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The main triangle projection in matrix spaces and its applications

by

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Introduction. The origin of this paper are the following three, at first appearance unrelated, problems:

1. Is the operator $S\colon l_1 \to l_\infty$ given by $S(a(n)) = \sum_{i\leqslant n} a(i)$ (p,q)-absolutely summing for $p > q \geqslant 1$? ([8], Problem 5).

2. Does there exist an unconditional basis in the space of all compact linear operators in an infinite-dimensional Hilbert space?

3. Is every unconditionally convergent series in l_1 of the form $\sum_n P^n x$, where $P^n(a(i)) = (a(i+n))$, absolutely convergent? (S. Mazur, Scottish Book, Problem 89).

It became clear that all these problems reduce to estimation of norms of "the main triangle projections" in corresponding matrix spaces. Let us consider, for example, the linear space of all matrices $a = (a(i,j))$ with the norm

$$\lambda_{2,2}(a) = \sup \sum_{i,j} s(i)t(j)a(i,j),$$

where the supremum is taken over all sequences $(s(i))$, $(t(j))$ of scalars such that $\sum_i |s^2(i)| \leqslant 1$, $\sum_j |t^2(j)| \leqslant 1$ ($\lambda_{2,2}(a)$ is equal to the norm of the operator in l_2 given by the matrix a). The main triangle projection is defined by

$$T_n(a)(i,j) = \begin{cases} a(i,j) & \text{if } i+j \leqslant n+1, \\ 0 & \text{otherwise.} \end{cases}$$

We prove that the norms of these projections grow the same as $\ln n$ when n becomes large. This order of growth is attained for the Hilbert matrices h_n, $h_n(i,j) = (n+1-i-j)^{-1}$ if $i+j \neq n+1$ and $i,j \leqslant n$, $h_n(i,j) = 0$ otherwise.

In the first section the concept of a matrix norm is introduced, and the norms of the main triangle projections with respect to some special matrix norms are estimated. The results of this section applied to the matrix norms σ_1 and σ_∞ are very closed to some theorems of I. C. Gochberg and M. G. Kreĭn concerning the Brodski integrals (cf. [4]).

In the second section the problem of the existence of unconditional bases in the matrix spaces is considered. The non-existence of unconditional bases in the space of compact operators on l_2 is proved. It is worth of mentioning that all important examples of matrix spaces can be constructed by means of tensor products in the sense of some cross-norm of Banach spaces with bases. For details see Section 3. Positive answer to Problem 1 is given in Section 4. In Section 5 we exhibit some relationships between the unconditional convergence of series in L_1 and the convergence almost everywhere. These results generalize the classical results on orthogonal series due to Menchoff and Rademacher (cf. [1]). At last, Section 6 contains the negative answer to Mazur's problem and a geometric interpretation of the main theorem of Section 1.

We would like to express our gratitude to Professor B. S. Mitjagin who brought to our notice the relationship between the boundedness of the main triangle projection and the existence of Brodski's integral in unitary ideals.

1. Matrix norm and the main triangle projections. Let M denote the linear space of all scalar-valued (real or complex) matrices $a = a(i,j)$ $(i,j = 1, 2, \ldots)$ such that $a(i,j) = 0$ for all but finitely many i, j. By a^* we denote the adjoint matrix of a, i.e. $a^*(i,j) = \overline{a(j,i)}$. For $a \in M$ we put

$$\operatorname{tr}(a) = \sum_i a(i,i).$$

For a, b in M, $a \circ b$ denotes the matrix defined by

$$(a \circ b)(i,j) = \sum_k a(i,k)\, b(k,j) \qquad (i,j = 1, 2, \ldots).$$

For $n, m = 1, 2, \ldots$ we define the matrix $u_{n,m}$ by $u_{n,m}(i,j) = 1$ for $i = n$, $j = m$ and $u_{n,m}(i,j) = 0$ otherwise.

Let $P_{n,m}(a) = \sum\limits_{\substack{i \leqslant n \\ j \leqslant m}} a(i,j) u_{i,j}$ for $a \in M$.

A non-negative function α on M is called a *matrix norm* if it satisfies the following conditions:

(i) $\alpha(a) = 0$ iff $a = 0$; $\alpha(ta) = |t|\,\alpha(a)$; $\alpha(a+b) \leqslant \alpha(a) + \alpha(b)$ for $a, b \in M$ and any scalar t.

(ii) $\alpha(u_{n,m}) = 1$ for $n, m = 1, 2, \ldots$

(iii) $\alpha\big(P_{n,m}(a)\big) \leqslant \alpha(a)$ for $a \in M$ $(n, m = 1, 2, \ldots)$.

A matrix norm is called *unconditional* if

(iv) $\alpha(a) = \alpha\big(s(i)t(j)a(i,j)\big)$ for $a \in M$ and for $|s(i)| = |t(j)| = 1$ $(i, j = 1, 2, \ldots)$.

An unconditional matrix norm is called *symmetric* if

(v) $\alpha(a) = \alpha\Big(a\big(\varphi(i), \psi(j)\big)\Big)$ for $a \in M$ and for all permutations φ, ψ of positive integers.

If α is a matrix norm, then the *conjugate norm* α^* is defined by

$$\alpha^*(a) = \sup_{b \in M,\, \alpha(b) \leqslant 1} \Big|\sum_{i,j} a(i,j)\, b(j,i)\Big| = \sup_{b \in M,\, \alpha(b) \leqslant 1} |\operatorname{tr}(a \circ b)|.$$

We have $\alpha^{**}(a) = \alpha(a)$.

Definition 1.1. Let us put for $a \in M$ $(n = 1, 2, \ldots)$

$$T_n(a) = \sum_{i+j \leqslant n+1} a(i,j)\, u_{i,j}.$$

The operator T_n is called the *n-th main triangle projection.*

In this section we are mainly interested in computing the quantities

$$t_n(\alpha) = \sup_{\alpha(a) \leqslant 1,\, a \in M} \alpha\big(T_n(a)\big) = \|T_n\|_\alpha \qquad (n = 1, 2, \ldots),$$

i.e. the norms of T_n with respect to a given matrix norm α.

For arbitrary matrix norm α we have

(1.1) $$t_n(\alpha) = t_n(\alpha^*) \qquad (n = 1, 2, \ldots).$$

If α is symmetric, then

(1.2) $$1 \leqslant t_1(\alpha) \leqslant t_2(\alpha) \leqslant \ldots$$

Less trivial is the following fact:

Proposition 1.1. *If α is an unconditional matrix norm, then*

(1.3) $$t_n(\alpha) \leqslant \log_2 2n.$$

Proof. Call a *chain* any set C of pairs of positive integers such that

$$C = \bigcup_{r=1}^{r(C)} A_r \times B_r,$$

where (A_r) and (B_r) are finite sequences of sets of positive integers such that if $r_1 \neq r_2$, then

$$A_{r_1} \cap A_{r_2} = \emptyset \quad \text{and} \quad B_{r_1} \cap B_{r_2} = \emptyset.$$

Let us put

$$P_C(a) = \sum_{(i,j) \in C} a(i,j)\, u_{i,j} \quad \text{for } a \in M.$$

Observe that for each $a \in M$ and $n = n(a, C)$ so large that $P_{n,n}(a) = a$ and $n \geqslant \max\limits_{r \leqslant r(C)}(\max\limits_{i \in A_r} i, \max\limits_{j \in B_r} j)$ we have the identity

$$P_C(a) = 2^{-r(C)} \sum_{(s(j)) \in S(C,n)} P_{n,n}\Big(\big(a(i,j)\, s(i)\, s(j)\big)\Big),$$

where $S(C, n)$ is the set of all sequences $(s(j))$ such that $s(j) = \pm 1$ for $j = 1, 2, \ldots$; $s(j) = 1$ for $j \geqslant n$; if $j_1 \in A_r$ and $j_2 \in B_r$, then $s(j_1)\cdot s(j_2) = 1$ for $r = 1, 2, \ldots, r(C)$. Since a is unconditional matrix norm, the last identity implies

$$a(P_C(a)) \leqslant a(a) \quad \text{for } a \in M. \tag{1.4}$$

Next put

$$\Delta_k = \{(i,j):\ i+j \leqslant k+1\} \quad (k = 1, 2, \ldots).$$

We shall show that ($[x]$ denotes the "entire" of x)

$$\Delta_k \text{ is a union of } S(k) = [\log_2 2k] \text{ chains.} \tag{1.5}$$

Assume that we have done this. Then combining (1.4) and (1.5) with the obvious identities $T_k(a) = \sum_{(i,j)\in\Delta_k} a(i,j)u_{i,j}$ we get (1.3).

We prove (1.5) by induction. For $k = 1$ it is trivial. Suppose that (1.5) holds for $1 \leqslant k \leqslant l$. Let $\Delta_l = \bigcup_{n=1}^{S([(l+1)/2])} C(n)$ for some chains $C(n)$ $(n = 1, 2, \ldots, [(l+1)/2])$. Let F_l and G_l be the "translations" defined by $F_l((i,j)) = (i+[(l+2)/2], j)$ and $G_l((i,j)) = (i, j+[(l+2)/2])$.

Put $C^*(n) = F_l(C(n)) \cup G_l(C(n))$ for $n = 1, 2, \ldots, [(l+1)/2]$. Since each $C(n)$ is a chain contained in Δ_l, one can easily see that $C^*(n)$ is also a chain. Moreover, we have

$$\Delta_{l+1} = I_l \cup F_l(\Delta_l) \cup G_l(\Delta_l) = I_l \cup \bigcup_{n=1}^{S([(l+1)/2])} C^*(n),$$

where $I_l = \{(i,j):\ 1 \leqslant i, j \leqslant [(l+2)/2]\}$. Hence Δ_{l+1} is a union of

$$S\left(\left[\frac{l+1}{2}\right]\right)+1 = S(l+1)$$

chains. This completes the induction and the proof of (1.3).

