, i_3 > k+1$), of $V^{\perp}$.

Now, we write

$$w = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n, \tag{22}$$

and define functions by $f_i(y_1, \ldots, y_n) = h(K_w^{i-1} w, w)$ Using Lemma 4, we get that the $(j+1) \times n$ matrix $\left[\frac{\partial f_i}{\partial y_\ell}\right]_{(1,0,\ldots,0)} = \left[m_{i\ell}\right]$ has the following properties:

(i) $m_{i\ell} = 0$, $\ell > k+1$,

(ii) $m_{i(k+2-i)} = (i+1)\alpha$,

(iii) $m_{i\ell} = 0$, $\ell < k+2-i$

Since the above system has maximal rank, the implicit function theorem shows that $M_j$ is an $(n-j-1)$-dimensional differentiable manifold.

q.e.d.

Therefore, if we denote by $q$ the greatest index such that $M_q$ is non empty and because all the considered spaces have different dimensions, we get the following inclusions,

$$\emptyset \subsetneq M_q \subsetneq M_{q-1} \subsetneq \cdots \subsetneq M_0 \tag{23}$$

It is clear that $q$ is limited by the index of the metric. Indeed, suppose that $M_{2s} \neq \emptyset$ and take $v \in M_{2s}$. Then as shown before, the space spanned by $v, K_v v, \ldots, K_v^s v$ determines an $s+1$-dimensional null space. This contradicts the fact that the index of the metric equals $s$.

We want to show now that $M_{2s-2} \neq \emptyset$. In order to do so, we consider the largest even number $2k$ such that $M_{2k} \neq \emptyset$. We define a series of functions by

$$f_i(v) = h(K_v^{i-1} v, v).$$

We now want to define a suitable function on a compact set. For this purpose we consider several cases.

(i) First we assume that $f_{2k+2}$ never vanishes on a connected component of $M_{2k}$. If so, we will restrict ourselves to that connected component and define a function $g$ on it by

$$g(v) = \frac{f_{2k+3}(v)}{f_{2k+2}(v)^{\frac{2k+4}{2k+3}}}$$

Since for $\lambda$ a positive number, we have that $g(\lambda v) = g(v)$ and since $g$ is continuous, it follows that $g$ attains an absolute maximum (and thus also a relative maximum) on our connected component.

(ii) If $f_{2k+3} > 0$ on a connected component of $M_{2k}$, we again restrict to this connected component and we define a function $g$ on it by

$$g(v) = \frac{f_{2k+2}(v)}{f_{2k+3}(v)^{\frac{2k+3}{2k+4}}}.$$

Similarly as before, we get that $g$ attains an absolute maximum and an absolute minimum on this connected component. Since the points for which $f_{2k+2}$ vanish form a lower dimensional differentiable manifold, at least one of those two numbers must be different from zero.

(iii) If $f_{2k+3} < 0$ on a connected component of $M_{2k}$, we again restrict to this connected component and we define a function $g$ on it by

$$g(v) = \frac{f_{2k+2}(v)}{(-f_{2k+3}(v))^{\frac{2k+3}{2k+4}}}.$$

Similarly as before, we get that $g$ attains an absolute maximum and an absolute minimum on this connected component. Since the points for which $f_{2k+2}$ vanish form a lower dimensional differentiable manifold, at least one of those two numbers must be different from zero.

(iv) If $f_{2k+3} \geq 0$ on a connected component of $M_{2k}$ and there exists a $v$ such that $f_{2k+3}(v) = 0$. Then $f_{2k+2}(v) \neq 0$. Choose $\epsilon = \pm 1$ such that $\epsilon f_{2k+2}(v) > 0$. On a neighborhood of $v$ in $M_{2k}$, we now define $g$ by

$$g(w) = \frac{f_{2k+3}(w)}{(\epsilon f_{2k+2}(w))^{\frac{2k+4}{2k+3}}}$$

Clearly $g$ has a relative minimum in $v$.

(v) If $f_{2k+3} \leq 0$ on a connected component of $M_{2k}$ and there exists a $v$ such that $f_{2k+3}(v) = 0$. Then $f_{2k+2}(v) \neq 0$. Choose $\epsilon = \pm 1$ such that $\epsilon f_{2k+2}(v) > 0$. On a neighborhood of $v$ in $M_{2k}$, we now define $g$ by

$$g(w) = \frac{f_{2k+3}(w)}{(\epsilon f_{2k+2}(w))^{\frac{2k+4}{2k+3}}}$$

Clearly $g$ has a relative maximum in $v$

(vi) We assume now that none of the above cases are satisfied. In particular, this implies that $M_{2k+1}$ is not the empty set. We also know that for every $v \in M_{2k+1}$, we have that $f_{2k+3}(v) \neq 0$. If $f_{2k+3}(v) > 0$ for every $v \in M_{2k+1}$, we define on the nonempty closed subset

$$A = \{v \in M_{2k} | f_{2k+3}(v) \leq 0\},$$

a function $g$ by

$$g(v) = \frac{f_{2k+3}(v)}{f_{2k+2}(v)^{\frac{2k+4}{2k+3}}}$$

Notice that if $f_{2k+2}(v) = 0$, it follows from our assumptions that $v \notin A$. Hence, the function $g$ is well defined on $A$. Therefore $g$ attains an absolute maximum and absolute minimum on $A$. In view of the previous cases, we know that there exist a $v$ such that $g(v) \neq 0$. Hence either the absolute maximum or the absolute minimum is different from zero, and thus occurs at an interior point of $A$. This implies that the function (defined on an open subset of $M_{2k}$ also has a relative maximum or minimum in that point. Similarly, if $f_{2k+3}(v) < 0$ for every $v \in M_{2k+1}$, we define on the nonempty closed subset

$$A = \{v \in M_{2k} | f_{2k+3}(v) \geq 0\},$$

a function $g$ by

$$g(v)=\frac{f_{2k+3}(v)}{f_{2k+2}(v)^{\frac{2k+4}{2k+3}}}$$

Notice that if $f_{2k+2}(v)=0$, it follows from our assumptions that $v\notin A$. Hence, the function $g$ is well defined on $A$. Therefore, $g$ attains an absolute maximum and absolute minimum on $A$. In view of the previous cases, we know that there exist a $v$ such that $g(v)\neq 0$. Hence, either the absolute maximum or the absolute minimum is different from zero, and thus occurs at an interior point of $A$. This implies that the function (defined on an open subset of $M_{2k}$) also has a relative maximum or minimum in that point.

(vii) Finally, we consider again that none of the above cases are satisfied. As before we have that then $M_{2k+1}$ is a differentiable manifold. We also have that $f_{2k+3}$ is nowhere zero on $M_{2k+1}$. By exclusion of the cases, we can write $M_{2k+1}$ as the disjoint union of two open (and closed) sets $A$ and $B$, where

$$A=\{v\in M_{2k+1}|af_{2k+3}(v)>0\},$$

and

$$B=\{v\in M_{2k+1}|af_{2k+3}(v)<0\}.$$

We define a function $g$ on $A$ by

$$g(v)=\frac{f_{2k+4}(v)}{(af_{2k+3}(v))^{\frac{2k+5}{2k+4}}}.$$

As before, $g$ attains an absolute maximum and an absolute minimum on $A$.

Remark that if $v\in M_{2k+1}$, that $v,\ldots,K_v^{2k+2}v$ are $2k+3$ linearly independent vectors whose metric components form an lowerdiagonal matrix with codiagonal entries given by $h(v,K_v^{2k+2}v)$, the sign of which determines the index of this subspace, which is different depending on whether $v\in A$ or $v\in B$. Therefore, in order for this case to occur, we must have that $n>2k+3$

Let $\ell=2k$ if Case (i) upto case (vi) is satisfied and let $\ell=2k+1$ if Case (vii) is satisfied. Denote by $v$ the vector which was constructed in all of the cases. Then, we have that $f_{\ell+2}(v)\neq 0$. Since $f_{\ell+2}(\lambda v)=\lambda^{\ell+3}f_{\ell+2}(v)$ it follows that we can rescale $v$ such that $f_{\ell+2}(v)=\epsilon$, where

$\epsilon = 1$, unless we have Case (vii) and $a$ is negative. In that case, we have $\epsilon = -1$.

We now proceed to construct a special basis. From Lemma 3, it follows that $v, K_v v, \dots, K_v^{\ell+1} v$ are linearly independent vectors which span a vector space $V$. As before, we introduce a basis $e_1, \dots, e_{\ell+2}$ by $e_i = K_v^{i-1} v$, for $i = 1, \dots, \ell+2$. As before, we get that $V$ is nondegenerate and we decompose $\mathbb{R}^n$ as the direct sum of $V$ and $V^\perp$. At the moment, we take for $e_{\ell+3}, \dots, e_n$ an arbitrary orthonormal basis. It now follows from the proof of Theorem 4 that $T_v M_\ell$ is spanned by $e_1, e_{\ell+3}, \dots, e_n$.

In the same way, we can show that there exists a neighborhood of $v$ in $M_\ell$ such that the vectors $u$ which belong to this neighborhood and satisfy $f_{\ell+2}(u) = \epsilon$ define an $(n-\ell+2)$-dimensional differentiable manifold (in a neighborhood of $v$). If we denote this manifold by $M_\ell^*$, then it is straightforward to check that $T_v M_\ell^*$ is spanned by $e_{\ell+3}, \dots, e_n$. Thus in a neighborhood of $v$, $M_\ell^*$ is a semi-Riemannian differentiable manifold.

We now need again another technical lemma of which the proof is straightforward.

