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THE MANDELBROT SET AND σ-AUTOMORPHISMS OF QUOTIENTS OF THE SHIFT

PAU ATELA

ABSTRACT. In this paper we study how certain loops in the parameter space of quadratic complex polynomials give rise to shift-automorphisms of quotients of the set Σ_2 of sequences on two symbols. The Mandelbrot set **M** is the set of parameter values for which the Julia set of the corresponding polynomial is connected. Blanchard, Devaney, and Keen have shown that closed loops in the complement of the Mandelbrot set give rise to shift-automorphisms of Σ_2, i.e., homeomorphisms of Σ_2 that commute with the shift map. We study what happens when the loops are not entirely in the complement of the Mandelbrot set. We consider closed loops that cross the Mandelbrot set at a single main bifurcation point, surrounding a component of **M** attached to the main cardioid. If n is the period of this component, we identify a period-n orbit of Σ_2 to a single point. The loop determines a shift-automorphism of this quotient space of Σ_2. We give these maps explicitly.

1. THE MANDELBROT SET AND THE EXTERNAL RAYS

We begin by recalling a few definitions and well-known facts. Let $\overline{\mathbb{C}}$ be the Riemann sphere $\mathbf{C} \cup \{\infty\}$, and consider the family $P_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of complex quadratic polynomials $P_c(z) = z^2 + c$ with the parameter $c \in \mathbf{C}$. Any complex quadratic polynomial is conjugate by an affine map $az + b$ to a unique one in this family. The polynomial P_c has a unique critical point $z = 0$. Since $P_c(0) = c$, the parameter value c is the critical value. For a fixed c we refer to the z-plane as the *dynamical plane* for P_c. In this plane, the orbit of a point z under iteration of P_c is the sequence $z, P(z), P^2(z) = P(P(z)), \ldots$. We refer to the c-plane as the *parameter plane.*

For a fixed c, let K_c be the set of points whose orbit under iteration of P_c remains bounded. That is, $K_c = \{ z | P_c^n(z) \not\to \infty \}$. K_c is called the filled-in Julia set. It is a compact set whose boundary is the Julia set J_c of P_c. J_c can be defined as the set of $z \in \mathbf{C}$ where the family of iterates $\{P_c^n\}_{n \in \mathbf{N}}$ fails to be normal. It turns out that J_c is the closure of the set of repelling periodic points (see, for example, [B, K]). It is known that if the orbit of the critical point is bounded ($0 \in K_c$), J_c is connected. Otherwise, if $0 \notin K_c$, J_c is totally disconnected and homeomorphic to a Cantor set [J, F, B]. By definition, the Mandelbrot set **M** is the set of parameter values c for which $0 \in K_c$ (J_c connected). For an excellent exposition of the Mandelbrot set see [Br]. As an introduction to **M** with beautiful pictures, one could see [PR] and therein

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the paper by A. Douady [Dou1]. Douady and Hubbard have shown that **M** is connected. It is still an open problem to prove that it is locally connected, as computer pictures seem to indicate.

The Mandelbrot set **M** appears as a bifurcation set in regions of parameter space in many other one-parameter families of maps. Whenever a family of maps "behaves like" a degree two polynomial, in parameter space, an **M**-like object will be present. This is the theory of polynomial-like mappings [DH], and in this sense the Mandelbrot set is a Universal object.

Douady and Hubbard show that **M** is connected by constructing an analytic homeomorphism $\mathbf{\Phi} : \overline{\mathbb{C}} - \mathbf{M} \to \overline{\mathbb{C}} - D_1$ tangent to the identity at ∞ (D_1 is the closed disc of radius 1) [DH1, DH2]. That is, the complement of **M** in the Riemann sphere is conformally equivalent to a disk (recall the Riemann mapping theorem). We briefly describe here the map $\mathbf{\Phi}$.

For every c, ∞ is a superattracting fixed point of P_c (fixed point with zero derivative). Therefore, in a neighborhood of ∞, P_c behaves like $z \mapsto z^2$. More specifically, there is a unique analytic map ϕ_c tangent to the identity at ∞ such that it conjugates P_c to z^2, i.e., $\phi_c \circ P_c \circ \phi_c^{-1} = P_0 : z \mapsto z^2$ in a neighborhood of ∞ (see, for example, [B]). These are sometimes called Böttcher coordinates [Bo].

Notice that 0 is the only critical point of P_c. For $c \in \mathbf{M}$, $P_c^n(0)$ lies in K_c for all n, so the conjugacy ϕ_c can be extended, by successively pulling back P_c, to an analytic homeomorphism $\overline{\mathbb{C}} - K_c \to \overline{\mathbb{C}} - D_1$.

For $c \notin \mathbf{M}$, if we take successive preimages of a neighborhood of ∞ we will eventually hit the critical value c. The next preimage of a "circle" containing c is a "figure 8". That is, if $c \notin \mathbf{M}$, ϕ_c extends to an analytic homeomorphism $\overline{\mathbb{C}} - L_c \to \overline{\mathbb{C}} - D_R$ (D_R is a closed disk of radius $R > 1$), where L_c is a compact set whose boundary is a curve homeomorphic to a lemniscate (see Figure 1). J_c is contained in the interior of L_c and $c \notin L_c$.

Douady and Hubbard define the map $\mathbf{\Phi} : \overline{\mathbb{C}} - \mathbf{M} \to \overline{\mathbb{C}} - D_1$ as

$$\mathbf{\Phi}(c) = \phi_c(c). \tag{1.1}$$

With $\mathbf{\Phi}(\infty) = \infty$, $\mathbf{\Phi}$ is an analytic homeomorphism.

Definition. For a fixed angle θ measured in turns, the *external ray* θ of **M** is $\mathbf{\Phi}^{-1}(\{re^{2\pi i\theta} \mid r > 1\})$.

Douady and Hubbard have proved that if θ is rational, the *external ray*

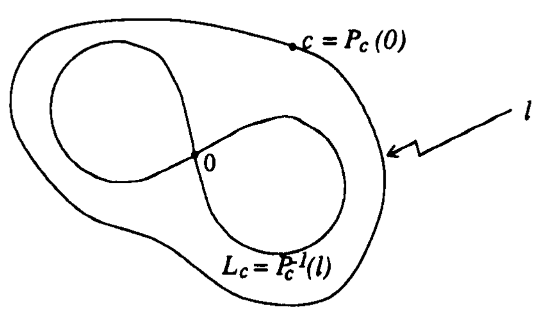

FIGURE 1. ϕ_c extends to the exterior of L_c

θ has a limit $c_0 \in \mathbf{M}$ when $r \to 1$ [DH2]. The behavior for a general θ irrational is still an open question.

Figure 2a pictures the Mandelbrot set with some of the *external rays* and where they land.

Definition (see Figure 2b). For a fixed parameter c, the *dynamic ray* of angle α is $\phi_c^{-1}(\{re^{2\pi i\alpha} \mid r > R\})$.

For $c \in \mathbf{M}$, $R = 1$; for $c \notin \mathbf{M}$, $R > 1$. In §4 we will extend the *dynamic rays* to the interior of L_c.

Notice that the *external rays* live in the parameter plane, while the *dynamic rays* live in the dynamic plane. From (1.1) it is clear that if the parameter $c \notin \mathbf{M}$ lies in the *external ray* θ of $\mathbf{M}$, then, in the dynamic plane, c lies in the *dynamic ray* of same angle θ.