Next we shall show that, in general, inequality (1.3) cannot be improved. We begin with the standard notation.

If $x = (x(i))$ is a sequence of scalars, then

$$\|x\|_p = \begin{cases} \left(\sum_i |x(i)|^p\right)^{1/p} & \text{for } 1 \leqslant p < \infty, \\ \sup_i |x(i)| & \text{for } p = \infty. \end{cases}$$

Let us set

$$p^* = \begin{cases} \infty & \text{for } p = 1, \\ p(p-1)^{-1} & \text{for } 1 < p < \infty, \\ 1 & \text{for } p = \infty. \end{cases}$$

Also if $p = \infty$, then by "$1/p$" we understand "0".

Definition 1.2. Let $1 \leqslant p,\ q \leqslant \infty$. Let us put

$$\lambda_{p,q}(a) = \sup_{\|x\|_{p^*}\leqslant 1, \|y\|_{q^*}\leqslant 1} \Big|\sum_{i,j} a(i,j)x(i)y(j)\Big| \quad (a \in M).$$

Clearly, $\lambda_{p,q}$ is a symmetric matrix norm. Using the Hölder inequality we get

Lemma 1.1. *If $1 \leqslant p \leqslant p_1 \leqslant \infty$ and $1 \leqslant q \leqslant q_1 \leqslant \infty$, then*

$$\lambda_{p,q}(P_{n,m}(a)) \leqslant \lambda_{p_1,q_1}(a)\cdot n^{1/p_1^*-1/p^*} m^{1/q_1^*-1/q^*} \tag{1.6}$$

$(a \in M;\ n, m = 1, 2, \ldots)$.

In the sequel an important role will play the following *Hilbert matrices* h_n $(n = 1, 2, \ldots)$ defined by

$$h_n(i,j) = \begin{cases} (n+1-i-j)^{-1} & \text{for } i+j \neq n+1 \text{ and } i, j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that for each p with $1 < p < \infty$ there exists a constant $K(p)$ such that

$$\lambda_{p,p^*}(h_n) \leqslant K(p) \quad \text{for } n = 1, 2, \ldots \tag{1.7}$$

(The proof of this fact may be found in [4], Chap. III, § 10, or may be simply derived from the Riesz theorem; cf. [2], Chap. XI, § 7. Historicaly the first proof is due to Titchmarsh [14].)

Proposition 1.2. *Let $p \neq \infty$, $q \neq \infty$ and let $1/p+1/q \geqslant 1$. Then*

$$t_n(\lambda_{p,q}) \geqslant C(p,q)\ln n \quad (n = 1, 2, \ldots), \tag{1.8}$$

where $C(p,q)$ is a universal constant.

Proof. Clearly, $P_{n,n}(h_n) = h_n$. By the assumption, $q \leqslant p^*$. Thus by (1.6) and (1.7) we get

$$\lambda_{p,q}(h_n) \leqslant \lambda_{p,p^*}(h_n)n^{1/p-1/q^*} \leqslant K(p)n^{1/p-1/q^*}.$$

On the other hand, by Definition 1.2, we have (for $n \geqslant 2$)

$$\lambda_{p,q}(T_n h_n) = \lambda_{p,q}\Big(\sum_{i+j\leqslant n} (n+1-i-j)^{-1}u_{i,j}\Big)$$
$$\geqslant n^{-1/p^*-1/q^*} \sum_{i+j\leqslant n} (n+1-i-j)^{-1}.$$

Since for some C independent of n we have

$$\sum_{i+j\leqslant n} (n+1-i-j)^{-1} = \sum_{i\leqslant n-1}\sum_{j\leqslant i}\frac{1}{j} \geqslant Cn\ln n,$$

we get

$$\lambda_{p,q}(T_n h_n) \geqslant C n^{1-1/p^*-1/q^*} \ln n = C n^{1/p-1/q^*} \ln n.$$

Thus

$$t_n(\lambda_{p,q}) \geqslant \frac{\lambda_{p,q}(T_n h_n)}{\lambda_{p,q}(h_n)} \geqslant \frac{C n^{1/p-1/q^*} \ln n}{K(p) n^{1/p-1/q^*}} \geqslant \frac{C}{K(p)} \ln n.$$

This completes the proof.

By (1.1) we get

COROLLARY 1.1. *If* $1/p+1/q \geqslant 1$ $(1 \leqslant p, q < \infty)$, *then* $t_n(\lambda_{p,q}^*) \geqslant C(p,q)\ln n$.

Problem 1. Estimate from below the numbers

$$t_n(\lambda_{p,q}) \quad \text{for } \frac{1}{p}+\frac{1}{q} < 1 \quad (p, q \neq \infty).$$

Observe that we have $\lambda_{\infty,q}(a) = \sup_i \big(\sum_j |a(i,j)|^q\big)^{1/q}$ and $\lambda_{p,\infty}(a) = \sup_j \big(\sum_i |a(i,j)|^p\big)^{1/p}$. Thus

$$t_n(\lambda_{p,\infty}) = t_n(\lambda_{\infty,q}) = 1 \quad \text{for } n = 1, 2, \ldots \ (1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty).$$

For each r with $1 \leqslant r \leqslant +\infty$ define the symmetric matrix norm σ_r by

$$\sigma_r(a) = \begin{cases} \big(\operatorname{tr}[(a \circ a^*)^{r/2}]\big)^{1/r} & \text{for } 1 \leqslant r < \infty, \\ \lambda_{2,2}(a) & \text{for } r = \infty. \end{cases}$$

It is well known that $\sigma_r^* = \sigma_{r^*}$ (cf. [4], Chap. III, § 1). Therefore, by (1.1) and Proposition 1.2, we get

COROLLARY 1.2. $t_n(\sigma_1) = t_n(\sigma_\infty) \geqslant C \ln n$ $(n = 1, 2, \ldots)$.

This corollary is also a consequence of a theorem proved by Gohberg and Kreĭn ([4], Chap. II, § 6). It follows from Macajev's results [9] (cf. also [4], Chap. III, § 6) that for $1 < p < +\infty$ the sequence $\big(t_n(\sigma_p)\big)_n$ is bounded.

COROLLARY 1.3. *We have*

$$\lim_{r=1} t_n(\sigma_r) = \lim_{r=\infty} t_n(\sigma_r) \geqslant C \ln n \quad (n = 1, 2, \ldots).$$

Proof. For each $a \in M$ we have $\lim_{r=\infty} \sigma_r(a) = \sigma_\infty(a)$. Thus, in particular, $\lim_{r=\infty} \sigma_r(h_n) = \sigma_\infty(h_n) = \lambda_{2,2}(h_n)$ and $\lim_{r=\infty} \sigma_r(T_n h_n) = \lambda_{2,2}(T_n h_n) \geqslant C \ln n$. Hence

$$\lim_{r=\infty} t_n(\sigma_r) \geqslant \lim_{r=\infty} \frac{\sigma_r(T_n h_n)}{\sigma_r(h_n)} \geqslant \frac{C}{K(2)} \ln n.$$

The identity $\lim_{r=1} t_n(\sigma_r) = \lim_{r^*=\infty} t_n(\sigma_{r^*})$ is an obvious consequence of (1.1).

For further application we shall need another property of the numbers $t_n(a)$.

Define the projection $D_n: M \to M$ $(n = 1, 2, \ldots)$ by

$$D_{2m-1}(a) = \sum_{k \leqslant m} \sum_{\max(i,j)=2k-1} a(i,j) u_{i,j},$$

$$D_{2m}(a) = \sum_{k \leqslant m} \sum_{\max(i,j)=2k} a(i,j) u_{i,j}.$$

PROPOSITION 1.3. *If a is a symmetric matrix norm, then*

$$t_n(a) = \sup_{a(a) \leqslant 1} a\big(D_n(a)\big) = \|D_n\|_a \quad (n = 1, 2, \ldots).$$

Proof. We consider only the case where n is an odd integer. The proof for n even is similar. For $n = 2m-1$ we define a permutation Φ_n of positive integers by

$$\Phi_n(i) = \begin{cases} \dfrac{n-i+2}{2} & \text{for } i \text{ odd and } i \leqslant n, \\ \dfrac{n+1+i}{2} & \text{for } i \text{ even and } i < n, \\ i & \text{for } i > n. \end{cases}$$

Next define an operator $U_n: M \to M$ by

$$U_n(a) = \sum_{i,j} a(i,j) u_{\Phi_n(i), \Phi_n(j)} \quad (a \in M).$$

One can easily check that

$$T_n(U_n a) = U_n(D_n a).$$

This identity together with the fact that U_n is an isometry (because a is a symmetric matrix norm) imply the desired conclusion.

2. Matrix spaces and bases.

Definition 2.1. If a is a matrix norm, then by M_a we denote the Banach space being the completion of the normed linear space M under the norm a.

The space M_a can be in a natural way identified with the subspace of all scalar-valued matrices. The norm in M_a will be also denoted by a.

We recall that a sequence (e_n) of elements of a Banach space E is a *basis* for E if for each e in E there exists a unique sequence of scalars (c_n) such that $e = \sum_n c_n e_n$.

