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The magnitude–redshift relation in a perturbed Friedmann universe

Misao Sasaki *Research Institute for Theoretical Physics, Hiroshima University, Takehara, Hiroshima 725, Japan*

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Summary. A general formula for the magnitude–redshift relation in a linearly perturbed Friedmann universe is derived. The formula does not assume any specific gauge condition, but the gauge-invariance of it is explicitly shown. Then the application of the formula to the spatially flat background model is considered and the implications are discussed.

1 Introduction

The origin, the evolution and the present features of large-scale structures of the universe have been lately a topic of great interest in cosmology. Modern theories of particle physics predict the existence of an inflationary stage in the very early universe and the spontaneous creation of density perturbations at that stage (see e.g. Brandenberger 1985). The spectrum of these primordial inhomogeneities is expected to be of Zel'dovich type and the best probe of it would be the observation of anisotropies in the cosmic microwave background (CMB) since they reflect most faithfully the structure of inhomogeneities at $z\sim1000$, i.e. the matter–radiation decoupling era when the perturbation amplitudes were still very small and retained the primordial information. Thus the recently reported upper bound on CMB anisotropies ($\delta T/T\lesssim4.5\times10^{-5}$ at 4$'\!.$5; Uson & Wilkinson 1984) gives a rather stringent constraint on the magnitude of primordial density perturbations.

On the other hand, the present features of large-scale structures can be seen by observing the distribution of galaxies up to the redshift $z\lesssim0.1$ and observations have become increasingly accurate owing to rapid progress in the observational technology, revealing features such as the presence of large voids and clusters (see e.g. Davis 1986).

Based on these facts, there have been a lot of theoretical works which attempt to link the allowable models of primordial spectra and the present features of large-scale structures (for a review, see e.g. Szalay 1984). Yet, none of them seems to give a sufficiently satisfactory description for the evolution of large-scale structures. It is perhaps due to some essential ingredient(s) in theories which we are missing. However, it is also true that the lack of our observational knowledge of the universe at $z\gtrsim1$ prevents us from saying anything definite. In this respect, it is hoped that further progress in the observational technology will make it possible to see the universe at high redshifts in the near future.

Assuming that this hope will be realized, we expect to see a number of galaxies just forming at around $z \sim 5$ and careful observations of them would resolve many yet unsettled problems. One important information we would obtain from such observations is the effect of large-scale inhomogeneities on the appearance of galaxies (or any other luminous astronomical bodies) at high redshifts. That is, the anisotropy in the magnitude–redshift (m–z) relation or distance–redshift (d–z) relation for objects at the same redshift with the same intrinsic properties, due to the effect of large-scale spacetime inhomogeneities on the propagation of light rays [the related, gravitational lens effect has been discussed by Press & Gunn (1973), Weinberg (1976), Peacock (1982), Turner, Ostriker & Gott (1984), Ehlers & Schneider (1986), Peacock (1986) and others].

It is generally regarded that the m–z relation is a poor indicator of the present state of the Universe because of the large evolutionary uncertainties and/or of the variations in the intrinsic luminosities of galaxies. However, if one is interested in the large angular scale anisotropy in the m–z relation and if one has statistically enough samples of galaxies at hand, one can obtain useful information on large-scale spacetime inhomogeneities since physics should be the same at the same redshift regardless of the direction one is looking at.

In view of the above considerations, in this paper we shall attempt to derive a general formula for the m–z relation in an inhomogeneous cosmological model with the metric described by a homogeneous and isotropic Friedmann–Robertson–Walker part and a small inhomogeneous part which can be treated as a linear perturbation on the homogeneous background. We consider this approximation to be reasonably reliable since there are enough observational indications that inhomogeneities on large scales are small and inhomogeneities on small scales would not affect the large angular scale anisotropy in the m–z relation even though they were non-linear.

The paper is organized as follows. In Section 2, we describe basic tools and equations adequate to the derivation of the m–z relation or, equivalently, the d_L–z relation (where d_L stands for the luminosity distance). In passing, we also give the basic equations for deriving the d_A–z relation (where d_A stands for the angular diameter distance). In Section 3, we derive a general formula for the d_L–z (or m–z) relation in a linearly perturbed Friedmann universe without fixing a coordinate-gauge of the metric perturbation. Then the gauge-invariance of the resulting formula is explicitly shown. In Section 4, we discuss the d_A–z relation and the relation between d_A and d_L. In Section 5, we apply the formula derived in Section 3 to the Einstein–de Sitter (i.e., spatially flat dust-dominated) background model and express the effects of density inhomogeneities in terms of corresponding inhomogeneities in the gravitational potential field. Finally, Section 6 is devoted to conclusions.

Throughout the paper we follow the sign conventions of Misner, Thorne & Wheeler (1973).

2 Basic tools and equations

In this section, we give basic tools and equations which will be used in the succeeding sections.

The metric of the spacetime is assumed to have the form

$$d\hat{s}^2 = a(\eta)^2\, ds^2 = a(\eta)^2 g_{\mu\nu}\, dx^\mu\, dx^\nu$$
$$= \hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu; \qquad x^\mu = (\eta, x^i), \tag{2.1}$$

where $\hat{g}_{\mu\nu}$ is the physical spacetime metric, $a(\eta)$ is a given function of time which will be taken to be the conventional scale factor function of a homogeneous and isotropic universe and $g_{\mu\nu}$ has the form

$$g_{\mu\nu} = g^{(b)}_{\mu\nu} + \delta g_{\mu\nu}; \tag{2.2a}$$

$$g^{(b)}_{\mu\nu}\, dx^\mu\, dx^\nu = -d\eta^2 + \gamma_{ij}\, dx^i\, dx^j, \tag{2.2b}$$

where $\delta g_{\mu\nu}$ is a small perturbation which represents spacetime inhomogeneities and γ_{ij} is the metric on the constant curvature 3-space. In what follows, we utilize the rescaled metric $g_{\mu\nu}$ (or $g^{(b)}_{\mu\nu}$) extensively, while all tensors defined on the physical spacetime will be indicated by a hat ($\hat{}$).

The propagation of light rays in the cosmological context is most conveniently described by geometric optics in curved spacetime (e.g. Ellis 1971; Misner *et al.* 1973). In geometric optics, the energy–momentum tensor of a congruence of photons with 4-momentum $\hat{k}^\mu$ is given by

$$\hat{T}^{\mu\nu}=\frac{1}{8\pi}\mathscr{A}^2\hat{k}^\mu\hat{k}^\omega, \tag{2.3}$$

where $\mathscr{A}$ is the scalar amplitude of the wave and k^μ is a hypersurface-orthogonal null vector tangent to a null geodesic;

$$\hat{k}^\mu=\frac{dx^\mu}{dv}=\hat{g}^{\mu\nu}\,\partial_\nu\varphi, \qquad \hat{k}_\mu\hat{k}^\mu=0, \tag{2.4a}$$

$$\hat{k}^\nu\,\hat{\nabla}_\nu\hat{k}^\mu=\frac{d^2x^\mu}{dv^2}+\hat{\Gamma}^\mu_{\alpha\beta}\,\frac{dx^\alpha}{dv}\,\frac{dx^\beta}{dv}=0, \tag{2.4b}$$

with φ representing the phase of the wave and v being the affine parameter along the ray. From the energy–momentum conservation law $\hat{\nabla}_\nu\hat{T}^{\mu\nu}=0$ and equation (2.4b), one obtains

$$\hat{\nabla}_\mu(\mathscr{A}^2\hat{k}^\mu)=2\mathscr{A}\left(\frac{d}{dv}\mathscr{A}+\frac{1}{2}\mathscr{A}\,\hat{\theta}\right)=0; \qquad \hat{\theta}=\hat{\nabla}^\mu\hat{k}_\mu. \tag{2.5}$$

