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THE MAIN EIGENVALUES OF A GRAPH: A SURVEY

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We survey results relating main eigenvalues and main angles to the structure of a graph. We provide a number of short proofs, and note the connection with star partitions. We discuss graphs with just two main eigenvalues in the context of measures of irregularity, and in the context of harmonic graphs.

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G) = \{1, 2, \dots, n\}$ and $(0, 1)$-adjacency matrix A. The eigenvalue μ of A is said to be a *main* eigenvalue of G if the eigenspace $\mathcal{E}(\mu)$ is not orthogonal to the all-1 vector $\mathbf{j}$. An eigenvector $\mathbf{x}$ is a *main* eigenvector if $\mathbf{x}^\top\mathbf{j} \neq 0$. The main eigenvalues of the connected graphs of order ≤ 5 are listed in [**12**, Appendix B], and those of all the connected graphs on 6 vertices are given in [**10**]. In this section we introduce notation and survey the basic results concerning main eigenvalues and main angles (as defined below). In Section 2, we provide a general context for the investigation of the main eigenvectors of G and its complement $\overline{G}$. We also extend the notion of star partition to a refined star partition that takes account of main eigenvalues. In Section 3, we discuss graphs with just two main eigenvalues in the context of measures of irregularity of a graph, and we note the connection with harmonic graphs. In Section 4, we deal with a simple instance of graphs with just three main eigenvalues.

Let A have spectral decomposition

$$A = \mu_1 P_1 + \mu_2 P_2 + \cdots + \mu_m P_m. \tag{1}$$

The *main angles* of G are the numbers $\beta_1, \beta_2, \dots, \beta_m$, where $\beta_i = \dfrac{1}{\sqrt{n}}\,||P_i\,\mathbf{j}||$ $(i = 1, 2, \dots, m)$. These are the cosines of the angles between $\mathbf{j}$ and the eigenspaces of A, and so μ_i is a main eigenvalue if and only if $\beta_i \neq 0$. Since $||\mathbf{j}||^2 = \sum\limits_{1=1}^{m} ||P_i\mathbf{j}||^2$,

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we have $\sum_{1=1}^{m} {\beta_i}^2 = 1$. The main eigenvalues include the index (largest eigenvalue) of G because there exists a corresponding eigenvector with no negative entries [**8**, Theorem 0.4]. The main angles of the connected graphs of order ≤ 5 are listed in [**12**, Appendix B].

We take the main eigenvalues of G to be $\mu_1, \mu_2, \ldots, \mu_s$, with μ_1 the index of G; no further ordering is assumed for $\mu_2, \ldots, \mu_m$. We write

$$m_G(x) = (x - \mu_1)(x - \mu_2) \cdots (x - \mu_s).$$

Note that if μ is a main eigenvalue of G then so is any algebraic conjugate μ^* of μ; for if $A\mathbf{x} = \mu\mathbf{x}$ where $\mathbf{j}^\top \mathbf{x} \neq 0$ then $A\mathbf{x}^* = \mu^* \mathbf{x}^*$ where $\mathbf{j}^\top \mathbf{x}^* \neq 0$. It follows that $m_G(x) \in \mathbb{Z}[x]$, a fact established by other means in [**10**].

Proposition 1.1 (cf. [**16**, Theorem 2.6]). *For $k \in \{1, 2, \ldots, s\}$, let $m_G(x) = (x - \mu_k) f_k(x)$ and $g_k(x) = f_k(x)/f_k(\mu_k)$. Then $f_k(A)\mathbf{j}$ is a main eigenvector of G corresponding to μ_k, and*

$$\beta_k = \frac{1}{\sqrt{n}} ||g_k(A)\mathbf{j}|| \quad (k = 1, 2, \ldots, s).$$

Proof. From Equation (1), we have $f_k(A)\mathbf{j} = \sum_{i=1}^{m} f_k(\mu_i) P_i \mathbf{j} = f_k(\mu_k) P_k \mathbf{j}$, a non-zero vector. □

The main eigenvalues and main angles of G are related to the structure of G as follows.

Proposition 1.2. *If N_k denotes the number of walks of length k in G then*

$$N_k = n \sum_{i=1}^{s} {\mu_i}^k {\beta_i}^2.$$

Proof. We have $N_k = \mathbf{j}^\top A^k \mathbf{j} = \sum_{i=1}^{m} {\mu_i}^k \mathbf{j}^\top P_i \mathbf{j} = \sum_{i=1}^{s} {\mu_i}^k ||P_i \mathbf{j}||^2$. □

Since the Vandermonde matrix (μ_j^{i-1}) is invertible the integers $N_0, \ldots N_{s-1}$ determine $\beta_1, \ldots, \beta_s$ and hence all N_k. The *walk generating function* $H_G(t)$ is defined by $H_G(t) = \sum_{k=0}^{\infty} N_k t^k$, and it follows from Proposition 1.2 that

$$H_G(t) = \sum_{i=1}^{s} \frac{n{\beta_i}^2}{1 - \mu_i t}. \tag{2}$$

As noted in [**24**], it also follows from Proposition 1.2 that

$$N_{2q+r} \leq {\mu_1}^r N_{2q} \tag{3}$$

for all positive integers q, r. If $N_{2q+r} = {\mu_1}^r N_{2q}$ then the main eigenvalues lie in the set $\{-\mu_1, 0, \mu_1\}$, a situation discussed in Sections 3 and 4.

Proposition 1.3. *$\overline{G}$ has exactly s main eigenvalues.*

Proof. The adjacency matrix of $\overline{G}$ is $J - I - A$, where J denotes the all-1 matrix and I denotes the identity matrix of size $n \times n$. Note that

$$\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp \subseteq \mathcal{E}_{J-I-A}(-1-\mu_i) \quad (i = 1, 2, \ldots, m),$$

where we allow $\mathcal{E}_{J-I-A}(-1-\mu_i)$ to be the zero subspace. By extending an orthonormal basis $\mathcal{B}_i$ of $\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp$ to one of $\mathcal{E}_A(\mu_i)$ $(i = 1, 2, \ldots, m)$, we see that

$$n - s = \sum_{i=1}^{m} \dim(\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp). \tag{4}$$

Since the $n - s$ vectors in $\bigcup_{i=1}^{m} \mathcal{B}_i$ are linearly independent eigenvectors of $\overline{G}$ in $\mathbf{j}^\perp$, we deduce that $\overline{G}$ has at most s main eigenvalues. Interchanging G and $\overline{G}$, we see that $\overline{G}$ has exactly s main eigenvalues. □

We call $\sum_{i=1}^{m} (\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp)$ the *tangent space* for G. From the proof of Proposition 1.3, we see that G and $\overline{G}$ share the same tangent space. Nevertheless the eigenvalues $-1-\mu_i$ $(i = s+1, \ldots, m)$ are not necessarily the non-main eigenvalues of $\overline{G}$; for example, if G is the cycle C_4 then its non-main eigenvalues are 0 and -2, but $\overline{G} = 2K_2$, with -1 as its unique non-main eigenvalue. However, it follows from Proposition 1.3 that if we denote the main eigenvalues of $\overline{G}$ by $\overline{\mu}_1, \ldots, \overline{\mu}_s$, then the characteristic polynomial of $\overline{G}$ is

$$P_{\overline{G}}(x) = \prod_{i=1}^{s} (x - \overline{\mu}_i) \prod_{i=1}^{s} (x + 1 + \mu_i)^{k_i - 1} \prod_{i=s+1}^{m} (x + 1 + \mu_i)^{k_i}, \tag{5}$$

where k_i denotes the multiplicity of μ_i $(i = 1, \ldots, m)$. It follows that $\sum_{i=1}^{s} (\mu_i + \overline{\mu_i}) = n - s$ [**19**, Proposition 7].