The following theorem is a slight generalization of a result of Gelbaum and Gil de Lamadrid [3]:

THEOREM 2.1. *The sequence* $(u_{i(k),j(k)})_{k=1}^{\infty}$ *is a basis for every matrix space* M_a, *where*

$$(2.1)\qquad i(k)=\begin{cases} m+1 & \text{for } k=m^2+s \text{ and } 1\leqslant s<m+1,\\ s-m & \text{for } k=m^2+s \text{ and } m+1\leqslant s\leqslant 2m+1;\end{cases}$$
$$j(k)=\begin{cases} s-m & \text{for } k=m^2+s \text{ and } 1\leqslant s<m+1;\\ m+1 & \text{for } k=m^2+s \text{ and } m+1\leqslant s\leqslant 2m+1\end{cases}$$
$$(m=0,1,2,\ldots).$$

Proof. Let us set

$$Q_k(a)=\sum_{r\leqslant k} a\big(i(r),j(r)\big)u_{i(r),j(r)}\qquad (a\in M;\ k=1,2,\ldots).$$

It follows from (2.1) that

$$(2.2)\qquad Q_k=\begin{cases} P_{m,m}+P_{m+1,s}-P_{m,s} & \text{for } k=m^2+s \text{ and } 1\leqslant s<m+1,\\ P_{m+1,m}+P_{s-m,m+1}-P_{s-m,m} & \text{for } k=m^2+s \text{ and } m+1\leqslant s\leqslant 2m+1\end{cases}$$
$$(m=0,1,2,\ldots).$$

(We put $P_{0,0}=P_{0,j}=P_{j,0}=0$). Thus $\|Q_k\|_a\leqslant 3$ for all k. Since by (2.2) $\lim_k Q_k(a)=a$ for each $a\in M$ and since M is dense in M_a, we infer that $\lim_k Q_k(a)=a$ for each $a\in M_a$. But this is equivalent to the assertion of the theorem.

We recall that a basis (e_n) in a Banach space E is called *unconditional* if the convergence of a series $\sum_n t_ne_n$ implies the convergence of every series $\sum_n s(n)t_ne_n$ for $s(n)=\pm 1$. Equivalently, (e_n) is an unconditional basis in E if for every permutation Φ of the indicies the sequence $(e_{\Phi(n)})$ is a basis for E.

Gelbaum and Gil de Lamadrid [3] observed that the double sequence $(u_{i,j})$ is not an unconditional basis for the space of compact operators in the Hilbert space l_2, i.e. in the space $M_{\lambda_{2,2}}$. In fact, those cases where the double sequence $(u_{i,j})$ is an unconditional basis for a matrix space (i.e. each ordering of $(u_{i,j})$ in a sequence is a basis) are rather exceptional and very often matrix spaces do not have any unconditional basis. First we consider the case of operator ideals on a Hilbert space.

Definition 2.2. A matrix norm a is called *unitary* if $a(u\circ a\circ v)=a(a)$ for every stable unitary matrices u and v and for $a\in M$.

A matrix u is unitary if $u^*=u^{-1}$, and u is *stable* if $u(i,j)=\delta_i^j$ for all but finitely many i and j, where $\delta_i^j=1$ for $i=j$ and $\delta_i^j=0$ otherwise.

THEOREM 2.2. *For every unitary matrix norm a the following conditions are equivalent*:

(2.3) *the double sequence* $(u_{i,j})$ *is an unconditional basis in* M_a;

(2.4) *the matrix space* M_a *consists of Hilbert-Schmidt matrices, i.e. the identity map* $a\to a$ *is an isomorphism from* M_a *onto* M_{σ_2}.

Proof. Clearly $(u_{i,j})$ is an unconditional basis in M_{σ_2}. Therefore (2.4) implies (2.3). Conversely, (2.3) implies that there is $K\geqslant 1$ such that

$$(2.5)\qquad K^{-1}a(b)\leqslant a(|b|)\leqslant Ka(b)\quad \text{for } b\in M,$$

where $|b|(i,j)=|b(i,j)|$ $(i,j=1,2,\ldots)$.

Next observe that for each $a\in M$ there exist stable unitary matrices u and v such that $u\circ a\circ v=\big((t_i\delta_i^j)\big)$, where

$$t_i=\sqrt{\sum_j |a(i,j)|^2}\quad \text{for } i=1,2,\ldots$$

(This is a consequence of the facts that every matrix has the polar representation (cf. [2], Chap. X) and that every self-adjoint matrix is unitary equivalent to a diagonal matrix.) Since $a\in M$, there is an index $n=n(a)$ such that $t_i=0$ for $i>n(a)$. Consider the unitary matrix w_n defined by

$$w_n(i,j)=\begin{cases}\dfrac{1}{\sqrt{n}}\exp\left(\sqrt{-1}\,\dfrac{2\pi}{n}ij\right) & \text{for } i,j\leqslant n,\\ \delta_i^j & \text{otherwise.}\end{cases}$$

Let $b=u\circ a\circ v\circ w_n=\big((t_i\delta_i^j)\circ w_n\big)=\big((t_iw_n(i,j)\big)$. Then $|b|(i,j)=t_i/\sqrt{n}$ $(i,j=1,2,\ldots)$. Thus choosing stable unitary matrices u_1 and v_1 so that $u_1\big((t_i)\big)=\sqrt{\sum_j t_j^2}(\delta_1^i)$ and $v_1\big((1,\underbrace{1,\ldots,}_{n\text{ times}}1,0,0,\ldots)\big)=\sqrt{n}(\delta_1^i)$ we have $u_1\circ|b|\circ v_1=\sqrt{\sum_i t_i^2}\,u_{1,1}$. Thus using the assumption that a is a unitary norm we get

$$(2.6)\qquad a(a)=a(b)\quad\text{and}\quad a(|b|)=\sqrt{\sum_i t_i^2}=\sqrt{\sum_{i,j}|a(i,j)|^2}=\sigma_2(a).$$

The desired conclusion follows now from (2.5), (2.6) and the fact that M is dense in M_a.

Remark 1. Mitjagin has observed that a similar argument shows that if a is a matrix such that $\lambda_{2,2}(|v\circ a\circ u|)<+\infty$ for all unitary matrices u and v, then $\sigma_2(a)<+\infty$.

Remark 2. Observe that for each matrix norm a condition (2.3) is equivalent to the following "elementary" condition which does not involve the notion of unconditional basis:

If $a\in M_a$, *then* $\big(s(i,j)a(i,j)\big)\in M_a$ *for every matrix* $\big(s(i,j)\big)$ *such that* $s(i,j)=\pm 1$ *for* $i,j=1,2,\ldots$

Our next result lies much deeper than Theorem 2.2. First we recall the following concept. Let (e_n) be a basis for a Banach space E, and let (e_n^*) denote the sequence of coefficient functionals of the basis (i.e. $e_n^*(x) = c_n$ for $x = \sum_n c_n e_n \in E$ $(n = 1, 2, \ldots)$). Let us put

$$K_{unc}\big((e_n)\big) = \sup_{\|x\| \leqslant 1} \sup \Big\| \sum_{j \leqslant s} e_{r_j}^*(x)\, e_{r_j} \Big\|,$$

where the second supremum is taken over all finite sequences of indices $r_1 < r_2 < \ldots < r_s$ $(s = 1, 2, \ldots)$.

It is well known that the basis (e_n) is unconditional if and only if $K_{unc}\big((e_n)\big) < +\infty$.

We begin with the following lemma:

LEMMA 2.1. *Let E be a Banach space with a basis (e_n). Let α be a symmetric matrix norm and let $U\colon M_\alpha \to E$ be an isometrically isomorphic embedding such that*

$$(2.7) \qquad \lim_i e_n^*(U u_{i,j}) = 0 \quad \textit{for } j, n = 1, 2, \ldots,$$

$$(2.8) \qquad \lim_j e_n^*(U u_{i,j}) = 0 \quad \textit{for } i, n = 1, 2, \ldots$$

Then

$$(2.9) \qquad K_{unc}\big((e_n)\big) \geqslant \sup_s t_s(\alpha).$$

Proof. Pick $\varepsilon > 0$ and fix an index s. Next, by Proposition 1.3, we choose a matrix $a = \big(a(i, j)\big)$ in M so that $\alpha(a) = 1$ and

$$(2.10) \qquad \alpha\big(D_s(a)\big) \geqslant t_s(\alpha) - \varepsilon.$$

We are going to show that

$$(2.11) \qquad K_{unc}\big((e_n)\big) \geqslant t_s(\alpha) - 2\varepsilon.$$

This clearly will imply the assertion of the lemma.

In the sequel we shall assume that $s = 2n-1$ is an odd positive number. The proof in the case of an even s is almost the same.

Using (2.7), (2.8) and the standard "gliding hump" procedure we define inductively three increasing sequences of indices $\big(m(k)\big)_{k=1}^{s+1}$, $\big(p(k)\big)_{k=1}^{s}$ and $\big(q(k)\big)_{k=1}^{s}$ so that for $b_k = \sum_{\max(i,j)=k} a(i, j)\, u_{p(i),q(j)}$ the following inequalities hold:

$$\sum_{r < m(k)} |e_r^*(Ub_k)|\, \|e_r\| < \frac{\varepsilon}{2s^2},$$

$$\Big\| \sum_{m(k+1) \leqslant r < m'} e_r^*(Ub_k)\, e_r \Big\| < \frac{\varepsilon}{2s^2}$$

for $k = 1, 2, \ldots, s$ and for each $m' > m(k+1)$.

These conditions imply:

$$(2.12) \qquad \Big\| \sum_{m(k) \leqslant r < m(k+1)} e_r^*(Ub_k)\, e_r - Ub_k \Big\| < \frac{\varepsilon}{s^2}$$

for $k = 1, 2, \ldots, s$,

$$(2.13) \qquad \Big\| \sum_{m(k) \leqslant r < m(k+1)} e_s^*(Ub_l)\, e_r \Big\| < \frac{\varepsilon}{s^2}$$

for $k \neq l$ $(k, l = 1, 2, \ldots, s)$.