It is known that the conformal transformation $\hat{g}_{\mu\nu}\rightarrow g_{\mu\nu}$ maps a null geodesic on $\hat{g}_{\mu\nu}$ to a null geodesic on $g_{\mu\nu}$ with the affine parameter transformed as $dv\rightarrow d\lambda=a^{-2}\,dv$ (e.g. Wald 1984, appendix D). Hence introducing the null vector k^μ conformal to $\hat{k}^\mu$, defined by

$$k^\mu=a^2\hat{k}^\mu=\frac{dx^\mu}{d\lambda}, \tag{2.6}$$

we may discuss the propagation of light in spacetime with the metric $g_{\mu\nu}$. Then, equations (2.4) and (2.5) are rewritten as

$$k_\mu=\partial_\mu\varphi, \qquad k_\mu k^\mu=0, \tag{2.7a}$$

$$k^\nu\,\nabla_\nu k^\mu=\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\,\frac{dx^\beta}{d\lambda}=0, \tag{2.7b}$$

$$\nabla_\mu(\mathscr{A}^2a^2k^\mu)=2\mathscr{A}\,a\left[\frac{d}{d\lambda}(\mathscr{A}\,a)+\frac{1}{2}\mathscr{A}\,a\theta\right]=0; \qquad \theta=\nabla^\mu k_\mu. \tag{2.7c}$$

By differentiating equation (2.7c) with respect to λ and using equations (2.7a, b and c) once again one obtains the propagation equation for θ;

$$\frac{d}{d\lambda}\theta=-R_{\mu\nu}k^\mu k^\nu-\frac{1}{2}\,\theta^2-2\sigma^2,$$

$$\sigma^2=\frac{1}{2}\left[k^{(\alpha;\beta)}k_{(\alpha;\beta)}-\frac{1}{2}\,\theta^2\right], \tag{2.8}$$

where σ is the shear of the congruence. Equations (2.7) and (2.8) are the basic equations used in the succeeding sections.

In addition to the above equations, we must know the precise meanings of the distance, magnitude and redshift which appear in the *d–z* and *m–z* relations. In order to do so, we first note that the energy flux measured by an observer with 4-velocity $\hat{u}^\mu$ is given by

$$\hat{f}^\alpha = -\hat{T}^\mu_\nu \hat{h}^\alpha_\mu \hat{u}^\nu$$

$$\equiv f\hat{n}^\alpha; \qquad f = \frac{1}{8\pi} \mathscr{A}^2\omega^2, \tag{2.9}$$

where

$$\hat{h}^\alpha_\mu = \delta^\alpha_\mu + \hat{u}^\alpha \hat{u}_\mu, \tag{2.10a}$$

$$\hat{n}^\alpha = \frac{1}{\omega}(\hat{k}^\alpha - \omega\hat{u}^\alpha); \qquad \hat{n}_\alpha \hat{n}^\alpha = 1, \tag{2.10b}$$

$$\omega = -\hat{k}_\mu \hat{u}^\mu. \tag{2.10c}$$

Here $\hat{h}^\alpha_\mu$ is the projection tensor orthogonal to $\hat{u}^\mu$, ω is the energy of a photon measured by the observer, $\hat{n}^\alpha$ is the unit space-like vector in the direction of the photon 3-momentum in the rest frame of the observer and f is the measured amplitude of the energy flux.

Now assuming that the photons are emitted by a spherically symmetric source with the proper radius R_s (this assumption is not crucial to our discussion of anisotropy as long as the source angular size is small compared with the angular scale of interest; hence we shall take the limit $R_s \to 0$ at the end of calculation), the intrinsic luminosity of the source is given by

$$L_s = 4\pi R_s^2 f(\lambda_s) \tag{2.11}$$

where λ_s is the value of the conformal affine parameter at the source (see Fig. 1). The distance which plays an essential role in the *m–z* relation is the one so-called the luminosity distance. The luminosity distance to the source as measured by the observer at $\lambda = 0$ is defined by

$$d_L^2 = \frac{L_s}{4\pi f(0)} = \frac{f(\lambda_s)}{f(0)} R_s^2. \tag{2.12}$$

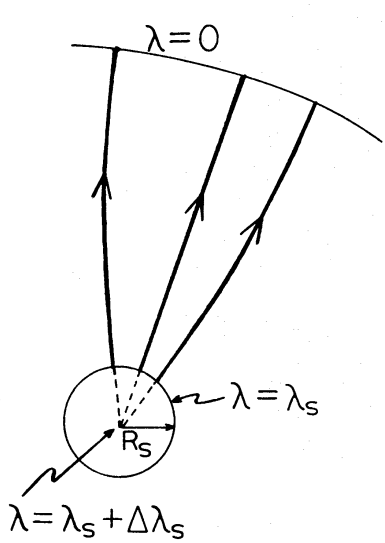

Figure 1. An expanding congruence of photons emitted by a spherically symmetric source with an infinitesimal radius R_s. $\Delta\lambda_s$ is the conformal affine distance corresponding to R_s.

Then recalling the definition of f and that of the redshift;

$$1+z(\lambda_s)=\frac{\omega(\lambda_s)}{\omega(0)}, \tag{2.13}$$

one obtains the following fundamental relation:

$$d_L=\frac{\mathscr{A}(\lambda_s)}{\mathscr{A}(0)}[1+z(\lambda_s)]R_s. \tag{2.14}$$

Thus the d_L–z relation is obtained if the right-hand side of equation (2.14) is expressed in terms of z and quantities describing the (perturbed) state of the universe. Once this is done, the m–z relation is easily obtained if one recalls the definition of the apparent magnitude;*

$$m=-2.5\log_{10}f(0)+\text{const}$$
$$=5\log_{10}d_L-2.5\log_{10}L_s+\text{const}, \tag{2.15}$$

where the second line follows from equation (2.12).