We may also express $P_{\overline{G}}(x)$ as follows [**11**, p. 90]:

$$P_{\overline{G}}(x) = \det\big((x+1)I + A - J\big) = \det\big((x+1)I + A\big) - \mathbf{j}^\top \mathrm{adj}\big((x+1)I + A\big)\mathbf{j}$$

$$= (-1)^n P_G(-x-1)\big(1 - \mathbf{j}^\top \big((x+1)I + A\big)^{-1}\mathbf{j}\big)$$

and so

$$P_{\overline{G}}(x) = (-1)^n P_G(-1-x)\left(1 - n\sum_{i=1}^{s} \frac{{\beta_i}^2}{x + 1 + \mu_i}\right). \tag{6}$$

From Equations (2) and (6), we obtain the relation [**6**]:

$$H_G(t) = \frac{1}{t}\left\{-1 + (-1)^n \frac{P_{\overline{G}}\left(-\dfrac{t+1}{t}\right)}{P_G\left(\dfrac{1}{t}\right)}\right\}. \tag{7}$$

From Equation (5) we see that

$$P_{\overline{G}}(-1-x)/P_G(x) = (-1)^{n-s} m_{\overline{G}}(-1-x)/m_G(x). \tag{8}$$

Equation (8) allows us to rewrite Equation (7) as:

$$H_G(t) = \frac{1}{t}\left\{-1 + (-1)^s \frac{m_{\overline{G}}\left(-\dfrac{t+1}{t}\right)}{m_G\left(\dfrac{1}{t}\right)}\right\}.$$

From Equations (5) and (8), we have

$$\begin{aligned}(x-\overline{\mu}_1)(x-\overline{\mu}_2)\cdots(x-\overline{\mu}_s)&\\ (x+1+\mu_1)(x+1+\mu_2)\cdots(x+1+\mu_s)&\left(1 - n\sum_{i=1}^{s}\frac{\beta_i^2}{x+1+\mu_i}\right).\end{aligned} \tag{9}$$

If we write the RHS of Equation (9) in polynomial form, we can see that none of $-1-\mu_i$ $(1 = 1,\dots,s)$ is a main eigenvalue of $\overline{G}$ [**16**, Theorem 3.1].

Using Equation (9) for both G and $\overline{G}$, we see that the main eigenvalues and main angles of $\overline{G}$ are determined by the main eigenvalues and main angles of G. In particular, it follows that $m_{\overline{G}}(x)$ is determined by $m_G(x)$ and $N_0, N_1, \dots, N_{s-1}$. The relation between the (integer) coefficients in $m_{\overline{G}}(x)$, the (integer) coefficients in $m_G(x)$ and the integers $N_0, N_1, \dots, N_{s-1}$ is determined in Corollary 2.5 below. The relation between main eigenvectors of G and main eigenvectors of $\overline{G}$ is described in Theorem 2.6.

The cone over G, usually denoted by $K_1 \nabla G$, can be constructed as $\overline{K_1 \dot\cup \overline{G}}$, and so its characteristic polynomial can be derived from two applications of Equation (6). Alternatively [**11**, p.90], we may note that

$$\begin{vmatrix} x & -\mathbf{j}^\top \\ -\mathbf{j} & xI - A \end{vmatrix} = x\det(xI-A) - \mathbf{j}^\top \mathrm{adj}(xI-A)\,\mathbf{j}$$

to obtain:

$$P_{K_1\nabla G}(x) = P_G(x)\left(x - n\sum_{i=1}^{s}\frac{{\beta_i}^2}{x-\mu_i}\right).$$

The next two observations relate the number of main eigenvalues to the structure of G.

Proposition 1.4. $s = 1$ *if and only if* G *is regular.*

Proof. $s = 1$ if and only if $\mathbf{j} \in (\mathcal{E}(\mu_2) + \cdots + \mathcal{E}(\mu_m))^{\perp}$, that is, $s = 1$ if and only if $\mathbf{j} \in \mathcal{E}(\mu_1)$. □

For $U \subseteq V(G)$, let $\mathbf{e}_U$ denote the characteristic vector of U, and write $\mathbf{e}_j$ for $\mathbf{e}_{\{j\}}$. For any permutation π in the symmetric group S_n, let M_π be the corresponding permutation matrix, so that $M_\pi(x_1, x_2, \ldots, x_n)^\top = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})^\top$. Then π is an automorphism of G if and only if $M_\pi A = AM_\pi$. The following result was established in [**9**] by a character-theoretic argument.

Proposition 1.5. *The automorphism group of* G *has at least* s *orbits in* $V(G)$*.*

Proof. Since P_i is a polynomial in A, we have $M_\pi P_i = P_i M_\pi$, and hence $M_\pi P_i\,\mathbf{j} = P_i\,\mathbf{j}$ for all $\pi \in \mathrm{Aut}(G)$. Now the subspace $S = \{\mathbf{x} \in \mathbb{R}^n : M_\pi \mathbf{x} = \mathbf{x}\ \forall\ \pi \in \mathrm{Aut}(G)\}$ has basis $\{\mathbf{e}_{U_1}, \ldots, \mathbf{e}_{U_r}\}$, where $U_1, \ldots, U_r$ are the orbits of $\mathrm{Aut}(G)$. Since $P_1\mathbf{j}, \ldots, P_s\,\mathbf{j}$ are linearly independent vectors in S, we have $s \leq r$ as required. □

Proposition 1.5 can be seen in the context of graph divisors, defined as follows. An *equitable partition* of G is a partition $U_1 \,\dot\cup\, \cdots \,\dot\cup\, U_r$ of $V(G)$ with the property that the number of edges from a vertex $u_i \in U_i$ to a vertex in U_j is independent of the choice of u_i from U_i. If we denote this number by d_{ij} then the $r \times r$ matrix $D = (d_{ij})$ is the adjacency matrix of a multigraph called a *divisor* (or *quotient graph*) of G. Note that for any $f(x) \in \mathbb{R}[x]$, we have $f(D)(y_1, \ldots, y_r)^\top = (z_1, \ldots, z_r)^\top$ if and only if $f(A)(y_1\mathbf{e}_{U_1} + \cdots + y_r\mathbf{e}_{U_r}) = z_1\mathbf{e}_{U_1} + \cdots + z_r\mathbf{e}_{U_r}$; in particular, $(y_1, \ldots, y_r)^\top$ is a λ-eigenvector of D if and only if $y_1\mathbf{e}_{U_1} + \cdots + y_r\mathbf{e}_{U_r}$ is a λ-eigenvector of G (cf. [**14**, Theorem 9.3.3]). Since the orbits of $\mathrm{Aut}(G)$ form an equitable partition of G, Proposition 1.5 is a corollary of the following result.