Let us put $x = \sum_{k \leqslant s} Ub_k$. Since U is an isometry and the matrix norm α is symmetric, we have

$$(2.14) \qquad \|x\| = \alpha\Big(\sum_{k \leqslant s} b_k\Big) = \alpha\Big(\sum_{i \leqslant s} \sum_{j \leqslant s} a(i, j)\, u_{p(i),q(j)}\Big)$$
$$= \alpha\Big(\sum_{i \leqslant s} \sum_{j \leqslant s} a(i, j)\, u_{i,j}\Big) = \alpha(P_{s,s} a) \leqslant \alpha(a) = 1.$$

Now we are going to estimate from below the number

$$\Big\| \sum_{l \leqslant n} \sum_{m(2l-1) \leqslant r < m(2l)} e_r^*(x)\, e_r \Big\|, \quad \text{where} \quad n = \frac{s+1}{2}.$$

We have

$$\Big\| \sum_{l \leqslant n} \sum_{m(2l-1) \leqslant r < m(2l)} e_r^*(x)\, e_r \Big\| \geqslant \Big\| \sum_{l \leqslant n} Ub_{2l-1} \Big\| - \sum_{l \leqslant n} \Big\| Ub_{2l-1} - \sum_{m(2l-1) \leqslant r < m(2l)} e_r^*(x)\, e_r \Big\|.$$

But, by (2.12) and (2.13), we have

$$\Big\| Ub_{2l-1} - \sum_{m(2l-1) \leqslant r < m(2l)} e_r^*(x)\, e_r \Big\|$$
$$\leqslant \Big\| Ub_{2l-1} - \sum_{m(2l-1) \leqslant r < m(2l)} e_r^*(Ub_{2l-1})\, e_r \Big\| + \sum_{\substack{k \leqslant s \\ k \neq 2l-1}} \Big\| \sum_{m(2l-1) \leqslant r < m(2l)} e_r^*(Ub_k)\, e_r \Big\|$$
$$\leqslant \frac{\varepsilon}{s^2} + (s-1)\frac{\varepsilon}{s^2} = \frac{\varepsilon}{s}.$$

Thus

$$(2.15) \qquad \Big\| \sum_{l \leqslant n} \sum_{m(2l-1) \leqslant r < m(2l)} e_r^*(x)\, e_r \Big\| \geqslant \Big\| \sum_{l \leqslant n} Ub_{2l-1} \Big\| - \frac{n}{s}\varepsilon.$$

Next observe that

$$(2.16) \qquad \Big\| \sum_{l \leqslant n} Ub_{2l-1} \Big\| = \alpha\Big(\sum_{l \leqslant n} b_{2l-1}\Big)$$
$$= \alpha\Big(\sum_{l \leqslant n} \sum_{\max(i,j)=2l-1} a(i, j)\, u_{p(i),q(j)}\Big)$$
$$= \alpha\Big(\sum_{l \leqslant n} \sum_{\max(i,j)=2l-1} a(i, j)\, u_{i,j}\Big) = \alpha\big(D_s(a)\big).$$

(Because U is an isometry and the matrix norm a is symmetric.) Combining (2.15) and (2.16) we get

$$(2.17)\qquad \Big\|\sum_{l\leqslant n}\ \sum_{m(2l-1)\leqslant r<m(2l)} e_r^*(x)e_r\Big\| \geqslant a\big(D_s(a)\big)-\varepsilon.$$

Comparing (2.17) and (2.10) with the definition of $K_{unc}(e_n)$ we obtain (2.11).

LEMMA 2.2. *Let (e_n^*) be a sequence of bounded linear functionals in a Banach space E, and let a be a symmetric matrix norm. If there exists an isomorphic embedding $\tilde{U}: M_a\to E$, then there exists another isomorphic embedding $U: M_a\to E$ such that conditions (2.7) and (2.8) are satisfied.*

Proof. Consider the "cubic matrix" $\{e_n^*(\tilde{U}u_{r,s})\}$. Since for each fixed pair of indices (n, r) the sequence $\big(e_n^*(\tilde{U}u_{r,s})\big)_{s=1}^\infty$ is bounded, one can extract, by the standard diagonal procedure, an increasing sequence of indices $\big(s(j)\big)_{j=1}^\infty$ such that there exist limits $\lim_j e_n^*(\tilde{U}u_{r,s(j)})$ for $n, r = 1, 2, \ldots$

Repeating the same arguments for the "cubic matrix" $\{e_n^*(\tilde{U}u_{r,s(j)})\}$ we extract an increasing sequence of indices $\big(r(i)\big)_{i=1}^\infty$ so that there exist limits $\lim_i e_n^*(\tilde{U}u_{r(i),s(j)})$ for $n, j = 1, 2, \ldots$

Next we put for $a\in M_a$

$$Va = \sum_{i,j} a(i,j)\big(u_{r(2i),s(2j)}+u_{r(2i-1),s(2j-1)}-u_{r(2i),s(2j-1)}-u_{r(2i-1),s(2j)}\big).$$

Since a is a symmetric matrix norm, for each two increasing sequences of indices $\big(p(i)\big)$ and $\big(q(j)\big)$, and each matrix $b\in M_a$ we have

$$a\Big(\sum_{i,j} b\big(p(i), q(j)\big)u_{i,j}\Big)\leqslant a(b) = a\Big(\sum_{i,j} b(i,j)u_{p(i),q(j)}\Big).$$

Applying this to the matrices a and Va we obtain

$$a(a)\leqslant a(Va)\leqslant 4a(a).$$

Thus $V: M_a\to M_a$ is an isomorphic embedding. Now it is easy to verify that $U = \tilde{U}V$ has the desired properties, which completes the proof.

THEOREM 2.3. *Let a be a symmetric matrix norm such that the sequence $\big(t_n(a)\big)$ is unbounded. Then M_a is not isomorphic to any linear subspace of a Banach space with an unconditional basis.*

Proof. Suppose on the contrary that $\tilde{U}: M_a\to E$ is an isomorphic embedding and (e_n) is an unconditional basis in E. Let (e_n^*) be a sequence of coefficient-functionals of the basis (e_n). By Lemma 2.2 there is another isomorphic embedding $U: M_a\to E$ which satisfies conditions (2.7) and (2.8). Now, according to [11], Proposition 1, we replace the original norm of E by an equivalent norm with the property that U with respect to the new norm is an isometrically isomorphic embedding. Clearly, (e_n) remains unconditional basis in the new norm. Now, by Lemma 2.1, we get $K_{unc}\big((e_n)\big) = +\infty$. Thus the basis (e_n) is not unconditional, a contradiction.

COROLLARY 2.1. *Let $1\leqslant p$, $q<\infty$ and $1/p+1/q\geqslant 1$. Then no of the spaces $M_{\lambda_{p,q}}$ and $M_{\lambda_{p,q}^*}$ is isomorphic to a linear subspace of a Banach space with an unconditional basis.*

Proof. This is an immediate consequence of Theorem 2.3 and Proposition 1.2 and Corollary 1.1.

Corollary 2.1 and Theorem 2.3 enable us to give various examples of Banach spaces without unconditional basis. These examples seem to be new from the point of view of the linear topological classification of Banach spaces.

Example 2.1. *The space $M_{\lambda_{1,1}}$ has the following properties:*

(2.18) *$M_{\lambda_{1,1}}$ is isomorphic to no subspace of a Banach space with an unconditional basis.*

(2.19) *In $M_{\lambda_{1,1}}$ weak and strong convergence of sequences coincide.*

(2.20) *$M_{\lambda_{1,1}}$ is isometrically isomorphic to a conjugate space of a Banach space.*

Proof. (2.18) follows from Corollary 2.1.

(2.19) Suppose that there exists in $M_{\lambda_{1,1}}$ a weak Cauchy sequence, say (a_m), which does not converge in the norm topology. Then there is $\delta>0$ and an increasing sequence of indices $\big(n(m)\big)_{m=1}^\infty$ such that

$$(2.21)\qquad \lambda_{1,1}(b_m)>\delta\quad \text{for } b_m = a_{n(2m)}-a_{n(2m-1)},\ m = 1, 2, \ldots$$

Clearly, the sequence (b_m) weakly converges to zero. For $a\in M_{\lambda_{1,1}}$ and for $p, q = 1, 2, \ldots$ we put

$$P_{p,\infty}(a) = \sum_{i\leqslant p}\sum_j a(i,j)u_{i,j},\qquad P_{\infty,q} = \sum_{j\leqslant q}\sum_i a(i,j)u_{i,j}.$$

Observe that the ranges of the projections $P_{p,\infty}$ (and $P_{\infty,q}$) are isomorphic to the Cartesian product of p (respectively q) copies of the space l_1. Since in the space l_1 norm and weak convergence for sequences coincide, we have

$$(2.22)\qquad \lim_m \lambda_{1,1}\big(P_{p,\infty}(b_m)\big) = \lim_m \lambda_{1,1}\big(P_{\infty,q}(b_m)\big) = 0\qquad (p, q = 1, 2, \ldots).$$

Using (2.21), (2.22) and applying again the "gliding hump" procedure, we define three increasing sequences of indices $\big(m(k)\big)$, $\big(i(k)\big)$ and $\big(j(k)\big)$ so that

$$(2.23)\qquad \lambda_{1,1}\Big(b_{m(k)}-\sum_{\substack{i(k)<i\leqslant i(k+1)\\ j(k)<j\leqslant j(k+1)}} b_{m(k)}(i,j)u_{i,j}\Big)<2^{-k}.$$

Next we define scalar sequences $(x(i))_{i=1}^{\infty}$ and $(y(j))_{j=1}^{\infty}$ so that $\sup_i |x(i)| = \sup_j |y(j)| = 1$ and

$$(2.24)\quad \lambda_{1,1}\Big(\sum_{\substack{i(k)<i\leqslant i(k+1)\\ j(k)<j\leqslant j(k+1)}} b_{m(k)}(i,j)u_{i,j}\Big) = \sum_{\substack{i(k)<i\leqslant i(k+1)\\ j(k)<j\leqslant j(k+1)}} b_{m(k)}(i,j)x(i)y(j).$$

It follows from the definition of the norm $\lambda_{1,1}$ that the sequences $(x(i))$ and $(y(j))$ determine by the formula

$$F(a) = \sum_{i,j} a(i,j)x(i)y(j)$$

a linear functional on $M_{\lambda_{1,1}}$ of norm 1. It follows from (2.21), (2.24) and (2.23) that

$$|F(b_{m(k)})| \geqslant \delta - 2^{-k} \quad (k = 1, 2, \ldots).$$

But this contradicts the fact that the sequence $(b_{m(k)})$ converges weakly to zero in $M_{\lambda_{1,1}}$. This completes the proof.

