The Manipulability of Voting Systems

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1. INTRODUCTION. When one speaks of a mathematical analysis of voting, two results spring to the forefront: the voting paradox of Condorcet [7] and Arrow's Impossibility Theorem [**1**]. In fact, most mathematicians—although perhaps unable to state either precisely—have heard of both, and these two results are finding their way into more and more undergraduate textbooks for non-majors; see [**6**], [**28**], or [**29**].

But Condorcet's and Arrow's contributions are, we feel, only the first two parts in a natural progression that is a trilogy—ending with the remarkable Gibbard-Satterthwaite Manipulability Theorem [**17**], [**25**]—or perhaps (as we might argue) a tetralogy, culminating in the striking generalization recently proved by Duggan and Schwartz [**9**], [**10**].

The basic voting-theoretic context in which we work has ballots that are lists (sometimes allowing ties, sometimes not) and elections whose outcome is a non-empty set of alternatives (again, sometimes allowing ties for the win, and sometimes not). A ballot in which there are no ties is called a *linear ballot*.

Following standard terminology in the field, a sequence P of ballots is called a *profile*. If P is a profile, then the set of winners, according to some specified voting system V, is denoted by $V(P)$.

Most people are aware of several examples of voting systems in this context. *Plurality*, for example, is the system in which the winner is the alternative with the most first-place votes. Scoring systems, on the other hand, assign points to alternatives based on where they appear on a ballot; the special case in which a first-place vote is worth $n - 1$ points, a second-place vote is worth $n - 2$, etc. is known as the *Borda count*. The *Hare system* (respectively, the *Coombs method*) proceeds by iteratively deleting the alternatives with the fewest first-place votes (respectively, the most last-place votes). All of these voting systems can produce ties for the win.

There are other voting systems that are less trivial mathematically, but not as well known. For example, assume for simplicity that we have n voters and n alternatives. Given a profile P, consider the voting system V in which an alternative a fails to be in $V(P)$ if and only if there exists a set X of voters and a set B of alternatives such that $|X| + |B| > n$, and every voter in X ranks every alternative in B higher on his ballot than a. The intuition here (roughly) is to give sets of voters the power to veto sets of alternatives if the set of alternatives is proportionately smaller than the set of voters, and to reject any alternative that, if chosen, would trigger the use of such veto power. A non-trivial theorem of Moulin (see [**21**] or [**22**, p. 122]) asserts (in part) that $V(P)$ is always non-empty.

It turns out that, as far as manipulability is concerned, the issue of whether we allow ties in the ballots or not is a relatively minor one. On the other hand, the issue of whether we allow ties in the outcome of an election is crucial. It is, in fact, what separates the Gibbard-Satterthwaite Theorem (where such ties are not allowed) and the Duggan-Schwartz Theorem (where they are allowed).

Of fundamental importance to our considerations is the question of what it means to say that a voter can manipulate a voting system. Intuitively, it means that there is at least one situation in which this voter prefers the election outcome resulting from his submission of a disingenuous ballot to the outcome resulting from his submission of a ballot corresponding to his true preferences. At an intuitive level, this is fine. But a few points require clarification.

First of all, what do we mean by a "situation"? This is easy—we simply mean a sequence of ballots cast by the other voters. Thus, we are assuming that the voter in question has complete knowledge of how everyone else voted (or perhaps better: will vote), and we are asking if he can take advantage of this knowledge to secure a better outcome—better, that is, from his point of view—by submitting an insincere ballot.

More to the point, however, is the question of what it means for our voter to "prefer" one election outcome to another. If we demand that our voting procedure produce a single winner—disallowing ties in the outcome of any election—things are easy. That is, given two ballots, one of which represents this voter's true preferences and one of which is disingenuous, we simply compare the two election results using the ballot representing his true preferences. These considerations give rise to the notion of manipulability on which the Gibbard-Satterthwaite Theorem is based.

The conclusion of the Gibbard-Satterthwaite Theorem is that (assuming there are three or more alternatives, each of which appears as a winner for at least one profile) the only voting system that is non-manipulable and always produces a single winner is a dictatorship. But this demand that an election never result in a tie is a weakness of sorts. For example, few such voting systems (manipulable or not) spring to mind—in part because we tend to want voting systems that treat all voters the same (a property called *anonymity*) and all alternatives the same (a property called *neutrality*), and these properties together certainly rule out single winners, as can be seen by considering two alternatives and two voters who rank them opposite ways, or three voters who rank three alternatives with ballots: abc, bca, cab.

Thus, when single winners are truly needed, one must resort to some kind of tie-breaking mechanism. Such a mechanism can be chance or deterministic, and, if deterministic, it can be democratic or not. One way to view the content of the Gibbard-Satterthwaite Theorem is that it rules out "democratically deterministic" as an option—one has to choose between chance resolutions and single-handedly imposed ones.

However, if we allow ties in the outcome of an election, then there certainly exist voting systems that treat all voters the same, all alternatives the same, and are intuitively non-manipulable. For example, one could declare every alternative to be tied for the win regardless of the ballots. Or one could take as a winner any alternative a that is not universally regarded as inferior to some other alternative b (the so-called Pareto-optimal set). Or one could declare an alternative to be a winner if and only if at least one voter ranked it first on his ballot.

But we have to be careful here. While the first example (everyone tied regardless of the ballots) is clearly non-manipulable via any reasonable definition, things are less clear with the latter two voting systems. For example, consider an election in which three voters have ballots $\langle a, b, c\rangle$, $\langle a, b, c\rangle$, and $\langle c, a, b\rangle$, and assume that these represent the voters' true preferences. The winning set is $\{a, c\}$ according to either of the latter two voting systems, but if voter one submits, instead, the insincere ballot $\langle b, a, c\rangle$, the winning set becomes $\{a, b, c\}$. If voter one feels that a and b are very close in value (with c much worse), one can imagine preferring $\{a, b, c\}$ to $\{a, c\}$—for example, if the ultimate winner is to chosen by a random draw from the winning set. For a related discussion, see [**16**].