Theorem 1.6. *The main eigenvalues of* G *are eigenvalues of every divisor of* G*.*

For a proof of Theorem 1.6 using walk-generating functions, see [**7**, Theorem 3]; we give an alternative proof in the next section, using an annihilator polynomial for $\mathbf{j}$.

If π denotes an equitable partition of G with r cells, and G/π denotes the corresponding quotient graph, then from Theorem 1.6 and Equation (8), we have [**29**, Theorem 3.3]:

$$H_G(t) = \frac{1}{t}\left\{-1 + (-1)^r \frac{m_{\overline{G}/\pi}\left(-\dfrac{t+1}{t}\right)}{m_{G/\pi}\left(\dfrac{1}{t}\right)}\right\}.$$

It follows from Theorem 1.6 that any equitable partition has at least s cells. It was conjectured in [**17**] that always there exists an equitable partition with exactly s cells. A counterexample was provided in [**26**, Section 3]: the graph obtained from $K_{1,3}$ by subdividing edges has 2 main eigenvalues (cf. Proposition 3.3), but no equitable partition with fewer than 3 cells.

2. THE GENERAL CASE

Here we include some observations (Propositions 2.1 and 2.2) which are instances of more general results on modules: note that $\mathbb{R}^n$ becomes an $\mathbb{R}[x]$-module if we define $f(x)\mathbf{v}$ as $f(A)\mathbf{v}$. We retain the notation of Section 1, and we write '$u \sim v$' to indicate that the vertices u, v are adjacent.

Proposition 2.1 [**29**, Theorem 2.5]. *For $f(x) \in \mathbb{R}[x]$, we have $f(A)\,\mathbf{j} = \mathbf{0}$ if and only if $m_G(x)$ divides $f(x)$.*

Proof. From (1) we have

$$f(A)\mathbf{j} = f(\mu_1)P_1\,\mathbf{j} + \cdots + f(\mu_s)P_s\,\mathbf{j},$$

and so $f(A)\,\mathbf{j} = \mathbf{0}$ if and only if $f(\mu_i)P_i\,\mathbf{j} = \mathbf{0}$ $(i = 1, \ldots, s)$; equivalently, $f(\mu_i) = 0$ $(i = 1, \ldots, s)$; equivalently, $m_G(x)$ divides $f(x)$. $\square$

To prove Theorem 1.6, let $c(x)$ be the characteristic polynomial of an $r \times r$ divisor matrix D, and let $\mathbf{j}_r$ be the all-1 vector in $\mathbb{R}^r$. By the Cayley-Hamilton theorem, $c(D)\mathbf{j}_r = \mathbf{0}$ and so $c(A)\,\mathbf{j} = \mathbf{0}$; by Proposition 2.1, $m_G(x)$ divides $c(x)$ as required.

From Proposition 2.1 we see that if the components of G are $G_1, \ldots, G_t$ then $m_G(x)$ is the least common multiple of $m_{G_1}(x), \ldots, m_{G_t}(x)$. Also, Proposition 2.1 accounts for most of the observations in [**28**, Section 1] concerning polynomials $f(x)$ such that $f(A)\mathbf{j} = \mathbf{0}$. For example, if $(A^k - \mu^k I)\mathbf{j} = \mathbf{0}$ then either G is μ-regular or k is even and $(A^2 - \mu^2 I)\mathbf{j} = \mathbf{0}$. The latter case is treated in Proposition 3.5.

Note that a linear relation among the integers N_k $(= \mathbf{j}^\top A^k\mathbf{j})$ has the form $\mathbf{j}^\top f(A)\mathbf{j} = 0$ for some $f(x) \in \mathbb{R}[x]$. As noted in [**29**, [Section 2], if $f(A)$ is positive semi-definite or negative semi-definite (e.g. if $f(A) = A^2 - \mu_1^2$) then the condition $\mathbf{j}^\top f(A)\mathbf{j} = 0$ is equivalent to the condition $f(A)\mathbf{j} = \mathbf{0}$.

The next observation is a variant of [**16**, Theorem 2.1].

Proposition 2.2. *The number of main eigenvalues of G is the largest integer k such that the vectors $\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^{k-1}\mathbf{j}$ are linearly independent.*

Proof. The vectors $\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^s\mathbf{j}$ are linearly dependent because $m_G(A)\mathbf{j} = \mathbf{0}$. On the other hand, if $\sum\limits_{i=0}^{s-1} c_iA^i\mathbf{j} = \mathbf{0}$ then by Proposition 2.1, $m_G(x)$ divides $\sum\limits_{i=1}^{s-1} c_ix^i$, whence $c_0 = c_1 = \cdots = c_{s-1} = 0$. $\square$

We mention one of the results from [**29**, Section 2] that can be deduced from Propositions 2.1 and 2.2. The *Hankel matrix H* of the sequence (N_k) is the infinite matrix (N_{i+j-2}), whose rows and columns are indexed by the positive integers.

Corollary 2.3 [**29**, Proposition 2.1]. *The number of main eigenvalues is equal to the rank of H.*

Proof. By Proposition 2.2, we may write $A^s\mathbf{j} = \sum_{i=0}^{s-1} h_i A^i\mathbf{j}$. Then $\mathbf{j}^\top A^{s+k}\mathbf{j} = \sum_{i=0}^{s-1} h_i\mathbf{j}^\top A^{i+k}\mathbf{j}$, that is, $N_{s+k} = \sum_{i=0}^{s-1} h_i N_{i+k}$, for all non-negative integers k. Hence each column of H beyond the s-th is a linear combination of its predecessors, and so $\operatorname{rank}(H) \leq s$. On the other hand, if H_s is the leading $s \times s$ submatrix of H then H_s is non-singular. To see this, we suppose that $H_s\mathbf{x} = \mathbf{0}$ and use a variant of the argument in [**29**]. Writing $\mathbf{u}_i = (1, \mu_i, \ldots, \mu_i^{s-1})^\top$ $(i = 1, \ldots, s)$, we have

$$0 = \mathbf{x}^\top H_s\mathbf{x} = n\sum_{i=1}^{s} \beta_i^2 \mathbf{x}^\top(\mathbf{u}_i\mathbf{u}_i^\top)\mathbf{x} = n\sum_{i=1}^{s} \beta_i^2(\mathbf{u}_i^\top\mathbf{x})^2,$$

whence $\mathbf{u}_i^\top\mathbf{x} = 0$ $(i = 1, \ldots, s)$. Since $\mathbf{u}_1, \ldots, \mathbf{u}_s$ are linearly independent, we have $\mathbf{x} = \mathbf{0}$ as required. $\square$

Let α denote the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ $(\mathbf{x} \in \mathbb{R}^n)$, and let $\overline{\alpha}$ denote the linear transformation $\mathbf{x} \mapsto (J - I - A)\mathbf{x}$ $(\mathbf{x} \in \mathbb{R}^n)$. We say that a subspace of $\mathbb{R}^n$ is *symmetric* if it is invariant under the transformation $\mathbf{x} \mapsto M_\pi\mathbf{x}$ $(\mathbf{x} \in \mathbb{R}^n)$ for all $\pi \in \operatorname{Aut}(G)$.