**Lemma 5.** *Assume that $u, w \in \{e_1, \dots, e_{\ell+2}\}^\perp$. Then, we have*

$$K_v^{i_2} K_u K_v^{i_1} v = K_v^{i_1+i_2+1} u, \tag{24}$$

$$h(u, K_v^{\ell+2}(v)) = h(v, K_v^{\ell+2}(u)) = h(v, K_v^{\ell+2-i_3} K_u K_v^{i_3-1}(v)), \tag{25}$$

$$h(u, K_v^i w) = h(w, K_v^i u), \tag{26}$$

$$= h(w, K_v^{i_4} K_u K_v^{i-i_4-1} v), \tag{27}$$

$$= h(u, K_v^{i_4} K_w K_v^{i-i_4-1} v), \tag{28}$$

$$= h(v, K_v^{i_4} K_u K_v^{i-i_4-1} w), \tag{29}$$

$$= h(v, K_v^{i_4} K_w K_v^{i-i_4-1} u), \tag{30}$$

$$= h(v, K_v^{i_5} K_w K_v^{i_6} K_u K_v^{i-i_5-i_6-2} v), \tag{31}$$

$$= h(v, K_v^{i_5} K_u K_v^{i_6} K_w K_v^{i-i_5-i_6-2} v), \tag{32}$$

**Theorem 5.** *We have that $K_v^{\ell+2} v$ is a linear combination of $e_1, \dots, e_{\ell+2}$, i.e., there exist numbers $a_1, \dots, a_{\ell+2}$ such that $K_v^{\ell+2} v = \sum_{i=1}^{\ell+2} a_i e_{\ell+3-i}$.*

*Proof.* We know that the tangent space to $M_\ell^*$ at $v$ is spanned by $e_{\ell+3}, \dots, e_n$. Let $u$ be a vector in the tangent space and $f$ be an arbitrary function, which locally extends to a function $\tilde{f}$ on $\mathbb{R}^n$. Elementary differential geometry gives that $u(f) = \frac{d}{dt} f(\alpha(t))|_{t=0} = \frac{d}{dt}\tilde{f}(v+tu)|_{t=0}$, where $\alpha$ is an arbitrary curve on $M_\ell$ with $\alpha(0) = v$ and $\alpha'(0) = u$.

Let $u \in \operatorname{span}\{e_{\ell+3}, \dots, e_n\}$. Then using Lemma 4 and 5, we obtain that

$$\begin{aligned} f_{\ell+2}(v+tu) &= f_{\ell+2}(v) + (\ell+3)t\, h(u, K_v^{\ell+1} v) + O(t^2) \\ &= f_{\ell+2}(v) + (\ell+3)t\, h(u, e_{\ell+2}) + O(t^2) \\ &= f_{\ell+2}(v) + O(t^2), \\ f_{\ell+3}(v+tu) &= f_{\ell+3}(v) + (\ell+4)t\, h(u, K_v^{\ell+2} v) + O(t^2), \end{aligned}$$

from which it follows that

$$u(f_{\ell+2}) = 0, \tag{33}$$

$$u(f_{\ell+3}) = (\ell+4) h(u, K_v^{\ell+2} v) \tag{34}$$

Now, applying a case by case analysis of the definition of $g$ it follows that $u(g)$ vanishes if and only if $h(u, K_v^{\ell+2} v) = 0$, which completes the proof of the theorem. q.e.d.

Recall that for $i_1, i_2, j_1, j_2 \in \{1, \dots, \ell+2\}$, we have that

$$h_{i_1 i_2} = h(e_{i_1}, e_{i_2}) = 0, \qquad i_1 + i_2 \le \ell + 2 \tag{35}$$

$$h_{i_1 i_2} = h(e_{i_1}, e_{i_2}) = \epsilon, \qquad i_1 + i_2 = \ell + 3 \tag{36}$$

$$h_{i_1 i_2} = h(e_{i_1}, e_{i_2}) = h_{j_1 j_2} = h(e_{j_1}, e_{j_2}), \quad i_1 + i_2 = j_1 + j_2 \tag{37}$$

Using the previous lemma, the other components of the matrix $h = [h_{i_1 i_2}]$ can be inductively defined as follows. For $i_1 + i_2 > \ell + 3$, we have that

$$h(e_{i_1}, e_{i_2}) = h(K_{e_1} e_{\ell+2}, e_{i_1+i_2-\ell-3}) \tag{38}$$

$$= \sum_{i=1}^{\ell+2} a_i h(e_{\ell+3-i}, e_{i_1+i_2-\ell-3})), \tag{39}$$

However, for our purposes, it is more convenient to introduce a matrix $H = [H_{i_1 i_2}]$, $i_1, i_2 \in \{1, \dots, \ell+2\}$ by

$$H_{i_1 i_2} = 0, \qquad i_1 + i_2 \ge \ell + 4 \tag{40}$$

$$H_{i_1 i_2} = \epsilon, \qquad i_1 + i_2 = \ell + 3 \tag{41}$$

$$H_{i_1 i_2} = -\epsilon a_{\ell+3-i_1-i_2}, \quad i_1 + i_2 \le \ell + 2 \tag{42}$$

Notice that whereas the matrix $h$ is an lowerdiagonal matrix, the matrix $H$ is an upperdiagonal one. It is easy to show that $h.H = Id$.

Remark that we already know $K_{e_1}e_{i_2}$, for $i_2 \in \{1, \ldots, \ell+2\}$. As in the case of the metric, using those it is possible to compute all $K_{e_{i_1}}e_{i_2}$, $i_1, i_2 \in \{1, \ldots, \ell+2\}$, by induction on $i_1 + i_2$. Indeed, we have for $i_1, i_2 \in \{2, \ldots, \ell+2\}$ that

$$\begin{aligned} K_{e_{i_1}}e_{i_2} &= K_{e_{i_1}}K_{e_1}e_{i_2-1} \\ &= K_{e_1}K_{e_{i_1}}e_{i_2-1} + [K_{e_{i_1}}, K_{e_1}]e_{i_2-1} \\ &= K_{e_1}K_{e_{i_1}}e_{i_2-1} - ah(e_{i_1}, e_{i_2-1})e_1, \end{aligned}$$

which allows us to determine $K_{e_{i_1}}e_{i_2}$, $i_1, i_2 \in \{1, \ldots, \ell+2\}$, explicitly in terms of $a, a_1, \ldots, a_{\ell+2}$. In particular, we get that $K_{e_{i_1}}e_{i_2} \in V$, for $i_1, i_2 \in \{1, \ldots, \ell+2\}$. Moreover, we also deduce that if $i_1 + i_2 \leq \ell + 3$ that

$$K_{e_{i_1}}e_{i_2} = K_{e_1}^{i_1+i_2-1}e_1$$

Of course, as was the case also with the components of $h$, these expressions can be quite complicated. Fortunately, we can avoid making these computations. For our purposes, it will be sufficient to know that it can be done. However, we still need to compute the traces of the operators $K_{e_{i_1}}$ restricted to the vector space $V$. For that purpose let $x \in \operatorname{span}\{e_1, \ldots, e_{\ell+2}\}$ and denote by $\alpha(x) = \operatorname{trace}_V K_x$, the trace of $K_x$ restricted to the vector space $V$. Since $H$ is the inverse matrix of $h$, we have that

$$\begin{aligned} \alpha(x) &= \sum_{i_1,i_2=1}^{\ell+2} h(K_x e_{i_1}, e_{i_2})H_{i_2 i_1} \\ &= \sum_{i_1,i_2=1}^{\ell+2} h(x, K_{e_{i_1}}e_{i_2})H_{i_2 i_1} && i_1 + i_2 \leq \ell + 3 \\ &= \sum_{i_1,i_2=1}^{\ell+2} h(x, K_{e_1}^{i_1+i_2-1}e_1)H_{i_2 i_1} && i_1 + i_2 \leq \ell + 3 \\ &= \sum_{i=1}^{\ell+2} ih(x, K_{e_1}^{i}e_1)H_{1i} && i = i_1 + i_2 - 1 \\ &= \sum_{i=1}^{\ell+1} ih(x, e_{i+1})H_{1i} + (\ell+2)\epsilon \sum_{i=1}^{\ell+2} a_i h(x, e_{\ell+3-i}) \end{aligned}$$

In particular, if we denote $\alpha_{i_1} = \alpha(e_{i_1})$, we get that

$$\alpha_{i_1} = \sum_{i=\ell+2-i_1}^{\ell+1} ih(e_{i_1}, e_{i+1})H_{1i} + (\ell+2)\epsilon \sum_{i=1}^{i_1} a_i h(e_{i_1}, e_{\ell+3-i})$$

In particular, since $H_{1\ell+1} = -\epsilon a_1$, it follows that

$$\alpha_1 = (\ell+1)\epsilon(-\epsilon a_1) + (\ell+2)\epsilon a_1 \epsilon = a_1 \tag{43}$$

In general, using the induction hypothesis for $h$, we get that

$$\begin{aligned}
\alpha_{i_1} = &\sum_{i=\ell+3-i_1}^{\ell+1} ih(e_{i_1}, e_{i+1})H_{1i} + (\ell+2)\epsilon \sum_{i=1}^{i_1-1} a_i h(e_{i_1}, e_{\ell+3-i}) \\
&+ (\ell+2-i_1)\epsilon H_{1\ell+2-i_1} + (\ell+2)\epsilon^2 a_{i_1} \\
= &\, i_1 a_{i_1} + \sum_{i=\ell+3-i_1}^{\ell+1} \sum_{\substack{i_2=1 \\ i+i_1-i_2 \geq \ell+2}}^{i+i_1-\ell-2} iH_{1i} a_{i_2} h(e_{\ell+3-i_2}, e_{i+i_1-\ell-2}) \\
&+ (\ell+2)\epsilon \sum_{i=1}^{i_1-1} \sum_{\substack{i_2=1 \\ \ell+2-i+i_1-i_2 \geq \ell+2}}^{i_1-i} a_i a_{i_2} h(e_{\ell+3-i_2}, e_{i_1-i}) \\
= &\, i_1 a_{i_1} + \sum_{i_2=1}^{i_1-1} a_{i_2} \Big( \sum_{i=\ell+2+i_2-i_1}^{\ell+1} iH_{1i} h(e_{i_1-i_2}, e_{i+1}) \\
&+ (\ell+2)\epsilon \sum_{i=1}^{i_1-i_2} a_i h(e_{i_1-i_2}, e_{\ell+3-i}) \Big) \Big) \\
= &\, i_1 a_{i_1} + \sum_{i_2=1}^{i_1-1} a_{i_1-i_2} \alpha_{i_2}.
\end{aligned}$$

**Remark 1.** The above derived induction relation is a well-known one and it is called the Newton-formula. We look at the equation

$$x^{\ell+3} - \sum_{i=1}^{\ell+2} a_i x^{\ell+3-i} + a\epsilon = 0, \tag{44}$$

and we denote by $s_{i_1} = \sum_{i=1}^{\ell+3} \lambda_i^{i_1}$, where $\lambda_i$, $i = 1, \ldots, \ell+3$ are the roots of (44). Then, it is well known that $s_1 = a_1$. However, perhaps less well known are the Newton formulas, which express the other $s_i$ by induction, see for example [5]. Indeed, we have that

$$s_{i_1} = i_1 a_{i_1} + \sum_{i_2=1}^{i_1-1} a_{i_1-i_2} s_{i_2}, \tag{45}$$

for $i_1 \leq \ell+2$. It, therefore, follows that $\alpha_{i_1} = s_{i_1}$ and that these numbers can be characterised as the sum of powers of roots of a polynomial equation.