Thus, if we allow ties in the outcome of an election, then there is a question as to what we mean by manipulation. Things are much stickier here, because we now must compare two *sets* of winners given only a preference ordering of single alternatives.

For example, if our voter ranks a over b over c over d, will he prefer an election outcome of $\{a, d\}$ to an election outcome of $\{b, c\}$, or vice versa?

The intuition behind the notion used in the Duggan-Schwartz generalization is the following. Let's assume that, when the dust settles, society needs to have a single winner, and that this single winner is selected in some way (randomly, by some committee, etc.) from those tied for the win according to our voting procedure.

Now, if a voter is sufficiently optimistic, and if he ranks a over b over c over d, then he prefers an election outcome of $\{a, d\}$ to an election outcome of $\{b, c\}$. This is because he assumes—optimistically—that a (his top choice overall) results from an election outcome of $\{a, d\}$, while b (his second choice overall) results from an election outcome of $\{b, c\}$. In general, a sufficiently optimistic voter compares two election outcomes (that is, two sets of alternatives) by asking which has a "larger max" according to his true preference ranking of the alternatives.

On the other hand, if a voter is sufficiently pessimistic, and if he ranks a over b over c over d, then he prefers an election outcome of $\{b, c\}$ to an election outcome of $\{a, d\}$. This is because he assumes—pessimistically—that d (his worst choice overall) results from an election outcome of $\{a, d\}$, while c (his third choice overall) results from an election outcome of $\{b, c\}$. In general, a sufficiently pessimistic voter compares two election outcomes (that is, two sets of alternatives) by asking which has a "larger min" according to his true preference ranking of the alternatives.

These considerations give rise to the notion of manipulability on which the Duggan-Schwartz Theorem is based. Other notions of manipulability in the context of elections with ties have also been considered; see **[2]**, **[3]**, **[4]**, **[11]**, **[12]**, **[15]**, **[18]**, **[19]**, **[20]**, **[27]**, **[30]**, and **[32]**. Formalizations of the kind of manipulability that we consider here occur in Section 2, with further intuitive justification for this notion in Section 6.

The rest of the paper is organized as follows. In Section 2, we prove the Duggan-Schwartz Theorem, pointing out, as we go, what parts of the proof can be ignored if one simply wants to obtain the Gibbard-Satterthwaite Theorem. This proof takes place in the context in which there are three or more alternatives, no ties in the ballots, and every alternative is the unique winner for at least one set of ballots. The Duggan-Schwartz conclusion is that there must, in this case, be a dictator in the sense that the alternative at the top of his ballot is always among the winners.

In Section 3, we present some easy consequences of the Gibbard-Satterthwaite and Duggan-Schwartz Theorems in which the dictatorship-like consequences of non-manipulability co-exist with a quasi-democratic ability of each voter to have a unilateral effect on the outcome of an election.

We achieve this co-existence of non-manipulability and quasi-democracy in the Gibbard-Satterthwaite setting by considering *social welfare functions* (wherein, by definition, the outcome of an election is a linear ordering of the alternatives instead of a single winning alternative). Here, we have a natural notion of manipulability based on lexicographic orderings: given a list L (which we think of as a ballot giving a voter's true preferences) and two other lists L_1 and L_2 (which we think of as possible election outcomes according to some social welfare function), we scan down the two lists until we reach the first place that they differ—at this point, we see which alternative is better according to the preference list L. It turns out that non-manipulability here is (with slight hedging) equivalent to the system being one in which some voter gets to pick which alternative is in first place, another voter then gets to specify which is in second place, and so on (allowing one voter to play more than one role).

We achieve a version of this co-existence of non-manipulability and quasi-democracy in the Duggan-Schwartz context by adding the additional restriction that the voting system treats all voters the same (anonymity). Unlike many theorems of its ilk, this leads not to an impossibility theorem, but to a characterization of a very natural voting system: an alternative is one of the winners if and only if it is ranked first by at least one voter.

In Section 4, we consider what happens if we delete the assumption that every alternative x is the unique winner in at least one election. For the Duggan-Schwartz Theorem, this extension is based on an embellishment of their result in [**10**], and the observation that this later Duggan-Schwartz result follows from the main theorem in Section 2, which is, in fact, an earlier unpublished result of theirs [**9**]. The main theorem in Section 4 generalizes a known result from the Gibbard-Satterthwaite context.

In Section 5, we extend the basic result in Section 2 to handle the case in which ties are allowed in the ballots, again generalizing a known result in the context of the Gibbard-Satterthwaite Theorem. Finally, in Section 6, we offer some concluding discussion.

2. THE DUGGAN-SCHWARTZ THEOREM.

As a context for a basic version of the Duggan-Schwartz Theorem, we take elections in which we have linear ballots, three or more alternatives, and in which the outcome of an election is—in contrast to what one has with the Gibbard-Satterthwaite Theorem—a non-empty set of winners. The kind of manipulation that we explore here is given by the following.

Definition 2.1. A voting system can be *manipulated by an optimistic voter* if there exists a profile $\langle B_1, \ldots, B_n \rangle$ (which we think of as giving the true preferences of the n voters) and another ballot C_i (which we think of as a disingenuous ballot from voter i) such that at least one of the winners from the profile

$$\langle B_1, \ldots, B_{i-1}, C_i, B_{i+1}, \ldots, B_n \rangle$$

is—according to B_i—preferred to the all of the winners from $\langle B_1, \ldots, B_n \rangle$. Similarly, a voting system can be *manipulated by a pessimistic voter* if there exists a profile $\langle B_1, \ldots, B_n \rangle$ (which we think of as giving the true preferences of the n voters) and another ballot C_i (which we think of as a disingenuous ballot from voter i) such that all of the winners from the profile

$$\langle B_1, \ldots, B_{i-1}, C_i, B_{i+1}, \ldots, B_n \rangle$$

are—according to B_i—preferred to at least one of the winners from $\langle B_1, \ldots, B_n \rangle$.