(2.20) is a particular case of the following fact:

PROPOSITION 2.1. *Let a be a matrix norm such that the space M_a has the following property: if a is a matrix such that*

$$\sup_{n,m} a\Big(\sum_{\substack{i\leqslant n\\ j\leqslant m}} a(i,j)u_{i,j}\Big) < +\infty,$$

then $a \in M_a$. Then M_a is isometrically isomorphic to the space $(M_{a*})^*$.

We omit the easy proof of this proposition.

Our next example shows that there exists a reflexive Banach space without an unconditional basis. We recall that if $(X_i)_{i=1}^{\infty}$ is a sequence of Banach spaces, then by $(\mathsf{P}_{1\leqslant i<\infty} X_i)_2$ we denote the Banach space of all sequences (x_i) such that $x_i \in X_i$ $(i = 1, 2, \ldots)$ and

$$\|(x_i)\| = \Big(\sum_i \|x_i\|^2\Big)^{1/2} < +\infty.$$

Example 2.2. *Let $(p(k))$ be a sequence of real numbers such that $1 < p(k) < +\infty$ and either $\lim_k p(k) = \infty$ or $\lim_k p(k) = 1$. Then the space $X = (\mathsf{P}_{1\leqslant k<\infty} M_{\sigma_{p(k)}})_2$ has the following properties:*

(2.25) *X is reflexive and separable.*

(2.26) *X is not isomorphic to any subspace of a Banach space with an unconditional basis.*

Proof. (2.25) follows from the fact that if $1 < r < \infty$, then the space M_{σ_r} is reflexive and separable (cf. [4], chap. III, § 1).

(2.26) Suppose that $V\colon X \to E$ is an isomorphic embedding of X into a Banach space E, and (e_n) is a basis in E. Then, by [11], a new equivalent norm on E may be given so that V is an isometric embedding with respect to this norm. Hence each $V_i = VJ_i$ is an isometric embedding of $M_{\sigma_{p(i)}}$ into E (J_i is the natural embedding of $M_{\sigma_{p(i)}}$ into $(\mathsf{P}_{1\leqslant k<\infty} M_{\sigma_{p(k)}})_2$). Since $\lim_k u_{k,l} = \lim_l u_{k,l} = 0$ in the weak topology of the space $M_{\sigma_{p(i)}}$, V_i satisfies conditions (2.7) and (2.8) of Lemma 2.1. Hence by this lemma $K_{unc}((e_n)) \geqslant \sup_k \sup_n t_n(a_{\sigma_{p(k)}}) = \infty$ (by Corollary 1.3). Thus the basis (e_n) is not unconditional. This completes the proof.

Problem 2. Does there exist an unconditional basis in the space M_{σ_p} for $1 < p < \infty$, $p \neq 2$?

3. Tensor products of Banach spaces and matrix spaces. In this section we restate the main results of Section 2 in terms of tensor products of Banach spaces.

If X, Y are Banach spaces, by $X \otimes Y$ we shall denote the algebraic tensor product of X and Y. A norm $\|\ \|_\lambda$ on $X \otimes Y$ is said to be *tensor norm* if

(3.1) $\|x\otimes y\| = \|x\|\cdot\|y\|$ for each $x \in X$ and $y \in Y$;

(3.2) $\|S\otimes T\| = \|S\|\cdot\|T\|$ for any two linear operators $S\colon X\to X$ and $T\colon Y\to Y$.

By $X \otimes_\lambda Y$ we shall denote the completion of $X \otimes Y$ with respect to the norm $\|\ \|_\lambda$.

We recall (cf. [5]) that if X and Y are Banach spaces, then by $X \hat{\otimes} Y$ (resp. $X \hat{\hat{\otimes}} Y$) we denote the *projective tensor product* (resp. the *weak tensor product*), i.e. the completion of $X \otimes Y$ with respect to the tensor norm

$$\|a\|_\wedge = \inf_{a=\sum x_i\times y_i} \sum \|x_i\|\,\|y_i\|$$

(resp.

$$\|a\|_{\mathbb{A}} = \sup_{\|x^*\|\leqslant 1;\ \|y^*\|\leqslant 1} \Big|\sum x^*(x_i)y^*(y_i)\Big| \quad \text{for } a = \sum x_i \otimes y_i).$$

Assume that (e_n) is a basis in X and (f_n) is a basis in Y. Then the space $X \otimes_\lambda Y$ is in a natural way isometric with some matrix space M_λ (this isometry is induced by the map $e_i \otimes f_j \to \|e_i\|\,\|f_j\| u_{i,j}$). Now Theorems 2.1 and 2.2 can be restated as follows:

THEOREM 3.1. *If λ is a tensor norm on $X \otimes Y$, then the sequence $e_1\otimes f_1$, $e_1\otimes f_2$, $e_2\otimes f_1$, $e_2\otimes f_2$, $e_1\otimes f_3$, $e_2\otimes f_3$, ... (in this particular order) is a basis in $X \otimes_\lambda Y$.*

THEOREM 3.2. *If (e_n) is a complete orthonormal system in l_2, then $(e_i\otimes e_j)$ is an unconditional basis in the space $l_2\otimes_\lambda l_2$ if and only if the tensor norm λ is equivalent to the Hilbert-Schmidt norm.*

In this section we shall mean by l_∞ the space c_0 and by L_∞ the space $C[0,1]$.

Since the space $l_p\hat{\hat{\otimes}} l_q$ corresponds to the matrix space $M_{\lambda_{p,q}}$, and $l_p\hat{\otimes} l_q$ to the space $M_{\lambda_{p^*,q^*}}$, Corollary 2.1 can be reformulated in the following way:

COROLLARY 3.1. *Let $1\leqslant p,\ q<\infty$ and $1\leqslant 1/p+1/q$. Then none of the spaces $l_p\hat{\hat{\otimes}} l_q$, $l_{p^*}\hat{\otimes} l_{q^*}$ is isomorphic to a subspace of a Banach space with an unconditional basis.*

For the function spaces L_p and L_q we have a rather complete result:

COROLLARY 3.2. *If $1\leqslant p, q\leqslant\infty$, then none of the spaces $L_p\hat{\otimes} L_q$, $L_p\hat{\hat{\otimes}} L_q$ is isomorphic with a subspace of a Banach space with unconditional basis.*

Proof. Corollary 3.1 implies that neither $l_2\hat{\otimes} l_2$ nor $l_2\hat{\hat{\otimes}} l_2$ is isomorphic to a subspace of a Banach space with an unconditional basis. If $1<r<+\infty$, then l_2 is isomorphic to a complemented subspace of L_r. Thus for each pair (p,q) with $1<p,\ q<+\infty$ the space $l_2\hat{\otimes} l_2$ (resp. $l_2\hat{\hat{\otimes}} l_2$) may be isomorphically embedded into the tensor product $L_p\hat{\otimes} L_q$, resp. $L_p\hat{\hat{\otimes}} L_q$. This completes the proof in the case where $1<p,\ q<+\infty$. In the remaining cases the tensor product $L_p\hat{\otimes} L_q$ (resp. $L_p\hat{\hat{\otimes}} L_q$) contains a subspace isometrically isomorphic either to L_1 or to L_∞. Since neither L_1 (cf. [12]) nor L_∞ (because L_∞ contains a subspace isomorphic to L_1) are isomorphic to subspaces of Banach spaces with unconditional bases, we get the desired conclusion, which completes the proof.

4. An application to (p,q)-absolutely summing operators.

In the sequel we shall need the following consequence of Proposition 1.1:

PROPOSITION 4.1. *If $(k(i))_{i=1}^n$ is a sequence of n positive integers and $a\in M$, then*

$$\sum_{i\leqslant n}\Big|\sum_{j\leqslant k(i)} a(i,j)\Big|\leqslant \log_2 2n\,\lambda_{1,1}(a). \tag{4.1}$$

Proof. The norm $\lambda_{1,1}(a)$ does not increase if we apply any of the following operations on the matrix a: alternation of order of columns or rows, multiplication of a column or a row by -1, addition any number of columns to the i-th one and in the same time replacing these columns (except the i-th one) by zeros, the same for the rows. Taking this into account, it is clear that we can transform the matrix a in a matrix a' such that $\lambda_{1,1}(a')\leqslant\lambda_{1,1}(a)$ and

$$\sum_{i\leqslant n}\Big|\sum_{j\leqslant k(i)} a(i,j)\Big| = \sum_{i\leqslant n}\ \sum_{i+j\leqslant n+1} a'(i,j).$$

Now (4.1) follows from Proposition 1.1, because

$$\sum_{i+j\leqslant n+1} a'(i,j)\leqslant\lambda_{1,1}\big(T_n(a')\big).$$

We recall that an operator $T\colon X\to Y$ (X, Y Banach spaces) is *(p,q)-absolutely summing* if there is a constant C such that

$$\Big(\sum_{i\leqslant n}\|Tx_i\|^p\Big)^{1/p}\leqslant C\sup_{\|x^*\|\leqslant 1}\Big(\sum_{i\leqslant n}|x^*(x_i)|^q\Big)^{1/q}$$

for each n and any sequence $(x_i)_{i=1}^n\subset X$.