Another commonly used notion of the distance is the angular diameter distance, which is defined by

$$d_A=d_s/\Delta\varphi, \tag{2.16}$$

where d_s is the actual diameter of the source at $\lambda=\lambda_s$ and $\Delta\varphi$ is the apparent angular size of it as measured by the observer at $\lambda=0$ (see Fig. 2). Now denoting the cross-sectional area of the light

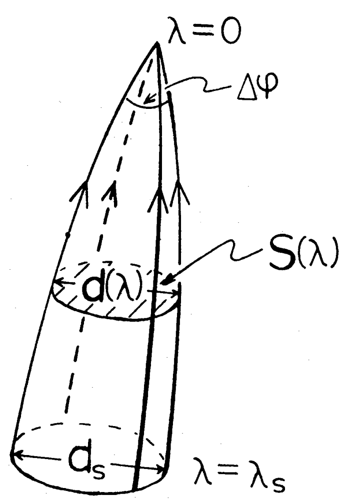

Figure 2 A congruence of photons appropriate for evaluating the angular diameter distance. Explanation of the symbols is given in the text.

rays received by the observer by $S(\lambda)$ and the corresponding diameter of the area by $d(\lambda)$, d_A is re-expressed as

$$d_A=\frac{d(\lambda_s)}{\Delta\varphi}=\frac{d(\lambda_s)}{d(\Delta\lambda)}\frac{d(\Delta\lambda)}{\Delta\varphi}$$
$$=\left[\frac{S(\lambda_s)}{S(\Delta\lambda)}\right]^{1/2}\frac{d(\Delta\lambda)}{\Delta\varphi}, \tag{2.17}$$

where $\Delta\lambda$ is an infinitesimal (conformal) affine distance from the observer, for which the limit

*If the magnitude at a specific frequency band is considered, the z-dependence of m will be different depending on the source spectrum. However, the anisotropy of m remains the same.

$\Delta\lambda \to 0$ will be taken at the end of calculation. Then using the famous theorem of geometric optics that states (e.g. Misner *et al.* 1973)

$$\frac{d}{d\lambda}(\mathscr{A}^2 S)=0, \tag{2.18}$$

and introducing the distance $\Delta\hat{r}$;

$$\Delta\hat{r}=d(\Delta\lambda)/\Delta\varphi, \tag{2.19}$$

which is the spatial distance corresponding to the affine distance $\Delta\lambda$, in the rest frame of the observer, d_A is further re-expressed as

$$d_A=\frac{\mathscr{A}(0)}{\mathscr{A}(\lambda_s)}\Delta\hat{r}. \tag{2.20}$$

This is the fundamental relation which will be used in Section 4 for the discussion of the d_A–z relation.

Finally for convenience, we introduce the 4-velocity u^μ on the conformally transformed spacetime by

$$u^\mu = a\hat{u}^\mu, \tag{2.21}$$

which apparently satisfies the normalization

$$g_{\mu\nu}u^\mu u^\nu = a^2 g_{\mu\nu}\hat{u}^\mu\hat{u}^\nu = -1. \tag{2.22}$$

3 The magnitude–redshift relation

As noted in the previous section (equation 2.15), the derivation of the m–z relation is theoretically equivalent to that of the d_L–z relation. Hence we focus on the latter throughout this section, but one should keep in mind that the actual observable is m and d_L is determined only after m is measured.

In order to derive the formula for the d_L–z relation in an inhomogeneous universe, we utilize the usual perturbation technique; we solve the unperturbed equations first and proceed to the first order thereafter. For notational convenience, we denote a perturbed quantity with a tilde ($\tilde{\ }$) on top and an unperturbed one without the tilde in what follows. For example, instead of equation (2.2a) we express the metric as

$$\tilde{g}_{\mu\nu}=g_{\mu\nu}+\delta g_{\mu\nu}. \tag{3.1}$$

Further, we introduce a new null vector $\tilde{K}^\mu$ equivalent to $\tilde{k}^\mu$ but with a different affine parametrization such that

$$\tilde{K}^\mu = -\frac{1}{\tilde{\omega}(\lambda_s)a[\tilde{\eta}(\lambda_s)]}\tilde{k}^\mu, \tag{3.2}$$

namely, $\tilde{K}^\mu$ is the past-directed null vector along a ray of light emitted by the source which is normalized as

$$(\tilde{g}_{\mu\nu}\tilde{K}^\mu\tilde{u}^\nu)_{\lambda_s}=\frac{1}{\tilde{\omega}(\lambda_s)a[\tilde{\eta}(\lambda_s)]}(-\tilde{g}_{\mu\nu}\tilde{k}^\mu\tilde{u}^\nu)_{\lambda_s}=1, \tag{3.3}$$

where, and in the following, λ denotes the new affine parameter associated with $\tilde{K}^\mu$. We note that the basic propagation equations (2.7) and (2.8) remain the same under this affine transformation.

First, let us consider the light propagation on the unperturbed background. From equation (2.2b) we have

$$R_{\mu\nu}=2K\gamma_{ij}\delta^i_\mu\delta^j_\nu, \tag{3.4}$$

where $K=\pm 1$ or 0 corresponding to a positive, negative or flat curvature 3-space. From equations (2.7a), (2.7b) and (3.3) we obtain

$$K^\mu=\frac{dx^\mu}{d\lambda}=(-1,\, \gamma^i), \tag{3.5}$$

where γ^i satisfies

$$\gamma_{ij}\gamma^i\gamma^j=1, \qquad \gamma^i_{|j}\gamma^j=0, \tag{3.6}$$

that is, γ^i is the unit vector tangent to a geodesic on the constant curvature 3-space. Then inserting equations (3.4) and (3.5) into equation (2.8) and using the fact that $\sigma=0$ for a spherically symmetric source, we find

$$\begin{aligned}\frac{d}{d\lambda}\theta&=-2K-\frac{1}{2}\theta^2\\ &=-\frac{1}{2}(\theta+2\sqrt{-K})(\theta-2\sqrt{-K}).\end{aligned} \tag{3.7}$$

This can be easily integrated to give

$$\begin{aligned}\theta_{\rm L}&=2\sqrt{-K}\coth\sqrt{-K}(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s})\\ &=\begin{cases}2\coth(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s}); & K=-1\\ \dfrac{2}{\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s}}; & K=0\\ 2\cot(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s}); & K=+1,\end{cases}\end{aligned} \tag{3.8}$$

where the suffix L of θ indicates the appropriate boundary condition for the light rays of interest and $\Delta\lambda_{\rm s}$ is the infinitesimal affine distance corresponding to the source radius $R_{\rm s}$ (see Fig. 1).

The relation between $R_{\rm s}$ and $\Delta\lambda_{\rm s}$ is determined as follows. Let $(d\hat{t},\, d\hat{x}^i)$ be the elements of the local Lorentz coordinates associated with the source's rest frame. Then from the normalization (3.3) of λ and the relation $dv\propto a^2 d\lambda$ one finds

$$(\tilde{g}_{\mu\nu}\tilde{K}^\mu\tilde{u}^\nu)_{\lambda_{\rm s}}=\frac{1}{a[\tilde{\eta}(\lambda_{\rm s})]}=\frac{-1}{a[\tilde{\eta}(\lambda_{\rm s})]^2}\left(\frac{d\hat{t}}{d\lambda}\right)_{\lambda_{\rm s}}, \tag{3.9}$$

where $\tilde{K}^\mu=a^{-2}\tilde{K}^\mu$. On the other hand, the null geodesic on the physical spacetime satisfies

$$d\hat{s}^2=-d\hat{t}^2+\delta_{ij}\, d\hat{x}^i\, d\hat{x}^j=0. \tag{3.10}$$

Hence from equations (3.9) and (3.10), $R_{\rm s}$ is expressed as

$$R_{\rm s}=\sqrt{\delta_{ij}\, d\hat{x}^i\, d\hat{x}^j}=|\Delta\hat{t}|=a[\tilde{\eta}(\lambda_{\rm s})]\Delta\lambda_{\rm s}. \tag{3.11}$$

