The Many Faces of Degeneracy in Conic Optimization

Dmitriy Drusvyatskiy
Department of Mathematics
University of Washington
ddrusv@uw.edu

Henry Wolkowicz
Faculty of Mathematics
University of Waterloo
hwolkowicz@uwaterloo.ca

the essence of knowledge
Boston — Delft

Foundations and Trends® in Optimization

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

D. Drusvyatskiy and H. Wolkowicz. *The Many Faces of Degeneracy in Conic Optimization* . Foundations and Trends® in Optimization, vol. 3, no. 2, pp. 77–170, 2016.

This Foundations and Trends® issue was typeset in LaTeX using a class file designed by Neal Parikh. Printed on acid-free paper.

ISBN: 978-1-68083-390-4

Foundations and Trends® in Optimization
Volume 3, Issue 2, 2016

Editorial Scope

Topics

Foundations and Trends® in Optimization publishes survey and tutorial articles on methods for and applications of mathematical optimization, including the following topics:

- Algorithm design, analysis, and implementation (especially on modern computing platforms)
- Models and modeling systems
- New optimization formulations for practical problems
- Applications of optimization in:
  - Machine learning
  - Statistics
  - Data analysis
  - Signal and image processing
  - Computational economics and finance
  - Engineering design
  - Scheduling and resource allocation
  - and other areas

Information for Librarians

Foundations and Trends® in Optimization, 2016, Volume 3, 4 issues. ISSN paper version 2167-3888. ISSN online version 2167-3918. Also available as a combined paper and online subscription.

Foundations and Trends® in Optimization
Vol. 3, No. 2 (2016) 77–170

DOI: 10.1561/2400000011

The Many Faces of Degeneracy in Conic Optimization

Dmitriy Drusvyatskiy
Department of Mathematics
University of Washington
ddrusv@uw.edu

Henry Wolkowicz
Faculty of Mathematics
University of Waterloo
hwolkowicz@uwaterloo.ca

Contents

Abstract

Slater's condition – existence of a "strictly feasible solution" – is a common assumption in conic optimization. Without strict feasibility, first-order optimality conditions may be meaningless, the dual problem may yield little information about the primal, and small changes in the data may render the problem infeasible. Hence, failure of strict feasibility can negatively impact off-the-shelf numerical methods, such as primal-dual interior point methods, in particular. New optimization modeling techniques and convex relaxations for hard nonconvex problems have shown that the loss of strict feasibility is a more pronounced phenomenon than has previously been realized. In this text, we describe various reasons for the loss of strict feasibility, whether due to poor modeling choices or (more interestingly) rich underlying structure, and discuss ways to cope with it and, in many pronounced cases, how to use it as an advantage. In large part, we emphasize the facial reduction preprocessing technique due to its mathematical elegance, geometric transparency, and computational potential.

D. Drusvyatskiy and H. Wolkowicz. *The Many Faces of Degeneracy in Conic Optimization* . Foundations and Trends® in Optimization, vol. 3, no. 2, pp. 77–170, 2016.
DOI: 10.1561/2400000011.

1

What this monograph is about

Conic optimization has proven to be an elegant and powerful modeling tool with surprisingly many applications. The classical *linear programming* problem revolutionized operations research and is still the most widely used optimization model. This is due to the elegant theory and the ability to solve in practice both small and large scale problems efficiently and accurately by the well known simplex method of Dantzig [37] and by more recent interior-point methods for convex and nonconvex problems, e.g., [151, 100, 27]. The size (number of variables) of linear programs that could be solved before the interior-point revolution was on the order of tens of thousands, whereas it immediately increased to millions for many applications. A large part of modern success is due to *preprocessing*, which aims to identify (primal and dual slack) variables that are identically zero on the feasible set. The article [98] is a good reference.

The story does not end with linear programming. Dantzig himself recounts in [38]: "the world is nonlinear". Nonlinear models can significantly improve on linear programs if they can be solved efficiently. Conic optimization has shown its worth in its elegant theory, efficient algorithms, and many applications e.g., [149, 10, 21]. Preprocessing to rectify possible loss of "strict-feasibility" in the primal or the dual problems is appealing for general conic optimization as well. In contrast to linear programming, however, the area of preprocessing for conic optimization is in its infancy; see e.g., [31, 140, 32, 109, 111] and Section 1.1, below. In contrast to linear programming, numerical error makes preprocessing difficult in full generality. This being said, surprisingly, there are many specific applications of conic optimization, where the rich underlying structure makes preprocessing possible, leading to greatly simplified models and strengthened algorithms. Indeed, exploiting structure is essential for making preprocessing viable. In this monograph, we present the background and the elementary theory of such regularization techniques in the framework of *facial reduction (FR)*. We focus on notable case studies, where such techniques have proven to be useful.