More briefly, a voting system can be manipulated by an optimist if there is at least one election in which some voter can file a disingenuous ballot and improve the max of the set of winners according to his true preferences. Similarly, a voting system can be manipulated by a pessimist if there is at least one election in which some voter can file a disingenuous ballot and improve the min of the set of winners according to his true preferences.

For the remainder of this section, we fix a context in which there are three or more alternatives, n voters for some fixed n, linear ballots (ties in the ballots are handled later), and—with the exception of Corollary 2.13—elections in which the outcome is a non-empty set of winners. If V is a voting system in this context, we say that an alternative x is *viable* if $V(P) = \{x\}$ for at least one profile P.

Theorem 2.2 (Duggan-Schwartz [9]). *If V is a voting system that cannot be manipulated by an optimist or a pessimist and in which every alternative x is viable, then there exists at least one voter whose top choice is always among the set of winners.*

Our proof of Theorem 2.2 (quite different from that of Duggan and Schwartz) requires several definitions and lemmas. Our starting point, however, is with a slightly strengthened version of one of their observations, although it is worth noting that the following discussion is not necessary if one is simply trying to prove the Gibbard-Satterthwaite Theorem. First, a piece of terminology: if P is a profile, then a set X of alternatives is said to be a *top set* (for P) if each voter prefers (according to his ballot) every alternative in X to every alternative not in X. For example, if every voter has x at the top of his ballot, then $\{x\}$ is a top set.

Suppose now that V is a voting system that cannot be manipulated by an optimist or a pessimist, and that P is a profile for which X is a top set. Assume that there is at least one profile P' for which $V(P') \subseteq X$. Then we claim that $V(P) \subseteq X$. If not, we could convert P to P', one ballot at a time, until the set of winners changes from not being a subset of X (which we are assuming is true with P) to being a subset of X (which we are assuming is true with P'). If this occurs as we change ballot B_i to C_i, then we can take B_i to be the true preferences of voter i and see that his insincere submission of C_i has improved the min (from something not in X to something in X). This proves the claim.

The key to our proof of the Duggan-Schwartz Theorem is the following definition.

Definition 2.3. A voting system V is said to satisfy *down-monotonicity for singleton winners* provided that the following always holds: if P is a profile and $|V(P)| = 1$, and if P' is the profile obtained from P by having one voter move one losing alternative down one spot on his ballot, then $V(P') = V(P)$.

From down-monotonicity for singleton winners, it follows that, if the outcome of an election is a singleton, then that outcome is unchanged if any number of voters move any number of losing alternatives down any number of spots on their ballots. In the Gibbard-Satterthwaite context, this means that whenever we have an election in which we are focussing on two alternatives, a and b, one of which is the winner, we can assume—with no loss of generality—that all other alternatives appear below a and b, and in (say) alphabetical order on all ballots.

The published proof of the Gibbard-Satterthwaite Theorem that appears to be closest to the one obtained by specializing what we give here for Theorem 2.2 to the case of singleton winners is in [**22**]. That proof is based on notions called strong positive association and strong monotonicity; see [**23**] and [**24**]. Both these notions are equivalent to down-monotonicity for voting systems in which winners are always singletons, but incomparable in the more general setting where ties are allowed in the outcome of an election and we delete the requirement that $|V(P)| = 1$ in the definition of down-monotonicity. For another proof of the Gibbard-Satterthwaite Theorem, see [**16**].

Lemma 2.4. *If a voting system cannot be manipulated by an optimist or a pessimist, then it satisfies down-monotonicity for singleton winners.*

Proof. If down-monotonicity for singleton winners fails, then there exist two elections, a single voter i, and two alternatives x and y such that:

In Election #1, voter i has ballot $B_i = \langle \ldots y, x \ldots \rangle$, and some $w \neq y$ is the only winner (that is, y is a non-winner).

In Election #2, voter i has ballot $C_i = \langle \ldots x, y \ldots \rangle$, all other ballots are the same as in Election #1, and some $Y \neq \{w\}$ is the set of winners.

Choose $v \in Y$ such that $v \neq w$. If v is preferred to w on both ballots, then we can regard B_i as the true preferences, and see that voter i's disingenuous submission of C_i improves the max (according to his true preferences) from w to at least v.

Similarly, if w is preferred to v on both ballots, then we can regard C_i as the true preferences, and see that voter i's disingenuous submission of B_i improves the min (according to his true preferences) from v or worse to w.

In the only remaining case, we must have $\{v, w\} = \{x, y\}$, and since $w \neq y$, we must have $x = w$ and $y = v$. But then we can regard B_i as the true preferences, and see that voter i's disingenuous submission of C_i improves the max (according to his true preferences) from $x = w$ to at least $y = v$. We could also have regarded C_i as the true preferences, and had voter i improve the min. ■

Definition 2.5. If V is a voting system, X is a set of voters, and a and b are distinct alternatives, then we write "aXb" to mean that $V(P) \neq \{b\}$ whenever P is a profile in which everyone in X has a over b on his ballot. We say that X is a *dictating set* if aXb for every pair of distinct alternatives a and b.

We really should include the name of the voting system V in the notation "aXb" and similarly speak of a "dictating set for V", but our suppression of the name V causes no confusion.

Lemma 2.6. *Assume that V is a voting system that satisfies down-monotonicity for singleton winners. Then, in order to show that aXb, it suffices to find a single profile P in which $\{a, b\}$ is a top set, everyone in X prefers a to b, everyone else prefers b to a, and in which $a \in V(P)$.*

In the Gibbard-Satterthwaite setting, one can omit the phrase "$\{a, b\}$ is a top set" and just use down-monotonicity to prove the resulting statement. This change then makes the upcoming Lemma 2.8 (a key element of the proof) trivial in the Gibbard-Satterthwaite setting.