The first order deviation of $\theta_{\rm L}$ due to the presence of inhomogeneities is given by

$$\delta\theta_{\rm L}(\lambda)=\tilde{\theta}_{\rm L}[\tilde{x}^\mu(\lambda)]-\theta_{\rm L}[x^\mu(\lambda)], \tag{3.12}$$

where

$$\tilde{x}^\mu(\lambda)=x^\mu(\lambda)+\delta x^\mu(\lambda) \tag{3.13}$$

is the perturbed null geodesic. Then the perturbation of equation (2.8) gives

$$\frac{d}{d\lambda}\,\delta\theta_{\rm L}=-\theta_{\rm L}\delta\theta_{\rm L}-\delta(R_{\mu\nu}K^{\mu}K^{\nu})_{\lambda}, \tag{3.14}$$

where

$$\delta(R_{\mu\nu}K^{\mu}K^{\nu})_{\lambda}=(\tilde{R}_{\mu\nu}\tilde{K}^{\mu}\tilde{K}^{\nu})_{\tilde{x}^{\mu}(\lambda)}-(R_{\mu\nu}K^{\mu}K^{\nu})_{x^{\mu}(\lambda)}$$

$$=(\delta R_{\mu\nu}K^{\mu}K^{\nu})_{x^{\mu}(\lambda)}+4K\,\frac{d}{d\lambda}\,(\gamma_i\delta x^i). \tag{3.15}$$

Integrating equation (3.14) with the boundary condition $\delta\theta_{\rm L}(\lambda_{\rm s})=0$, we obtain

$$\delta\theta_{\rm L}(\lambda)=\frac{1}{{\rm sh}^2\sqrt{-K}(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s})}\int_{\lambda}^{\lambda_{\rm s}}d\lambda'\ {\rm sh}^2\sqrt{-K}(\lambda'-\lambda_{\rm s}-\Delta\lambda_{\rm s})\delta(R_{\mu\nu}K^{\mu}K^{\nu})_{\lambda'}. \tag{3.16}$$

With the aid of equation (3.8), equation (2.7c) can be integrated to give

$$\frac{\mathscr{A}(\lambda_{\rm s})a[\tilde{\eta}(\lambda_{\rm s})]}{\mathscr{A}(0)a[\tilde{\eta}(0)]}=\frac{{\rm sh}\sqrt{-K}(\lambda_{\rm s}+\Delta\lambda_{\rm s})}{{\rm sh}\sqrt{-K}\Delta\lambda_{\rm s}}\exp\left[-\frac{1}{2}\int_0^{\lambda_{\rm s}}d\lambda\ \delta\theta_{\rm L}(\lambda)\right], \tag{3.17}$$

where one should keep in mind that the last exponential factor is actually equivalent to $1-$(*the exponent*) in the accuracy of our approximation. Inserting equations (3.11) and (3.17) into equation (2.14) and taking the limit $\Delta\lambda_{\rm s}\to 0$, the luminosity distance is expressed in the form,

$$\tilde{d}_{\rm L}(\lambda_{\rm s})=a[\tilde{\eta}(0)]\,\frac{{\rm sh}\sqrt{-K}\lambda_{\rm s}}{\sqrt{-K}}\,[1+\tilde{z}(\lambda_{\rm s})]\,\exp\left(-\frac{1}{2}\int_0^{\lambda_2}\delta\theta_{\rm L}\,d\lambda\right), \tag{3.18}$$

where $\delta\theta_{\rm L}$ is given by equation (3.16) with $\Delta\lambda_{\rm s}=0$.

Although equation (3.18) is perfectly correct, it is not quite adequate since $d_{\rm L}$ is expressed as a function of $\lambda_{\rm s}$ (and tacitly the solid angle as seen by the observer) but $\lambda_{\rm s}$ is not directly observable. Rather we should express $\tilde{d}_{\rm L}$ as a function of the redshift. In order to do so, we first note that

$$1+\tilde{z}=\frac{\tilde{\omega}(\lambda_{\rm s})}{\tilde{\omega}(0)}=\frac{a[\tilde{\eta}(0)]}{a[\tilde{\eta}(\lambda_{\rm s})]}\,\frac{(\tilde{K}_{\mu}\tilde{u}^{\mu})_{\lambda_{\rm s}}}{(\tilde{K}\tilde{u}^{\mu})_0}$$

$$=\frac{a[\eta(0)]}{a[\eta(\lambda_{\rm s})]}\left\{1-\left[\frac{a'}{a}\,\delta\eta+\frac{d}{d\lambda}\,\delta\eta-A-(\beta_i+v_i)\gamma^i\right]_0^{\lambda_{\rm s}}\right\}, \tag{3.19}$$

where $[\ldots]_0^{\lambda_{\rm s}}$ is the difference of the quantity inside the square brackets evaluated at $\lambda=\lambda_{\rm s}$ and $\lambda=0$ and A, β_i and v_i are perturbation variables defined by

$$\delta g_{0_\mu}=(-2A,\ \beta_i), \tag{3.20a}$$

$$\delta g_{ij}=H_{ij} \tag{3.20b}$$

$$\delta u^{\mu}=(-A,\ v^i). \tag{3.20c}$$

Here the variable H_{ij} is also introduced for later use and all the spatial indices are to be raised or lowered with γ^{ij} or γ_{ij}. Note also that from the normalization of λ, equation (3.3), one has

$$\delta(\tilde{K}_{\mu}\tilde{u}^{\mu})_{\lambda_{\rm s}}=\left[-\frac{d}{d\lambda}\,\delta\eta+A+(\beta_i+v_i)\gamma^i\right]_{\lambda_{\rm s}}=0. \tag{3.21}$$

Next we replace $\lambda_{\rm s}$ in equations (3.18) and (3.19) by $\lambda_{\rm s}(z)+\delta\lambda_{\rm s}$ where $\lambda_{\rm s}(z)$ is the affine distance corresponding to the redshift z on the unperturbed background, i.e., $\lambda_s(z)$ is defined implicitly by the relation,

$$1+z=\frac{a[\eta(0)]}{a[\eta(\lambda_s)]}=\frac{a(\eta_0)}{a(\eta_0-\lambda_s)}; \qquad \eta_0=\eta(0). \tag{3.22}$$

Then equation (3.18) becomes

$$\tilde{d}_L(z,\lambda^i)=d_L(z)\left[1+\left(\frac{a'}{a}\,\delta\eta\right)_0+\coth\sqrt{-K}\lambda_s\sqrt{-K}\,\delta\lambda_s-\frac{1}{2}\int_0^{\lambda_s}\delta\theta_L(\lambda)\,d\lambda\right], \tag{3.23}$$

where, and in what follows, $\lambda_s=\lambda_s(z)$ as defined by equation (3.22), $d_L(z)$ is the luminosity distance on the unperturbed background;

$$d_L(\eta)=a(\eta_0)\,\frac{\mathrm{sh}\sqrt{-K}\lambda_s}{\sqrt{-K}}\,(1+z), \tag{3.24}$$

and $\delta\lambda_s$ is given by

$$\delta\lambda_s=\frac{1}{(a'/a)_{\lambda_s}}\left[\frac{a'}{a}\,\delta\eta+\frac{d}{d\lambda}\,\delta\eta-A-(\beta_i+v_i)\gamma^i\right]_0^{\lambda_s}$$

$$=\delta\eta(\lambda_s)-\frac{1}{(a'/a)_{\lambda_s}}\left[\frac{a'}{a}\,\delta\eta+\frac{d}{d\lambda}\,\delta\eta-A-(\beta_i+v_i)\gamma^i\right]_0, \tag{3.25}$$

the first line of which follows from equation (3.19) with $\tilde{z}=z$ and λ_s replaced by $\lambda_s+\delta\lambda_s$, and the second line from equation (3.21).