Theorem 2.4. *Let $\mathcal{B} = \{\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^{s-1}\mathbf{j}\}$, let V be the subspace of $\mathbb{R}^n$ with basis $\mathcal{B}$, and let $W = \sum_{i=1}^{m}(\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp)$. Then*

(i) *V is orthogonal to W and $\mathbb{R}^n = V \oplus W$;*

(ii) *each of V and W is α-invariant, $\overline{\alpha}$-invariant and symmetric.*

Proof. (i) If $\mathbf{x} \in \mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp$ then $\mathbf{x}^\top A^k\mathbf{j} = \mu_i^k\mathbf{x}^\top\mathbf{j} = 0$ for every non-negative integer k. Hence $V \perp W$. Using Equation (4) and comparing dimensions, we see that $\mathbb{R}^n = V \oplus W$.
(ii) The subspace V is α-invariant by construction, since $A^s\mathbf{j}$ is a linear combination of $\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^{s-1}\mathbf{j}$. It is invariant under the transformation $\mathbf{x} \mapsto J\mathbf{x}$ $(\mathbf{x} \in \mathbb{R}^n)$ since $\mathbf{j} \in V$. Hence V is also $\overline{\alpha}$-invariant. Now W is invariant under the symmetric transformations α and $\overline{\alpha}$ because $W = V^\perp$. The subspace V is symmetric because, for every $\pi \in \operatorname{Aut}(G)$, M_π fixes each element of $\mathcal{B}$. And W is symmetric because $\mathbf{j}^\perp$ and all eigenspaces are M_π-invariant. $\square$

The subspace V features in [**26**, Section 2], and the following remarks concerning $\alpha|_V$ and $\overline{\alpha}\,|_V$ can be found in [**26**, Section 4].

The matrix of $\alpha|_V$ with respect to $\mathcal{B}$ is the companion matrix of $m_G(x)$, while $JA^k\mathbf{j} = N_k\mathbf{j}$. Hence if $m_G(x) = x^s + a_{s-1}x^{s-1} + \cdots + a_1x + a_0$, then the matrix of $\overline{\alpha}\,|_V$ with respect to $\mathcal{B}$ is

$$M = \begin{pmatrix} N_0-1 & N_1 & N_2 & \cdots & N_{s-2} & N_{s-1}+a_0 \\ -1 & -1 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & -1 & \cdots & 0 & a_2 \\ 0 & 0 & -1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & -1 & a_{s-2} \\ 0 & 0 & 0 & \cdots & -1 & a_{s-1}-1 \end{pmatrix}. \tag{10}$$

Since W is the tangent space for $\overline{G}$, $m_{\overline{G}}(x)$ is the characteristic polynomial of M. To determine this polynomial, suppose that $M\mathbf{x} = \overline{\mu}\mathbf{x}$, where $\mathbf{x} = (x_1, \dots, x_s)^\top$, and let $\nu = -1 - \overline{\mu}$. Then we have:

$$\nu x_1 + \sum_{j=1}^{s} N_{j-1}x_j + a_0 x_s = 0, \tag{11}$$

$$x_{i-1} = a_{i-1}x_s + \nu x_i \quad (i = 2, \dots, s). \tag{12}$$

If we define polynomials $m_0(x), m_1(x), \dots, m_s(x) \in \mathbb{Z}[x]$ by

$$m_G(x) = m_0(x), \quad m_{i-1}(x) = m_{i-1}(0) + x m_i(x) \quad (i = 1, \dots, s),$$

then Equation (12) yields:

$$x_i = m_i(\nu)x_s \quad (i = 1, \dots, s-1).$$

Now Equation (11) becomes

$$(m_G(\nu) + \sum_{j=1}^{s} N_{j-1}m_j(\nu))x_s = 0.$$

Hence $x_s \neq 0$ and we deduce the following:

Corollary 2.5. $(-1)^s m_{\overline{G}}(x) = m_G(-1-x) + \sum_{j=1}^{s} N_{j-1}m_j(-1-x)$.

In order to discuss further the main eigenvectors of $\overline{G}$, we note first that $\sum_{i=1}^{s} \mathcal{E}_{J-I-A}(\overline{\mu}_i) = \sum_{i=1}^{s} \mathcal{E}_A(\mu_i)$, because these subspaces have the same orthogonal complement in $\mathbb{R}^n$. Thus if $\mathbf{x}_1, \dots, \mathbf{x}_s$ are orthonormal eigenvectors with corresponding eigenvalues $\mu_1, \dots, \mu_s$ then any main eigenvector $\mathbf{y}$ of $\overline{G}$ is a linear combination of $\mathbf{x}_1, \dots, \mathbf{x}_s$. For an explicit formulation due to Hagos [**16**], suppose that

$$(J - I - A)\mathbf{y} = \overline{\mu}\mathbf{y}, \quad \mathbf{y} = c_1\mathbf{x}_1 + \dots + c_s\mathbf{x}_s.$$

Since also $A\mathbf{y} = c_1\mu_1\mathbf{x}_1 + \cdots + c_s\mu_s\mathbf{x}_s$, we have

$$J\mathbf{y} = \sum_{i=1}^{s} c_i(\mu_i + 1 + \overline{\mu})\mathbf{x}_i.$$

It follows that

$$c_i(\mu_i + 1 + \overline{\mu}) = \mathbf{x}_i^\top J\mathbf{y} = \mathbf{x}_i^\top \mathbf{j}\,\mathbf{j}^\top \mathbf{y} = \sum_{j=1}^{s} \mathbf{j}^\top \mathbf{x}_i\, \mathbf{j}^\top \mathbf{x}_j c_j \ (i = 1, \ldots, m).$$

Thus if E is the $s \times s$ matrix with (i, j)-entry $\mathbf{j}^\top \mathbf{x}_i\, \mathbf{j}^\top \mathbf{x}_j$ then

$$(E - I - \mathrm{diag}(\mu_1, \ldots, \mu_s))\,\mathbf{c} = \overline{\mu}\,\mathbf{c},$$

where $\mathbf{c} = (c_1, \ldots, c_s)^\top$. We deduce the following.