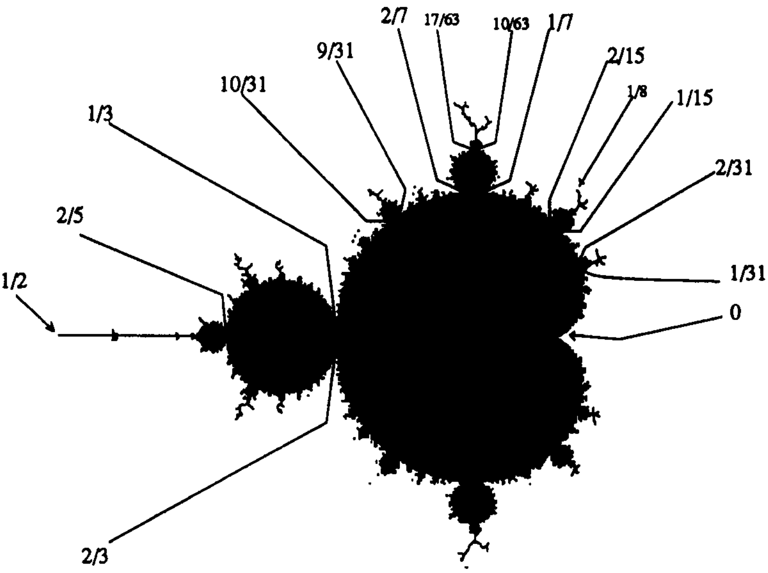

FIGURE 2a. The Mandelbrot set and some external rays (see [DH1])

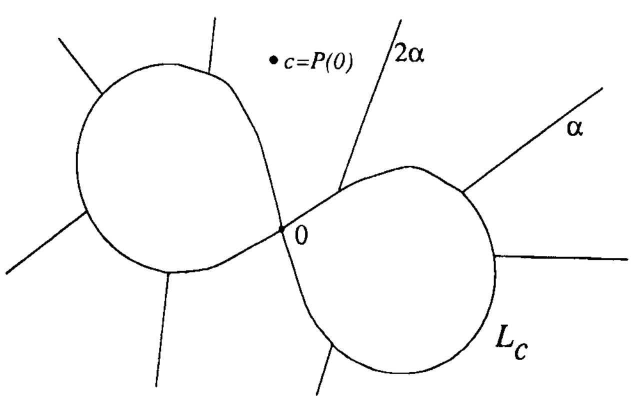

FIGURE 2b. The *dynamic rays* and L_c, for $c \notin \mathbf{M}$

2. Coding the Julia set J_c

We have mentioned that if $c \notin \mathbf{M}$, the Julia set J_c is totally disconnected, homeomorphic to a Cantor set. In this section we will show this and the standard way of associating to each point of J_c a sequence of 0 and 1 (symbolic dynamics).

Let Σ_2 be the set of all binary sequences,

$$\Sigma_2 = \{\{a_i\}_{i=0}^{\infty} \,|a_i = 0 \text{ or } 1\}.$$

Σ_2 has a metric defined by

$$d(\mathbf{a},\mathbf{b}) = \sum_{i=0}^{\infty} \delta_i 2^{-i}, \qquad \delta_i = \begin{cases} 0 & \text{if } a_i = b_i, \\ 1 & \text{if } a_i \neq b_i, \end{cases}$$

which makes it a Cantor set.

Let $\sigma\colon \Sigma_2 \to \Sigma_2$ be the *shift map*:

$$\sigma\left(\{a_i\}_{i=0}^{\infty}\right) = \{a_{i+1}\}_{i=0}^{\infty}$$

(σ just drops the first digit of the sequence). Periodic points, density of periodic points, sensitivity to initial conditions, and existence of dense orbits (in short, "*chaos*"), are very easily seen for σ (see, for example, [D]).

If $c \notin \mathbf{M}$, there is a homeomorphism $h_c\colon J_c \to \Sigma_2$ (see construction below) which we will call a "coding of J_c," such that $h_c \circ P_c|_{J_c} = \sigma \circ h_c$, i.e., P_c is topologically conjugate to σ on J_c [B], their dynamics are then completely equivalent.

We briefly sketch the construction of h_c. The interior of the lemniscate L_c (Figure 1) has two connected components which we will arbitrarily label U_0 and U_1. The first preimage of L_c under P_c consists of two more "figure 8" curves (see Figure 3), one in each of U_0 and U_1, and whose interiors again we will label according to what their image under P_c is.

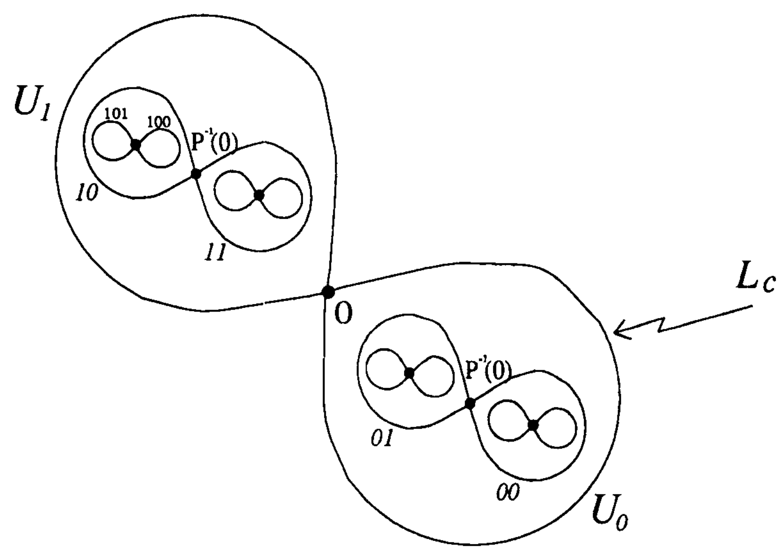

Figure 3. L_c and its preimages

There are four of these regions now. We label them $U_{a_0a_1}$ with $a_0, a_1 \in \{0, 1\}$ according to the rule

$$U_{a_0a_1} \subset U_{a_0}, \qquad P_c(U_{a_0a_1}) = U_{a_1}. \tag{2.1}$$

It is easy to see that with an induction procedure, one can continue to label 2^{n+1} such regions in step n of the process. They will be labeled $U_{a_0a_1a_2\cdots a_n}$, $a_i \in \{0, 1\}$, again according to the rule

$$U_{a_0a_1\cdots a_n} \subset U_{a_0\cdots a_{n-1}}, \qquad P_c(U_{a_0a_1\cdots a_n}) = U_{a_1\cdots a_n}.$$

Notice that everything outside these 2^{n+1} regions will escape to ∞, so the Julia set is confined in the interior of them. By a nesting sets argument (P_c is expanding in J_c) [B], these regions limit down to single points in the Julia set J_c (which is invariant under P_c):

$$\bigcap_{n=0}^{\infty} U_{a_0a_1\cdots a_n} = \{x\}.$$

Therefore, each $x \in J_c$ has a unique sequence $\{a_i\} \in \Sigma_2$ assigned to it:

Definition. Let $h_c\colon J_c \to \Sigma_2$ be defined by

$$h_c(x) = \{a_i\}_{i=0}^{\infty} \Leftrightarrow \forall i\ P_c^i(x) \in U_{a_i}.$$

That is, the point $x \in J_c$ is in U_{a_0}, then under the map P_c it jumps to U_{a_1}, then to U_{a_2}, to U_{a_3}, etc.

This is a standard construction of h_c, which is then a homeomorphism that conjugates $P_c|J_c$ to the shift map.

3. σ-AUTOMORPHISMS OF Σ_2

Recently, Blanchard, Devaney, and Keen [BDK] have shown that a closed loop γ in the exterior of the Mandelbrot set leads to a σ-automorphism of Σ_2, i.e., a homeomorphism $\Sigma_2 \to \Sigma_2$ that commutes with the shift map σ. They also show the following generalized result: A closed loop in the shift locus (subset of the space of polynomials of degree d consisting of those which have all critical points escaping to infinity under iteration) induces an automorphism of the shift on d symbols. Moreover, any such automorphism can be obtained in this way. In a recent paper, Boyle, Franks, and Kitchens [BFK] studied the group of automorphisms of the d-shift. The group is infinitely generated with very rich structure.

In this section, we sketch an argument in the case of quadratic polynomials.

Let $\gamma : [0, 1] \to \mathbf{C} - \mathbf{M}$ be a path in the complement of the Mandelbrot set and denote by J_t the Julia set for $\gamma(t) \in \mathbf{C} - \mathbf{M}$.

Let us see what is happening in the dynamic plane. The Julia set J_t is a Cantor set. Since P is expanding on J_t and J_t depends continuously on $\gamma(t) \in \mathbf{C} - \mathbf{M}$ [B], the points in the Julia set will be moving continuously without colliding (remember that $\forall t\ J_t$ is totally disconnected), each of them describing a path in the dynamic plane.