Proof of Lemma 2.6. Assume that aXb fails, and choose a profile P' in which everyone in X prefers a to b and for which $V(P') = \{b\}$. Using down-monotonicity for singleton winners, we can convert P' into the profile P that is assumed to exist, and get $V(P) = \{b\}$. But this is a contradiction since $a \in V(P)$. ■

Lemma 2.7. *Assume that V is a voting system that satisfies down-monotonicity for singleton winners, and for which every alternative is viable. Then the set of all voters is a dictating set.*

Proof. Suppose that P is a profile in which every voter has a over b on his ballot, but $V(P) = \{b\}$. Choose a profile P' such that $V(P') = \{a\}$. Now, using down-monotonicity for singleton winners, we can first move b to the bottom of every ballot in P' and then repeat this for each of the other losing alternatives (in some fixed order—picture it as being alphabetical: $c, d, e, \dots$). Similarly, we can move all alternatives other than a and b to the bottom (in this same fixed order) of all the ballots in P. But then we have identical profiles with two different election outcomes. ■

We can reach the conclusion of Lemma 2.7 with "$V(P) = \{x\}$" replaced by the weaker assumption "$x \in V(P)$" if we replace down-monotonicity with the direct assumption that the system cannot be manipulated by an optimist or a pessimist. That is, if every voter has a over b and $V(P) = \{b\}$, then we can use down-monotonicity to make $\{a, b\}$ a top set. Now choose P' such that $a \in V(P')$. Convert P to P', one ballot at a time, until a becomes a winner. At this point, the voter who just changed his ballot has improved his max to his most preferred alternative a.

For the next four lemmas, we assume that V is a voting system that cannot be manipulated by an optimist or a pessimist, and for which every alternative x is viable.

Lemma 2.8. *Suppose that X is a set of voters, a and b are alternatives, and aXb. Now assume that $c \neq a$ and $c \neq b$, and suppose that X is partitioned into disjoint sets Y and Z (one of which may be empty). Then either aYc or cZb.*

Proof. This proof is quite trivial in the Gibbard-Satterthwaite context where we always have singleton winners. But here, consider the election in which the profile P is as follows:

Everyone in Y has ballot $\langle a, b, c, \ldots \rangle$.
Everyone in Z has ballot $\langle c, a, b, \ldots \rangle$.
Everyone else has ballot $\langle b, c, a, \ldots \rangle$.

Because $\{a, b, c\}$ is a top set, our previous discussion guarantees that $V(P) \subseteq \{a, b, c\}$. Because aXb, $V(P) \neq \{b\}$, and so either $a \in V(P)$ or $c \in V(P)$.

Case 1: $a \in V(P)$.

For each voter in Y, we one-by-one move b just below c. As we do this—changing a ballot from B_i to C_i—a remains a winner (or else we could regard C_i as the true preferences and then have voter i improve his max from something other than his top choice to his top choice a). Now, for every voter not in Y or Z ("Everyone else"), we one-by-one move b just below a. Again, as we do this—changing a ballot from B_i to C_i—a remains a winner (or else we could regard B_i as the true preferences and then have voter i improve his min from a to b or c). But now we have produced a profile P' in which $\{a, c\}$ is a top set, everyone in Y prefers a to c, everyone else prefers c to a, and in which $a \in V(P')$. Thus, Lemma 2.6 ensures that aYc, as desired.

Case 2: $c \in V(P)$.

For each voter in Z, we one-by-one move a just below b. As we do this—changing a ballot from B_i to C_i—c remains a winner (or else we could regard C_i as the true preferences and then have voter i improve his max from something other than his top choice to his top choice c). Now, for every voter in Y, we one-by-one move a just below c. Again, as we do this—changing a ballot from B_i to C_i—c remains a winner (or else we could regard B_i as the true preferences and then have voter i improve his min from c to a or b). But now we have produced a profile P' in which $\{b, c\}$ is a top set, everyone in Z prefers c to b, everyone else prefers b to c, and in which $c \in V(P')$. Thus, Lemma 2.6 ensures that cZb, as desired. ∎

Lemma 2.9. *Suppose X is a set of alternatives and that aXb for some a and b. Then*

(i) for all $c \neq a$, we have aXc, and
(ii) for all $c \neq b$, we have cXb.

Proof. We never have $x \oslash y$ for any x and y, or else, for every profile P, we would have $V(P) \neq \{y\}$. Hence, (i) follows from Lemma 2.8 with $Z = \oslash$, and (ii) follows from Lemma 2.8 with $Y = \oslash$. ∎

Lemma 2.10. *Suppose X is a set of alternatives and that aXb for some a and b. Then X is a dictating set.*

Proof. Assume that x and y are distinct alternatives. Using Lemma 2.9, we have:

If $y \neq a$, then aXb implies aXy implies xXy.

If $x \neq b$, then aXb implies xXb implies xXy.

If $y = a$ and $x = b$, then choose some $z \neq a, b$. Then aXb implies yXx implies yXz implies xXz implies xXy. ∎

Lemma 2.11. *Suppose that X is a dictating set and that X is partitioned into disjoint sets Y and Z. Then either Y is a dictating set or Z is a dictating set.*

Proof. This is immediate from Lemmas 2.8 and 2.10. ∎

Lemma 2.12. *For the kind of voting system that we are considering, there is a voter whose top choice is the unique winner whenever the winner is a singleton.*

Proof. It follows from Lemmas 2.7 and 2.11 that there is a voter i such that $\{i\}$ is a dictating set. But this means that the only singleton winner can be the alternative at the top of voter i's ballot. ∎

In the Gibbard-Satterthwaite context, the proof of Theorem 2.2 is complete at this point. In the present context, however, we need one additional observation. Assume, then, that V is a voting system that cannot be manipulated by an optimist or a pessimist, and for which every alternative x is viable. Suppose voter i's top choice is the unique winner whenever the winner is a singleton (as guaranteed by Lemma 2.12). Then, we claim that voter i's top choice is always among the set of winners.