Theorem 2.6 [**16**, Theorem 3.4]. *Let $\mu_1, \ldots, \mu_s$ be the main eigenvalues of the graph G, and let $\mathbf{x}_1, \ldots, \mathbf{x}_s$ be corresponding orthonormal eigenvectors. Let E be the $s \times s$ matrix whose (i, j)-entry is $\mathbf{j}^\top \mathbf{x}_i\, \mathbf{j}^\top \mathbf{x}_j$, and let $N = E - I - \mathrm{diag}(\mu_1, \ldots, \mu_s)$. The eigenvalues of N are precisely the main eigenvalues of $\overline{G}$. Moreover, if $\mathbf{c} = (c_1, \ldots, c_s)^\top$ is an eigenvector of N corresponding to the eigenvalue $\overline{\mu}$ then $\sum_{i=1}^{s} c_i\mathbf{x}_i$ is an eigenvector of $\overline{G}$ corresponding to $\overline{\mu}$.*

We may take the vector $\mathbf{x}_i$ above to be a unit vector $\mathbf{v}_i$ orthogonal to $\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp$ $(i = 1, \ldots, s)$. Then V has basis $\mathcal{B}' = \{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$, and if $\mathbf{j} = \sum_{i=1}^{s} \gamma_i\mathbf{v}_i$ then $P_i\mathbf{j} = \gamma_i\mathbf{v}_i$ $(i = 1, \ldots, s)$; hence the signs of the unit vectors $\mathbf{v}_i$ may be chosen so that

$$\frac{1}{\sqrt{n}}\mathbf{j} = \beta_1\mathbf{v}_1 + \cdots + \beta_s\mathbf{v}_s.$$

Now $N = (n\beta_i\beta_j)$, and this is the matrix of $\overline{\alpha}\,|_V$ with respect to $\mathcal{B}'$.

Next we turn to star complements and star partitions. Let μ be an eigenvalue of G of multiplicity k. Recall that the subset X of $V(G)$ is a *star set* for μ in G if $|X| = k$ and μ is not an eigenvalue of $G - X$. Here $G - X$ denotes the subgraph induced by the complement $\overline{X}$ of X; such a subgraph is called a *star complement* for μ in G.

For any subset X of $V(G)$ of size k, let

$$A = \begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then (see [**12**, Theorems 7.4.1 and 7.4.4]) X is a star set for μ in G if and only if μ is not an eigenvalue of C and

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{13}$$

In this situation, the eigenspace of μ consists of the vectors

$$\begin{pmatrix} \mathbf{v} \\ (\mu I - C)^{-1}B\mathbf{v} \end{pmatrix} \qquad (\mathbf{v} \in \mathbb{R}^k),$$

and we define a bilinear form on $\mathbb{R}^t$ $(t = n - k)$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top (\mu I - C)^{-1}\mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^t).$$

If we denote the columns of B by $\mathbf{b}_u$ $(u = 1, \ldots, k)$, then it follows from Equation (13) that, for all vertices u, v of X,

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \begin{cases} \mu & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases} . \tag{14}$$

Now we can establish the following result, noted in [**27**, Section 1]. Here, $\mathbf{j}_t$ denotes the all-1 vector in $\mathbb{R}^t$.

Proposition 2.7. *The eigenvalue μ is a non-main eigenvalue if and only if $\langle \mathbf{j}_t, \mathbf{b} \rangle = -1$ for every column $\mathbf{b}$ of B.*

Proof. Since $B\mathbf{e}_i$ is the i-th column of B, $\mathcal{E}_A(\mu)$ has a basis consisting of the vectors

$$\begin{pmatrix} \mathbf{e}_i \\ (\mu I - C)^{-1}\mathbf{b}_i \end{pmatrix} \quad (i = 1, \ldots, k).$$

Now μ is a non-main eigenvalue if and only if $\mathbf{j}$ is orthogonal to each of these vectors, that is, if and only if $\langle \mathbf{j}_t, \mathbf{b}_i \rangle = -1$ $(i = 1, \ldots, k)$. $\square$

Recall that a *star partition* for G is a partition $V(G) = X_1 \,\dot{\cup}\, \cdots \,\dot{\cup}\, X_m$ such that X_i is a star set for μ_i $(i = 1, \ldots, m)$. Every graph has a star partition [**12**, Theorem 7.1.3]; we observe here that always there exists a star partition which has a refinement determined by the main eigenvalues of G. First, let

$$\mathcal{E}_A(\mu_i) = V_i \oplus (\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp) \quad (i = 1, \ldots, s),$$

where V_i is spanned by the vector $\mathbf{v}_i$ (as defined above). Then we have an orthogonal decomposition

$$\mathbb{R}^n = V_1 \oplus (\mathcal{E}_A(\mu_1) \cap \mathbf{j}^\perp) \oplus \cdots \oplus V_s \oplus (\mathcal{E}_A(\mu_s) \cap \mathbf{j}^\perp) \oplus \mathcal{E}_A(\mu_{s+1}) \oplus \cdots \oplus \mathcal{E}_A(\mu_m).$$

Let P'_i, P''_i denote the orthogonal projection of $\mathbb{R}^n$ onto V_i, $\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp$, respectively. (If $\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp$ is the zero subspace then P''_i is the zero map.) All the summands in our decomposition are A-invariant, and so A commutes with each of these projections.

We may label vertices so that $X'_i = \{i\}$ $(i = 1, \ldots, s)$. Now the argument used to prove [**12**, Theorem 7.1.3] may be extended to show that there exists a partition

$$V(G) = X'_1 \,\dot{\cup}\, X''_1 \,\dot{\cup} \cdots \dot{\cup}\, X'_s \,\dot{\cup}\, X''_s \,\dot{\cup}\, X_{s+1} \,\dot{\cup} \cdots \dot{\cup}\, X_m \tag{15}$$

such that $P_i'\mathbf{e}_i$ spans V_i $(i = 1, \ldots, s)$, $\mathcal{E}_A(\mu_i) \cap \mathbf{j}^\perp$ has basis $\{P_i''\mathbf{e}_j : j \in X_i''\}$ $(i = 1, \ldots, s)$ and $\mathcal{E}_A(\mu_i)$ has basis $\{P_i\mathbf{e}_j : j \in X_i\}$ $(i = s+1, \ldots, m)$. Let $X_i = X_i' \,\dot\cup\, X_i'$ $(i = 1, \ldots, s)$. Since $P_i = P_i' + P_i''$ $(i = 1, \ldots, s)$, the partition $V(G) = X_1 \,\dot\cup\, \ldots \,\dot\cup\, X_m$ is a star partition for G.

We call the partition (14) a *refined* star partition for G, and we call the cells $X_1'', \ldots, X_s'', X_{s+1}, \ldots, X_m$ *tangent* star cells. (Note that if the main eigenvalue μ_i is simple then $X_i'' = \emptyset$.) We may now state our conclusion as follows.

Theorem 2.8. *Every graph has a refined star partition.*

We provide an example and then prove one result that relates tangent cells to graph structure.