Let $S\colon l_1\to l_\infty$ be the "sum operator", i.e. let S map the sequence $(a(i))_1^\infty$ in l_1 into the sequence of its partial sums $\big(\sum_{k\leqslant i} a(k)\big)$ in l_∞.

The following proposition answers Problem 5 of [8]:

PROPOSITION 4.2. *If $p>q\geqslant 1$, then S is a (p,q)-absolutely summing operator.*

Proof. First observe that according to statements (0.4)-(0.7) of [7] it is enough to prove that the operator S is $(p,1)$-absolutely summing for each $p>1$. Let $(x_i)_{i=1}^n$ be a sequence of n vectors in l_1. Without loss of generality we may assume that $\|Sx_1\|\geqslant\|Sx_2\|\geqslant\ldots\geqslant\|Sx_n\|$ and that each x_i has almost all coordinates equal to zero. For $m=1,2,\ldots,n$ we define a matrix a_m by

$$a_m(i,j)=\begin{cases} x_i(j) & \text{if } i\leqslant m,\\ 0 & \text{otherwise.}\end{cases}$$

One can easily show that

$$\lambda_{1,1}(a_m)\leqslant\sup_{\|x^*\|\leqslant 1}\sum_{i\leqslant n}|x^*(x_i)|\qquad (m=1,2,\ldots,n). \tag{4.2}$$

Since $x_i(j)=0$ for all but finitely many j, there is for each $i\leqslant m$ ($m=1,2,\ldots,n$) an index $k(i)$ such that for l_∞-norm of Sx_i we have

$$\|Sx_i\|=\sup_k\Big|\sum_{j=1}^{k} a_m(i,j)\Big| = \Big|\sum_{j=1}^{k(i)} a_m(i,j)\Big|.$$

Hence, by Proposition 4.1 and by (4.2), we get for each $m=1,2,\ldots,n$

$$\sum_{i\leqslant m}\|Sx_i\|\leqslant\log_2 2m\,\lambda_{1,1}(a_m)\leqslant\log_2 2m\sup_{\|x^*\|\leqslant 1}\ \sum_{i\leqslant n}|x^*(x_i)|.$$

Hence for $m=1,\ldots,n$ we have

$$\|Sx_m\|\leqslant\frac{\log_2 2m}{m}\sup_{\|x^*\|\leqslant 1}\sum_{i\leqslant n}|x^*(x_i)|.$$

This implies

$$\Big(\sum_{i\leqslant n}\|Sx_i\|^p\Big)^{1/p}\leqslant C\sup_{\|x^*\|\leqslant 1}\sum_{i\leqslant n}|x^*(x_i)|,$$

where

$$C=\Big(\sum_m\Big(\frac{\log_2 2m}{m}\Big)^p\Big)^{1/p}<\infty.$$

This completes the proof.

Since the operator S is not weakly compact, we have

COROLLARY 4.1. *For each pair (p,q) such that $p>q\geqslant 1$ there exists a (p,q)-absolutely summing operator which is not weakly compact.*

5. An application to unconditionally convergent series in L_1. For $(x_i)_{i=1}^n\subset X$ (X is a Banach space) we shall write

$$\boldsymbol{l}_1\big((x_i)\big)=\sup_{\|x^*\|\leqslant 1}\sum_{i\leqslant n}|x^*(x_i)|=\sup_{|s(i)|\leqslant 1}\Big\|\sum_{i\leqslant n}s(i)x_i\Big\|.$$

In the sequel we put for sake of brevity $I=[0,1]$ and $L_1=L_1[0,1]$.

PROPOSITION 5.1. *Let $(f_i)_{i=1}^n\subset L_1$ and let $(E_i)_{i=1}^n$ be a decreasing or increasing sequence of mesurable subsets of the interval I. Then*

$$\sum_{i\leqslant n}\int_{E_i}f_i(s)\,ds\leqslant\log_2 2n\,\boldsymbol{l}_1\big((f_i)\big). \tag{5.1}$$

Proof. Suppose that the sequence (E_i) is decreasing (the proof for an increasing sequence being essentially the same). Let $F_1=E_n$ and $F_j=E_{n+1-j}-E_{n+2-j}$ $(j=2,3,\dots,n)$. Let a be a matrix defined by

$$a(i,j)=\begin{cases}\int_{F_j}f_i(s)\,ds & \text{for } i,j=1,2,\dots,n,\\ 0 & \text{if } i>n \text{ or } j>n.\end{cases}$$

Then

$$\sum_{i\leqslant n}\int_{E_i}f_i(s)\,ds=\sum_{i+j\leqslant n+1}a(i,j)\leqslant\lambda_{1,1}(T_n a).$$

On the other hand,

$$\lambda_{1,1}(a)=\sup_{|t(i)|\leqslant 1}\sum_{j\leqslant n}\Big|\sum_{i\leqslant n}t(i)a(i,j)\Big|=\sup_{|t(i)|\leqslant 1}\sum_{j\leqslant n}\Big|\int_{F_j}\sum_{i\leqslant n}t(i)f_i(s)\,ds\Big|$$
$$\leqslant\sup_{|t(i)|\leqslant 1}\int_{E_1}\Big|\sum_{i\leqslant n}t(i)f_i(s)\Big|\,ds\leqslant\boldsymbol{l}_1\big((f_i)\big).$$

Thus (5.1) is a consequence of Proposition 1.1.

Remark 3. Let $S\colon L_1\to C$ be an operator defined by

$$(Sf)(t)=\int_0^t f(s)\,ds\qquad (t\in[0,1]).$$

Using (5.1) and argument similar to that of Proposition 4.2, one can prove that S is (p,q)-absolutely summing for $p>q\geqslant 1$.

PROPOSITION 5.2. *Let $(f_i)_{i=1}^n\subset L_1$ and let*

$$g(s)=\max_{i\leqslant n}\Big|\sum_{k\leqslant i}f_k(s)\Big|\quad \textit{for } s\in I.$$

Then

$$\int_I g(s)\,ds\leqslant 2\log_2 2n\,\boldsymbol{l}_1\big((f_i)\big). \tag{5.2}$$

Proof. Let us put for $i=1,2,\dots,n$

$$A_i=\Big\{s\in I\colon\ g(s)=\sum_{k\leqslant i}f_k(s),\ g(s)>\sum_{k\leqslant j}f_k(s)\quad\text{for } j=1,\dots,i-1\Big\},$$

$$B_i=\Big\{s\in I\colon\ -g(s)=\sum_{k\leqslant i}f_k(s),\ -g(s)<\sum_{k\leqslant j}f_k(s)\quad\text{for } j=1,\dots,i-1\Big\}$$

and let

$$E_k=\bigcup_{i\geqslant k}A_i,\qquad F_k=\bigcup_{i\geqslant k}B_i\qquad\text{for } k=1,2,\dots,n.$$

Then

$$\int_I g(s)\,ds\leqslant\sum_{i\leqslant n}\int_{A_i}\sum_{k\leqslant i}f_k(s)\,ds-\sum_{i\leqslant n}\int_{B_i}\sum_{k\leqslant i}f_k(s)\,ds$$
$$=\sum_{k\leqslant n}\int_{E_k}f_k(s)\,ds-\sum_{k\leqslant n}\int_{F_k}f_k(s)\,ds.$$

Since the sequences of subsets (E_k) and (F_k) are decreasing, by Proposition 5.2, we get

$$\int_I g(s)\,ds\leqslant 2\log_2 2n\,\boldsymbol{l}_1\big((f_1)\big),$$

which completes the proof.

THEOREM 5.1. *Let $\sum_i f_i$ be an unconditional convergent series in L_1 and let $(t_i)_{i=1}^\infty$ be a sequence of real numbers such that $t_i=O(\ln^{-\varepsilon}i)$ for some $\varepsilon>1$ or $(t_i)\in l_p$ for some $p<\infty$. Then $\sum_i t_if_i(s)$ converges almost everywhere on I.*

Proof. Let

$$g_n(s)=\sup_{n\leqslant l\leqslant k}\Big|\sum_{l\leqslant i\leqslant k}t_if_i(s)\Big|.$$

We have to prove that $\big(g_n(s)\big)$ converges almost everywhere on I to zero. Since (g_n) is a decreasing sequence of positive functions, it is enough to show that

$$\lim_n\int_I g_n(s)\,ds=0. \tag{5.3}$$

We have $g_n(s) \leqslant 2\lim_m g_{n,m}(s)$, where

$$g_{n,m}(s) = \max_{n\leqslant k\leqslant m}\Big|\sum_{n\leqslant i\leqslant k} t_i f_i(s)\Big|.$$

For each n, the sequence of positive functions $(g_{n,m})_{m=n}^{\infty}$ is increasing. Thus

$$\int_I g_n(s)\,ds \leqslant 2\lim_m \int_I g_{n,m}(s)\,ds. \tag{5.4}$$

For fixed $g_{n,m}$ we define two decreasing sequences $(E_i)_{i=n}^{m}$ and $(F_i)_{i=n}^{m}$ of measurable subsets of I, in the same way as the sequences (E_k) and (F_k) for the function $g(s)$ in the proof of Proposition 5.2, such that

$$\int_I g_{n,m}(s)\,ds \leqslant \sum_{n\leqslant i\leqslant m} t_i\Big(\int_{E_i} f_i(s)\,ds - \int_{F_i} f_i(s)\,ds\Big). \tag{5.5}$$

Assume now that $(t_i)\in l_p$ for some $1\leqslant p<\infty$. Because for each decreasing sequence $(A_i)_{i=n}^{m}$ of measurable subsets of I

$$\Big(\sum_{n\leqslant i\leqslant m}\Big|\int_{A_i} f_i(s)\,ds\Big|^{p^*}\Big)^{1/p^*} \leqslant C\boldsymbol{l}_1\big((f_i)_{i=n}^{m}\big),$$

where C is a constant which depends only on p (compare with Proposition 4.2 and Remark 3), by (5.5) and the Hölder inequality we get

$$\int_I g_{n,m}(s)\,ds \leqslant \Big(\sum_{n\leqslant i\leqslant m}|t_i|^p\Big)^{1/p} C\boldsymbol{l}_1\big((f_i)_{i=n}^{m}\big). \tag{5.6}$$

Since the series $\sum_i f_i$ is unconditionally convergent in L_1, $\boldsymbol{l}_1\big((f_i)_{i=n}^{m}\big) \leqslant \boldsymbol{l}_1\big((f_i)\big) < \infty$. This together with (5.6) and (5.4) implies (5.3).