1.1 Related work

To put this text in perspective, it is instructive to consider nonlinear programming. Nontrivial statements in constrained nonlinear optimization always rely on some regularity of the constraints. To illustrate, consider a minimization problem over a set of the form $\{x : f(x) = 0\}$ for some smooth f. How general are such constraints? A celebrated result of Whitney [146] shows that *any* closed set in a Euclidean space can written as a zero-set of some C^∞-smooth function f. Thus, in this generality, there is little difference between minimizing over arbitrary closed sets and sets of the form $\{x : f(x) = 0\}$, for smooth f. Since little can be said about optimizing over arbitrary closed sets, one must make an assumption on the equality constraint. The simplest one, eliminating Whitney's construction, is that the gradient of f is nonzero on the feasible region – the earliest form of a constraint qualification. There have been numerous papers, developing weakened versions of regularity (and optimality conditions) in nonlinear programming; some good examples are [64, 26, 23].

The Slater constraint qualification, we discuss in this text, is in a similar spirit, but in the context of (convex) conic optimization. Some good early references on the geometry of the Slater condition, and weakened variants, are [59, 95, 96, 147, 20]. The concept of facial reduction for general convex programs was introduced in [24, 25], while an early application to a semi-definite type best-approximation problem was given in [148]. Recently, there has been a significant renewed interest in facial reduction, in large part due to the success in applications for graph related problems, such as Euclidean distance matrix completion and molecular conformation [78, 77, 48, 6] and in polynomial optimization [112, 113, 76, 144, 143]. In particular, a more modern explanation of the facial reduction procedure can be found in [89, 106, 109, 138, 145].

We note in passing that numerous papers show that strict feasibility holds "generically" with respect to unstructured perturbations. In contrast, optimization problems appearing in applications are often highly structured and such genericity results are of little practical use.

1.2 Outline of the monograph

The monograph is divided into two parts. In Part I, we present the necessary theoretical grounding in conic optimization, including basic optimality and duality theory, connections of Slater's condition to the *distance to infeasibility* and sensitivity theory, the facial reduction procedure, and the singularity degree. In Part II, we concentrate on illustrative examples and applications, including matrix completion problems (semi-definite, low-rank, and Euclidean distance), relaxations of hard combinatorial problems (quadratic assignment and max-cut), and sum of squares relaxations of polynomial optimization problems.

1.3 Reflections on Jonathan Borwein and FR

These are some reflections on Jonathan Borwein and his role in the development of the facial reduction technique, by Henry Wolkowicz. Jonathon Borwein passed away unexpectedly on Aug. 2, 2016. Jon was an extraordinary mathematician who made significant contributions in an amazing number of very diverse areas. Many details and personal memories by myself and many others including family, friends, and colleagues, are presented at the memorial website `jonborwein.org`.

This was a terrible loss to his family and all his friends and colleagues, including myself. The facial reduction process we use in this monograph originates in the work of Jon and the second author (myself). This work took place from July of 1978 to July of 1979 when I went to Halifax to work with Jon at Dalhousie University in a lectureship position. The optimality conditions for the general abstract convex program using the facially reduced problem is presented in the two papers [24, 23]. The facial reduction process is then derived in [25].

References

[1] A. Alfakih. Graph rigidity via Euclidean distance matrices. *Linear Algebra Appl.*, 310(1-3):49–165, 2000.

[2] A. Alfakih, M.F. Anjos, V. Piccialli, and H. Wolkowicz. Euclidean distance matrices, semidefinite programming, and sensor network localization. *Portug. Math.*, 68(1):53–102, 2011.

[3] A. Alfakih, A. Khandani, and H. Wolkowicz. Solving Euclidean distance matrix completion problems via semidefinite programming. *Comput. Optim. Appl.*, 12(1-3):13–30, 1999. A tribute to Olvi Mangasarian.

[4] A. Alfakih and H. Wolkowicz. Matrix completion problems. In *Handbook of semidefinite programming*, volume 27 of *Internat. Ser. Oper. Res. Management Sci.*, pages 533–545. Kluwer Acad. Publ., Boston, MA, 2000.