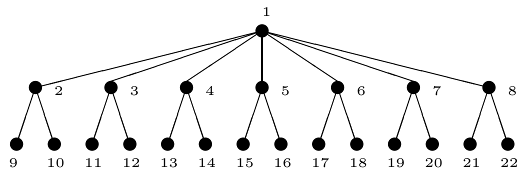

Fig. 1

Example 2.9. The graph G shown in Fig. 1 has spectrum $3^{(1)}, \sqrt{2}^{\,(6)}, 0^{(8)}, -\sqrt{2}^{\,(6)}, -3^{(1)}$, and the main eigenvalues are $\mu_1 = 3, \mu_2 = 0$ (see Proposition 3.3). Let $X = \{1, 9, 11, 13, 15, 17, 19, 21\}$; then X is a star set for 0 since $G - X = 7K_2$. By [**12**, Theorem 7.4.5], G has at least one star partition $X_1 \,\dot\cup\, X_2 \,\dot\cup\, \cdots \,\dot\cup\, X_5$ with $X_2 = X$. Such a star partition can be refined because $X_1'' = \emptyset$ and we may take $X_2' = \{1\}$, $X_2'' = \{9, 11, 13, 15, 17, 19, 21\}$. To see this, note that $\mathcal{E}_A(0)$ has a basis consisting of the orthogonal vectors $\mathbf{w}, \mathbf{w}_1, \ldots, \mathbf{w}_7$, where

$$\begin{aligned}
\mathbf{w}^\top &= (2; 0, 0, \ldots, 0; -1, -1, -1, -1, \ldots, -1, -1, -1, -1),\\
\mathbf{w}_1^\top &= (0; 0, 0, \ldots, 0; 1, -1, 0, 0, \ldots, 0, 0, 0, 0),\\
\mathbf{w}_7^\top &= (0; 0, 0, \ldots, 0; 0, 0, 0, 0, \ldots, 0, 0, 1, -1).
\end{aligned}$$

Thus $\mathcal{E}_A(0) \cap \mathbf{j}^\perp$ has basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_7\}$. If we normalize $\mathbf{w}, \mathbf{w}_1, \ldots, \mathbf{w}_7$ to obtain an orthonormal basis $\{\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_7\}$ of $\mathcal{E}_A(0)$, then $P_2' = \mathbf{u}\mathbf{u}^\top$ and $P_2'' = \sum_{i=1}^{7} \mathbf{u}_i\mathbf{u}_i^\top$. Now we find that $P_2'\mathbf{e}_1 = \frac{1}{9}\mathbf{w}$ and $P_2''\mathbf{e}_9 = \frac{1}{2}\mathbf{w}_1$, $P_2''\mathbf{e}_{11} = \frac{1}{2}\mathbf{w}_2$, $\ldots$, $P_2''\mathbf{e}_{21} = \frac{1}{2}\mathbf{w}_7$. □

Proposition 2.10. *Let μ be a main eigenvalue of G and let X be the associated tangent cell in a refined star partition for G. Then no vertex in X is adjacent to every vertex outside X.*

Proof. We take $\mu = \mu_i$ and $X = X_i''$. For $j \in X_i''$, we have

$$\mu_i P_i''\mathbf{e}_j = AP_i''\mathbf{e}_j = P_i''A\mathbf{e}_j = P_i'' \sum_{k \sim j} \mathbf{e}_k.$$

Let $\Delta(j)$ denote the set of neighbours of j and suppose, by way of contradiction, that $\overline{X_i''} \subseteq \Delta(j)$. Then we obtain:

$$\mu_i P_i''\mathbf{e}_j = \sum\{P_i''\mathbf{e}_k : k \in \Delta(j) \cap X_i''\} + \sum\{P_i''\mathbf{e}_k : k \in \overline{X_i''}\}.$$

Since $P_i''\mathbf{j} = \mathbf{0}$, this equation may be rewritten:

$$\mu_i P_i''\mathbf{e}_j = \sum\{P_i''\mathbf{e}_k : k \in \Delta(j) \cap X_i''\} - \sum\{P_i''\mathbf{e}_k : k \in X_i''\}.$$

Now the vectors $P_i''\mathbf{e}_k$ $(k \in X_i'')$ are linearly independent; hence $\mu_i = -1$ and $\Delta(j) \cap X_i'' = X_i'' \setminus \{j\}$. Thus j is adjacent to every other vertex of G. Now we apply Proposition 2.7 with $\mu = \mu_i$ and $X = X_i$. Note first that $\mathbf{j}_t$ is a column of B because $\overline{X_i''} \subseteq \Delta(j)$, and so $\langle \mathbf{j}_t, \mathbf{j}_t \rangle = -1$ by Equation (14). Secondly, since j is adjacent to every other vertex of X_i'', we know from Equation (14) that $\langle \mathbf{j}_t, \mathbf{b} \rangle = -1$ for every column $\mathbf{b}$ of B different from $\mathbf{j}_t$. By Proposition 2.7, μ_i is a non-main eigenvalue, a contradiction. $\square$

The above argument shows that if μ_i is a non-main eigenvalue, and j is a vertex of X_i adjacent to every vertex outside X_i then $\mu_i = -1$ and j has degree $n-1$.

3. THE CASE $s = 2$

In view of Proposition 1.4, the case $s = 2$ is the first non-trivial case. Here, G is non-regular and $(A^2 - aA + bI)\mathbf{j} = \mathbf{0}$, where $a = \mu_1 + \mu_2$ and $b = \mu_1\mu_2$. Let $\mathbf{d} = (d_1, \ldots, d_n)^\top$, where d_i is the degree of vertex i, so that $\mathbf{d} = A\mathbf{j}$ and $A\mathbf{d} = a\mathbf{d} - b\mathbf{j}$. It follows that if u, v are vertices of different degree, and $s(u) = \sum\limits_{i \sim u} d_i$, then [**16**, Theorem 2.4]:

$$a = \frac{s(u) - s(v)}{d_u - d_v}, \quad b = \frac{d_u s(u) - d_v s(v)}{d_u - d_v}.$$

The main eigenvalues μ_1, μ_2 may be found as roots of $x^2 - ax + b$.

Secondly, when $s = 2$ we have $\beta_1{}^2 + \beta_2{}^2 = 1$ and Equation (9) yields

$$\overline{\mu}_{1,2} = \frac{1}{2}\left(n - 2 - \mu_1 - \mu_2 \pm \sqrt{(\mu_1 - \mu_2 + n)^2 - 4n\beta_1{}^2(\mu_1 - \mu_2)}\right), \tag{16}$$

as noted by Lepović [**20**]. In that paper, it is shown that, for various perturbations G^* of a graph G with just two main eigenvalues, if G_1^* and G_2^* are cospectral then $\overline{G_1^*}$ and $\overline{G_2^*}$ are cospectral.

Thirdly, it follows from Proposition 1.1 that the following are unit eigenvectors of G corresponding to μ_1 and μ_2 [**16**, Corollary 2.7]:

$$\mathbf{x_1} = \frac{(A - \mu_2 I)\mathbf{j}}{\sqrt{(\mu_1 - \mu_2)\mathbf{j}^\top(A - \mu_2 I)\mathbf{j}}}\ , \quad \mathbf{x_2} = \frac{(A - \mu_1 I)\mathbf{j}}{\sqrt{(\mu_2 - \mu_1)\mathbf{j}^\top(A - \mu_1 I)\mathbf{j}}}\ .$$

The simplest examples of graphs with just two main eigenvalues are the graphs $G(h,a;k,b) = hK_a \dot{\cup} kK_b$ $(a \neq b)$. In view of Proposition 1.3, the graphs $\overline{G(h,a;k,b)}$ and $\overline{\ell\overline{G(h,a;k,b)}}$ share this property; those with integer eigenvalues have been investigated by Lepović [**22**, **23**].