It is clear that for each t there is an induced bijective map $J_t \xrightarrow{\psi_t} J_0$. If we fix a coding $h\colon J_0 \to \Sigma_2$ for J_0, i.e., we make a choice for the labels

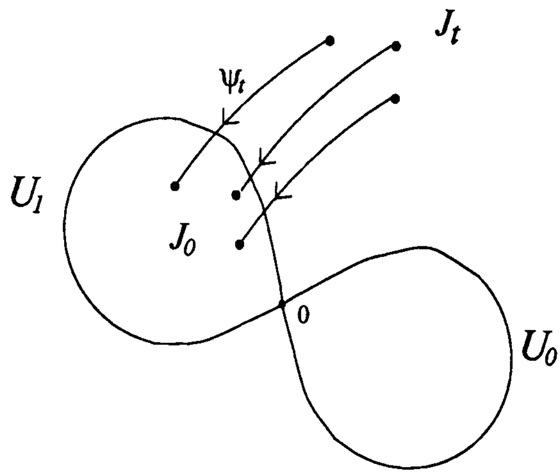

FIGURE 4

U_0 and U_1, then the composition $h \circ \psi_t \colon J_t \to \Sigma_2$ gives a coding of J_t (see Figure 4). If the path γ is a closed loop, then $J_1 = J_0$, and the map

$$h^* = h \circ \psi_1 : J_0 \to \Sigma_2$$

will be a second coding of J_0. That is, we now have two codings of the initial Julia set J_0:

$$h \colon J_0 \to \Sigma_2, \qquad h^* \colon J_0 \to \Sigma_2.$$

The discrepancy, the map $h^* \circ h^{-1} \colon \Sigma_2 \to \Sigma_2$, is then a σ-automorphism. It commutes with σ because both h and h^* do. Hedlund has proved in [H] that there are only two such automorphisms, the one that interchanges the symbols 0 and 1, and the identity. In this context they beautifully correspond to whether the loop γ goes around **M** or not [BDK].

In the next sections we will see what the corresponding map $H \colon \Sigma_2 \to \Sigma_2$ is when the loop γ is not entirely in the complement of **M**. We will take γ crossing the Mandelbrot set **M** through one of the main bifurcation points, which we will define later on (see Figure 5).

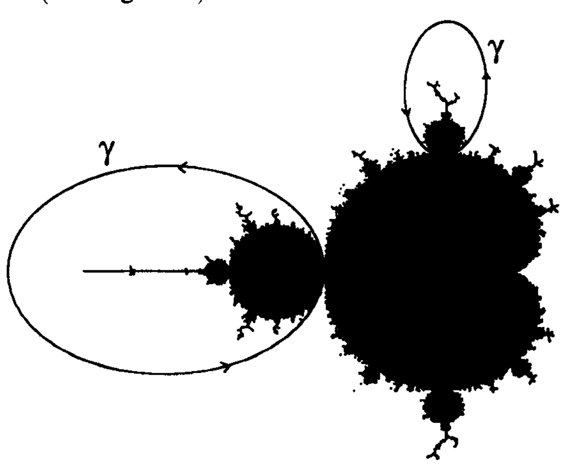

FIGURE 5. Two of the loops γ through main bifurcation points

4. Extension of the dynamic rays

We denote by S^1 the unit circle. We will measure the angles in S^1 in turns. Let $c \in \mathbb{C} - \mathbf{M}$ and let $\theta_c \in S^1$ be the angle of the external ray of the Mandelbrot set containing c. Remember that (1.1) says that also in the dynamic plane $c \in dynamic\ ray(\theta_c)$.

In §1 we mentioned that the conjugacy ϕ_c of P_c to $z \mapsto z^2$ extends, by successively pulling back P_c, analytically up to the exterior of a lemniscate L_c (Figure 1). For a given angle $\alpha \in S^1$, the *dynamic ray* (α) is so far well defined (in §1) from ∞ to the boundary of L_c (see Figure 6). For the moment, we will refer to these as *unextended rays.* Notice that they land on L_c.

Proposition 4.1 (see Figure 6). *For the unextended rays*:

(1) $P_c(ray\ \alpha) \subset ray(2\alpha)$ (mod 1).

(2) *ray* $\left(\frac{\theta_c}{2}\right)$ *and ray* $\left(\frac{\theta_c}{2} + \frac{1}{2}\right)$ *hit the origin.*

Proof. (1) Directly from the definition of *dynamic ray.*

(2) Just remember that 0 is the critical point, $P_c(0) = c$ and $c \in ray(\theta_c)$. □

It is our intention now to extend the *dynamic rays* to the interior of L_c up to the Julia set.

Definition. Let $f : S^1 \to S^1$ be the map $\alpha \mapsto 2\alpha$ (mod 1). Fix $c \in \mathbf{C} - \mathbf{M}$. Let θ_c be the *external ray* of $\mathbf{M}$ containing c. A *dynamic ray* α will be called *branched* if the angle α is any of the preimages $f^{-k}(\theta_c)$ of θ_c for $k \geq 1$. All the other rays will be called *unbranched.*

We will see the justification of this definition in the following process.

Extension process. Fix $c \in \mathbf{C} - \mathbf{M}$. So far, the *dynamic rays* are well defined from ∞ up to L_c. We now extend them, step by step, taking preimages of P_c. For the moment, let α be an *unbranched ray.* As Figure 7 illustrates, in the first step we extend the *ray* (α) from the boundary of L_c up to one of the regions $U_{a_0 a_1}$ by taking, out of the two preimages of the *ray* (2α), the one that contains *ray* (α). The *ray* (2α) is in turn extended by taking a preimage of the *ray* (4α).

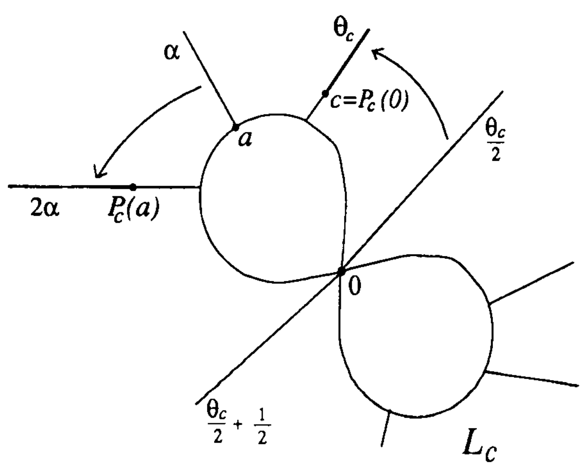

Figure 6. The *dynamic rays* from ∞ to L_c

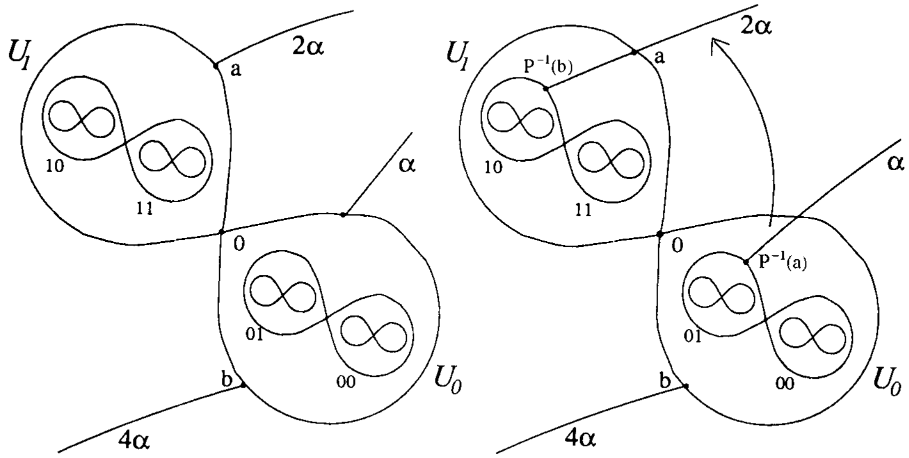

FIGURE 7. First extension of the rays α and 2α

Notice that, by (2.1), if the *unextended rays* α and 2α are landing on the first regions U_{a_0} and U_{a_1} respectively, then the *extended ray* α will go into the inner region $U_{a_0 a_1}$.

Now we have extended all the *unbranched rays* from ∞ up to the inner second regions $U_{a_0 a_1}$. Observe that only the second preimages of θ, that is, $\frac{\theta}{4}$, $\frac{\theta}{4}+\frac{1}{4}$, $\frac{\theta}{4}+\frac{1}{2}$, and $\frac{\theta}{4}+\frac{3}{4}$, would hit the center of a figure 8. This is so because these centers are preimages of zero and the first preimages of ray θ hit zero (Proposition 4.1). This justifies the above definition.