Now suppose that $t_i = O(\ln^{-\varepsilon} i)$ for some $\varepsilon > 1$.

Using Abel's transformation, the right-hand side of (5.5) is replaced by

$$\sum_{n\leqslant i\leqslant m}\big(\ln^{-\varepsilon} i - \ln^{-\varepsilon}(i+1)\big)\sum_{n\leqslant k\leqslant i} t_k \ln^{\varepsilon} k\Big(\int_{A_i} f_i(s)\,ds - \int_{B_i} f_i(s)\,ds\Big) +$$
$$+\ln^{-\varepsilon} m \sum_{n\leqslant k\leqslant m} t_k\ln^{\varepsilon} k\Big(\int_{A_i} f_i(s)\,ds - \int_{B_i} f_i(s)\,ds\Big).$$

Let

$$C' = \sup_k |t_k \ln^{\varepsilon} k|;$$

then, by Proposition 5.2, we get

$$\sum_{n\leqslant k\leqslant i} t_k\ln^{\varepsilon} k\Big(\int_{A_i} f_i(s)\,ds - \int_{B_i} f_i(s)\,ds\Big)$$
$$\leqslant 2\log_2 2(i+1-n)\,\boldsymbol{l}_1\Big(\big(t_k(\ln^{\varepsilon} k) f_k\big)_{k=n}^{i}\Big) \leqslant C''(\ln i)\,C'\boldsymbol{l}_1\big((f_i)\big).$$

Thus the right-hand side of (5.5) does not exceed

$$C'''\boldsymbol{l}_1\big((f_i)\big)\sum_{n\leqslant i<m}\big(\ln^{-\varepsilon} i - \ln^{-\varepsilon}(i+1)\big)\ln i + C'''\boldsymbol{l}_1\big((f_i)\big)\ln^{-\varepsilon} m\cdot \ln m.$$

Since the series $\sum_i\big(\ln^{-\varepsilon} i - \ln^{-\varepsilon}(i+1)\big)\ln i$ is convergent and $\lim_m \ln^{-\varepsilon+1} m = 0$, inequalities (5.5) and (5.4) imply (5.3). This completes the proof.

Remark 4. Let (φ_n) be an orthonormal sequence in L_2 and $(s_n)\in l_r$ for some $1\leqslant r<2$. Then putting $(f_n) = (|s_n|^{r/2}\varphi_n)$ and $(t_n) = (|s_n|^{1-r/2})$, we obtain from Proposition 5.1 a well-known theorem of Rademacher and Menchoff (cf. [1], Theorem 2.5.4). The fundamental Menchoff theorem (cf. [1], Theorem 2.4.2) suggests the following problem:

Problem 3. Is the assertion of Theorem 5.1 valid for $(t_n) = (\ln^{-1} n)$?

6. Two other applications. The following argument shows that the answer to the Mazur's question (cf. Scottish Book, Problem 83) is negative. Namely

There exists a real sequence $(c(i))_{i=1}^{\infty}$ *such that*

$$\sup_{|s(i)|\leqslant 1;\,|t(j)|\leqslant 1}\Big|\sum_{i,j} s(i)t(j)c(i+j-1)\Big| < +\infty$$

but $\sum_i i|c(i)| = +\infty$.

Proof. For each n let $(c_n(i))$ be the sequence defined by $c_n(i) = (n-i+1)^{-1}$ for $i<2n+1$ and $i\neq n+1$; $c_n(i)=0$ otherwise. Then $c_n(i+j-1) = h_n(i,j)$ for $i,j\leqslant n$ (where h_n denotes the n-th Hilbert matrix defined in Section 1). By a simple computation, we get

$$\sup_{|s(i)|\leqslant 1;\,|t(j)|\leqslant 1}\Big|\sum_{i,j} s(i)t(j)c_n(i+j-1)\Big| \leqslant \sup_{|s(i)|\leqslant 1;\,|t(j)|\leqslant 1}\Big|\sum_{i,j} s(i)t(j)h_n(i,j)\Big| +$$
$$+\sum_{j>n+1}\sum_i |c_n(i+j-1)| + \sum_{i>n+1}\sum_j c_n(i+j-1) \leqslant \lambda_{1,1}(h_n) + 2n.$$

Now using (1.6) and (1.7) we infer that

$$\lambda_{1,1}(h_n) \leqslant \lambda_{2,2}(h_n)n \leqslant K(2)n.$$

Hence

$$\sup_{|s(i)|\leqslant 1;\,|t(j)|\leqslant 1}\Big|\sum_{i,j} s(i)t(j)c_n(i+j-1)\Big| \leqslant \big(2+K(2)\big)n,$$

while

$$\sum_i i|c_n(i)| \geqslant \sum_{i\leqslant n} i(n+1-i)^{-1} > Cn\cdot\ln n \quad (n=1,2,\ldots).$$

The existence of a sequence $(c(i))$ with the desired properties is a simple consequence of the Banach-Steinhaus theorem.

Our last result gives a geometric interpretation of Proposition 1.1 in the case of the norm $\lambda_{1,1}$. By an *ellipsoid* in the n-dimensional (either real or complex) vector space R^n we shall mean the image of the Euclidean unit ball $B_n = \{x \in R^n : \|x\| \leqslant 1\}$ by an arbitrary non-degenerated linear transformation of R^n. Here $\|x\| = \big(\sum_{i \leqslant n} |x(i)|^2\big)^{1/2}$. By the *size* of a set W in R^n we mean the quantity

$$s(W) = \sup_{x \in W} \max_{i \leqslant n} |x(i)|.$$

Furthermore, let $\mathscr{S}_n$ denote the family of all ellipsoids $\mathscr{E}$ in R^n such that

(6.1) the points $(1, 0, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1)$ belong to $\mathscr{E}$.

We are going to prove the following fact:

PROPOSITION 6.1. *There are positive constants C_1 and C_2 (which do not depend on n) such that*

$$C_1 \ln(n+1) \leqslant \inf_{\mathscr{E} \in \mathscr{S}_n} s(E) \leqslant C_2 \ln(n+1). \tag{6.2}$$

This proposition is an obvious consequence of the next three lemmas and Proposition 1.1 in the case of the norm $\lambda_{1,1}$.

LEMMA 6.1. *If b_n is the matrix defined by*

$$b_n(p, q) = \begin{cases} 1 & \text{for } n \geqslant p \geqslant q \geqslant 1, \\ 0 & \text{otherwise}, \end{cases}$$

then $\lambda_{1,1}^*(b_n) = t_n(\lambda_{1,1})$.

Proof. Since $\lambda_{1,1}^*$ is a symmetric matrix norm, $\lambda_{1,1}^*(b_n) = \lambda_{1,1}^*(\tilde{b}_n)$, where $\tilde{b}_n(p, q) = 1$ for $p+q \leqslant n+1$ and $\tilde{b}_n(p, q) = 0$ otherwise. Next, taking into account that $\lambda_{1,1}(a) = \lambda_{1,1}(a^*)$, we have

$$\lambda_{1,1}^*(\tilde{b}_n) = \sup_{\lambda_{1,1}(a) \leqslant 1} \Big|\sum_{p,q} \tilde{b}_n(p, q) a(p, q)\Big| = \sup_{\lambda_{1,1}(a) \leqslant 1} \Big|\sum_{p+q \leqslant n+1} a(p, q)\Big|.$$

One can easily derive from the definition of the norm $\lambda_{1,1}$ that

$$\sup_{\lambda_{1,1}(a) \leqslant 1} \Big|\sum_{p+q \leqslant n+1} a(p, q)\Big| = \sup_{\lambda_{1,1}(a) \leqslant 1} \lambda_{1,1}\big(T_n(a)\big) = t_n(\lambda_{1,1}).$$

This completes the proof of the Lemma.

LEMMA 6.2. *There are positive constants $\tilde{C}_1$ and $\tilde{C}_2$ (which do not depend on n) such that*

$$\tilde{C}_1 \cdot \lambda_{1,1}^*(b_n) \leqslant \inf_{(y_q)_{q \leqslant n} \in \mathscr{Y}} \sup_{q \leqslant n} \|y_q\| \leqslant \tilde{C}_2 \lambda_{1,1}^*(b_n), \tag{6.3}$$

where $\mathscr{Y}$ is the set of all such sequences $(y_q)_{q \leqslant n}$ of elements of R^n that there is a sequence $(x_p)_{p \leqslant n}$ such that the following conditions are satisfied:

$$\|x_p\| \leqslant 1 \quad \text{for } p = 1, 2, \ldots, n, \tag{6.4}$$

$$(x_p, y_q) = b_n(p, q) \quad \text{for } p, q = 1, 2, \ldots, n. \tag{6.5}$$

(We use the notation $(x, y) = \sum_{i \leqslant n} x(i)\overline{y(i)}$ for $x, y \in R^n$.)

Proof. We apply the following inequality due to Grothendieck [6] (cf. also [8], Theorem 2.1):

(6.6) There is a universal positive constant K_G such that

$$\Big|\sum_{p,q} a(p, q)(x_p, y_q)\Big| \leqslant K_G \lambda_{1,1}(a) \sup_p \|x_p\| \sup_q \|y_q\|$$

for $x_p, y_q \in R^n$ and for $a \in M$.