Other examples include the graphs obtained from a strongly regular graph G by deleting a vertex i. To see this, let G have spectrum $r^{(1)}, \mu_2^{(m_2)}, \mu_3^{(m_3)}$ (where G is regular of degree r). Then G has $(m_2 - 1) + (m_3 - 1)$ linearly independent eigenvectors in $\mathbf{j}^\perp$ with i-th entry equal to 0. Deletion of the i-th entry yields $m_2 + m_3 - 2$ linearly independent eigenvectors of $G - i$ orthogonal to $\mathbf{j}$. Since $G - i$ has order $m_2 + m_3$, it has at most two main eigenvalues; and exactly two since $G - i$ is not regular.

For any graph with just μ_1, μ_2 as main eigenvalues, it follows from Proposition 1.1 that $\mathbf{d} - \mu_1\mathbf{j}$ is a μ_2-eigenvector, while $\mathbf{d} - \mu_2\mathbf{j}$ is a μ_1-eigenvector. Hence $(\mathbf{d} - \mu_1\mathbf{j})^\top(\mathbf{d} - \mu_2\mathbf{j}) = 0$. Since $\frac{1}{n}\mathbf{j}^\top\mathbf{d}$ is the average degree $\overline{d}$, this orthogonality relation may be written as follows:

Proposition 3.1. *If G has mean degree $\overline{d}$ and just two main eigenvalues, μ_1 and μ_2, then*

$$\frac{1}{n}\sum_{i=1}^{n}(d_i - \overline{d})^2 = (\mu_1 - \overline{d})(\overline{d} - \mu_2).$$

Both the variance of the degrees and the difference $\mu_1 - \overline{d}$ have been used as measures of irregularity of a graph; see [**1**] and [**5**] respectively. In what follows, δ and Δ denote the smallest and largest degrees, respectively.

Corollary 3.2. *Let G be a connected graph with just two main eigenvalues, μ_1 and μ_2, where $\mu_1 > \mu_2$. Then $\mu_2 < \delta < \overline{d} < \mu_1 < \Delta$.*

Proof. Since G is connected, $\mathcal{E}_A(\mu_1)$ is spanned by a vector $\mathbf{x} = (x_1, \dots, x_n)^\top$ such that all x_i are positive. Without loss of generality, $x_1 \geq x_2 \geq \cdots \geq x_n > 0$. Then $\mu_1 x_1 = \sum\limits_{j\sim 1} x_j \leq \Delta x_1$, whence $\mu_1 \leq \Delta$. If $\mu_1 = \Delta$ then the remaining eigenvalue equations $\Delta x_i = \sum\limits_{j\sim i} x_j$ show that G is regular (and $x_1 = x_2 = \cdots = x_n$), contrary to Proposition 1.4. Hence $\mu_1 < \Delta$, as is well known. Always $\mu_1 \geq \mathbf{j}^\top A\mathbf{j}/\mathbf{j}^\top\mathbf{j} = \overline{d}$, and so we know from Proposition 3.1 that $\mu_1 > \overline{d}$ (and $\mu_2 < \overline{d}$). The eigenvector $\mathbf{d} - \mu_2\mathbf{j}$ has an entry $\Delta - \mu_2 > \mu_1 - \mu_2 > 0$, and so $\mathbf{d} - \mu_2\mathbf{j} = \alpha\mathbf{x}$ for some $\alpha > 0$; in particular, $\delta > \mu_2$. Finally, $\delta < \overline{d}$ because G is non-regular. $\square$

When $s = 2$, the matrix M of Equation (10) is $\begin{pmatrix} n-1 & 2e + \mu_1\mu_2 \\ -1 & -1 - \mu_1 - \mu_2 \end{pmatrix}$, where e is the number of edges of G. Thus

$$m_{\overline{G}}(x) = x^2 + (\mu_1 + \mu_2 + 2 - n)x + 2e + \mu_1\mu_2 - (n-1)(1 + \mu_1 + \mu_2).$$

Using Proposition 3.1, we recover Equation (16).

Now we turn to two special cases of graphs with just two eigenvalues, namely the cases $\mu_2 = 0$ and $\mu_2 = -\mu_1$.

The graph G is said to be *harmonic* if it has $\mathbf{d}$ as an eigenvector. (An exercise for the reader: both G and $\overline{G}$ are harmonic if and only if G is regular.) If $A\mathbf{d} = \mu\mathbf{d}$ then we say that G is *μ-harmonic*; in this situation, μ is an integer and $\mu = \mu_1$. In [**24**], a harmonic graph without isolated vertices is called *pseudo-regular*; for any vertex v in such a graph, the mean degree of the neighbours of v is μ_1.

An example of a non-regular 6-harmonic graph is the graph obtained from the Clebsch graph [**4**, p. 35] by switching with respect to a vertex [**4**, p. 59]. The unique connected 2-harmonic graph is the tree T_2 of order 7 obtained from $K_{1,3}$ by subdividing edges. For $\mu > 2$, let T_μ be the tree obtained from $\mu^2 - \mu + 1$ disjoint stars $K_{1,\mu-1}$ by adding a vertex adjacent to the central vertex of each star. Then T_μ is the only μ-harmonic tree, and the only connected μ-harmonic graph with $\Delta \geq \mu^2 - \mu + 1$ [**15**]. The tree T_3 is shown in Fig. 1. For a given integer $c \geq 3$, there are only finitely many connected non-regular harmonic graphs G in which the number of edges is $|V(G)| + c - 1$ [**3**, Theorem 11]; those graphs for which $c \leq 4$ are discussed in [**2**] and [**3**].

The relation $A\mathbf{d} = \mu\mathbf{d}$ may be written $(A^2 - A\mu I)\mathbf{j} = \mathbf{0}$. If G is not regular, then $A\mathbf{j} \neq \mathbf{0}$ and $(A-\mu I)\mathbf{j} \neq \mathbf{0}$, and so from Proposition 2.1. we have the following (cf. [**24**, Theorem 8]):

Proposition 3.3. *Let G be a connected non-trivial graph with index μ. Then G is harmonic and non-regular if and only if the main eigenvalues of G are μ and 0.*

It follows that for harmonic graphs, we have $N_k = n{\beta_1}^2{\mu_1}^2$ $(k \geq 1)$ by Proposition 1.2. By Proposition 1.1, $n{\beta_1}^2 = \mu_1^{-2}||\mathbf{d}||^2$ and so $N_k = \mu^{k-2}\sum_{i=1}^{n} d_i^{\,2}$ $(k \geq 1)$.