We repeat this process to further extend the *unbranched rays* up to the third regions $U_{a_0 a_1 a_2}$ and so on. Notice that in the extension process, an *unbranched* ray α is successively penetrating nested regions U_{a_0}, $U_{a_0 a_1}$, $U_{a_0 a_1 a_2}$, $\ldots$, and that these indices are given by one of the two big regions U_j which the *unextended rays* $(2^i\alpha)$, $i \geq 0$, are landing on. That is, we have proved the following

Lemma 4.2. *The ray* α *extended* n *times lands on the region* $U_{a_0 a_1 \cdots a_n} \Leftrightarrow$ *for each* $i \in \{0, 1, \ldots, n\}$ *the unextended ray* $(2^i\alpha)$ *lands on the first region* U_{a_i}.

Definition. The *extended dynamic ray* (α) will be the infinite union of all the finite extensions. From now on we will refer to the *extended dynamic ray* (α) simply as *ray* (α).

By the above discussion and definition, we have

Proposition 4.3. *If* α *is an unbranched ray,* $P_c(ray\,\alpha) = ray\,(2\alpha)$.

(Compare with Proposition 4.1.)

Theorem 4.4. *The unbranched rays land directly on a single point of the Julia set.*

Proof. The nested regions $U_{a_0 a_1 \cdots a_n}$ that the *ray* (α) is penetrating in the extension process determine this unique point. □

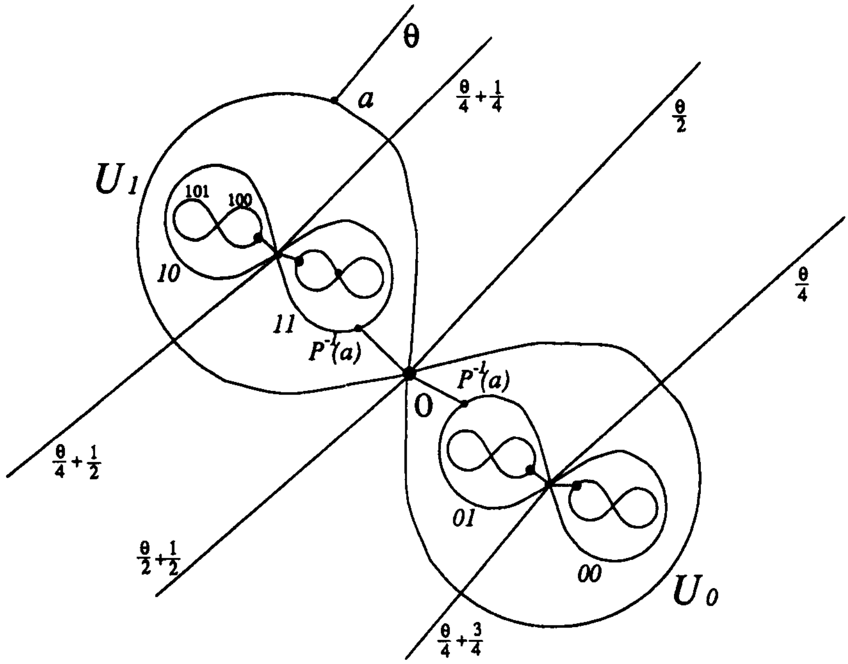

FIGURE 8a. The branched dynamic rays

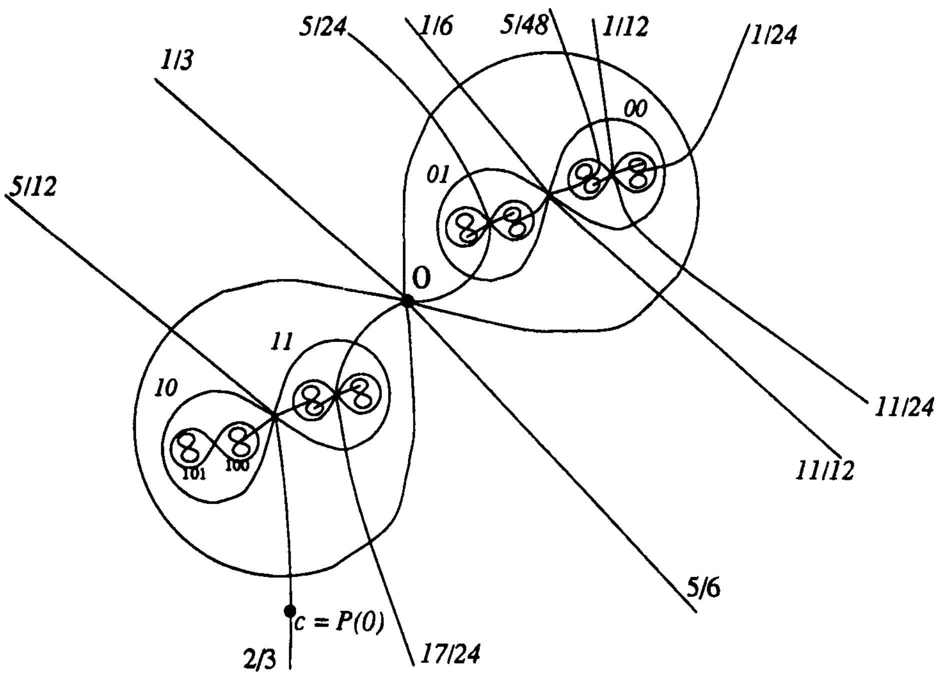

FIGURE 8b. The branched dynamic rays when c on the external ray $\frac{2}{3}$

To extend the *branched rays* defined earlier, the process is basically the same one. Note that the centers of the nested figure 8 curves are the preimages of 0 under P_c. Therefore, at each of these points, two preimages $f^{-k}(\theta)$ of the *ray* θ will fuse and, thus, more than one angle corresponds to the same (connected) *branched ray* (see Figure 8a). If θ_c is not periodic under doubling, a *branched ray* will branch only once, the two branches going then directly to points in the Julia set. Just remember that an irrational angle is not periodic under $\alpha \mapsto 2\alpha$.

If θ_c is periodic, each *branched ray* "branches" infinitely many times at preimages of 0. See, for example, the *branched dynamic rays* for $c \in ray\, \theta_c = 2/3$ shown in Figure 8b. Recall that θ is periodic under doubling if and only if it is a rational $\theta = p/q$ with $(p, q) = 1$ and q odd. Any such rational can be written in the form $l/(2^k - 1)$ for some $l \in \mathbb{N}$, where k is the period.

5. Coding the dynamic rays

In §2 we saw how the points in the Julia set are coded. In this section we will define a coding of the *dynamic rays*.

As in §4, fix $c \in \mathbb{C} - \mathbf{M}$ with $c \in ray(\theta)$. For convenience, we will sometimes write θ instead of θ_c. We know that in the dynamic plane, the *rays* $\frac{\theta}{2}$ and $\frac{\theta}{2} + \frac{1}{2}$ hit the origin.

We divide S^1 into two arcs that we label with marks 0 and 1. Let $V_j \subset S^1$, $j = 0, 1$, be the closed arcs of S^1 divided by $\frac{\theta}{2}$ and $\frac{\theta}{2} + \frac{1}{2}$ (see Figure 9). Of the two possibilities, we will decide on the one that corresponds to the choice made for the labels U_j in §2.

We will code the *unbranched rays* $\alpha \in S^1$ following the dynamics, just as we coded J_c in §2. Since $P_c(ray\, \alpha) = ray(2\alpha) \pmod 1$, we give the following

Definition (Figure 10). For c on the *external ray* θ, we define $S_\theta : S^1 \to \Sigma_2$ by

(1) If α is an *unbranched ray* (not a preimage of θ),

$$S_\theta(\alpha) = \{a_i\}_{i=0}^{\infty} \Leftrightarrow \forall i \quad 2^i \alpha \in V_{a_i} \pmod 1.$$

(2) If α is a *branched ray*, $S_\theta(\alpha)$ consists of exactly two sequences:

$$S_\theta^+(\alpha) = \lim_{\beta \to \alpha^+} S_\theta(\beta) \quad \text{and} \quad S_\theta^-(\alpha) = \lim_{\beta \to \alpha^-} S_\theta(\beta),$$

where, of course, the limit should be taken over unbranched β's.