**Remark 2.** Since $v, \dots, K_v^{\ell+1} v$ are linearly independent, we must have that $n \geq \ell + 2$. Moreover, as explained before, if $n = \ell + 2$, then we must be in Case (i) to Case (vi), and thus $\ell = 2k$. In that case $v, \dots, K_v^{2k+1} v$ form a basis for the entire tangent space and we have seen how we can express both the components of the metric and the multilinear map $K$ in terms of $a, a_1, \dots, a_{\ell+2}$. Of course, in this case the apolarity conditions would imply that $\alpha_{i_1} = s_{i_1} = 0$. Since $s_1 = a_1$, this implies that $a_1 = 0$ and rewriting (45) as $i_1 a_{i_1} = s_{i_1} - \sum_{i_2=1}^{i_1-1} a_{i_1-i_2} s_{i_2}$, it also follows that $a_2, \dots, a_{\ell+2}$ vanish. Therefore, we have found a basis in which everything can be expressed in terms of the non-zero number $a$, and the index which since the matrix with respect to the basis $e_1, \dots, e_{2k+2}$ is an underdiagonal matrix with entries 1 on the codiagonal has to equal $k + 1$.

We now may assume that $n > \ell+2$, and therefore $V^\perp$ is a non-empty invariant subspace of $K_v$. This means that if necessary by complexifying, we can find a Jordan form for $K_v$ on $V^\perp$. So, we can divide $V^\perp$ up into say $i_1$ Jordan blocks of say length $r_{i_1}$. Assume that $u_1, \dots, u_{r_i}$ form a Jordan basis for one of those blocks (with eigenvalue $\lambda$), i.e., we have

$$K(v, u_{i_2}) = u_{i_2+1} + \lambda u_{i_2},$$

where we put $u_{r_i+1} = u_{r_i+2} = \cdots = 0$. Then, by induction we can prove the following lemma:

**Lemma 6.** *We have*

$$K_{e_{i_1}} u_i = \sum_{i_2=0}^{i_1} \binom{i_1}{i_2} \lambda^{i_1-i_2} u_{i+i_2},$$

*where* $i_1 \in \{1, \dots, \ell + 2\}$.

**Lemma 7.** *The eigenvalues of the linear operator $K_v$, restricted to $V^\perp$ are roots of the equation (44). Moreover, a Jordan block of dimension greater then or equal to two can only exist if (44) has a double root.*

*Proof.* Using the previous lemma, we see that

$$\begin{aligned} a\epsilon u_i &= [K_{u_i}, K_{e_{\ell+2}}]e_1 \\ &= K_{u_i} K_{e_1} e_{\ell+2} - K_{e_{\ell+2}} K_{e_1} u_i \\ &= K_{u_i}(\sum_{i_1=1}^{\ell+2} a_{i_1} e_{\ell+3-i_1}) - K_{e_{\ell+2}}(\lambda u_i + u_{i+1}) \\ &= \sum_{i_1=1}^{\ell+2} a_{i_1} \sum_{i_2=0}^{\ell+3-i_1} \binom{\ell+3-i_1}{i_2} \lambda^{\ell+3-i_1-i_2} u_{i+i_2} \\ &\quad - \lambda \sum_{i_2=0}^{\ell+2} \binom{\ell+2}{i_2} \lambda^{\ell+2-i_2} u_{i+i_2} - \sum_{i_2=0}^{\ell+2} \binom{\ell+2}{i_2} \lambda^{\ell+2-i_2} u_{i+i_2+1}. \end{aligned}$$

Taking $i = 1$ and looking at the $u_1$-component we get that

$$-\sum_{i_1=1}^{\ell+2} a_{i_1} \lambda^{\ell+3-i_1} + a\epsilon + \lambda^{\ell+3} = 0 \tag{46}$$

which shows that $\lambda$ is a root of the equation (44). If there exists a Jordan block of length greater then or equal to 2, we also have an $u_2$-component for this block which yields

$$0 = \sum_{i_1=1}^{\ell+2} a_{i_1}(\ell+3-i_1)\lambda^{\ell+2-i_1} - (\ell+3)\lambda^{\ell+2}, \tag{47}$$

and therefore implies that $\lambda$ is a double root of (44). q.e.d.

Denote by $\lambda_1, \dots, \lambda_{\ell+3}$ the roots of the equation (44) which are possibly complex numbers and let $m_{i_1}$ be the multiplicity that each root $\lambda_{i_1}$ of (44) appears as an eigenvalue of the linear operator $K_v$ restricted to $V^\perp$. Then, the apolarity conditions state that for $i \in \{1, \dots, \ell+2\}$, we have that

$$0 = \operatorname{trace} K_{e_i} = \alpha_i + \sum_{i_1=1}^{\ell+3} m_{i_1} \lambda_{i_1}^i = \sum_{i_1=1}^{\ell+3} (m_{i_1}+1)\lambda_{i_1}^i \tag{48}$$

**Lemma 8.** *The equation* (44) *has no double roots and therefore* $K_v$ *restricted to* $V^\perp$ *can be diagonalized over* $\mathbb{C}$.

*Proof.* Notice that $\lambda = 0$ is not a root of the equation (44). Let us assume that the equation (44) has $q$ different roots, which we may assume to be $\lambda_1, \dots, \lambda_q$, where $q < \ell+3$. Therefore, we can express (48) using only those different roots. Therefore, there exist positive natural numbers such that $\sum_{i_1=1}^{p} \tilde{m}_{i_1} \lambda_{i_1}^{i} = 0$, for $i = 1, \dots, \ell+2$. Notice that since $q < \ell + 3$, we can interprete this as a system of linear equations in $\tilde{m}_i$ with at least as many equations as we have unknowns. The determinant of the first $q$ equations is a determinant of Vandermonde, which since zero is not a root and all remaining $\lambda$'s are different is different from zero. Therefore, this system should only have the trivial zero solution. This contradicts the fact that the $\tilde{m}_i$ are positive natural numbers. q.e.d.

**Lemma 9.** *If* $\ell = 2k$, *the equation* (44) *has only* 1 *real root, whereas if* $\ell = 2k+1$, *the equation* (44) *has no real roots.*

*Proof.* Denote by $2q$ the number of complex roots of the equations (44). Therefore, we can arrange the roots in such a way that $\lambda_{2i-1} = \bar{\lambda}_{2i}$, where $i = 1, \dots, q$ are the conjugate complex roots and $\lambda_{2q+1}, \dots, \lambda_{\ell+3}$ are the real roots of the equation. Notice also that, since $K_{e_1}\bar{v} = \overline{K_{e_1}v}$, the multiplicity with which a complex root and its conjugate occur are the same, i.e., we have that $m_{2i-1} = m_{2i}$. We consider now on $\mathbb{R}^{\ell+3}$, given by

$$\mathbb{R}^{\ell+3} = \{(x_1 + ix_2, x_3 + ix_4, \dots, x_{2q-1} + ix_{2q}, x_{2q+1}, \dots, x_{\ell+3})\}$$

the following metric:

$$< x, y >= \sum_{i_1=1}^{q} 2(m_{2i_1}+1)(x_{2i_1-1}y_{2i_1-1} - x_{2i_1}y_{2i_1}) + \sum_{i_1=2q+1}^{\ell+3} (m_{i_1}+1)x_{i_1}y_{i_1}.$$

The signature of this metric equals $q$ and therefore the maximal dimension of a nullspace is $q$. On the other hand, the vectors $z_{i_1}$ defined by

$$z_{i_1} = (\lambda_1^{i_2}, \lambda_3^{i_2}, \dots, \lambda_{2p-1}^{i_2}, \lambda_{2p+1}^{i_2}, \lambda_{2p+2}^{i_2}, \dots, \lambda_{\ell+3}^{i_2}),$$

where $i_2 = 1, \dots, \ell+2$ are linearly independent, since the determinant is a determinant of Vandermonde. By the apolarity conditions, see (48), we have that

$$< z_{i_1}, z_{i_2} >= \sum_{i_3=1}^{\ell+3} (m_{i_3}+1)\lambda_{i_3}^{i_1+i_2} = 0,$$

provided $i_1 + i_2 \leq \ell + 2$.

Now, we consider two cases. If $\ell = 2k$, we have that $z_1, \dots, z_{k+1}$ span a $k+1$-dimensional null-space. Since the metric has signature $q$, we have that $q \geq k+1$. On the other hand, we have that $\ell + 3 - 2q = 2k + 3 - 2p \geq 0$. Combining these, we get that $q = k+1$ and thus $\ell + 3 - 2p = 2k + 3 - 2k - 2 = 1$. Therefore, (44) has only 1 real root.