The argument here runs as follows. Suppose not, and choose a profile P such that the alternative x that is at the top of voter i's ballot is not in $V(P)$, and such that $|V(P)|$ is as small as possible. We can't have $|V(P)| = 1$ by our assumption that voter i's top choice is the unique winner whenever the winner is a singleton.

Assume that $V(P) = \{s_1, \ldots, s_t\}$ with $t \geq 2$ and $x \notin V(P)$, and assume that voter i ranks s_1 over s_2 over ... over s_t. Let P' be any profile in which voter i's ballot is the same as in P, but in which all the other voters have $s_1, \ldots, s_t$ as a top set in that order. Now change P to P' one ballot at a time.

We first claim that as we change a ballot from B_j to C_j, no new alternative w gets added to the set $V(P)$ of winners, since we could then regard C_j as the true preferences of that voter, and the disingenuous submission of B_j would then improve the min from w or worse to s_t. This argument covers $x = w$ as well.

Moreover, no s_i can be lost from $V(P)$ by the minimality of $|V(P)|$—this is why we needed to observe that x is not added to $V(P)$. But now, starting with P', voter i can bring s_1 to the top of his ballot and make the set of winners a singleton $\{s_1\}$ (because $\{s_1\}$ is then a top set), thus improving his minimum because $t \geq 2$. This completes the proof of Theorem 2.2.

Of course, we immediately have the following, still in the context of linear ballots and three or more alternatives.

Corollary 2.13 (Gibbard-Satterthwaite). *Suppose V is a voting system in which the outcome of an election is always a single winner, and for which every alternative is* *viable. Then V is non-manipulable if and only if there exists a single voter whose top choice is always the unique winner.*

3. THE CO-EXISTENCE OF NON-MANIPULABILITY AND QUASI-DEMOCRACY. Our use of the term "quasi-democracy" in the title of this section (and in our remarks in Section 1) is meant to be informal. Roughly, what we have in mind is a situation wherein every voter can unilaterally affect the outcome, although not necessarily on an equal basis. Let's begin with the Gibbard-Satterthwaite Theorem.

Our approach here is to shift contexts from social choice functions, where the outcome of an election is a single alternative, to social welfare functions, where the outcome of an election is a linear ordering of the set of alternatives—what we call a "final list." We still assume that ballots involve no ties, and we likewise disallow ties in the final list produced by a social welfare function.

Suppose L is a linear ordering of the alternatives that represents a voter's true preferences, and suppose that L_1 and L_2 are two linear orderings that can arise in the final list using some social welfare function. What does it mean to say that this voter prefers one list to the other? The answer that we make use of is the one alluded to in Section 1; it is the analogue of a lexicographic ordering.

Definition 3.1. A social welfare function can be *manipulated by a voter* if there exists a profile $\langle B_1, \ldots, B_n\rangle$ (which we think of as giving the true preferences of the n voters) and another ballot C_i (which we think of as a disingenuous ballot from voter i) such that:

(1) $V(\langle B_1, \ldots, B_n\rangle) = \langle x_1 \ldots x_p a \ldots\rangle$.
(2) $V(\langle B_1, \ldots, B_{i-1}, C_i, B_{i+1}, \ldots, B_n\rangle) = \langle x_1 \ldots x_p b \ldots\rangle$.
(3) b is ranked above a on the ballot B_i giving voter i's true preferences.

While dictatorships (wherein we fix one of the voters and the final list is simply taken to be his ballot) are certainly non-manipulable in this sense, there are more interesting examples, of which we consider two. For the first, assume that voter 1 gets to specify which alternative is at the top of the final list, then voter 2 gets to specify which of the remaining alternatives is second, then voter 3 gets a similar say, and so on. Of course, we could modify this by returning to voter 1 for a decision as to which alternative, for example, is third on the final list.

But the example that best illustrates the general case is the following. Suppose we have three voters and three alternatives: a, b, and c. Voter 1 gets to choose which of the three alternatives is at the top of the final list. If voter 1 chooses a, then voter 2 gets to choose which alternative is second on the final list. On the other hand, if voter 1 chooses b, then voter 3 gets to choose which alternative is second. But, if voter 1 chooses c, then which of the remaining two alternatives is second on the final list is determined by majority vote based on the ballots cast.

There are two ways in which the latter example is more complicated than the former. First, not all of the alternatives are treated the same in the latter example—the question of which voter picks what is second on the final list depends on which alternative voter 1 has at the top of his ballot. Thus, the system is not neutral. Second, the ordering of the bottom two alternatives in the final list in the latter example is not decided by a particular voter, but by majority rule. This has no effect on the question of manipulability—in spite of the Gibbard-Satterthwaite Theorem—because we are dealing with only *two* alternatives at this point in the process.

As to the first complication, we opt for simplicity in what follows and consider only social welfare functions that are neutral. There is, however, no getting around the second complication, and the general case requires a definition.

Definition 3.2. A *simple game* G is a pair (N, W) in which N is a non-empty set and W is a collection of subsets of N that is closed under the formation of supersets; G is said to be *constant sum* if for each set $X \subseteq N$, exactly one of X and $N - X$ is in W.

Simple games are often associated with voting systems in which a single alternative is pitted against the status quo. In this context, sets in W are called *winning coalitions*, with the intuition being that a set X is a winning coalition if and only if the issue at hand passes when the voters in X are precisely the ones who vote in favor of the issue. For more on this, see [**13**], [**29**], and [**31**].

But constant-sum simple games can also be used to select a winner from two alternatives based on a profile indexed by N. This is done by declaring that

$$a \text{ is the winner if } \{i \in N\colon \text{voter } i \text{ ranks } a \text{ over } b\} \in W,$$

and

$$b \text{ is the winner if } \{i \in N\colon \text{voter } i \text{ ranks } b \text{ over } a\} \in W.$$

This system is non-manipulable precisely because the collection of winning coalitions is closed under the formation of supersets, and it always produces a unique winner because exactly one of X and $N - X$ is in W.

With these preliminaries at hand, we can now state the consequence of the Gibbard-Satterthwaite Theorem that shows how non-manipulability can co-exist with a kind of quasi-democracy.