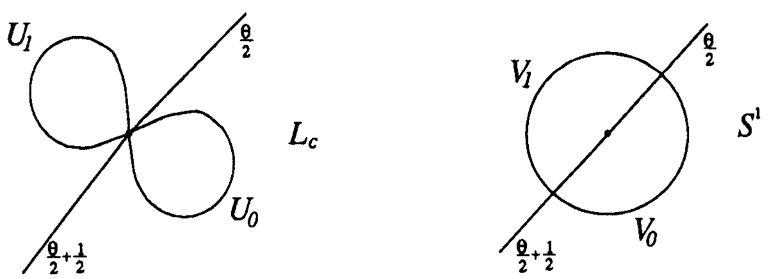

Figure 9

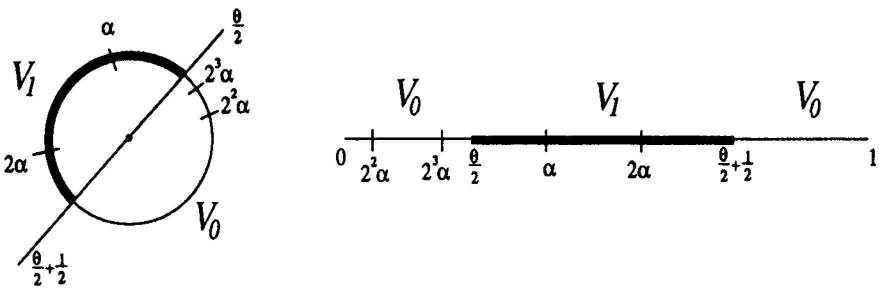

FIGURE 10. Coding of a ray α. $S_\theta(\alpha) = 1100\ldots$

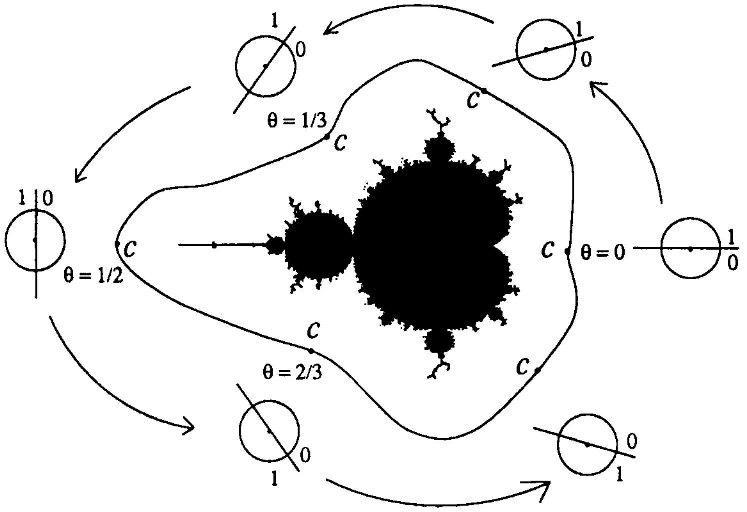

FIGURE 11. Change of the codings S_{θ_c} of S^1 as c moves around $\mathbf{M}$

Note. Part (2) of the above definition makes S_θ a multivalued function. Do not be horrified, this is precisely the beauty of it.

It is easy to check that, by construction, the following holds.

Proposition 5.1. *Let α be a (branched or unbranched) dynamic ray. Then*

(1) $S_\theta(2\alpha) = \sigma S_\theta(\alpha)$,
(2) $S_\theta(2^i\alpha) = \sigma^i S_\theta(\alpha)$.

From Theorem 4.4, we know that an *unbranched ray* lands on a unique point in the Julia set J_c. The following extremely useful result gives us the relation between the codings h_c of J_c and S_θ of S^1.

Theorem 5.2. *A ray lands on its itinerary. Let $c \in$ external ray θ of* $\mathbf{M}$. *Let $h\colon J_c \to \Sigma_2$ be the corresponding coding of J_c (§4), and $S_\theta\colon S^1 \to \Sigma_2$ be the compatible coding of the dynamic rays (Figure 9). Let α be an unbranched ray. Then*

$$\textit{ray } \alpha \textit{ lands on the point } x \in J_c \Leftrightarrow S_{\theta_c}(\alpha) = h_c(x).$$

Proof. From Theorem 4.4 we know that *ray* α does land on a unique $x \in J_c$.

This point is determined by the nested sequence of regions

$$U_{a_0}, \ U_{a_0a_1}, \ U_{a_0a_1a_2}, \ \ldots, \ U_{a_0a_1\cdots a_i}$$

which the *ray* α is successively penetrating. From the definition of h_c, $h_c(x) = \{a_i\}_{i=0}^{\infty}$. On the other hand, by Lemma 4.2, the *unextended rays* $2^i\alpha$ land on the first regions U_{a_i}. From Figure 9 it is clear that then $2^i\alpha \in V_{a_i}$, and this means that $S_{\theta_c}(\alpha) = \{a_i\}$. $\square$

Remark. If c moves along the *external ray* θ, the coding $S_\theta(\alpha)$ of the angle α remains the same although the *dynamic ray* α itself (as a subset in the complex plane) moves.

Figure 11 illustrates how the codings S_{θ_c} of S^1 change as c moves around the Mandelbrot set along a loop γ.

Notice that now it is easy to see, after a complete turn around **M**, the interchange between 0 and 1 referred to in §3 of the coding of the Julia set.

6. The maps $H_n: \Sigma_2 \to \Sigma_2$

In this and the next sections, we will consider only external rays of **M** of the form $\theta_n = \frac{1}{2^n-1}$ $(n \geq 2)$.

The main cardioid in **M** corresponds to the values of the parameter c for which there is a fixed point with derivative $|\lambda| = 1$. Therefore, a parametrization of the cardioid is given by

$$c(w) = z - z^2, \quad \text{where} \quad z = \tfrac{1}{2}e^{2\pi i w}, \ w \in [0, 1].$$

We will call the c values on the cardioid for $w = 1/n$ $(n \geq 2)$ *the main bifurcation points.* It is known that there are two external rays, θ_n and $2\theta_n$, landing on each of these (see Figure 12), and a period-n component of **M** is attached to the cardioid at the point c [DH2], [Dou]. This component contains values of c for which P_c has a period-n attracting orbit. Again we refer the reader to [Br] for an extensive introduction to the Mandelbrot set.

When the parameter $c \in$ **M** crosses the main bifurcation point from the interior of the cardioid to the attached component, in the dynamic plane a fixed point gives birth to a period-n attracting orbit [DH].

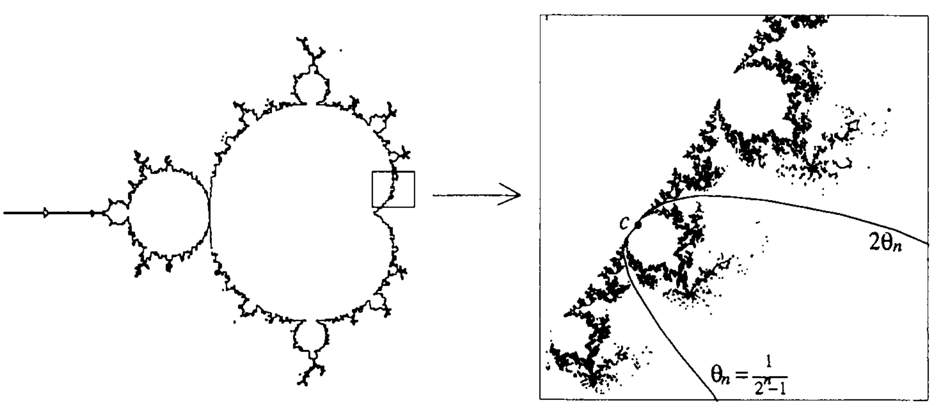

FIGURE 12

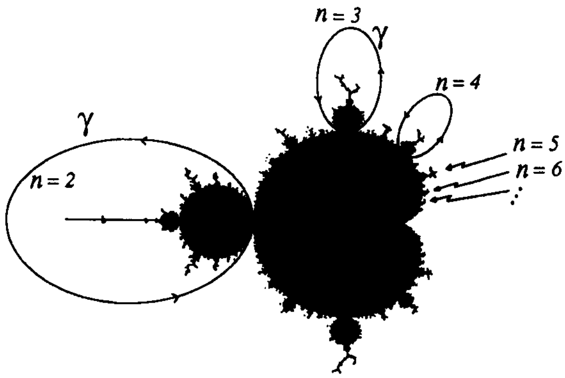

FIGURE 13. The loop γ through one of the main bifurcation points

Now fix $n \geq 2$ and let $c_0 \in \mathbf{M}$ be the above corresponding bifurcation point, having θ_n and $2\theta_n$ as external rays.