Equation (3.23) is the general formula for the d_L–z relation in a perturbed Friedmann universe. The formula does not assume any specific gauge in which it should be evaluated. Since the d_L–z relation is a gauge-independent concept, this implies that the right-hand side of equation (3.23) is gauge-invariant. Therefore, as a verification of the formula, let us show its gauge-invariance explicitly.

Under a gauge transformation induced by an infinitesimal coordinate transformation

$$\bar{x}^\mu=x^\mu+\xi^\mu, \tag{3.26}$$

$\delta g_{\mu\nu}$ and δu^μ transform as

$$\bar{A}=A-T'+\frac{a'}{a}\,T, \tag{3.27a}$$

$$\bar{\beta}_i=\beta_i+T_{|i}+L_i', \tag{3.27b}$$

$$\bar{H}_{ij}=H_{ij}-2\,\frac{a'}{a}\,T\gamma_{ij}-(L_{i|j}+L_{j|i}), \tag{3.27c}$$

$$\bar{v}^i=v^i-L^{i\prime}, \tag{3.27d}$$

where a vertical bar denotes covariant differentiation with respect to γ_{ij}, a prime partial differentiation with respect to η and ξ^μ is represented as

$$\xi^\mu=(T,L^i). \tag{3.28}$$

As for the gauge transformation properties associated with the perturbation of the null geodesic, one should be slightly careful about deriving them, since the affine parameter λ does get affected by gauge transformations. To find how λ is affected, we note that equation (3.3) implies

$$\tilde{\omega}(\lambda_s+\delta\lambda_s)\ dv=-\frac{a[\tilde{\eta}(\lambda)]^2}{a[\tilde{\eta}(\lambda_s+\delta\lambda_s)]}\ d\lambda, \tag{3.29}$$

where λ_s has been replaced by $\lambda_s+\delta\lambda_s$ as mentioned before. Now, since $\omega(\lambda_s+\delta\lambda_s)$ is the proper frequency of a light ray emitted by the source on the rest frame of the source and v is the physical affine parameter along the ray as fixed by equation (2.4a), the left-hand side is gauge-invariant. Hence

$$\frac{a[\bar{\tilde{\eta}}(\bar{\lambda})]^2}{a[\bar{\tilde{\eta}}(\lambda_s+\overline{\delta\lambda}_s)]}\ d\bar{\lambda}=\frac{a[\tilde{\eta}(\lambda)]^2}{a[\tilde{\eta}(\lambda_s+\delta\lambda_s)]}\ d\lambda, \tag{3.30}$$

where from equation (3.26) one has

$$\bar{\tilde{\eta}}(\bar{\lambda})=\tilde{\eta}(\lambda)+T[x^\mu(\lambda)], \tag{3.31a}$$

or in general,

$$\bar{\tilde{x}}^\mu(\bar{\lambda})=x^\mu(\lambda)+\xi^\mu(\lambda). \tag{3.31b}$$

From equations (3.30) and (3.31a) we obtain

$$d\bar{\lambda}=(1-\tau)\ d\lambda, \tag{3.32a}$$

$$\overline{\delta\lambda}_s=\delta\lambda_s-\int_0^{\lambda_s}\tau\ d\lambda, \tag{3.32b}$$

where

$$\tau(\lambda)=\left(2\frac{a'}{a}T\right)_\lambda-\left(\frac{a'}{a}T\right)_{\lambda_s}. \tag{3.33}$$

Then it is easy to see that

$$\overline{\delta x}^\mu(\lambda)=\delta x^\mu(\lambda)+\xi^\mu(\lambda)+\int_0^\lambda \tau(\lambda')\ d\lambda' K^\mu(\lambda), \tag{3.34}$$

or

$$\frac{D}{d\lambda}\overline{\delta x}^\mu=\frac{D}{d\lambda}\delta x^\mu+\frac{D}{d\lambda}\xi^\mu+\tau K^\mu. \tag{3.35}$$

From the above gauge transformation properties, one can find how each term in the right-hand side of equation (3.23) transforms:

$$\left(\frac{a'}{a}\overline{\delta\eta}\right)_0=\left(\frac{a'}{a}\delta\eta\right)_0+\left(\frac{a'}{a}T\right)_0, \tag{3.36a}$$

$$\coth\sqrt{-K}\lambda_s\sqrt{-K}\,\overline{\delta\lambda}_s=\coth\sqrt{-K}_s\sqrt{-K}\left(\delta\lambda_s-\int_0^{\lambda_s}\tau\ d\lambda\right), \tag{3.36b}$$

$$-\frac{1}{2}\int_0^{\lambda_s}\overline{\delta\theta}_L\ d\lambda=-\frac{1}{2}\int_0^{\lambda_s}\delta\theta_L\ d\lambda+\coth\sqrt{-K}\lambda_s\sqrt{-K}\int_0^{\lambda_s}\tau\ d\lambda+\frac{1}{2}\,[\tau(\lambda_s)+\tau(0)]. \tag{3.36c}$$

Recalling equation (3.33), it is now apparent that the sum of these three terms are gauge-invariant as a whole. Hence we verified the gauge-invariance of equation (3.23).

4 The angular diameter distance

The basic method for deriving the d_A–z relation is the same as that for deriving the d_L–z relation. However there are some minor differences; one of them is the fact that the light rays to be considered for the d_A–z relation are those expanding from the observer's world line, as opposed to the case of the d_L–z relation in which the light rays are expanding from the source. Hence instead of equation (3.8), the expansion of the rays on the unperturbed background is given by

$$\theta_A = 2\sqrt{-K}\coth\sqrt{-K}\lambda, \tag{4.1}$$

and the perturbation of it is expressed as

$$\delta\theta_A = \frac{-1}{\mathrm{sh}^2\sqrt{-K}\lambda}\int_0^\lambda d\lambda'\ \mathrm{sh}^2\sqrt{-K}\lambda'\,\delta(R_{\mu\nu}K^\mu K^\nu)_{\lambda'}, \tag{4.2}$$

where the overall minus sign comes from the fact the rays are pointing toward the past. Then instead of equation (3.17), the integration of equation (2.7c) gives

$$\frac{\mathcal{A}(\lambda_s)a[\tilde{\eta}(\lambda_s)]}{\mathcal{A}(\Delta\lambda)a[\tilde{\eta}(\Delta\lambda)]} = \frac{\mathrm{sh}\sqrt{-K}\Delta\lambda}{\mathrm{sh}\sqrt{-K}\lambda_s}\exp\left[-\frac{1}{2}\int_{\Delta\lambda}^{\lambda_s}\delta\theta_A(\lambda)\,d\lambda\right], \tag{4.3}$$

where $\Delta\lambda$ is the infinitesimal affine distance introduced in equation (2.17).