Proposition 3.4. *Let G be a non-trivial graph with index μ_1. Then G is harmonic if and only if $N_3 = \mu_1 N_2$.*

Proof. We have $\mathbf{d}^\top A\mathbf{d} = \mu_1\mathbf{d}^\top\mathbf{d}$ if and only if $A\mathbf{d} = \mu_1\mathbf{d}$; that is, $N_3 = \mu_1 N_2$ if and only if $A\mathbf{d} = \mu_1\mathbf{d}$. $\square$

If the main eigenvalues of G are μ and $-\mu$ $(\mu > 0)$ then $\mu = \mu_1$, $m_G(x) = x^2 - \mu^2$ and G is bipartite. This last fact follows from a theorem of Frobenius (see [**13**, Chapter 13] or [**8**, Theorem 0.5]).

Examples of graphs with $m_G(x) = x^2 - \mu^2$ include the *semi-regular bipartite* graphs, i.e. the non-regular graphs in which every edge joins a vertex of degree δ to a vertex of degree Δ. For if the corresponding bipartition of $V(G)$ is $U \dot\cup V$, then $\delta\mathbf{e}_U \pm \Delta\mathbf{e}_V$ is an eigenvector corresponding to $\pm\mu$, and $\mathbf{j} \in \mathcal{E}(\mu) \oplus \mathcal{E}(-\mu)$; in this situation, $\delta\Delta = \mu^2$. In fact, these are the only examples among connected graphs [**25**] (see also [**29**, Proposition 2.7]), and we provide a short proof. Recall that a symmetric matrix M is *reducible* if, for some permutation matrix P, $P^\top MP$ has the form $\begin{pmatrix} M_1 & O \\ O & M_2 \end{pmatrix}$. Further, the largest eigenvalue of an irreducible symmetric matrix is a simple eigenvalue, and there exists a corresponding eigenvector whose entries are all positive (see [**13**, Chapter 13] or [**8**, Theorem 0.3]).

Proposition 3.5. *Let G be a non-trivial connected graph with index μ. Then G is a semi-regular bipartite graph if and only if the main eigenvalues of G are μ and $-\mu$.*

Proof. It remains to show that if the main eigenvalues of G are μ and $-\mu$ then G is semi-regular. Let $A = \begin{pmatrix} O & B^\top \\ B & O \end{pmatrix}$, where $B^\top$ has size $r \times s$. Then $A^2 = \begin{pmatrix} B^\top B & O \\ O & BB^\top \end{pmatrix}$. Now $A^2\mathbf{j} = \mu^2\mathbf{j}$, whence $B^\top B\mathbf{j}_r = \mu^2\mathbf{j}_r$ and $BB^\top\mathbf{j}_s = \mu^2\mathbf{j}_s$. We have

$$\mathbf{j}_s^\top B(B^\top B)B^\top\mathbf{j}_s = \mathbf{j}_s(BB^\top)^2\mathbf{j}_s = s\mu^4 \ = \mu^2\mathbf{j}_s^\top BB^\top\mathbf{j}_s.$$

Now μ^2 is the largest eigenvalue of $B^\top B$, and so $B^\top\mathbf{j}_s$ is an eigenvector of $B^\top B$ corresponding to μ^2. Since G is connected, $B^\top B$ is irreducible, and so μ^2 is a simple eigenvalue of $B^\top B$. Hence $B^\top\mathbf{j}_s = \alpha\mathbf{j}_r$ for some $\alpha \in \mathbb{N}$. Similarly, $B\mathbf{j}_r = \beta\mathbf{j}_s$ for some $\beta \in \mathbb{N}$. Thus each edge of G joins a vertex of degree α to a vertex of degree β; moreover, $\alpha \neq \beta$ because G has two main eigenvalues. $\square$

The graphs for which $(A^2 - \mu^2 I)\mathbf{j} = \mathbf{0}$ are precisely those for which $\sum_{u\sim v} d_u$ is the same for all $v \in V(G)$; such graphs were called Γ-*regular* by Plonka [**25**]. In view of Proposition 3.5, a graph is Γ-regular if and only if each component is μ-regular or semi-regular bipartite with $\delta\Delta = \mu^2$.

4. CONCLUDING REMARKS

We have discussed the cases $m_G(x) = x - \mu$, $m_G(x) = x(x-\mu)$ and $m_G(x) = x^2 - \mu^2$. The 'next' case is $m_G(x) = x(x^2 - \mu^2)$. This case is investigated in [**24**] using relations on the integers N_k (cf. Proposition 1.2), but in some proofs the eigenvalue 0 is overlooked as a main eigenvalue. The graph numbered 103 in [**10**] is a counterexample to [**24**, Theorem 5], and the graph T_3 in Fig. 1 is a counterexample to [**24**, Theorem 11]. (The principal assertion of [**24**, Theorem 5], namely the inequality (3) above, remains correct; the graphs in question are those for which equality is attained in (3).) As noted by the original author at `http://arxiv.org/abs/math.CO/0506259`, the results can be corrected by first defining a *pseudo-semi-regular* graph as a bipartite graph, without isolated vertices, in which neighbours of vertices from the same part have the same mean degree. (Thus semi-regular bipartite graphs are pseudo-semi-regular.) Now if $A(A^2 - \mu^2 I)\mathbf{j} = \mathbf{0}$, each component of G is μ-regular or pseudo-semi-regular. To establish this, we may assume that G is connected and $m_G(x) = x(x^2 - \mu^2)$. As before, G is bipartite, and so we can give a direct proof.

Proposition 4.1. *Let G be a connected graph for which $m_G(x) = x(x^2 - \mu^2)$. Then G is pseudo-semi-regular.*

Proof. We may argue as in the proof of Proposition 3.5, replacing $\mathbf{j} = \begin{pmatrix} \mathbf{j}_r \\ \mathbf{j}_s \end{pmatrix}$ with $\mathbf{d} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}$. Since $A^2\mathbf{d} = \mu^2\mathbf{d}$, we find that $B^\top \mathbf{d}_2 = \alpha\mathbf{d}_1$ for some $\alpha > 0$ and $B\mathbf{d}_1 = \beta\mathbf{d}_2$ for some $\beta > 0$. Thus the neighbours of the i-th vertex have mean degree β if $1 \le i \le r$, and mean degree α if $r+1 \le i \le n$. $\square$

In Section 1, we compared the eigenvalues of A with those of $J - I - A$. In similar fashion we may compare the eigenvalues of A with those of $J-I-2A$, which is the *Seidel adjacency matrix* of G. In particular, the characteristic polynomial $S_G(x)$ of $J - I - 2A$ is given by

$$S_G(x) = (-2)^n P_G\Big(-\frac{1}{2}(x+1)\Big)\left(1 - n\sum_{i=1}^{m} \frac{\beta_i^{\,2}}{x+1+2\mu_i}\right),$$

obtained in exactly the same way as equation (6) above. Main eigenvalues and main eigenvectors of $J - I - 2A$ are defined just as before. A relation between the main eigenvectors of A and those of $J - I - 2A$ is given in **21**].

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