Let γ be a closed loop $\gamma = \{c_t\}$ in $\mathbb{C} - \mathbf{M}$ through c_0 oriented as shown in Figure 13, going along the *ray* $2\theta_n$ "before" c_0 and along the *ray* θ_n "after" c_0. That is, $c_t \in ray(2\theta_n)$ for $t < 0$, and $c_t \in ray(\theta_n)$ for $t > 0$.

The Julia set J_{c_t} is totally disconnected except when $c = c_0$. We choose a labeling (U_0 and U_1) (see §2) for a certain $c_t \neq c_0$ in the loop so that we have continuous codings $h_{c_t}\colon J_{c_t} \to \Sigma_2$, $t \neq 0$.

It is clear that, in a sense, J_{c_0} inherits a coding when c_t approaches c_0 along the external ray $2\theta_n$ and another one when c_t approaches c_0 from the other side, i.e., along the external ray θ_n (see Figure 11).

These codings are different and our aim is to describe the change via a map $H_n\colon \Sigma_2 \to \Sigma_2$.

To be precise, as c_t moves in the parameter plane along $2\theta_n$ towards c_0, in the *dynamic plane* we follow an *unbranched dynamic ray* of angle α with (unique) endpoint $\alpha_{x_t} \in J_{c_t}$.

By Theorem 5.2, the sequence $S_{2\theta_n}(\alpha)$ is the coding $h_{c_t}(\alpha_{x_t})$ of the point $\alpha_{x_t} \in J_t$. Moreover

$$S_{2\theta_n}(\alpha) = h_{c_t}(\alpha_{x_t}) \quad \text{is constant as } t \to 0^-,$$

and similarly

$$S_{\theta_n}(\alpha) = h_{c_t}(\alpha_{x_t}) \quad \text{is constant as } t \to 0^+.$$

The endpoint α_{x_t} of the ray α for c_t (*unbranched*) is continuous with respect to c_t, even when c_t passes through c_0 [L, Theorem, p. 77]. Therefore,

$$\text{ray } \alpha \text{ of } J_{c_0} \text{ lands on } \alpha_{x_0} = \lim_{t \to 0^-} \alpha_{x_t} = \lim_{t \to 0^+} \alpha_{x_t}.$$

From the above observation, we have that $S_{2\theta_n}(\alpha)$ is the coding that $\alpha_{x_0} \in J_{c_0}$ inherits from the left and $S_{\theta_n}(\alpha)$ is the coding that α_{x_0} inherits from the right. Thus, by the above argument, the map $H_n\colon \Sigma_2 \to \Sigma_2$ can be written as (i.e., we define it as) (see Figures 11 and 14):

$$S_{2\theta_n}(\alpha) \xmapsto{H_n} S_{\theta_n}(\alpha).$$

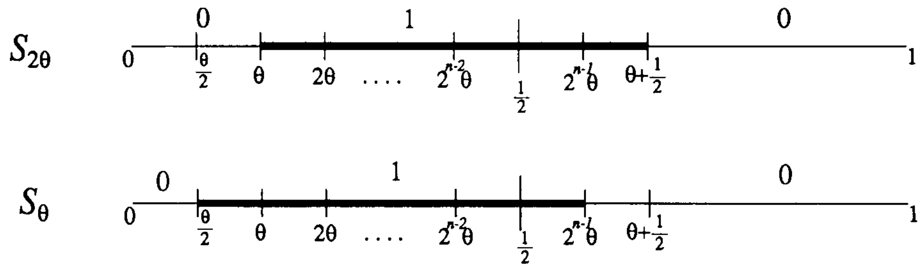

FIGURE 14. The codings $S_{2\theta}$ and S_θ of S^1

Note. We remark that H_n could be multivalued. We will deal with this later. We shall see that H_n is multivalued only at $111\ldots$, which corresponds to a fixed point, and its preimages by σ. The image under H_n of the fixed point is a group of n sequences: a period-n orbit.

7. THE DYNAMIC GRAPHS

For convenience, we will drop the index and write θ (instead of θ_n) for the angle $\theta = \frac{1}{2^n-1}$ in the rest of the paper.

Our goal now is to find explicitly the map

$$S_{2\theta}(\alpha) \overset{H_n}{\longmapsto} S_\theta(\alpha).$$

The angle θ is periodic of period n: $2^n\theta = \theta \pmod 1$. The angles $\theta = \frac{1}{2^n-1}$, 2θ, $2^2\theta$, $\ldots$, $2^{n-1}\theta$, $2^n\theta = \theta$ form a period-n cycle, and they are distributed on the interval $[0, 1]$ as illustrated in Figure 14. That is, $2^{n-2}\theta < 1/2 < 2^{n-1}\theta < \theta + 1/2$ and

$$length(\theta/2, \theta) = length(2^{n-1}\theta, \ \theta + 1/2) = \theta/2.$$

From Figure 14 we can see, for example, that

$$\begin{aligned}
S^-_{2\theta}(\theta) &= \underbrace{0111\ldots1}_{n}\underbrace{0111\ldots1}_{n}\underbrace{0111\ldots1}_{n}\ldots, \\
S^+_{2\theta}(\theta) &= 1111111\ldots, \\
S^-_{\theta}(\theta) &= 111111\ldots, \\
S^+_{\theta}(\theta) &= \underbrace{111\ldots10}_{n}\underbrace{111\ldots10}_{n}\underbrace{111\ldots10}_{n}\ldots.
\end{aligned}$$

Definition. As Figure 15 shows, we define the following partition of $S^1 \cong [0, 1] \pmod 1$:

$B_i = [2^{i-1}\theta, 2^i\theta]$, $i = 1, \ldots, n-1$,
$B_n = [2^{n-1}\theta, \theta + \frac{1}{2}]$,
$A = [\theta + \frac{1}{2}, \frac{\theta}{2}]$, i.e., $A = [\theta + \frac{1}{2}, 1] \cup [0, \frac{\theta}{2}]$,
$C = [\frac{\theta}{2}, \theta]$.

Notice that $A \cup C = V_0$ and $\bigcup_i B_i = V_1$ for $S_{2\theta}$, and $A \cup B_n = V_0$ and $\bigcup_{i<n} B_i \cup C = V_1$ for S_θ. Note also that C and B_n are the only two preimages of B_1.

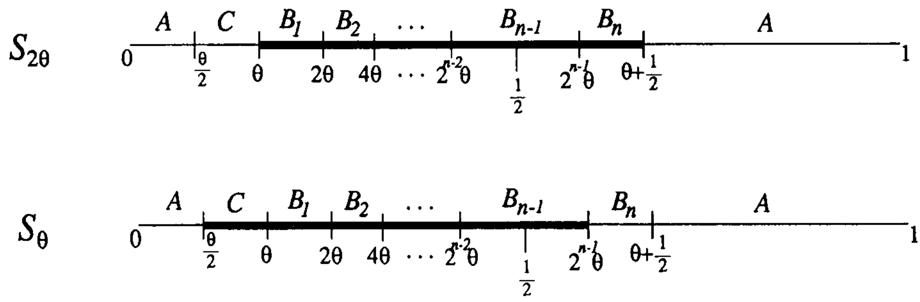

FIGURE 15. Partition of S^1

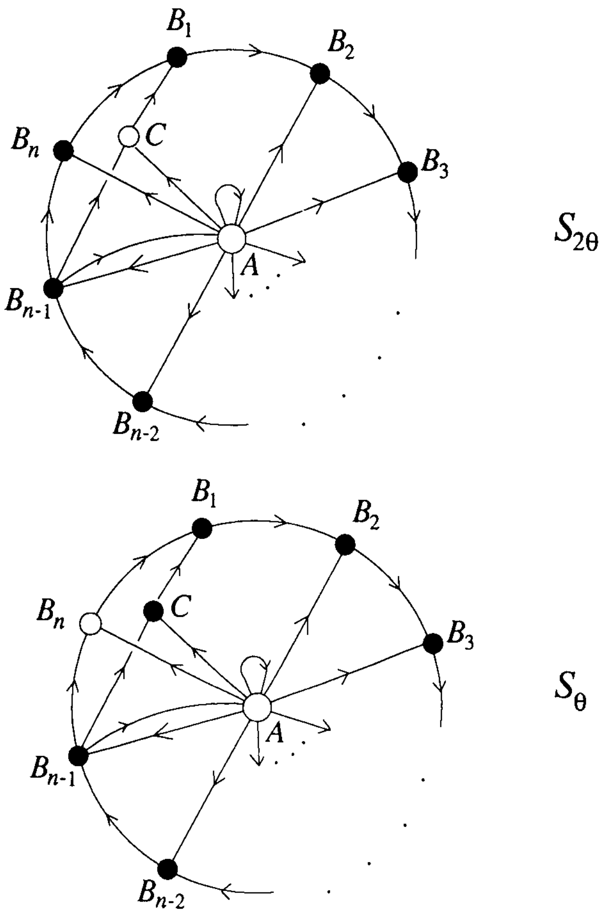

FIGURE 16. The dynamic graphs for $S_{2\theta}$ and S_θ

Fundamental Remark. Since $length(C) = length(B_n) = \frac{\theta}{2}$, the difference in the codings $S_{2\theta}$ and S_θ is only that C and B_n interchange marks.