In the case that $\ell = 2k+1$, we proceed in the same way. However, in this case, we use that if $i_1 + i_2 = 2k + 4 = \ell + 3$, that we have that

$$\begin{aligned} < z_{i_1}, z_{i_2} > &= \sum_{i_3=1}^{\ell+3} (m_{i_3}+1)\lambda_{i_3}^{\ell+3} \\ &= \sum_{i_3=1}^{\ell+3} (m_{i_3}+1)(\sum_{i_2=1}^{\ell+2} a_{i_2}\lambda_{i_3}^{\ell+3-i_2} - a\epsilon) \\ &= -a\epsilon \sum_{i_3=1}^{\ell+3} (m_{i_3}+1) \\ &= -a\epsilon(n+1). \end{aligned}$$

So, we see that, with respect to the vectors $z_1, \dots, z_{2k+3}$, the metric is a lowerdiagonal matrix with negative entries (because of our choice of $\epsilon$) on the codiagonal. This means that the signature of this subspace equals $k+2$ and thus $k+2 \leq q$. On the other hand, $\ell+3-2q = 2k+4-2q \geq 0$, so we deduce that $q = k+2$ and that our equation does not have any real roots. q.e.d.

So far, we have only used the fact that the function $g$ had a critical value in our vector $v$. However, we also know that this critical value has to be a local minimum or a local maximum. This would imply that the matrix obtained by taking the second derivatives cannot be indefinite. Since $g$, considered as a function on $M_\ell$ attains a relative minimum or relative maximum in the vector $v$, it is clear that the function $g$, considered as a function on the semi-Riemannian manifold $M_\ell^*$ also attains a relative minimum or maximum at the vector $v$. Since the function $f_{\ell+2}$ is constant on $M_\ell^*$ this together with the definition of $g$ in the different cases implies that the function $f_{\ell+3}$, considered as a function on $M_\ell^*$ attains a relative minimum or maximum at the vector $v$.

We now use the exponential map and denote by

$$\kappa(y_{\ell+3},\ldots,y_n) = exp_v(\sum_{i_1=\ell+3}^{n} y_{i_1}e_{i_1}). \tag{49}$$

Then elementary differential geometry shows that $\kappa$ defines a local diffeomorphism between a neighborhood of $v$ in $M_\ell^*$ and an open part around the origin of $\mathbb{R}^{n-\ell-2}$. Since $\kappa$ is a local diffeomorphism, we know that $f_{\ell+3}\circ\kappa$ attains a relative minimum or relative maximum at the origin. In order to compute the type of the critical point, we now have to compute

$$\frac{\partial^2}{\partial y_{i_1}\partial y_{i_2}}(f_{\ell+3}\circ\kappa)|_0,$$

where $i_1, i_2 \in \{\ell+3,\ldots,n\}$.

Looking at $\kappa$ as a vector in $\mathbb{R}^n$, we know that

$$f_{i_1}(\kappa(y_{\ell+3},\ldots,y_n)) = 0, \tag{50}$$

$$f_{\ell+2}(\kappa(y_{\ell+3},\ldots,y_n)) = \epsilon, \tag{51}$$

for $i_1 = 1,\ldots,\ell+1$. Deriving the above equation, denoting by $\kappa^{i_1}$ the $i_1$-th component of the vector $\kappa$, we obtain that

$$\sum_{i_2=1}^{n} \frac{\partial f_{i_1}}{\partial x_{i_2}}\frac{\partial \kappa^{i_2}}{\partial y_\alpha} = 0,$$

and

$$\sum_{i_2,i_3=1}^{n} \frac{\partial^2 f_{i_1}}{\partial x_{i_2}\partial x_{i_3}}\frac{\partial \kappa^{i_2}}{\partial y_\alpha}\frac{\partial \kappa^{i_3}}{\partial y_\beta} + \sum_{i_2=1}^{n} \frac{\partial f_{i_1}}{\partial x_{i_2}}\frac{\partial^2 \kappa^{i_2}}{\partial y_\alpha \partial y_\beta} = 0$$

where $\alpha, \beta \in \{\ell+3,\ldots,n\}$.

Since by the definition of the exponential map, $\kappa(t,0,\ldots,0)$ is the geodesic through $v$ in the direction of $e_{\ell+3}$, we get that $\frac{\partial\kappa^{i_2}}{\partial y_\alpha}|_0 = \delta_{i_2\alpha}$, and since the exponential map provides parallel coordinates at $v$, we also have that $\frac{\partial^2\kappa}{\partial y_\alpha\partial y_\beta}|_0$ is normal to $T_vM_\ell^*$. Substituting these values in the previously obtained equations, we find that

$$\frac{\partial^2 f_{i_1}}{\partial x_\alpha \partial x_\beta}|_v = -\sum_{i_2=1}^{\ell+2} \frac{\partial f_{i_1}}{\partial x_{i_2}}|_v \frac{\partial^2 \kappa^{i_2}}{\partial y_\alpha \partial y_\beta}|_0 \tag{52}$$

Completely similarly, denoting by $F = f_{\ell+3} \circ \kappa$, we find that

$$\frac{\partial^2 F}{\partial y_\alpha \partial y_\beta}|_0 = \frac{\partial^2 f_{\ell+3}}{\partial x_\alpha \partial x_\beta}|_v + \sum_{i_2=1}^{\ell+2} \frac{\partial f_{\ell+3}}{\partial x_{i_2}}|_v \frac{\partial^2 \kappa^{i_2}}{\partial y_\alpha \partial y_\beta}|_0 \tag{53}$$

Therefore, in order to obtain $\frac{\partial^2 F}{\partial y_\alpha \partial y_\beta}$, we need to compute $\frac{\partial f_{i_1}}{\partial x_{i_2}}|_v$, $\frac{\partial f_{\ell+3}}{\partial x_{i_2}}|_v$, $\frac{\partial^2 f_{i_1}}{\partial x_\alpha \partial x_\beta}|_v$ and $\frac{\partial^2 f_{\ell+3}}{\partial x_\alpha \partial x_\beta}|_v$, where $i_2 = 1, \ldots, n$, $\alpha, \beta = \ell+3, \ldots, n$ and $i_1 = 1, \ldots, \ell+2$. This can be done in a straightforward way using Lemma 4 and Lemma 5. It follows

$$\begin{aligned}
\frac{\partial f_{i_1}}{\partial x_{i_2}}|_v &= (i_1+1) h(e_{i_2}, e_{i_1}),\\
\frac{\partial f_{\ell+3}}{\partial x_{i_2}}|_v &= (\ell+4) \sum_{i_1=1}^{\ell+2} a_{i_1} h(e_{i_2}, e_{\ell+3-i_1})\\
\frac{\partial^2 f_{i_1}}{\partial x_\alpha \partial x_\beta}|_v &= (i_1+1) i_1 h(e_\alpha, K_{e_1}^{i_1-1} e_\beta)\\
\frac{\partial^2 f_{\ell+3}}{\partial x_\alpha \partial x_\beta}|_v &= (\ell+4)(\ell+3) h(e_\alpha, K_{e_1}^{\ell+2} e_\beta).
\end{aligned}$$

Therefore, we get that (52) reduces to

$$i_1 h(e_\alpha, K_{e_1}^{i_1-1} e_\beta) = -\sum_{i_2=1}^{\ell+2} h(e_{i_2}, e_{i_1}) \frac{\partial^2 \kappa^{i_2}}{\partial y_\alpha \partial y_\beta}|_0. \tag{54}$$

Since $h$ and $H$ are each others inverse, we can still rewrite (54) as

$$\frac{\partial^2 \kappa^{i_3}}{\partial y_\alpha \partial y_\beta}|_0 = -\sum_{i_1=1}^{\ell+2} i_1 H_{i_1 i_3} h(e_\alpha, K_{e_1}^{i_1-1} e_\beta) \tag{55}$$

Combining now all of these previous results, we find that

$$\begin{aligned}
\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}|_0 &= (\ell+4)\{(\ell+3) h(e_\alpha, K_{e_1}^{\ell+2} e_\beta)\\
&\quad - \sum_{i_2=1}^{\ell+2} \sum_{i_1=1}^{\ell+2} a_{i_1} h(e_{i_2}, e_{\ell+3-i_1}) (\sum_{i_3=1}^{\ell+2} i_3 H_{i_2 i_3} h(e_\alpha, K_{e_1}^{i_3-1} e_\beta))\}\\
&= (\ell+4)\{(\ell+3) h(e_\alpha, K_{e_1}^{\ell+2} e_\beta)\\
&\quad - \sum_{i_3=1}^{\ell+2} i_3 h(e_\alpha, K_{e_1}^{i_3-1} e_\beta)) (\sum_{i_1, i_2=1}^{\ell+2} a_{i_1} h_{i_2 \ell+3-i_1} H_{i_3 i_2})\}
\end{aligned}$$

$$= (\ell+4)\{(\ell+3)h(e_\alpha, K_{e_1}^{\ell+2}e_\beta) \\ - \sum_{i_3=1}^{\ell+2} i_3 h(e_\alpha, K_{e_1}^{i_3-1}e_\beta))a_{\ell+3-i_3}\}$$

Now, we can obtain the following lemmas by choosing the remaining part of the basis $e_{\ell+3}, \dots, e_n$ appropriately.

**Lemma 10.** *The linear operator $K_v$ restricted to $V^\perp$ cannot have a complex eigenvalue*

*Proof.* We suppose that it has a complex eigenvalue $\mu = \mu_1 + i\mu_2$. It is easy to check that then there exists real orthogonal vectors $v_1$ and $v_2$ with $h(v_1, v_1) = 1 = -h(v_2, v_2)$ and such that

$$K_v v_1 = \mu_1 v_1 - \mu_2 v_2$$
$$K_v v_2 = \mu_2 v_1 + \mu_1 v_2$$

We choose $e_{\ell+3} = v_1$, $e_{\ell+4} = v_2$. In $(V \oplus \{e_{\ell+3}, e_{\ell+4}\})^\perp$ we take an arbitrary orthonormal basis $e_{\ell+5}, \dots, e_n$. Then it follows that

$$\frac{\partial^2 F}{\partial x_{\ell+3}\partial x_\beta} = 0, \frac{\partial^2 F}{\partial x_{\ell+4}\partial x_\beta} = 0,$$

if $\beta > \ell + 4$. Moreover, denoting by $q(x)$ the lefthandside of (44) and by $q'(x)$ its derivative, we have that

$$\frac{\partial^2 F}{\partial x_{\ell+3}\partial x_{\ell+3}} = (\ell+4)Re(q'(\mu))$$
$$\frac{\partial^2 F}{\partial x_{\ell+3}\partial x_{\ell+4}} = (\ell+4)Im(q'(\mu))$$
$$\frac{\partial^2 F}{\partial x_{\ell+4}\partial x_{\ell+4}} = -(\ell+4)Re(q'(\mu))$$

Since $\mu$ is an eigenvalue of $K_v$, we know that $\mu$ is a root of (44). Hence $q(\mu) = 0$. Since (44) has no double roots, it follows that $q'(\mu)$ is different from zero. This implies that the $\ell+3, \ell+4$-block of the matrix is indefinite, which is a contradiction. q.e.d.