[5] A.Y. Alfakih. A remark on the faces of the cone of Euclidean distance matrices. *Linear Algebra Appl.*, 414(1):266–270, 2006.

[6] B. Alipanahi, N. Krislock, A. Ghodsi, H. Wolkowicz, L. Donaldson, and M. Li. Determining protein structures from NOESY distance constraints by semidefinite programming. *J. Comput. Biol.*, 20(4):296–310, 2013.

[7] B. Alipanahi, N. Krislock, A. Ghodsi, H. Wolkowicz, L. Donaldson, and M. Li. Determining protein structures from NOESY distance constraints by semidefinite programming. *J. Comput. Biol.*, 20(4):296–310, 2013.

[8] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Optim.*, 5:13–51, 1995.

[9] E.D. Andersen and K.D. Andersen. The Mosek interior point optimizer for linear programming: an implementation of the homogeneous algorithm. In *High performance optimization*, volume 33 of *Appl. Optim.*, pages 197–232. Kluwer Acad. Publ., Dordrecht, 2000.

[10] A.F. Anjos and J.B. Lasserre, editors. *Handbook on Semidefinite, Conic and Polynomial Optimization.* International Series in Operations Research & Management Science. Springer-Verlag, 2011.

[11] M. F. Anjos and H. Wolkowicz. Strengthened semidefinite programming relaxations for the max-cut problem. In *Advances in convex analysis and global optimization (Pythagorion, 2000)*, volume 54 of *Nonconvex Optim. Appl.*, pages 409–420. Kluwer Acad. Publ., Dordrecht, 2001.

[12] M.F. Anjos. *New Convex Relaxations for the Maximum Cut and VLSI Layout Problems.* PhD thesis, University of Waterloo, 2001.

[13] M.F. Anjos and H. Wolkowicz. Strengthened semidefinite relaxations via a second lifting for the Max-Cut problem. *Discrete Appl. Math.*, 119(1-2):79–106, 2002. Foundations of heuristics in combinatorial optimization.

[14] K.M. Anstreicher. Recent advances in the solution of quadratic assignment problems. *Math. Program.*, 97(1-2, Ser. B):27–42, 2003. ISMP, 2003 (Copenhagen).

[15] K.M. Anstreicher and N.W. Brixius. A new bound for the quadratic assignment problem based on convex quadratic programming. *Math. Program.*, 89(3, Ser. A):341–357, 2001.

[16] M. Bakonyi and C.R. Johnson. The Euclidean distance matrix completion problem. *SIAM J. Matrix Anal. Appl.*, 16(2):646–654, 1995.

[17] M. Bakonyi and H.J. Woerdeman. *Matrix completions, moments, and sums of Hermitian squares.* Princeton University Press, Princeton, NJ, 2011.

[18] A. Barvinok. *A course in convexity*, volume 54 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2002.

[19] A. Ben-Israel. Theorems of the alternative for complex linear inequalities. *Israel J. Math.*, 7:129–136, 1969.

[20] A. Ben-Israel, A. Ben-Tal, and S. Zlobec. *Optimality in nonlinear programming: a feasible directions approach.* John Wiley & Sons, Inc., New York, 1981. A Wiley-Interscience Publication.

[21] A. Ben-Tal and A.S. Nemirovski. *Lectures on modern convex optimization.* MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Analysis, algorithms, and engineering applications.

[22] J.M. Borwein and A.S. Lewis. *Convex analysis and nonlinear optimization.* Springer-Verlag, New York, 2000. Theory and examples.

[23] J.M. Borwein and H. Wolkowicz. Characterization of optimality for the abstract convex program with finite-dimensional range. *J. Austral. Math. Soc. Ser. A*, 30(4):390–411, 1980/81.

[24] J.M. Borwein and H. Wolkowicz. Facial reduction for a cone-convex programming problem. *J. Austral. Math. Soc. Ser. A*, 30(3):369–380, 1980/81.

[25] J.M. Borwein and H. Wolkowicz. Regularizing the abstract convex program. *J. Math. Anal. Appl.*, 83(2):495–530, 1981.

[26] J.M. Borwein and H. Wolkowicz. Characterizations of optimality without constraint qualification for the abstract convex program. *Math. Programming Stud.*, 19:77–100, 1982. Optimality and stability in mathematical programming.

[27] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, Cambridge, 2004.