Now, we have to express $\Delta\hat{r}$, the proper spatial distance corresponding to $\Delta\lambda$ as given by equation (2.19), in terms of $\Delta\lambda$. This is done in the same way as in the case of R_s discussed around equations (3.9)–(3.11). Thus

$$\begin{aligned}(\tilde{g}_{\mu\nu}\tilde{K}^\mu\tilde{u}^\nu)_0 &= \frac{1}{[1+\tilde{z}(\lambda_s)]a[\tilde{\eta}(\lambda_s)]}\\ &= \frac{-1}{a[\tilde{\eta}(0)]^2}\left(\frac{d\hat{t}}{d\lambda}\right)_0.\end{aligned} \tag{4.4}$$

Consequently we find

$$\Delta\hat{r} = |\Delta\hat{t}| = \frac{a[\tilde{\eta}(0)]^2\Delta\lambda}{[1+\tilde{z}(\lambda_s)]a[\tilde{\eta}(\lambda_s)]}. \tag{4.5}$$

Inserting equations (4.3) and (4.5) into equation (2.20) and taking the limit $\Delta\lambda \to 0$, we obtain

$$\tilde{d}_A(\lambda_s) = \frac{a[\tilde{\eta}(0)]}{1+\tilde{z}(\lambda_s)}\,\frac{\mathrm{sh}\sqrt{-K}\lambda_s}{\sqrt{-K}}\exp\left[+\frac{1}{2}\int_0^{\lambda_s}\delta\theta_A(\lambda)\,d\lambda\right]. \tag{4.6}$$

This is the fundamental formula for the angular diameter distance, which corresponds to equation (3.18) for the luminosity distance. Comparing equation (4.6) with equation (3.18), one finds the relation

$$\tilde{d}_A(z,\gamma^i) = \frac{\tilde{d}_L(z,\gamma^i)}{(1+z)^2}\exp\left[\frac{1}{2}\int_0^{\lambda_s}(\delta\theta_L+\delta\theta_A)\,d\lambda\right], \tag{4.7}$$

where we have used the fact that $\tilde{z}(\lambda_s+\delta\lambda_s)=z(\lambda_s)$ as discussed in the previous section.

The only remaining task is then to evaluate the exponential factor in equation (4.7). In general, one might expect that it depends on specific features of inhomogeneities as well as on background models. However, there is a theorem, known as the reciprocity theorem, which states that the relation $d_A = d_L/(1+z)^2$ holds in an arbitrary spacetime (Ellis 1971)★. Hence the exponent in

★This theorem may be viewed as a consequence of the invariance of the photon phase-space density.

equation (4.7) should vanish identically. For completeness, let us demonstrate this fact in the following.

Let $\delta(R_{\mu\eta}K^\mu K^\nu)_\lambda$ be an arbitrary function of λ and let us introduce the functionals

$$I_L[f]=\int_0^{\lambda_s}\frac{d\lambda}{\mathrm{sh}^2\sqrt{-K}(\lambda-\lambda_s)}\int_\lambda^{\lambda_s}d\lambda'\ \mathrm{sh}^2\sqrt{-K}(\lambda'-\lambda_s)f(\lambda'), \tag{4.8a}$$

$$I_A[f]=-\int_0^{\lambda_s}\frac{d\lambda}{\mathrm{sh}^2\sqrt{-K}\lambda}\int_0^{\lambda}d\lambda'\ \mathrm{sh}^2\sqrt{-K}\lambda' f(\lambda'). \tag{4.8b}$$

These correspond to the integrals

$$\int_0^{\lambda_s}\delta\theta_L\,d\lambda \quad\text{and}\quad \int_0^{\lambda_s}\delta\theta_A\,d\lambda,$$

respectively. Then, assuming that $f(\lambda)$ is regular at $0\leq\lambda\leq\lambda_s$, one can perform integration by parts and reduce the double integrals in I_L and I_A to single integrals. The results are

$$I_L=\frac{-1}{\sqrt{-K}}\int_0^{\lambda_s}d\lambda[\coth\sqrt{-K}\lambda_s+\coth\sqrt{-K}(\lambda-\lambda_s)]\ \mathrm{sh}^2\sqrt{-K}(\lambda-\lambda_s)f(\lambda), \tag{4.9a}$$

$$I_A=\frac{1}{\sqrt{-K}}\int_0^{\lambda_s}d\lambda(\coth\sqrt{-K}\lambda_s-\coth\sqrt{-K}\lambda)\ \mathrm{sh}^2\sqrt{-K}\lambda f(\lambda). \tag{4.9b}$$

Now, using the addition theorem for exponential functions, it is straightforward to see that $I_L=-I_A$.

Thus regardless of the nature of inhomogeneities and of models for the unperturbed background, one has

$$\tilde{d}_A(z,\gamma^i)=\frac{\tilde{d}_L(z,\gamma^i)}{(1+z)^2}. \tag{4.10}$$

Since d_A, m (or d_L) and z can be determined by respectively independent observations, the above relation could become important when one would try to exclude effects on the m–z relation and/or the d_A–z relation other than purely geometrical effects such as those considered here.

5 Application to the Einstein–de Sitter background model

On the Einstein–de Sitter background, equation (3.23) reduced to

$$\tilde{d}_L(z,\gamma^i)=d_L(z)\left\{1+\frac{2}{\eta_0}\,\delta\eta_0+\frac{\delta\lambda_s}{\lambda_s}-\frac{1}{2}\,I_L[\delta(R_{\mu\nu}K^\mu K^\nu)_\lambda]\right\}, \tag{5.1}$$

where

$$d_L(z)=\frac{2\lambda_s}{H_0\eta_0}(1+z)=\frac{2}{H_0}\sqrt{1+z}(\sqrt{1+z}-1), \tag{5.2a}$$

$$\delta\lambda_s=\delta\eta_s-\frac{\eta_s}{2}\left[\frac{2}{\eta_0}\,\delta\eta_0+\left(\frac{d}{d\lambda}\,\delta\eta\right)_0-A_0-(\beta_i+v_i)_0\gamma^i\right], \tag{5.2b}$$

$$\frac{1}{2}\,I_L=\frac{1}{2\lambda_s}\int_0^{\lambda_s}d\lambda(\lambda-\lambda_s)\lambda\delta(R_{\mu\nu}K^\mu K^\nu)_\lambda. \tag{5.2c}$$

In the above the suffix (0) denotes the value evaluated at $\lambda=0$, the suffix (s) at $\lambda=\lambda_s$, $\eta_s=\eta_0-\lambda_s$ and $H_0=(a'/a^2)_0=2/(a_0\eta_0)$ is the Hubble constant. Note that only the time coordinate of the geodesic appears in the present case.