As a consequence of the above lemma, since in Case 7 there are only complex eigenvalues, we get that that case cannot occur. Hence only Case 1 upto Case 6 are possible and therefore from now on, we may assume that $\ell = 2k$. However, in that case, as we discovered before, (44) has only one real root. So, we get that $K_v$ restricted to $V^\perp$ is a multiple of the identity.

**Lemma 11.** *The metric, restricted to the space $V^\perp$ is positive definite*

*Proof.* Let us suppose that $V^\perp$ is not positive definite. Hence, there exists a vector which we can choose to be $e_{\ell+3}$ such that $h(e_{\ell+3}, e_{\ell+3}) = -1$. Since $\ell = 2k$, it follows that the index of the metric restricted to the $2k+2$-dimensional space $V$ is $k+1$, i.e., on $V$ the number of $+$ signs equals the number of $-$ signs. Since the index was at most half of the dimension, it follows that there must exist also a vector, orthogonal to $V \oplus \{e_{\ell+3}\}$ such that $h(e_{\ell+4}, e_{\ell+4}) = 1$. In $(V \oplus \{e_{\ell+3}, e_{\ell+4}\})^\perp$ we take an arbitrary orthonormal basis $e_{\ell+5}, \dots, e_n$. Since $K_v$ on $V^\perp$ is a multiple of the identity, it follows that

$$\frac{\partial^2 F}{\partial x_{\ell+3} \partial x_\alpha} = 0,$$
$$\frac{\partial^2 F}{\partial x_{\ell+4} \partial x_\beta} = 0,$$

where $\ell + 2 < \alpha \neq \ell + 3$ and $\ell + 2 < \beta \neq \ell + 4$

If we denote by $\lambda$ the eigenvalue, it follows that

$$\frac{\partial^2 F}{\partial x_{\ell+3} \partial x_{\ell+3}} = -(\ell+4)(q'(\lambda))$$
$$\frac{\partial^2 F}{\partial x_{\ell+4} \partial x_{\ell+4}} = (\ell+4)(q'(\lambda))$$

Since $\lambda$ is an eigenvalue of $K_v$, we know that $\lambda$ is a root of (44). Hence $q(\lambda) = 0$. Since (44) has no double roots, it follows that $q'(\lambda)$ is different from zero. This implies that the $\ell+3, \ell+4$-block of the matrix is indefinite, which is a contradiction. q.e.d.

Since the components of the metric restricted to the space spanned by $v, \dots, K_v^{2k+1} v$ form a lowerdiagonal matrix with 1 entries on the diagonal, we have that the index of the space $V$ equals $k+1$. This together with the previous lemma shows that $k = s-1$, where $s$ denotes the index of the metric. In particular, we get that $M_{2s-2} \neq \emptyset$.

Notice that we already have that

$$K(v_1, v_2) \in V \tag{56}$$
$$K(e_i, w_1) = \lambda^i w_1, \tag{57}$$

for $i \in \{1, \dots, 2s\}$, $v_1, v_2 \in V$ and $w_1 \in V^\perp$. This means that we can apply the reduction theorem. However, before doing so, we first want to investigate the apolarity conditions once more, taking into account that only 1 eigenvalue occurs, that $k = s - 1$ and that $\ell = 2k$. Then, we have that

$$s_i + (n - 2s)\lambda^i = 0,$$

for $i = 1, \ldots, (n - 2s)$. From this, using the Newton formulas and an induction argument, it is possible to determine the coefficients $a_i$ explicitely.

**Lemma 12.** *We have that*

$$a_i = -\binom{n - 2s + i - 1}{i}\lambda^i.$$

Since $\lambda$ itself has to be a real root of (44), it follows that

**Lemma 13.** *We have* $a = -\lambda^{2s+1}\binom{n}{2s}$.

The above lemmas show that on the space $V$ everything is completely determined by the value of $a$. Now, we apply the reduction theorem and obtain a similar problem on a positive definite space $V^{\perp}$ with number $a^*$, where $a^*$ is given by

$$\begin{aligned} a^* &= a - \operatorname{trace}_V \alpha^2 \\ &= a - \sum_{i_1, i_2 = 1}^{2s} \alpha(e_{i_1})\alpha(e_{i_2}) H_{i_1 i_2} \\ &= a - \sum_{i_1, i_2 = 1}^{2s} \lambda^{i_1 + i_2} H_{i_1 i_2} \\ &= a + \sum_{i_1, i_2 = 1}^{2s} \lambda_{i_1 + i_2} a_{2s+1-i_1-i_2} \\ &= a + \sum_{i=1}^{2s-1} i\lambda^{i+1} a_{2s-i} - 2s\lambda^{2s+1} \\ &= -\lambda^{2s+1}(\binom{n}{2s} + 2s + \sum_{i=1}^{2s-1} i\binom{n-i-1}{2s-i}). \end{aligned}$$

The above formula implies that the signs $a$ and $a^*$ correspond. In particular, $a^*$ is different from zero. Since we have shown that our problem on an $n - 2s$-dimensional positive definite vector space has a unique solution, unless $n - 2s > 1$ and $a^*$ is positive in which case there are no solutions, we obtain the uniqueness part of our claim.

In order to get the existence part, all we have to do is to write down an explicit solution. This can be done as follows. We take a basis of our vector space $\{u_1, \dots, u_s, v_1, \dots, v_s, e_1, \dots, e_r\}$ and define a metric $h$ by

$$h(u_i, u_j) = h(v_i, v_j) = 0, \tag{58}$$
$$h(e_k, e_\ell) = \delta_{k\ell}, \tag{59}$$
$$h(u_i, v_j) = \delta_{ij}, \tag{60}$$
$$h(u_i, e_k) = h(v_i, e_k) = 0, \tag{61}$$

where $i, j = 1, \dots, s$ and $k, \ell = 1, \dots, r$. Assuming now that $\ell < k$ and $i < j$, we introduce a multilinear map $K$ by

$$K(u_i, u_j) = \lambda_i u_j \tag{62}$$
$$K(u_i, v_j) = \lambda_i v_j \tag{63}$$
$$K(u_i, e_k) = \lambda_i e_k \tag{64}$$
$$K(v_i, u_j) = \lambda_i(\lambda_i - \alpha_i) u_j \tag{65}$$
$$K(v_i, v_j) = \lambda_i(\lambda_i - \alpha_i) v_j \tag{66}$$
$$K(v_i, e_k) = \lambda_i(\lambda_i - \alpha_i) e_k \tag{67}$$
$$K(u_i, u_i) = \alpha_i u_i + v_i \tag{68}$$
$$K(u_i, v_i) = \sum_{j=1}^{i-1} (\lambda_j v_j + (\lambda_j - \alpha_j)\lambda_j u_j) + \beta_i u_i + \alpha_i v_i \tag{69}$$
$$K(v_i, v_i) = \gamma_i u_i + \beta_i v_i \tag{70}$$
$$K(e_k, e_\ell) = -\mu_\ell e_k \tag{71}$$
$$K(e_k, e_k) = \sum_{j=1}^{s} (\lambda_j v_j + (\lambda_j - \alpha_j)\lambda_j u_j) - \sum_{\ell=1}^{k-1} \mu_\ell e_\ell + (r-k)\mu_k e_k, \tag{72}$$

where the $\alpha_i, \beta_i, \gamma_i, \mu_k$ and $\lambda_i$ are constants determined by

$$\lambda_i^3 = \lambda_1^3 \tfrac{n(n+1)(n-1)}{(n-2i+2)(n-2i+1)(n-2i+3)}, \tag{73}$$
$$\alpha_i = -\tfrac{n-2i}{2}\lambda_i, \tag{74}$$
$$\beta_i = -\tfrac{1}{4}(n-2i)(n-2i+2)\lambda_i^2, \tag{75}$$
$$\gamma_i = \tfrac{\lambda_i^3}{8}(n-2i+2)^3, \tag{76}$$
$$a_1 = -\tfrac{\lambda_1^3}{2}(n-2i+2)(n-2i+1), \tag{77}$$
$$\mu_1^2 = -\tfrac{a_1(n+1)}{r(r+1)}, \tag{78}$$
$$\mu_{\ell+1}^2 = \tfrac{(r-\ell+2)}{(r-\ell)}\mu_\ell^2. \tag{79}$$

Remark from (72) that the numbers $\mu_i$ are only needed in case that $r > 1$. It is also clear that if $r > 1$, it is possible to define those numbers provided that $a < 0$, which is exactly the case that we are considering at the moment.

It can also be verified straightforwardly that $h$ and $K$ as defined above satisfy (5), (3) and (4).

## 4. Introducing flat coordinates

In this section we want to show first that the basis which we constructed at one point can be extented differentiably such that at each point of a neighborhood $h$ and $K$ have the same expresion, i.e., we want to show that given $p \in M$, there exist differentiable vector fields $\{U_1, V_1, \ldots, U_s, V_s, E_1, \ldots, E_r\}$, defined on a neighborhood of the point $p$ which satisfy (58) to (79) in a neighborhood of $p$.