Theorem 3.3. *Suppose we have a set $A = \{x_1, \dots, x_k\}$ of alternatives, and a social welfare function V that treats all alternatives the same (i.e., that is neutral). Then the following are equivalent:*

(1) V is non-manipulable and for every linear ordering L of the set of alternatives, if $P = \langle L, \dots, L\rangle$, then $V(P) = L$.

(2) Either:

(i) there exists a sequence (repetitions allowed) $\langle i_1, \dots, i_{k-1}\rangle$ such that for each $p \le k - 1$, the pth alternative on the final list $\langle x_1, \dots, x_k\rangle$ is the alternative in $A - \{x_1, \dots, x_{p-1}\}$ that is ranked highest by voter i_p.

or

(ii) there exists a sequence (repetitions allowed) $\langle i_1, \dots, i_{k-2}\rangle$ and a constant-sum simple game $G = \langle N, W\rangle$ such that for each $p \le k - 2$, the pth alternative on the final list $\langle x_1, \dots, x_k\rangle$ is the alternative in $A - \{x_1, \dots, x_{p-1}\}$ that is ranked highest by voter i_p and the order of the last two alternatives in the final list is determined by G (i.e., $\{i\colon$ voter i ranks x_{k-1} over $x_k\} \in W$).

Proof. Clearly, (2) implies (1). We derive the converse from the Gibbard-Satterthwaite Theorem. Given a social welfare function V, we begin by inductively constructing a sequence $\langle i_1, \ldots, i_{k-2}\rangle$ of voters. To obtain i_1, consider the voting system V' obtained by setting $V'(P)$ equal to the top alternative on the list $V(P)$. If $|A| \geq 3$, then V' satisfies the hypotheses of the Gibbard-Satterthwaite Theorem in Section 2. Hence, there is a voter i_1 such that the top alternative on his ballot is always the top alternative on the list $V(P)$.

Now, fix one alternative a and let V'' be the voting system for the set $A - \{a\}$ of alternatives that is defined as follows. If P is a profile for $A - \{a\}$, then let P' be the profile for A that is obtained by placing a at the top of all ballots in P. Now let $V''(P)$ be the second alternative on the list $V(P')$. Note that $V''(P) \neq a$ since a is the top alternative in the list $V(P')$ because voter i_1 has a at the top of his ballot.

Again, if $|A - \{a\}| \geq 3$, we claim that V'' satisfies the hypotheses of the Gibbard-Satterthwaite Theorem in Section 2. To see that it is not manipulable, assume that the profile P represents the true preferences over $A - \{a\}$ of the voters, that $V''(P) = b$, that Q is a profile that results from a change by voter j alone, and that $V''(Q) = c$, where c is preferred to b on voter j's ballot in P. Let P' and Q' be obtained by placing a at the top of all the ballots. Then voter j can change the outcome with V from $\langle ab\ldots\rangle$ to $\langle ac\ldots\rangle$, and he prefers the second list to the first according to our lexicographic definition.

Hence, there is a voter i_2 such that—if all voters have alternative a at the top of their ballots—the second alternative of the final list is the alternative that he (voter i_2) ranks highest among those in $A - \{a\}$. In the general case, i_2 is a function of a (as in our three-voter, three-alternative example). But with our assumption of neutrality, i_2 must be independent of which alternative voter i_1 has at the top of his ballot.

We now claim that if P is a profile in which voter i_1 has a at the top, then voter i_2's top-ranked alternative in $A - \{a\}$ is in second place on the final list $V(P)$ regardless of where any of the other voters (except voter i_1) place a. To see this, suppose P' is a profile showing otherwise; thus, voter i_2 has b as the highest ranked alternative in $A - \{a\}$, but the outcome is a list $\langle ac\ldots\rangle$ with $c \neq b$. One-by-one move a to the top of each ballot in P' until the final list changes so that it begins $ax\ldots$ with $x \neq c$ (and there must be such a point because it is true when everyone has moved a to the top). But the last voter to make this change had x and c ranked the same way on both ballots, and so he has succeeded in manipulating the outcome according to our lexicographic definition.

We can now consider two fixed alternatives a and b and repeat this argument with ballots for $A - \{a, b\}$, and an election outcome being the third-ranked alternative in the final list arrived at by placing a first and b second on all these ballots. This yields voter i_3, and we can continue this process until we have only two alternatives left. Then, the Gibbard-Satterthwaite Theorem no longer applies. But at this point we can obtain the constant-sum simple game G by saying that a set X is in W if and only if the final ordering of these two alternatives agrees with the way they are ordered by voters in X whenever all the voters in X have them ordered one way and everyone else has them ordered the opposite way. A special case of this is when a set is winning if and only if it contains some voter i_k. This gives us conclusion (i) instead of conclusion (ii). ■

Having non-manipulability co-exist with a kind of quasi-democracy in the setting of the Duggan-Schwartz Theorem is considerably easier (and, in some ways, more satisfying) than in the Gibbard-Satterthwaite context that we have just considered. The key here is again to add an assumption about equal treatment—but now with reference to the equal treatment of voters (anonymity) instead of to equal treatment of alternatives (neutrality).

Theorem 3.4. *Suppose that a voting system is anonymous, cannot be manipulated by an optimist or a pessimist, and has every alternative viable. Then the set of winners includes the top ranked alternative of each voter.*

Proof. This is immediate from Theorem 2.2. ■

All other things being equal, one would like the set of winners in an election to be as small as possible. For this reason, let's say that one voting system *dominates* another if they are distinct and the set of winners using the former is always a subset of the set of winners using the latter. Then a corollary of Theorem 3.4 is the following.