Combining (6.4), (6.5) with (6.6) we get

$$K_G^{-1} \lambda_{1,1}^*(b_n) = K_G^{-1} \sup_{\lambda_{1,1}(a^*) \leqslant 1} \Big|\sum_{p,q} a(p, q)(x_p, y_q)\Big| \leqslant \sup_{q \leqslant n} \|y_q\|.$$

This yields the left-hand side inequality of (6.3) with $\tilde{C}_1 = K_G^{-1}$.

To prove the right-hand side inequality of (6.3), we define the linear operator $\hat{b}_n$ from l_1^n (i.e. the space R^n equipped with the norm $\| \|_1$) into the space l_∞^n (i.e. the space R^n equipped with the norm $\| \|_\infty$) by

$$(\hat{b}_n x)(q) = \sum_{p \leqslant n} b(p, q) x(p) \quad \text{for } x \in l_1^n \text{ and } q = 1, 2, \ldots, n.$$

Then the nuclear norm of $\hat{b}_n$ (cf. [13], p. 45, for the definition) is equal to $\lambda_{1,1}^*(b_n)$ (because the space of $n \times n$ matrices with the norm $\lambda_{1,1}^*$ is in a natural way isometrically isomorphic to the projective tensor product $l_\infty^n \hat{\otimes} l_\infty^n$ which is isometrically isomorphic to the space of all nuclear operators from l_1^n into l_∞^n). Therefore for each $\varepsilon > 0$ there are a Hilbert space H and linear operators $u \colon l_1^n \to H$ and $v \colon H \to l_\infty^n$ such that

$$\hat{b}_n = vu, \quad \|u\| = 1, \quad \lambda_{1,1}^*(b_n) + \varepsilon > \|v\| \tag{6.7}$$

(cf. [10], p. 73, proof of Proposition 3). Since $\hat{b}_n$ is an isomorphism, one can assume without loss of generality that $H = l_2^n$ (i.e. R^n equipped with the norm $\|\cdot\| = \|\cdot\|_2$). Indeed, replace (if necessary) H by $(\ker v)^\perp$ — the orthogonal complement of the kernel of v, the operator u by Pu, where P is the orthogonal projection from H onto $(\ker v)^\perp$ and v by its restriction to $(\ker v)^\perp$, and use the fact that each n-dimensional Hilbert space is isometrically isomorphic to l_2^n.

Now we put $x_p = ue_p$ for $p = 1, 2, \ldots, n$ and $y_q = v^* f_q^*$ for $q = 1, 2, \ldots, n$, where $e_p = (\delta_p^i)_{i\leqslant n}$ is the p-th unit vector, v^* denotes the adjoint operator of v and f_q^* is the q-th coordinate functional, i.e. $f_q^*(y) = y(q)$ for $y \in l_\infty^n$. Then clearly identity (6.5) holds. Using the formulas

$$\|u\| = \max_{p\leqslant n} \|x_p\|, \qquad \|v\| = \max_{q\leqslant n} \|y_q\|$$

we derive from (6.7) condition (6.4) and the following inequality:

$$\max_{q\leqslant n} \|y_q\| < \lambda_{1,1}^*(b_n) + \varepsilon.$$

Letting ε tend to zero we get the right-hand side inequality of (6.3). This completes the proof.

LEMMA 6.3. *For each ellipsoid $\mathscr{E}$ in $\mathscr{S}_n$ there are sequences $(x_p)_{p\leqslant n}$ and $(y_q)_{q\leqslant n}$ satisfying* (6.4) *and* (6.5) *and such that*

$$s(\mathscr{E}) = \max_{q\leqslant n} \|y_q\|. \tag{6.8}$$

Conversely, each pair of sequences satisfying (6.4) *and* (6.5) *determines an ellipsoid $\mathscr{E}$ in $\mathscr{S}_n$ such that* (6.8) *holds.*

Proof. Let $\mathscr{E} \in \mathscr{S}_n$ and let $u\colon R^n \to R^n$ be a non-degenerated linear operator such that $\mathscr{E} = u(B_n)$. Let us put $x_p = u^{-1}\big((\underbrace{1, 1, \ldots, 1}_{p \text{ times}}, 0, \ldots, 0)\big)$ for $p = 1, 2, \ldots, n$ and define y_q by the relation

$$(x, y_q) = (ux)(q), \qquad x \in R^n,\ q = 1, 2, \ldots, n. \tag{6.9}$$

Then clearly we have (6.5), and (by (6.1)) inequality (6.4). Further, we have

$$s(\mathscr{E}) = \sup_{z\in\mathscr{E}} \max_{q\leqslant n} |z(q)| = \sup_{x\in B_n} \max_{q\leqslant n} |(ux)(q)| = \sup_{x\in B_n} \max_{q\leqslant n} |(x, y_q)| = \max_{q\leqslant n} \|y_q\|.$$

Conversely, if the sequences $(x_p)_{p\leqslant n}$ and $(y_q)_{q\leqslant n}$ satisfy (6.4) and (6.5), then there is the unique linear operator $u\colon R^n \to R^n$ satisfying (6.9). We put $\mathscr{E} = u(B_n)$. Then (6.8) holds. This completes the proof.

Added in proof. J. Lindenstrauss has pointed out to us that our Theorem 2.3 can be strengthened as follows:

Let a be a symmetric matrix norm. Then

A. *If $\sup_n t_n(a) = +\infty$, then M_a is not isomorphic to any subspace of a Banach space with an unconditional basis of finite dimensional subspaces.*

B. *If $\sup_n t_n(a) = K < +\infty$, then M_a has an unconditional basis of finite dimensional subspaces.*

Proof. A. Replace everywhere on p. 52-54 the "unconditional basis of E" by "an unconditional basis (E_n) of finite-dimensional subspaces of E" and the "one-dimensional projectors $e_n^*(\cdot)e_n$" by the "coordinate projectors $\pi_n\colon E \to E_n$". Conditions (2.7) and (2.8) replace by the condition

$$\lim_i \|\pi_n(Uu_{i,j})\| = \lim_j \|\pi_n(Uu_{i,j})\| = 0.$$

In Lemma 2.2 replace $\{e_n^*(Uu_{r,s})\}$ by the matrix $\{\pi_n(Uu_{r,s})\}$ and use the fact that for a fixed pair (n, r) the finite dimensionality of E_n implies the total boundedness of the sequence $\big(\pi_n(Uu_{r,s})\big)_{s=1}^\infty$.

B. The subspaces E_n spanned by $u_{i,j}$ with $\max(i, j) = n$ form the unconditional decomposition of M_a. The coordinate projectors are $\pi_n = P_{n,n} - P_{n-1,n-1}\colon M_a \to E_n$. We show that

$$K_{unc}\big((E_n)\big) = \sup_{a(a)\leqslant 1} \sup_{1\leqslant r_1 < r_2 < \ldots < r_s} a\Big(\sum_{j\leqslant s} \pi_{r_j}(a)\Big) \leqslant 2K.$$

Fix $r_1 < r_2 < \ldots < r_s$ and $a \in M$. Then $P_{m,m}(a) = a$ for some m. Put

$$a' = \sum_{j\leqslant m} \sum_{i\leqslant j} a(i, j) u_{i,j} \quad \text{and} \quad a'' = a - a'.$$

Since $t_m(a) \leqslant K$ and the matrix norm a is symmetric, $\max\big(a(a'), a(a'')\big) \leqslant Ka(a)$. Pick a permutation of indices F so that $F(r_j) = r_j$ for $j \leqslant s$, and if $k \leqslant m$ and $k \neq r_j$ for $j \leqslant s$, then $F(k) > m$. Let U and V denote the isometries of M_a induced by this permutation of columns and rows respectively. Then

$$\sum_{j\leqslant s} \pi_{r_j}(a) = P_{m,m}(Ua' + Va'').$$

Hence

$$a\Big(\sum_{j\leqslant s} \pi_{r_j}(a)\Big) \leqslant 2K\,a(a).$$

Since M is dense in M_a, this completes the proof.

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La fonction de Green d'un processus de Galton-Watson

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1. Introduction. Je me propose d'étudier le comportement asymptotique de la fonction de Green d'un processus de Galton-Watson dont la moyenne est finie et est plus grande que 1. Je serai alors en mesure de signaler quelques propriétés des solutions harmoniques extrémales associées au processus.

Soit $\{p(n)\}_{n=0}^{\infty}$ une suite de nombres positifs dont la somme est 1; on définit une matrice infinie $P = (p(x, y))$, $x = 0, 1, 2, \ldots$, $y = 0, 1, 2, \ldots$ de façon récurrente par rapport à x: $p(0, y) = \delta(0, y)$,

$$p(x+1, y) = \sum_{z=0}^{y} p(z)p(x, y-z).$$

La puissance matricielle $n^{\text{ème}}$ de P donne la matrice $P^n = (p_n(x, y))$. On introduit la fonction de Green

$$G(x, y) = \sum_{n=0}^{\infty} p_n(x, y) \leqslant +\infty.$$

On introduit également les fonctions génératrices

$$f_n(z) = \sum_{y=0}^{\infty} p_n(1, y) z^y$$

où z est un nombre complexe dont le module ne dépasse pas 1. On a $|f(z)| \leqslant 1$ et $f_{r+s}(z) = f_r(f_s(z))$. De plus

$$\sum_{y=0}^{\infty} p_n(x, y) z^y = (f_n(z))^x.$$

Ces diverses matrices permettent de considérer pour chaque entier x une suite de variables aléatoires indépendantes $\{Z_n^x\}_{n=0}^{\infty}$ où $P[Z_n^x = y] = p_n(x, y)$. Lorsque $x = 1$, on note plus simplement $Z_n^1 = Z_n$. Ceci