First, we show that we can define $U_1$ and $V_1$ differentiably. We take a point $p \in M$ and we take the frame constructed at $p$ previously. We can extend this frame to local vector fields $\{\tilde{U}_1, \ldots, \tilde{U}_s, \tilde{V}_1, \ldots, \tilde{V}_s, \tilde{E}_1, \ldots, \tilde{E}_r\}$ such that $h$ has the desired form and such that $\tilde{U}_i(p) = u_i$, $\tilde{V}_i(p) = v_i$ and $\tilde{E}_k(p) = e_k$. Then, the nulldirections $V$, different from $\tilde{V}_1$, can be parameterized by

$$\begin{aligned} V(q, (x_2, \ldots, x_s, y_2, \ldots, y_s, z_1, \ldots, z_r)) \\ = \tilde{U}_1(q) - (\tfrac{1}{2}\sum_{k=1}^{r} z_k^2 + \sum_{i=2}^{s} x_i y_i)\tilde{V}_1(q) \\ + \sum_{k=1}^{r} z_k \tilde{E}_k(q) + \sum_{i=2}^{s}(x_i \tilde{U}_i(q) + y_i \tilde{V}_i(q)). \end{aligned}$$

Let $f_2, f_3$ be as defined in the previous section. Since $f_3(u_1) = 2\alpha_1 \neq 0$, we have that $g \circ V = \frac{(f_2 \circ V)^{\frac{4}{3}}}{(f_3 \circ V)}$ is a well defined function on a neighbourhood of $(p, (0, \ldots, 0))$. Since

$$\begin{aligned} f_2(V(p, (x_2, \ldots, x_s, y_2, \ldots, y_s, z_1, \ldots, z_k)) \\ = 1 + 3(\sum_{i=2}^{s} x_i y_i + \tfrac{1}{2}\sum_{k=1}^{r} z_k^2)(6\lambda_1 - 3\alpha_1) + o(3) \\ = 1 + \tfrac{3}{2}(n+2)\lambda_1(\sum_{i=2}^{s} x_i y_i + \tfrac{1}{2}\sum_{k=1}^{r} z_k^2) + o(3) \end{aligned}$$

and

$$
\begin{aligned}
f_3(V(p,(x_2,\dots,x_s,y_2,\dots,y_s,z_1,\dots,z_k))\\
&= 2\alpha_1 + (\sum_{i=2}^{s} x_i y_i + \tfrac{1}{2}\sum_{k=1}^{r} z_k^2)(12\lambda_1^2 - 4\beta_1 - 4\alpha_1^2) + o(3)\\
&= 2\alpha_1 + \lambda_1^2(2n+8)(\sum_{i=2}^{s} x_i y_i + \tfrac{1}{2}\sum_{k=1}^{r} z_k^2) + o(3),
\end{aligned}
$$

we see that $g|_{\{p\}\times\mathbb{R}^{n-2}}$ has a critical value at $(p,(0,\dots,0))$ with value

$$g(u_1)^3 = -\tfrac{n(n-1)}{2(n-2)^3 a}.$$

Moreover. a straightforward computation yields that

$$
\begin{aligned}
\tfrac{\partial^2}{\partial x_j \partial x_i} g|_{(p,(0,\dots,0))} &= 0,\\
\tfrac{\partial^2}{\partial y_j \partial y_i} g|_{(p,(0,\dots,0))} &= 0\\
\tfrac{\partial^2}{\partial x_j \partial y_i} g|_{(p,(0,\dots,0))} &= \tfrac{4}{3}\tfrac{f_2^{\frac{1}{3}}}{f_3}\tfrac{\partial^2 f_2}{\partial x_j \partial y_i} - \tfrac{f_2^{\frac{4}{3}}}{f_3^2}\tfrac{\partial^2 f_3}{\partial x_j \partial y_i}|_{(p,(0,\dots,0))}\\
&= ((n+2)\tfrac{\lambda_1}{\alpha_1} - \tfrac{1}{2}(n+4)\tfrac{\lambda_1^2}{\alpha_1^2})\delta_{ij}\\
&= -\tfrac{2n(n+1)}{(n-2)^2}\delta_{ij}\\
\tfrac{\partial^2}{\partial x_i \partial z_k} g|_{(p,(0,\dots,0))} &= 0\\
\tfrac{\partial^2}{\partial y_i \partial z_k} g|_{(p,(0,\dots,0))} &= 0\\
\tfrac{\partial^2}{\partial z_k \partial z_\ell} g|_{(p,(0,\dots,0))} &= -\tfrac{2n(n+1)}{(n-2)^2}\delta_{k\ell}
\end{aligned}
$$

Hence, the implicit function theorem shows that we can find local functions $x_2, y_2, \dots, x_s, y_s, z_1, \dots, z_r$ on $M$ such that $g|_{\{q\}\times\mathbb{R}^{n-2}}$ attains an critical value at every point $q$ in a neighborhood of $p$. Denote the obtained vector field by $U_1$, if necessary after rescaling to ensure that $h(K(U_1,U_1),U_1) = 1$ and take $V_1$ as the null vector in $\operatorname{span}\{U_1, K(U_1,U_1)\}$ such that $h(U_1,V_1) = 1$. In order to show that $U_1$ and $V_1$ are the desired vector fields, we first need to prove the following lemma:

**Lemma 14.** *We consider the function*

$$g(u) = \tfrac{h(K(u,u),u)^{\frac{4}{3}}}{h(K(u,u),K(u,u))},$$

*defined on an open set of nullvectors at a point of $M$. Then there exists a finite set containing all possible critical values. Moreover, if $g$ attains a critical value in $v$ with*

$$g(v)^3 = -\tfrac{n(n-1)}{2(n-2)^3 a},$$

*then $K_v$ restricted to the space orthogonal to $v$ and $K(v,v)$ is a uniquely determined multiple of the identitity.*

*Proof.* We take a point of $M$. Remark that all lemmas upto Lemma 9 of the previous section only used that $v = e_1$ is a vector in which the function $g$ attains a non-zero critical value. Assuming that this is the case, we can rescale $v$ such that $h(K(v,v),v) = 1$ and we use the same basis as constructed before. So we have $e_1 = v$ and $e_2 = K(e_1,e_1)$. We also know that

$$K(e_1,e_2) = a_2 e_1 + a_1 e_2,$$

and that $K_{e_1}$ restricted to the orthogonal complement of $\{e_1,e_2\}$ is complex diagonalisable. Moreover, each eigenvector has to be a solution of the equation:

$$x^3 - a_1 x^2 - a_2 x + a = 0,$$

which as we have seen has 1 real root and 2 complex conjugate roots. Denote by $k$ the multiplicity with which the real root $\mu_1$ occurs as an eigenvalue and by $j$ the multiplicity with which the complex conjugate roots $\mu_2 \pm i\mu_3$ occur as eigenvalues. Then $k + 2j = n - 2$. Clearly, we have

$$a_1 = \mu_1 + 2\mu_2, \tag{80}$$

$$a_2 = -2\mu_1\mu_2 - (\mu_2^2 + \mu_3^2), \tag{81}$$

On the other hand, using the apolarity, we have

$$(k+1)\mu_1 + 2(j+1)\mu_2 = 0, \tag{82}$$

$$(k+1)\mu_1^2 + 2(j+1)(\mu_2^2 - \mu_3^2) = 0. \tag{83}$$

The last two equations imply that

$$\mu_2 = -\tfrac{(k+1)}{2(j+1)}\mu_1, \tag{84}$$

$$(\mu_2^2 - \mu_3^2) = -\tfrac{(k+1)}{2(j+1)}\mu_1^2. \tag{85}$$

Therefore, we obtain that

$$\mu_2^2+\mu_3^2=\left(\tfrac{(k+1)}{2(j+1)}+\tfrac{(k+1)^2}{2(j+1)^2}\right)\mu_1^2=\tfrac{(k+1)(k+j+2)}{2(j+1)^2}.$$

Using these equations, we now can solve for $a_1$ and $a_2$.

$$a_1=\tfrac{(j-k)}{j+1}\mu_1 \tag{86}$$

$$a_2=\tfrac{(k+1)}{(j+1)}\mu_1^2-\tfrac{(k+1)(k+j+2}{2(j+1)^2}\mu_1^2=\tfrac{(k+1)(j-k)}{2(j+1)^2}\mu_1^2 \tag{87}$$

Of course, we also know that $\mu_1$ is a real root of our equation. Therefore, we also have that

$$a=-\mu_1^3+a_1\mu_1^2+a_2\mu_1 \tag{88}$$

$$=\mu_1^3\left(-1+\tfrac{(j-k)}{j+1}+\tfrac{(k+1)(j-k)}{2(j+1)^2}\right) \tag{89}$$

$$=-\mu_1^3\tfrac{(j+k+2)(k+1)}{2(j+1)^2}. \tag{90}$$

Since $g(v)=\frac{1}{a_1}$, the first part of the lemma is clear.

In order to obtain the second part, we investigate the function

$$\eta(j)=g(v)^3a \tag{91}$$

$$=-\tfrac{(j+1)^3}{(j-k)^3}\tfrac{(j+k+2)(k+1)}{2(j+1)^2} \tag{92}$$

$$=-\tfrac{(j+1)(n-j)(n-1-2j)}{2(3j-n+2)^3}. \tag{93}$$

A straightforward computation shows that the derivative of this function is given by

$$\eta'[j]=-\tfrac{(n+1)(n^2+2n-2-6j-3j^2)}{2(n-3j-2)^4}=-\tfrac{(n+1)((n+1)^2-3(j+1)^2)}{2(n-3j-2)^4}. \tag{94}$$

Since $k+2j=n-2$, we have $2(j+1)<(n+1)$. Hence, on the domain we are interested in, the function $(\frac{1}{\eta})'=-\frac{\eta'}{\eta^2}$ does not change sign. This means that the function $\frac{1}{\eta}$ is strictly increasing. Therefore, if $g(v)=\eta(0)$, as the assumption of the lemma claims, we must have that $j=0$ and thus that $K_v$ restricted to the orthogonal complement of $\{e_1,e_2\}$ is a uniquely determined multiple of the identity. q.e.d.