Corollary 3.5. *Let V be the voting system in which the winners are precisely the alternatives that receive at least one first-place vote. Then V cannot be manipulated by either an optimist or a pessimist, is anonymous, produces each alternative as a singleton winner for at least one profile, and dominates every other voting procedure that satisfies these three properties.*

4. CHARACTERIZATIONS. If there are no ties in the ballots and if the winner is a single alternative, the version of the Gibbard-Satterthwaite Theorem presented in Section 2 (as a corollary of the Duggan-Schwartz Theorem) characterizes dictatorships using manipulability and the assumption that every alternative is viable. If we could delete this latter assumption, then we'd have a characterization of non-manipulable voting systems. In this section, we derive a weakened version of such a result in the Duggan-Schwartz context that nevertheless generalizes the known characterization in the Gibbard-Satterthwaite context.

Our starting point is the following definition from [**10**].

Definition 4.1. A voting system V is said to satisfy *residual resoluteness* (RR) provided that $|V(P)| = 1$ whenever P is a profile in which there are two alternatives x and y such that $\{x, y\}$ is a top set and all but at most one voter has y over x.

For the next theorem, we are again in the context of linear ballots, three or more alternatives, and election outcomes that are non-empty sets of winners.

Theorem 4.2 (Duggan-Schwartz [10]). *Assume that V is a voting system that satisfies RR, cannot be manipulated by an optimist or a pessimist, and for which every alternative x is among the winners (but not necessarily a singleton winner) for at least one profile. Then there exists a single voter such that, in every election, the alternative at the top of his ballot is the unique winner.*

Proof. We first claim that if P' is a profile in which every voter has x at the top and y second, then $V(P') = \{x\}$. To see this, choose P such that $x \in V(P)$, and note that if $V(P')$ did not contain x, then we could change P' to P one ballot at a time until x appeared as a winner—thus allowing some voter to improve his max to his most preferred alternative x. But our assumption now guarantees that $|V(P')| = 1$. Thus, $V(P') = \{x\}$.

Theorem 2.2 now guarantees that there is a voter i whose top choice is among the winners. Suppose that there is a profile P such that voter i has x at the top of his ballot, but $V(P) \neq \{x\}$. Fixing the ballots of the other voters, choose a ballot for voter i such that an alternative y occurs in $V(P)$ that is as low on his ballot as possible. Let P' be any profile in which voter i has x at the top of his ballot and y second, and every other voter has y at the top and x second. By our assumption, $|V(P')| = 1$, and, since x is at the top of voter i's ballot, $x \in V(P')$. Thus, $V(P') = \{x\}$.

Now change P' to P one ballot at a time for every voter except voter i. If y appears at some point then that voter has improved his max to his most preferred alternative y. If y never appears, let P'' be the resulting profile, and note that P'' and P differ only because of voter i's ballot. But now, if voter i's ballot in P represents his true preferences, then he can use the disingenuous ballot in P'' to improve his min from y to something better (it is better because we chose y to be as low as possible). ■

Corollary 4.3. *A voting system V is non-manipulable by optimists and pessimists and satisfies RR if and only if one of the following holds:*

(1) There is a single alternative x for which $\{x\}$ is the winner regardless of the ballots.

(2) There are two alternatives x and y and two simple games $G_x = (N, W_x)$ and $G_y = (N, W_y)$ that are "pairwise proper" in the sense that if $X \in W_1$ and $Y \in W_2$, then $X \cap Y \neq \emptyset$, and for which every singleton set is winning in one of the games or its complement is winning in the other, and such that $\{x\}$ wins if the set of voters who rank x over y is a winning coalition in G_x, $\{y\}$ wins if the set of voters who rank y over x is a winning coalition in G_y, and $\{x, y\}$ wins otherwise.

(3) There is a set B containing three or more alternatives, and a particular voter such that the unique winner of the election is the element of B that is ranked highest by this voter.

Proof. It is easy to see that the voting systems described in Corollary 4.3 are all non-manipulable by optimists and pessimists and satisfy RR. For the converse, let B be the set of "viable" alternatives in the sense that there is at least one sequence of ballots that yields it as one of the winners. If B is a singleton, then the system is as described in (1).

If B has exactly two elements then (2) holds, but the verification of this requires using non-manipulability to show that the placement of other alternatives on the ballots has no effect on whether x wins or y wins.

Suppose now that B has at least three alternatives. We first claim that if $x \in B$, then $V(P) = \{x\}$ whenever every voter ranks x first and ranks some other common element y second. This is because RR guarantees that $|V(P)| = 1$, and if the result were $\{y\}$, we could convert ballots one-by-one until the outcome included x, thus improving the max for that voter.

Let V' be the voting system on the set B obtained by applying the original voting system to the result of placing all the alternatives not in B at the bottom (in some fixed predetermined order) of all the ballots. Then V' is still non-manipulable, RR still holds, and the argument in the previous paragraph shows that for every x in B there is at least one profile P such that $V'(P) = \{x\}$.

It now follows from the Duggan-Schwartz Theorem that there is a dictator for V' in the sense that the top-ranked alternative on his ballot is among the winners. We claim that the winner in the original system is a singleton set consisting of the element of B that is ranked highest by the dictator for V'. Suppose not. Then the set of winners includes some alternative x, necessarily in B, that is ranked lower on the dictator's list than some other element y of B.

Now move all the alternatives not in B below x. If the winner switches to $\{y\}$, then the dictator has improved his min. Otherwise, the new min is some x' that is no higher on the dictator's list than x. The result is that y is now at the top of the dictator's list. Now we can, one ballot at a time, move x to the top of each of the other ballots. Then x remains a winner, or undoing the last change before x became a loser would improve the max for that voter. Similarly, if we now move y into second place on each of these ballots (other than the dictator's), then x stays a winner. Finally, move x into the second spot on the dictator's list. By RR, the winning set is now a singleton. If $V(P) = \{y\}$, then the dictator has improved his min, and that's impossible. So $V(P) = \{x\}$, and we can then use down-monotonicity for singleton winners (Lemma 2.4) to place all alternatives not in B at the bottom of all the ballots in the correct order. But now $V(P) = \{y\}$, and this is a contradiction. ■

In the Gibbard-Satterthwaite context, the following consequence of Corollary 4.3 is a known result.