We build now the "dynamic graphs" associated to $S_{2\theta}$ and S_θ. We will have one vertex for each of the partition intervals A, C, B_i. An arrow from vertex I to vertex J ($I \to J$) means that $2I \supset J$ (mod 1) (i.e., the image of interval I under multiplication by 2 (mod 1) contains the entire interval J). Figure 16 shows these two graphs. A vertex marked $\circ$ means that the corresponding interval is marked with a 0, and a vertex marked • corresponds to an interval marked with a 1.

Every vertex has two incoming arrows, they correspond to the two preimages of the corresponding interval.

8. Sequences and paths

Let **P** be the set of all infinite paths (without coloring) of the dynamic graph of §7 with any of the vertices as a starting point. (**P** is the same for both $S_{2\theta}$ and S_θ.)

Proposition 8.1. (1) *In both cases* ($S_{2\theta}$ *and* S_θ) *every path in* **P** *generates a unique sequence in* Σ_2.

(2) *Every path* $\{I_i\}_{i=0}^{\infty} \in \mathbf{P}$ *represents a unique point* $x \in [0, 1] \cong S^1$ *satisfying* $2^i x \in I_i \ \forall i \geq 0$.

Proof. A path in **P** is a sequence of vertices $I_0, I_1, I_2, \ldots$. The marks (0 or 1) on these vertices give a distinct element in Σ_2. For (2), we give a nested intervals argument. From the oriented graph, it is clear that in the sequence $I_0, I_1, I_2, \ldots$ at least one of the vertices B_{n-1} or A appears infinitely many times. Define

$$J_k = \{x \in I_0 |\ 2^i x \in I_i \quad \forall i \in \{1, \ldots, k\}\}.$$

We have that $J_k \neq \varnothing$ and $J_1 \supset J_2 \supset J_3 \supset \cdots$ is a nested sequence with perhaps equalities but each time that $I_k \in \{B_{n-1}, A\}$ (which happens infinitely many times), the length of J_k is reduced by a factor $0 < \delta_k < 1$ with respect to the length of J_{k-1}. This is so because both B_{n-1} and A are covering more than one vertex and the map is linear. There are finitely many of these δ_k since the graph is finite. This guarantees that $length(J_k) \to 0$ as $k \to \infty$, therefore $\bigcap J_k$ is a single point $x \in I_0$. Notice that $2^i x \in I_i \ \forall i \geq 0$. □

Note. Since the vertices of the graph correspond to closed intervals, the unique point of part (2) could be one of the partition points.

Definition. As Proposition 8.1 allows us, we define the maps

$$F_{2\theta}\colon \mathbf{P} \to \Sigma_2, \quad F_\theta\colon \mathbf{P} \to \Sigma_2, \quad \text{and} \quad G\colon \mathbf{P} \to S^1.$$

Theorem 8.2. *The maps* $F_{2\theta}, F_\theta\colon \mathbf{P} \to \Sigma_2$ *and* $G\colon \mathbf{P} \to S^1$ *are onto. Moreover,*

$$S_{2\theta} \circ G = F_{2\theta} \quad \text{and} \quad S_\theta \circ G = F_\theta.$$

That is, the following diagram commutes:

$$\begin{tikzcd}
& \mathbf{P} \arrow[d, "G"] \arrow[ddl, bend right, "F_{2\theta}"'] \arrow[ddr, bend left, "F_\theta"] & \\
& S^1 \arrow[dl, "S_{2\theta}"'] \arrow[dr, "S_\theta"] & \\
\Sigma_2 \arrow[rr, "H"'] & & \Sigma_2
\end{tikzcd}$$

Note: If $G(p)$ is *unbranched*, the assertion is $(S_{2\theta} \circ G)(p) = F_{2\theta}(p)$. If $G(p)$ is *branched*, $F_{2\theta}(p)$ is one of the two codings of $G(p)$ by $S_{2\theta}$, i.e., $(S_{2\theta} \circ G)(p) \ni F_{2\theta}(p)$.

Proof. We will prove first that $F_{2\theta}$ is onto, a completely analogous proof is valid for F_θ. Let $\mathbf{a} \in \Sigma_2$. A preimage of $\mathbf{a}$ under $F_{2\theta}$ is a path of the graph of $S_{2\theta}$ (Figure 16) that produces this sequence as we go along it. Divide the sequence $\mathbf{a}$ by its *elements* (portion of the sequence from the first digit or a zero to the last digit before next zero).

Example:

$$1111\,|\,0111\,|\,011111\,|\,0\,|\,01\,|\,0\ldots.$$

We will find the portion of the path that each element produces. Let d be the number of 1's in the element:

$$\ldots\,|\,0\,\overbrace{111\ldots1}^{d}\,|\,0\ldots$$

Notice that we could have $d = 0$ or $d = \infty$. If $d = \infty$, there are clearly n choices for a generating path (we will see them in detail in the next section). Thus, these n different paths are all the preimages of the sequence under $F_{2\theta}$. If $d \neq \infty$, we will see that the portion of the path "producing" the element is uniquely determined. The last digit 1 of the element is followed by a digit 0 of the next element. From Figure 16, it is clear that in the dynamic graph of $S_{2\theta}$ the *only* way to go from a 1 to a 0 is from vertex B_{n-1}. Therefore, this last digit 1 corresponds to vertex B_{n-1}. The previous digit 1 must then correspond to vertex B_{n-2} and so on. So we can easily find the *unique* portion of the path p that produces this element. (If $d = 0$ the vertex is certainly vertex A.)

Repeating this process element after element will give us a path p that produces the whole sequence $\mathbf{a}$, i.e., $F_{2\theta}(p) = \mathbf{a}$. Notice that the path p is determined uniquely except if $\mathbf{a}$ ends with a string of 1's, in which case there are exactly n choices. We have then proved that the map $F_{2\theta}$ is onto.

We will see now that $(S_{2\theta} \circ G)(p) = F_{2\theta}(p)$ (a completely analogous proof is valid for $(S_\theta \circ G)(p) = F_\theta(p)$). A path $p \in \mathbf{P}$ is a sequence of vertices $\{I_i\} = I_0, I_1, I_2, \ldots$, each with a mark $a_i \in \{0, 1\}$. So $I_i \subset V_{a_i}$. By Proposition 8.1 and the subsequent definition, $F_{2\theta}(p)$ is the sequence $a_0, a_1, a_2, \ldots$, and the unique point $x = G(p)$ satisfies $2^i x \in I_i$ $\forall i \geq 0$. As $I_i \subset V_{a_i}$, we then have $2^i x \in V_{a_i}$. In case $G(p) = x$ is *unbranched*, this is enough to assure $S_{2\theta}(x) = a_0 a_1 a_2 \cdots$. If x is *branched*, it is one of the preimages of 2θ. By Proposition 5.1 we can assume that $x = \theta$. Therefore $2^i\theta \in I_i$ $\forall i \geq 0$. This can happen only if the sequence a_i ends with a string of 1's or if it ends with the period-n block $0111\ldots1$ repeating. Therefore, we are in one of the two following cases:

(1) The sequence I_i is $B_1 B_2 \cdots B_n B_1 B_2 \cdots B_n B_1 B_2 \cdots B_n \cdots$ (period-n group of vertices $B_1 B_2 \cdots B_n$ repeating).

(2) The sequence I_i is $C B_1 B_2 \cdots B_{n-1} C B_1 B_2 \cdots B_{n-1} \cdots$ (period-n group of vertices $C B_1 B_2 \cdots B_{n-1}$ repeating).