When space–time inhomogeneities are due to density perturbations, perturbations of all quantities can be described by functions which are scalar with respect to spatial indices. Specifically, in addition to $A=-1/2\delta g_{00}$ which is a scalar from the beginning, we have

$$\beta_i=B_{|i}, \tag{5.3a}$$

$$H_{ij}=2\mathscr{R}\gamma_{ij}+2H_{\mathrm{T}|ij}, \tag{5.3b}$$

$$v_i=-v_{|i}, \tag{5.3c}$$

where we have assumed that the velocities of both the source and the observer are of cosmological origin, i.e., correlated to A, β_i and H_{ij}. Then $\delta(R_{\mu\nu}K^\mu K^\nu)_\lambda$ is expressed as

$$\delta(R_{\mu\nu}K^\mu K^\nu)_\lambda=[A-\mathscr{R}-(H'_{\mathrm{T}}-B)']^{|i}_{|i}-2\mathscr{R}''+4\mathscr{R}_{|i}\gamma^i-[A+\mathscr{R}-(H'_{\mathrm{T}}-B)']_{|ij}\gamma^i\gamma^j, \tag{5.4}$$

and the time component of the geodesic equation gives

$$\frac{d^2}{d\lambda^2}\,\delta\eta=\frac{d}{d\lambda}\left\{-\frac{d}{d\lambda}\,(H'_{\mathrm{T}}-B)+2[A-(H'_{\mathrm{T}}-B)']\right\}+[A-\mathscr{R}-(H'_{\mathrm{T}}-B)']'. \tag{5.5}$$

Up to now we have not specified a gauge to work with. In order to find a gauge which is most convenient for the evaluation of equation (5.1), we resort to the method of gauge-invariant cosmological perturbation theory (Bardeen 1980; Kodama & Sasaki 1984). According to it, the metric perturbations are most conveniently described by the following two mutually independent variables;

$$\Psi=A-\frac{1}{a}\,[a(H'_{\mathrm{T}}-B)]', \tag{5.6a}$$

$$\Phi=\mathscr{R}-\frac{a'}{a}\,(H'_{\mathrm{T}}-B), \tag{5.6b}$$

and the gauge-invariant velocity field is given by

$$V=v-H'_{\mathrm{T}}. \tag{5.7}$$

In terms of Ψ, Φ and V, the general growing mode solution is known to be

$$\Psi=-\Phi=\text{a function of } x^i \text{ only}, \tag{5.8a}$$

$$V=\frac{1}{3}\,\eta\Psi. \tag{5.8b}$$

We note that Ψ is just the Newtonian potential in the present case. In the following, all the effects of perturbations are expressed in terms of Ψ.

Taking the above solution into account, an inspection of equations (5.4) and (5.5) suggests immediately that the gauge $B=H'_{\mathrm{T}}=0$, known as the longitudinal gauge, is most convenient. Denoting a quantity evaluated in the longitudinal gauge by the subscript $_*$ equations (5.4) and (5.5) become

$$\delta_*(R_{\mu\nu}K^\mu K^\nu)_\lambda=2\overset{(3)}{\Delta}\Psi, \tag{5.9}$$

$$\frac{d^2}{d\lambda^2}\,\delta_*\eta=2\,\frac{d}{d\lambda}\,\Psi, \tag{5.10}$$

and equations (5.2b) and (5.2c) become

$$\delta_*\lambda_s=\delta_*\eta_s-\frac{\eta_s}{2}\left[\frac{2}{\eta_0}\,\delta_*\eta_0+\left(\frac{d}{d\lambda}\,\delta_*\eta\right)_0-\Psi_0+\frac{1}{3}\,\eta_0(\Psi_{|i}\gamma^i)_0\right], \tag{5.11a}$$

$$\frac{1}{2}\,I_L=\frac{1}{\lambda_s}\int_0^{\lambda_s}d\lambda(\lambda-\lambda_s)\lambda\overset{(3)}{\Delta}\Psi, \tag{5.11b}$$

where $\Psi=A_*=-\mathscr{R}_*$ as is clear from equations (5.6) and (5.8a).

To evaluate $\delta_*\lambda_s$ we just have to integrate equation (5.10);

$$\left(\frac{d}{d\lambda}\,\delta_*\eta\right)_s=\left(\frac{d}{d\lambda}\,\delta_*\eta\right)_0+2(\Psi_s-\Psi_0), \tag{5.12a}$$

$$\delta_*\eta_s=\delta_*\eta_0+\left(\frac{d}{d\lambda}\,\delta_*\eta\right)_0\lambda_s+2\int_0^{\lambda_s}(\Psi-\Psi_0)\,d\lambda. \tag{5.12b}$$

Here we note that the value of $(d/d\lambda\ \delta_*\eta)_0$ is determined from equation (3.21) which reads

$$\left(\frac{d}{d\lambda}\,\delta_*\eta\right)_s=\Psi_s-\frac{1}{3}\,\eta_s(\Psi_{|i}\gamma^i)_s. \tag{5.13}$$

Hence equations (5.12a) and (5.13) give

$$\left(\frac{d}{d\lambda}\,\delta_*\eta\right)_0=-\Psi_s+2\Psi_0-\frac{1}{3}\,\eta_s(\Psi_{|i}\gamma^i)_s. \tag{5.14}$$

Then equations (5.12b) and (5.14) give

$$\delta_*\eta_s=\delta_*\eta_0+\lambda_s\left[-\Psi_s+2\Psi_0-\frac{1}{3}\,\eta_s(\Psi_{|i}\gamma^i)_s\right]+2\int_0^{\lambda_s}(\Psi-\Psi_0)\,d\lambda. \tag{5.15}$$

Equations (5.14) and (5.15) are just what we need to express $\delta_*\lambda_s$ in terms of Ψ (and of $\delta_*\eta_0$, determination of which will be discussed later).

Inserting equations (5.14) and (5.15) into equation (5.11a), arranging terms and inserting the resulting expression for $\delta_*\lambda_s$ and equation (5.11b) into equation (5.1), we obtain

$$\begin{aligned}\frac{\Delta d_L(z,\gamma^i)}{d_L(z)}=&\frac{1}{\lambda_s}\int_0^{\lambda_s}d\lambda\left[(\lambda-\lambda_s)\lambda\overset{(3)}{\Delta}\Psi+2(\Psi-\Psi_0)\right]+\frac{1}{2}\left(\frac{\eta_0}{\lambda_s}-3\right)(\Psi_s-\Psi_0)\\&+\frac{1}{6}\left(\frac{\eta_0}{\lambda_s}-3\right)[\eta_s(\psi_{|i}\gamma^i)_s-\eta_0(\psi_{|i}\gamma^i)_0]-\frac{1}{3}\,\eta_0(\psi_{|i}\gamma^i)_0+\Psi_0+\frac{3}{\eta_0}\,\delta_*\eta_0,\end{aligned} \tag{5.16}$$