Applying now the first part of the previous lemma, it follows that the critical value obtained at the vector field $U_1$ must be constant and equal to the value obtained at $U_1(p)$. It follows that we can write:

$$K(U_1(q),U_1(q))=\alpha_1U_1(q)+V_1(q).$$

Since $g$ attains a relative extremum, we again obtain that the space spanned by $U_1$ and $V_1$ is invariant under $K$. Therefore, using the symmetries of $K$, we can write:

$$K(U_1(q), V_1(q)) = \beta_1(q)U_1(q) + \alpha_1 V_1(q)$$
$$K(V_1(q), V_1(q)) = \gamma_1(q)U_1(q) + \beta_1(q)V_1(q).$$

Using the second part of the lemma, it follows that, for $X$ orthogonal to $\operatorname{span}\{U_1, V_1\}$, we can write

$$K(U_1, X) = \lambda_1 X(q).$$

From (5) it then follows that

$$K(V_1, Y) = \lambda_1(\lambda_1 - \alpha_1)Y(q).$$

Since $\alpha_1$ is constant it follows from the apolarity conditions that $\beta_1$ is constant too. Therefore, since $\gamma_1$ can be determined using (5), it follows that $\gamma_1$ is constant too.

It now follows that we can apply the reduction theorem on a neighborhood of $p$, introducing $K^*$ and a constant $a^*$. We now proceed by induction to construct $U_2, V_2, \dots, U_s, V_s$ following the above procedure. The construction of $E_1, \dots, E_k$ is completely similar to the one described in the proof of Lemma 4.1 of [7].

Proceeding now similarly as in Lemma 4.2 of [7] we can show the following lemma:

**Lemma 15.** *Let $\{U_1, V_1, \dots, U_s, V_s, W_1, \dots, W_r\}$ be the frame constructed before. Then all connection coefficients (with respect to $\widehat{\nabla}$) vanish. In particular, $M$ has flat affine metric.*

From the previous lemma, we know that there exists coordinates $u_1, v_1, \dots, u_s, v_s$ on $M^n$ such that

$$U_i = \frac{\partial}{\partial u_i} \tag{95}$$
$$V_i = \frac{\partial}{\partial v_i} \tag{96}$$
$$E_k = \frac{\partial}{\partial w_k} \tag{97}$$

where $i = 1, \dots, s$ and $k = 1, \dots, r$. We denote the immersion of $M^n$ into $\mathbb{R}^{n+1}$ by $x$. We have that upto a translation $\xi = -Hx = -ax$. Therefore, we get that an affine hypersphere with constant sectional curvature $c$ and non zero Pick invariant $J$ is characterized by the following system of differential equations:

$$x_{u_iu_j} = \lambda_i x_{u_j}, \quad j > i \tag{98}$$
$$x_{u_iv_j} = \lambda_i x_{v_j}, \quad j > i \tag{99}$$
$$x_{u_iw_k} = \lambda_i x_{w_k} \tag{100}$$
$$x_{v_iu_j} = \lambda_i(\lambda_i - \alpha_i)x_{u_j}, \quad j > i \tag{101}$$
$$x_{v_iv_j} = \lambda_i(\lambda_i - \alpha_i)x_{v_j}, \quad j > i \tag{102}$$
$$x_{v_iw_k} = \lambda_i(\lambda_i - \alpha_i)x_{w_k} \tag{103}$$
$$x_{u_iu_i} = \alpha_i x_{u_i} + x_{v_i} \tag{104}$$
$$\begin{aligned} x_{u_iv_i} = &\sum_{j=1}^{i-1}(\lambda_j x_{v_j} + (\lambda_j - \alpha_j)\lambda_j x_{u_j}) \\ &+ \beta_i x_{u_i} + \alpha_i x_{v_i} - ax \end{aligned} \tag{105}$$
$$x_{v_iv_i} = \gamma_i x_{u_i} + \beta_i x_{v_i} \tag{106}$$
$$x_{w_kw_\ell} = -\mu_\ell x_{w_k}, \quad k > \ell \tag{107}$$
$$\begin{aligned} x_{w_kw_k} = &\sum_{j=1}^{s}(\lambda_j x_{v_j} + (\lambda_j - \alpha_j)\lambda_j x_{u_j}) \\ &- \sum_{\ell=1}^{k-1}\mu_\ell x_{w_\ell} + (r-k)\mu_k x_{w_k} - ax, \end{aligned} \tag{108}$$

where the $a_i, \alpha_i, \beta_i, \gamma_i, \mu_k$ and $\lambda_i$ are the constants defined earlier.

In particular, we have that

$$x_{u_1u_1} = \alpha_1 x_{u_1} + x_{v_1}, \tag{109}$$
$$x_{u_1v_1} = \beta_1 x_{u_1} + \alpha_1 x_{v_1} - ax, \tag{110}$$
$$x_{v_1v_1} = \gamma_1 x_{u_1} + \beta_1 x_{v_1}, \tag{111}$$

From these equations we deduce that

$$\begin{aligned} x_{u_1u_1u_1} &= \alpha_1 x_{u_1u_1} + x_{u_1v_1} \\ &= \alpha_1 x_{u_1u_1} + \beta_1 x_{u_1} + \alpha_1(x_{u_1u_1} - \alpha_1 x_{u_1}) - ax \\ &= 2\alpha_1 x_{u_1u_1} + (\beta_1 - \alpha_1^2)x_{u_1} - a_1 x \end{aligned}$$

We now look at the corresponding equation of degree 3,

$$t^3 - 2\alpha_1 t^2 - (\beta_1 - \alpha_1^2)t + a_1 = 0. \tag{112}$$

It is easy to see that (112) has one real root, namely $\lambda_1$ and two complex roots $\eta_{11} + i\eta_{12}$ and $\eta_{11} - i\eta_{12}$ which are determined by

$$\begin{aligned}\eta_{11} &= -\tfrac{1}{2}\lambda_1(n-1)\\ \eta_{12} &= \tfrac{1}{2}\lambda_1\sqrt{(n-1)(n+1)}.\end{aligned}$$

Using now once more our system of differential equations, it follows that we can write

$$x = A(u_i, v_j, w_k)e^{\lambda_1 u_1} + C_{11}(v_1)z_{11} + C_{12}(v_1)z_{12},$$

where we have written

$$\begin{aligned}z_{11} &= e^{\eta_{11}u_1 + (\eta_{11}^2 - \eta_{12}^2 - \alpha_1\eta_{11})v_1}\cos(\eta_{12}u_1 + (2\eta_{11}\eta_{12} - \alpha_1\eta_{12})v_1)\\ z_{12} &= e^{\eta_{11}u_1 + (\eta_{11}^2 - \eta_{12}^2 - \alpha_1\eta_{11})v_1}\sin(\eta_{12}u_1 + (2\eta_{11}\eta_{12} - \alpha_1\eta_{12})v_1).\end{aligned}$$

Substituting now the above expression for $x$ into

$$x_{v_1} = x_{u_1u_1} - \alpha_1 x_{u_1},$$

and using the fact that $e^{\lambda_1 u_1}$, $z_{11}$ and $z_{12}$ are linearly independent functions, we obtain the following system of differential equations for $A$, $C_{11}$ and $C_{12}$:

$$\begin{aligned}A_{v_1} &= (\lambda_1 - \alpha_1)\lambda_1 A,\\ (C_{11})_{v_1} &= 0,\\ (C_{12})_{v_1} &= 0,\end{aligned}$$

from which it follows that

$$\begin{aligned}A &= x^2(u_2, v_2, \ldots, u_s, v_s, w_1, \ldots, w_r)e^{(\lambda_1 - \alpha_1)\lambda_1 v_1}\\ C_{11}(v_1) &= C_{11}\\ C_{12}(v_1) &= C_{12}.\end{aligned}$$

A straightforward computation shows that

$$a_2 = a - 2\lambda_1^2(\lambda_1 - \alpha_1)$$

and

$$\begin{aligned}0 &= \lambda_1(\eta_{11}^2 - \eta_{12}^2) - \alpha_1\eta_{11} + \lambda_1(\lambda_1 - \alpha_1)\eta_{11} - a\\ 0 &= \lambda_1(2\eta_{11}\eta_{12} - \alpha_1\eta_{12}) + \lambda_1(\lambda_1 - \alpha_1)\eta_{12}.\end{aligned}$$

Therefore, we get that

$$(\lambda_1 x_{v_1} + (\lambda_1 - \alpha_1)\lambda_1 x_{u_1}) - ax) = -a_2 x^2 e^{\lambda_1 u_1 + (\lambda_1 - \alpha_1)\lambda_1 v_1}.$$

Using the above, we obtain by substituting the found expression of $x = x^1$ into the system of differential equations, that $x^2$ satisfies a similar system of differential equations. Therefore, proceeding by induction we can define vector valued functions $x^{j_1+1}$ and constant vectors $C_{j_1 1}$ and $C_{j_1 2}$ for all indices $j_1$ where $1 \leq j_1 \leq s$ which satisfy

$$x^{j_1} = x^{j_1+1} e^{\lambda_{j_1} u_{j_1}} + C_{j1}(v_{j_1}) z_{j1} + C_{j2}(v_{j_1}) z_{j2}$$

where we have written

$$\begin{aligned} z_{j1} &= e^{\eta_{j_1 1} u_{j_1} + (\eta_{j_1 1}^2 - \eta_{j_1 2}^2 - \alpha_{j_1}\eta_{j_1 1}) v_{j_1}} \cos(\eta_{j_1 2} u_{j_1} + (2\eta_{j_1 1}\eta_{j_1 2} - \alpha_{j_1}\eta_{j_1 2}) v_{j_1}) \\ z_{j2} &= e^{\eta_{j_1 1} u_{j_1} + (\eta_{j_1 1}^2 - \eta_{j_1 2}^2 - \alpha_{j_1}\eta_{j_1 1}) v_{j_1}} \sin(\eta_{j_1 2} u_{j_1} + (2\eta_{j_1 1}\eta_{j_1 2} - \alpha_{j_1}\eta_{j_1 2}) v_{j_1}) \end{aligned}$$

where $i \geq j_1 + 1$ and where the numbers $\eta_{j_1 1}$ and $\eta_{j_1 2}$ are respectively defined by

$$\begin{aligned} \eta_{j_1 1} &= -\tfrac{1}{2}\lambda_{j_1}(n - 2j_1 + 1) \\ \eta_{j_1 2} &= \tfrac{1}{2}\lambda_{j_1}\sqrt{(n - 2j_1 + 1)(n - 2j_1 + 3)}. \end{aligned}$$

Moreover, $x^{j_1+1}$ depends only on $u_{j_1+1}, v_{j_1+1}, \ldots, u_s, v_s, w_1, \ldots, w_k$.