In case (1), the sequence $\{a_i\}$ is the right-hand limit $S_{2\theta}^+(x)$. In case (2), it is $S_{2\theta}^-(x)$.

Now, as $F_{2\theta}$ is onto, $S_{2\theta}$ is also onto. To see that G is onto, let $x \in S^1$ and look at the sequence $S_{2\theta}(x)$. From the above discussion it is clear that for a path p producing this sequence, $G(p) = x$. $\square$

Remarks. (1) The only sequences that can be generated by different paths are those ending with a string of 1's, i.e., $1111\ldots$ and its preimages under σ.

(2) Moreover, there are exactly n different paths generating a sequence that ends with a string of 1's and this corresponds to the fact that in the external "wake" of **M** enclosed by the rays θ and 2θ (parameter plane), the *dynamic rays* $\theta, \ldots, 2^{n-1}\theta$ are attached to the fixed point $1111\ldots$.

9. Realization of the map H_n

Remember that $\theta = \frac{1}{2^n-1}$. By Theorem 8.2, the translation of sequences (codes) given by the map

$$S_{2\theta}(\alpha) \overset{H_n}{\longmapsto} S_\theta(\alpha)$$

can now be realized as the following **rule**:

$$H_n(\mathbf{a}) = F_\theta\big(F_{2\theta}^{-1}(\mathbf{a})\big).$$

Given a sequence, find on the graph of $S_{2\theta}$ the path that produces it, then follow the same path on the graph of S_θ (which is the same one except for the marks of vertices C and B_n) and see what new sequence is being produced.

In the last section, to find the path that produces a given sequence, we broke the sequence into its elements. We do the same to "translate" the sequence, i.e., to find its image under H_n. We will be translating it element by element.

Translation of an element (Image under H_n). Take an element of the sequence. Let d be the number of digits 1 in it. It can be of two types:

$$\text{Type I:} \quad \ldots\big|0\overbrace{111\ldots1}^{d}\big|0\ldots \qquad \text{Type II:} \quad \overbrace{111\ldots1}^{d}\big|0\ldots$$

Type II can only occur as the first element of the sequence. To "translate" an element (i.e., to find its image under H_n), we first find the path producing it in the graph of $S_{2\theta}$. We then follow the same path in the graph of S_θ.

Case (1): $d = \infty$. If the element is of Type I, it is $011111\ldots$. There are exactly n paths producing it under $S_{2\theta}$. These are:

$$C\,B_1\,B_2\cdots \qquad \text{and} \qquad A\,B_i\,B_{i+1}\ldots \quad \text{for } n \geq i > 1.$$

Therefore the image of this element consists of a group of n sequences:

$$01111\ldots \overset{H_n}{\longmapsto} \left\{\begin{array}{l} \overbrace{111\ldots1}^{n}\overbrace{0111\ldots1}^{n}\overbrace{0111\ldots1}^{n}\ldots \\ 0\overbrace{111\ldots1}^{k}\overbrace{0111\ldots1}^{n}\overbrace{0111\ldots1}^{n}\ldots \quad k = 0, \ldots, n-2 \end{array}\right\}$$

Similarly, if it is of type II:

$$1111\ldots \overset{H_n}{\longmapsto} \left\{\overbrace{111\ldots1}^{k}\overbrace{0111\ldots1}^{n}\overbrace{0111\ldots1}^{n}\ldots \quad k = 0, \ldots, n-1\right\}$$

Notice that $0111\ldots1$ is a period-n block.

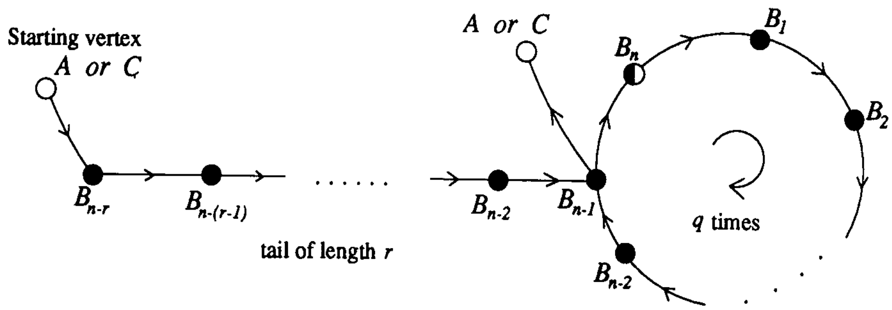

FIGURE 17. The path that produces the element

Case (2): $d < \infty$. Divide d by n to get $d = qn + r$, with $0 \leq r < n$. If $d = 0$, the element is $|0|0\ldots$. Therefore it corresponds to vertex A and thus the translation is to a digit 0. If $d \neq 0$, we know that the last digit 1 of the element corresponds to vertex B_{n-1}. Since $d \equiv r \pmod{n}$, following the graph backwards, we will go around it q times until we find that the first digit 1 in the element corresponds to vertex B_{n-r} (see Figure 17).

To translate it, we need to notice that the vertex B_n is black (marked with "1") for $S_{2\theta}$ and white (marked "0") for S_θ. For a Type II element, there are d digits to translate:

$$\underbrace{111\ldots1}_{r}\overbrace{\underbrace{0111\ldots1}_{n}\underbrace{0111\ldots1}_{n}\ldots\ldots\underbrace{0111\ldots1}_{n}}^{q\text{ period-}n\text{ blocks}}$$

For a Type I element, there are $d+1$ digits to translate. If $r = n-1$, the starting 0 vertex of the element must be C. If $r < n-1$, the starting 0 vertex must be A. Therefore the translations are:

$$r = n-1 \qquad 1\,\underbrace{111\ldots1}_{r}\overbrace{\underbrace{0111\ldots1}_{n}\underbrace{0111\ldots1}_{n}\ldots\ldots\underbrace{0111\ldots1}_{n}}^{q\text{ blocks}}$$

$$r < n-1 \qquad 0\,\underbrace{111\ldots1}_{r}\overbrace{\underbrace{0111\ldots1}_{n}\underbrace{0111\ldots1}_{n}\ldots\ldots\underbrace{0111\ldots1}_{n}}^{q\text{ blocks}}$$

10. THE QUOTIENT SPACES

By the symmetry of the graphs, the (multivalued) map H_n satisfies

$$H_n^2 = F_\theta \circ F_{2\theta}^{-1} \circ F_\theta \circ F_{2\theta}^{-1} = \mathrm{Id}.$$

Notice, for example, that

$$1111\ldots \overset{H_n}{\longmapsto} \left\{ \overbrace{111\ldots1}^{k}\overbrace{0111\ldots1}^{n}\overbrace{0111\ldots1}^{n}\ldots \quad k = 0, \ldots, n-1 \right\}$$

$$\overset{H_n}{\longmapsto} \quad 1111\ldots$$

This tells us that the image of the fixed point $111\ldots$ under H_n is the whole period-n orbit $0111\ldots1$ and vice versa. This is important. The *external rays* 2θ and θ are surrounding a period-n hyperbolic component of $\mathbf{M}$ attached to the cardioid. The bifurcation taking place when the parameter c moves, within $\mathbf{M}$, from the cardioid to the component is the following: While c is in the cardioid, the Julia set J_c has an attracting fixed point. As c goes into the Hyperbolic component crossing the boundary of the cardioid, this fixed point changes from being attractive to repelling, and a cycle of period-n changes from being repelling to being attractive (see, for example, [Br]).

Definition. The above allows us to pass naturally to a quotient space $\Sigma_2/\sim$, where $\sim$ is defined as:

$$\mathbf{a} \sim \mathbf{b} \Leftrightarrow H_n(\mathbf{a}) = H_n(\mathbf{b}).$$

The equivalence relation identifies the period-n orbit $0111\ldots1\ldots$, which is the image of the fixed point $111\ldots$ to a single point. The loop γ considered leads us then to this quotient space and to an induced map $\widehat{H}_n$ in it. We have

Theorem. $\widehat{H}_n : \Sigma_2/\sim \;\to\; \Sigma_2/\sim$ *is a σ-automorphism.*

Proof. $\widehat{H}_n$ is clearly continuous, $\widehat{H}_n^2 = \mathrm{Id}$, and as $H_n \circ \sigma = \sigma \circ H_n$, it also commutes with σ. □

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Program in Applied Mathematics, University of Colorado, Boulder, Colorado 80309-526

Current address: Department of Mathematics, Smith College, Northampton, Massachusetts 01063

E-mail address: patela@sophia.smith.edu