where $\Delta d_L=\tilde{d}_L-d_L$. We note that from equation (2.15) one has $\Delta m=(5/\log_{10})\Delta d_L/d_L$. The above is the final formula that describes the effects of density (or gravitational potential) inhomogeneities on the d_L-z (or $m-z$) relation except for the ambiguity associated with the last term involving $\delta_*\eta_0$. However, it is an irrelevant term for the discussion of anisotropy in the m–z relation since it depends neither on γ^i nor on z. Nevertheless, it would be nice if we could eliminate the ambiguity from equation (5.16). Fortunately, this may be done by noting that $\delta_*\eta_0$ is simply a constant shift in the observer's time coordinate or, equivalently, a slight change in the Hubble constant H_0. Hence a natural resolution to the problem is to impose the condition that $\Delta d_L/d_L$, when averaged over solid angles, vanishes in the limit $z\to 0$. Consequently from

$$\frac{\Delta d_L}{d_L}\xrightarrow[z\to 0]{}\frac{3}{\eta_0}\,\delta_*\eta_0+\Psi_0+\frac{\eta_0^2}{6}\,(\Psi_{|ij}\gamma^i\gamma^j)_0, \tag{5.17}$$

one has

$$\delta_* \eta_0 = -\frac{\eta_0}{3}\left[\Psi_0 + \frac{\eta_0^2}{18}\overset{(3)}{\Delta}\Psi_0\right]. \tag{5.18}$$

Having obtained the final formula for the d_L–z (or m–z) relation under the presence of density inhomogeneities, let us now consider its implications when compared with observational data. For this purpose, we decompose equation (5.16) into multipole components as usually done for analysing anisotropies of a physical quantity. Then

$$\frac{\Delta d_L}{d_L} = \sum_{l,m} C_{l,m}(z) Y_{lm}(\Omega), \tag{5.19}$$

with $Y_{lm}(\Omega)$ being the conventional spherical harmonic and

$$\begin{aligned} C_{l,m} &= \int \frac{\Delta d_L}{d_L} Y_{lm}^*(\Omega)\, d\Omega \\ &= 4\pi i^l \left[\frac{1}{y_s}\int_0^{y_s} dy\{y(y_s-y)j_l(y)+2[j_l(y)-j_l(0)]\}+\frac{1}{2}\left(\frac{\eta_0}{\lambda_s}-3\right)[j_l(y_s)-j_l(0)]\right. \\ &\quad +\frac{1}{6}\left(\frac{\eta_0}{\lambda_s}-3\right)\left[(y_0-y_s)j_l'(y_s)-y_0 j_l'(0)\right] \\ &\quad \left. -\frac{1}{3}y_0 j_l'(0)+\frac{y_0^2}{18}j_l(0)\right]\Psi_k Y_{lm}^*(\Omega_k)\exp(i\mathbf{k}\cdot\mathbf{x}_0), \end{aligned} \tag{5.20}$$

where Ψ has been expressed in terms of its Fourier components;

$$\Psi = \sum_k \Psi_k \exp(i\mathbf{k}\cdot\mathbf{x}), \tag{5.21}$$

Ω_k denotes the solid angle of $\mathbf{k}$, $j_l(y)$ is the spherical Bessel function of order l, $j_l'(y)$ is its derivative, $\mathbf{x}_0$ is the spatial coordinate of the observer, $y_s = k\lambda_s$, $y_0 = k\eta_0$ and $k = |\mathbf{k}|$.

In reality, only a first few multipoles would be observationally meaningful owing to the limited number of source objects. Further the monopole would be of little interest since it would only give rise to small fluctuations in the solid-angle-averaged m–z relation which would be well hidden in the effect of intrinsic variations of sources. Hence we concentrate our discussion on the dipole and quadrupole components.

The dipole components are usually regarded as representing the effect of peculiar velocities of sources and the observer. However, it is apparent that $C_{1,m}$ ($m = -1, 0, 1$) contains terms due to effects other than peculiar velocities, which may be regarded as intrinsically gravitational effects. In fact, if the anisotropy were due only to peculiar velocities, one would have

$$\begin{aligned} \frac{\Delta d_L^{(\mathrm{P.V.})}}{d_L} &= -\frac{\eta_s}{2\lambda_s}[(v_i\gamma^i)_s-(v_i\gamma^i)_0] \\ &= \frac{\eta_s}{2\lambda_s}[(V_{|i}\gamma^i)_s-(V_{|i}\gamma^i)_0], \end{aligned} \tag{5.22}$$

and, instead of equation (5.20), one would obtain

$$C_{l,m}^{(\mathrm{P.V.})} = 4\pi i^l \frac{\eta_s}{6\lambda_s}[(y_0-y_s)j_l'(y_s)-y_0 j_l'(0)]\Psi_k Y_{lm}^*(\Omega_k)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}_0), \tag{5.23}$$

which looks quite different from $C_{l,m}$. In particular, comparison of $C_{l,m}^{(\mathrm{P.V.})}$ with $C_{l,m}$ shows that $C_{l,m}^{(\mathrm{P.V.})}$ is a good approximation to $C_{l,m}$ only at $z \ll 1$ $(\lambda_s/\eta_0 \ll 1)$; at $z \gtrsim 1$, $C_{l,m}$ is totally different from what one might naively expect. This is an important point which should be taken into account when one attempts to determine the peculiar velocity field at a large redshift by observations.

The quadrupole components are of great interest since they are the only ones that remain finite at $z \to 0$ as can be seen from equation (5.17). Specifically, the dominant term of $C_{2,m}$ at $z \ll 1$ is given by

$$C_{2,m} \sim -4\pi \frac{j_2'(y_s)}{6y_s} y_0^2 \Psi_k Y_{2m}^*(\Omega_k) \exp(i\boldsymbol{k}\cdot\boldsymbol{x}_0), \tag{5.24}$$

where we note that $y_0^2\Psi_k = -6(\Delta\varrho_0/\varrho_0)_k$ from the Newtonian Poisson equation. Hence $C_{2,m}$ directly represents the amplitude of density inhomogeneities in the present universe and could perhaps be evaluated from observational data already available at present.

6 Conclusions

We have derived a general formula for the m–z relation in an inhomogeneous universe which can be described by a Friedmann–Robertson–Walker metric plus small perturbations. In passing, we have also discussed the relation between the luminosity distance (d_L) and the angular diameter distance (d_A) and reconfirmed the reciprocity theorem, $d_L = (1+z)^2 d_A$, in our approximation scheme.

Then we have applied the formula to the Einstein–de Sitter universe with small density inhomogeneities and expressed the anisotropy in the m–z relation in terms of the corresponding gravitational potential inhomogeneities. In particular, the anisotropy has been decomposed into multipole components and the significance of dipole and quadrupole components has been discussed. It has been stressed that not only do peculiar velocities of source objects and the observer contribute to the dipole but also other (gravitational) effects as well; the latter become important at $z \gtrsim 1$. Further the importance of observing the quadrupole has been argued since it is the unique multipole that remains finite at $z \to 0$ and that is directly related to the amplitude of density inhomogeneities in the present universe.

Thus our formula for the m–z relation could play a powerful role in identifying large-scale inhomogeneities in the near future when a large enough number of luminous sources at high redshifts are observed.

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