Corollary 4.4. *If ballots are linear orderings of a set of three or more alternatives, and if the outcome of every election is a single winner, then a voting system V is non-manipulable if and only if one of the following holds:*

(1) There is a single alternative x and it is the winner regardless of the ballots.
(2) There are two alternatives x and y and a simple game G such that x wins if and only if the set of voters who rank x over y is a winning coalition in G, and otherwise y wins.
(3) There is a set B containing three or more alternatives, and there is a voter such that the winner of the election is the element of B that is ranked highest by this voter.

5. TIES IN THE BALLOTS. In the Gibbard-Satterthwaite context, it is known that if ties are allowed in the ballots (but not in the outcome), and if every alternative is viable, then there is a dictator in the sense that the winner must be one of the alternatives that is tied for top position on his ballot. In the Duggan-Schwartz context, we can do a similar thing—again assuming that there are three or more alternatives and that every alternative is viable.

Theorem 5.1. *Assume that V is a voting system that cannot be manipulated by an optimist or a pessimist and in which every alternative is viable. Then—even if ties in the ballots are allowed—there exists a voter such that $V(P)$ always contains at least one of the alternatives that is tied for top position on his ballot.*

Proof. If V is the system postulated let V' be the restriction of V to profiles in which no ties occur. Then V' also cannot be manipulated by either an optimist or a pessimist. Moreover, we claim that if P is a profile in which every voter has x at the top of his ballot, then $V(P) = \{x\}$. To see this, choose P' (consisting of ballots that may have ties) such that $V(P') = \{x\}$. Change P to P' one ballot at a time until the winner becomes $\{x\}$. At this point some voter has improved the min to be his most preferred alternative x.

It now follows from Theorem 2.2 that if all the ballots are linear orderings, then the top alternative on voter i's ballot is among the winners. Assume that P is a profile (in which ties occur) such that no alternative from voter i's top block is among the winners, and choose such P so that $|V(P)|$ is as small as possible. One ballot at a time, move the set $V(P)$ to the top of everyone's (except voter i's) ballot and break all ties in these ballots. No new winner w is added as we do this, or we could regard the ballots with $V(P)$ at the top (and no ties) as the true preferences, and see that the min has been improved from w to something in $V(P)$. Moreover, nothing in $V(P)$ is lost by the minimality of $|V(P)|$.

Finally, we can break all the ties in voter i's ballot, in which case some alternative that was in his top block becomes one of the winners. At this point, voter i has improved his max from something not in his top block to something in his top block. ■

Corollary 4.4 characterizes the collection of all non-manipulable voting systems in the context of no ties in the ballots and no ties in the outcome of an election. One can generalize Corollary 4.4 to allow ties in the ballots, but this involves the construction of a somewhat elaborate tree.

6. CONCLUSION. The notion of manipulability by an optimist or a pessimist has, in our opinion, a nice feeling of mathematical naturality while still respecting the intuitions from the real-world problems of voting theory. But some might object to a framework wherein the success of a disingenuous ballot depends on the psychological state of a voter (his being sufficiently optimistic or pessimistic), as opposed to how he feels about the relative value of the alternatives.

Indeed, our use of the adjectives "optimistic" and "pessimistic" certainly bring to mind a context in which ties are randomly broken and the voter in question has a state of mind that is rather extreme in one of two ways. There is nothing inherently wrong with this context—the oft-invoked concept of "risk-averse" coincides (roughly) with our notion of "pessimism". But there are also interpretations grounded in the reality of preferences.

For example, consider a typical academic setting in which a department is trying to decide which of five candidates (all of whom have been interviewed by the department and the dean) to hire. One can imagine using a voting system of the kind we have considered here and agreeing that, if the outcome based on the ballots of the department members is a tie, then the dean (who did not vote) breaks the tie. In this context, an optimist is someone who feels that his own values (regarding the importance of teaching versus research, etc.) are shared by the dean; a pessimist is someone who feels just the opposite.

Duggan and Schwartz also deal with this issue in **[10]**, where they point out the following. Suppose that a voter can manipulate an election to secure an outcome of A rather than B, and that A has either a better max or min for this voter. Then, for any probability distribution over A and for any probability distribution over B—assuming that every element of each set gets non-zero probability—there is a function f mapping $A \cup B$ to the reals that is consistent with this voter's ordinal preferences ($f(x) > f(y)$ if and only if he prefers x to y) and is such that the expected utility of A is greater than the expected utility of B.

Comparing sets based on an ordinal preference of elements is a fairly well-traveled road, but the voting-theoretic context is somewhat special; see **[5]**, **[14]**, or **[26]**. For example, one axiom that often arises in such situations is called deFinetti's (additivity) axiom **[8]**. It asserts that if A is preferred to B, and C is disjoint from both A and B, then $A \cup C$ should be preferred to $B \cup C$. But if we prefer x to y and y to z, then—in the voting context—we clearly prefer $\{x\}$ to $\{x, y\}$ and $\{y, z\}$ to $\{z\}$, and so this axiom would imply that we prefer $\{x, z\}$ to $\{x, y, z\}$ while—at the same time—we prefer $\{x, y, z\}$ to $\{x, z\}$.

There are, however, other notions of when a voter might prefer one set of winners to another. Perhaps the most natural is to say that a voter prefers A to B if A can be written as the disjoint union of A' and A'', B can be written as the disjoint union of B' and B'', there is a bijection $f : A' \rightarrow B'$ such that a is equal to or preferred to $f(a)$ for every $a \in A'$, every element of A'' is preferred to every element of A', and every element of B' is preferred to every element of B''. Exactly which voting systems are non-manipulable in this sense seems to be an open question; but see **[20]** for some related work.

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> The man who solicits votes to obtain any office is deprived completely of the hope of holding any office at all.... They have very few laws because very few are needed for persons so educated.... Moreover, they absolutely banish from their country all lawyers, who cleverly manipulate cases and cunningly argue legal points.
>
> ——St. Thomas More, *Utopia*, Book II