Therefore, we may assume that we have obtained constant vectors $C_{11}, C_{12}, \ldots, C_{s1}, C_{s2}$ and a vector valued function which depends only on $w_1, \ldots, w_r$, $x^{s+1}$ satisfying the following system of differential equations:

$$\begin{aligned} x^{s+1}_{w_k w_\ell} &= -\mu_\ell x^{s+1}_{w_k}, \quad k > \ell \\ x^{s+1}_{w_k w_k} &= -\sum_{\ell=1}^{k-1} \mu_\ell x^{s+1}_{w_\ell} + (r - k)\mu_k x^{s+1}_{w_k} - a_{s+1} x^{s+1}, \end{aligned}$$

where

$$a_{s+1} = a_s - 2\lambda_s^3 + 2\alpha_s \lambda_s^2 = a_1 \tfrac{(n+1)}{(r+1)}.$$

Now, we have to consider different cases depending on the value of $n - 2s$. If $n - 2s = 0$, we end up with constant vectors $C_{11}, C_{12}, \ldots, C_{s1}, C_{s2}$ and $x^{s+1}$. Applying now an affine tranformation, we may assume that $C_{11}, C_{12}, \dots, C_{s1}, C_{s2}$ and $x^{s+1}$ is the standard basis of $\mathbb{R}^{n+1}$. From the previous formulas, it follows that

$$\begin{aligned}
(x_1^2 + x_2^2) &= e^{2(\eta_{11}u_1 + (\eta_{11}^2 - \eta_{12}^2 - \alpha_1\eta_{11})v_1)}\\
(x_3^2 + x_4^2) &= e^{2(\lambda_1 u_1 + (\lambda_1 - \alpha_1)\lambda_1 v_1 + \eta_{21}u_2 + (\eta_{21}^2 - \eta_{22}^2 - \alpha_2\eta_{21})v_2)}\\
&\dots\dots\\
(x_{2i-1}^2 + x_{2i}^2) &= e^{2(\sum_{j=1}^{i-1}(\lambda_j u_j + (\lambda_j - \alpha_j)\lambda_j v_j) + \eta_{i1}u_i + (\eta_{i1}^2 - \eta_{i2}^2 - \alpha_i\eta_{i1})v_i)}\\
&\dots\dots\\
(x_{2s-1}^2 + x_{2s}^2) &= e^{2(\sum_{j=1}^{s-1}(\lambda_j u_j + (\lambda_j - \alpha_j)\lambda_j v_j) + \eta_{s1}u_s + (\eta_{s1}^2 - \eta_{s2}^2 - \alpha_s\eta_{s1})v_s)}\\
x_{2s+1} &= e^{(\sum_{j=1}^{s}(\lambda_j u_j + (\lambda_j - \alpha_j)\lambda_j v_j)}.
\end{aligned}$$

Since

$$\begin{aligned}
&2\eta_{i1} + (n - 2i + 1)\lambda_i = 0\\
&2(\eta_{i1}^2 - \eta_{i2}^2 - \alpha_i\eta_{i1}) + (n - 2i + 1)\lambda_i(\lambda_i - \alpha_i) = 0,
\end{aligned}$$

we obtain that in this case $M^n$ is affine equivalent with

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2)\dots(x_{2s-1}^2 + x_{2s}^2)x_{2s+1} = 1$$

In case that $n - 2s = 1$, we obtain that $x^{s+1}$ which depends only on the variable $w_1$ satisfies the following differential equation:

$$x^{s+1}_{w_1 w_1} = -a_{s+1}x$$

In order to solve the above equation, we have to consider two subcases, depending on the sign of $a_{s+1}$. If $\lambda_1 > 0$, we introduce a number $b$ such that $b^2 = -a_{s+1}$. Hence, it follows that there exists constant vectors such that $x^{s+1}(w_1) = C_{s+11}e^{bw_1} + C_{s+12}e^{-bw_1}$. Applying now an affine tranformation, we may assume that $C_{11}, C_{12}, \dots, C_{s+11}, C_{s+12}$ is the standard basis of $\mathbb{R}^{n+1}$. Similarly as in the previous case, we get that $M$ is affine equivalent with

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2)\dots(x_{2s-1}^2 + x_{2s}^2)x_{2s+1}x_{2s+2} = 1.$$

If $\lambda_1 < 0$, we introduce a number $b$ such that $b^2 = a_{s+1}$. Hence, it follows that there exists constant vectors such that $x^{s+1}(w_1) = C_{s+11}\cos(bw_1) + C_{s+12}\sin(bw1)$. Applying now an affine tranformation, we may assume that $C_{11}, C_{12}, \dots, C_{s+11}, C_{s+12}$ is the standard basis of $\mathbb{R}^{n+1}$. Therefore, we get that $M$ is affine equivalent with

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2)\dots(x_{2s-1}^2 + x_{2s}^2)(x_{2s+1}^2 + x_{2s+2}^2) = 1.$$

Finally, we consider the case that $n-2s>1$. In this case, we know that $a$, and thus also $a_{s+1} = a_1\frac{n+1}{r+1}$ is negative. We also have that $a_{s+1} = -r\mu_1^2$. We write $y^1$ for the vector valued function $x^{s+1}$. In this casse we are left with the following system of differential equations:

$$\begin{aligned} y^1_{w_k w_\ell} &= -\mu_\ell y^1_{w_k}, \quad k>\ell \\ y^1_{w_k w_k} &= -\sum_{\ell=1}^{k-1} \mu_\ell y^1_{w_\ell} + (r-k)\mu_k y^1_{w_k} + r\mu_1^2 y^1, \end{aligned}$$

In particular, we have that

$$y^1_{w_1 w_1} = (r-1)\mu_1 y^1_{w_1} + r\mu_1^2 y^1.$$

This together with the previous equation implies that there exists a constant vector $D_1$ and a vector valued function $y^2$ such that

$$y^1(w_1,\ldots,w_r) = y^2(w_2,\ldots,w_r)e^{-\mu_1 w_1} + D_1 e^{r\mu_1 w_1}.$$

Since

$$\begin{aligned} -\mu_1 y^1_{w_1} + r\mu_1^2 y^1 &= (r+1)\mu_1^2 y^2 e^{-\omega_1 w_1} \\ &= (r-1)\mu_2^2 y^2 e^{-\omega_1 w_1}, \end{aligned}$$

we get that the function $y^2$ satisfies the following system of differential equations:

$$\begin{aligned} y^2_{w_k w_\ell} &= -\mu_\ell y^2_{w_k}, \quad k>\ell \\ y^2_{w_k w_k} &= -\sum_{\ell=2}^{k-1} \mu_\ell y^2_{w_\ell} + (r-k)\mu_k y^2_{w_k} + (r-1)\mu_2^2 y^2. \end{aligned}$$

Proceeding again by induction we get that there exists maps $y^2,\ldots,y^{r+1}$ and constant vectors $D_1, \ldots, D_r$ such that

$$y^{k_1}(w_{k_1},\ldots,w_r) = y^{k_1+1}(w_{k_1+1},\ldots,w_r)e^{-\mu_{k_1} w_{k_1}} + D_k e^{(r-k_1+1)\mu_{k_1} w_{k_1}},$$

for every $k_1$ between 1 and $r$, satisfying moreover the following system of differential equations:

$$\begin{aligned} y^{k_1+1}_{w_k w_\ell} &= -\mu_\ell y^{k_1+1}_{w_k}, \quad k>\ell \\ y^{k_1+1}_{w_k w_k} &= -\sum_{\ell=k_1+1}^{k-1} \mu_\ell y^2_{w_\ell} + (r-k)\mu_k y^{k_1+1}_{w_k} + (r-k_1)\mu_{k_1+1}^2 y^{k_1+1}. \end{aligned}$$

for every $k, \ell \geq k_1 + 1$.

Therefore, we may assume that we have constructed the above vectors and constants. It speaks for itself that $y^{r+1}$ is a constant too. Applying now an affine tranformation, we may assume that $C_{11}, C_{12}, \dots, C_{s1}, C_{s2}$, $D_1, \dots, D_r$ and $y^{r+1}$ is the standard basis of $\mathbb{R}^{n+1}$. From the previous formulas, it follows that

$$
\begin{aligned}
(x_1^2 + x_2^2) &= e^{2(\eta_{11} u_1 + (\eta_{11}^2 - \eta_{12}^2 - \alpha_1 \eta_{11}) v_1)} \\
(x_3^2 + x_4^2) &= e^{2(\lambda_1 u_1 + (\lambda_1 - \alpha_1)\lambda_1 v_1 + \eta_{21} u_2 + (\eta_{21}^2 - \eta_{22}^2 - \alpha_2 \eta_{21}) v_2)} \\
&\dots\dots \\
(x_{2i-1}^2 + x_{2i}^2) &= e^{2(\sum_{j=1}^{i-1} (\lambda_j u_j + (\lambda_j - \alpha_j)\lambda_j v_j) + \eta_{i1} u_i + (\eta_{i1}^2 - \eta_{i2}^2 - \alpha_i \eta_{i1}) v_i)} \\
&\dots\dots \\
(x_{2s-1}^2 + x_{2s}^2) &= e^{2(\sum_{j=1}^{s-1} (\lambda_j u_j + (\lambda_j - \alpha_j)\lambda_j v_j) + \eta_{s1} u_s + (\eta_{s1}^2 - \eta_{s2}^2 - \alpha_s \eta_{s1}) v_s)} \\
x_{2s+1} \dots x_{n+1} &= e^{(r+1)(\sum_{j=1}^{s} (\lambda_j u_j + (\lambda_j - \alpha_j)\lambda_j v_j)}
\end{aligned}
$$

implying that $M$ is affine equivalent with

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) \dots (x_{2s-1}^2 + x_{2s}^2) x_{2s+1} \dots x_{n+1} = 1.$$

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