[28] R.E. Burkard, S. Karisch, and F. Rendl. QAPLIB – a quadratic assignment problem library. *European J. Oper. Res.*, 55:115–119, 1991. anjos.mgi.polymtl.ca/qaplib/.

[29] E.J. Candès and B. Recht. Exact matrix completion via convex optimization. *Found. Comput. Math.*, 9(6):717–772, 2009.

[30] E. Çela. *The quadratic assignment problem*, volume 1 of *Combinatorial Optimization.* Kluwer Academic Publishers, Dordrecht, 1998. Theory and algorithms.

[31] Y.-L. Cheung. *Preprocessing and Reduction for Semidefinite Programming via Facial Reduction: Theory and Practice.* PhD thesis, University of Waterloo, 2013.

[32] Y-L. Cheung, S. Schurr, and H. Wolkowicz. Preprocessing and regularization for degenerate semidefinite programs. In D.H. Bailey, H.H. Bauschke, P. Borwein, F. Garvan, M. Thera, J. Vanderwerff, and H. Wolkowicz, editors, *Computational and Analytical Mathematics, In Honor of Jonathan Borwein's 60th Birthday*, volume 50 of *Springer Proceedings in Mathematics & Statistics*, pages 225–276. Springer, 2013.

[33] R. Connelly and S.J. Gortler. Iterative universal rigidity. *Discrete Comput. Geom.*, 53(4):847–877, 2015.

[34] G. Cravo. Recent progress on matrix completion problems. *J. Math. Sci. Adv. Appl.*, 1(1):69–90, 2008.

[35] G. Cravo. Matrix completion problems. *Linear Algebra Appl.*, 430(8-9):2511–2540, 2009.

[36] G.M. Crippen and T.F. Havel. *Distance geometry and molecular conformation*, volume 15 of *Chemometrics Series*. Research Studies Press Ltd., Chichester, 1988.

[37] G. Dantzig. *Linear Programming and Extensions*. Princeton University Press, Princeton, New Jersey, 1963.

[38] G. Dantzig. Linear programming. In *History of Mathematical Programming: A Collection of Personal Reminiscences*. CWI North-Holland, Amsterdam, 1991.

[39] J. Dattorro. Convex optimization & Euclidean distance geometry. `https://ccrma.stanford.edu/~dattorro/mybook.html`, Meboo Publishing, 2005.

[40] E. de Klerk, M. E.-Nagy, and R. Sotirov. On semidefinite programming bounds for graph bandwidth. *Optim. Methods Softw.*, 28(3):485–500, 2013.

[41] E. de Klerk, C. Roos, and T. Terlaky. Infeasible–start semidefinite programming algorithms via self–dual embeddings. In *Topics in Semidefinite and Interior-Point Methods*, volume 18 of *The Fields Institute for Research in Mathematical Sciences, Communications Series*, pages 215–236. American Mathematical Society, 1998.

[42] E. de Klerk and R. Sotirov. Exploiting group symmetry in semidefinite programming relaxations of the quadratic assignment problem. *Math. Program.*, 122(2, Ser. A):225–246, 2010.

[43] E. de Klerk, R. Sotirov, and U. Truetsch. A new semidefinite programming relaxation for the quadratic assignment problem and its computational perspectives. *INFORMS J. Comput.*, 27(2):378–391, 2015.

[44] Y. Ding, N. Krislock, J. Qian, and H. Wolkowicz. Sensor network localization, Euclidean distance matrix completions, and graph realization. *Optim. Eng.*, 11(1):45–66, 2010.

[45] I. Dokmanic, R. Parhizkar, J. Ranieri, and M. Vetterli. Euclidean distance matrices: A short walk through theory, algorithms and applications. *CoRR*, abs/1502.07541, 2015.

[46] A.L. Dontchev, A.S. Lewis, and R.T. Rockafellar. The radius of metric regularity. *Trans. Amer. Math. Soc.*, 355(2):493–517 (electronic), 2003.

[47] A.W.M. Dress and T.F. Havel. The fundamental theory of distance geometry. In *Computer aided geometric reasoning, Vol. I, II (Sophia-Antipolis, 1987)*, pages 127–169. INRIA, Rocquencourt, 1987.

[48] D. Drusvyatskiy, N. Krislock, Y-L. Cheung Voronin, and H. Wolkowicz. Noisy Euclidean distance realization: robust facial reduction and the Pareto frontier. *SIAM Journal on Optimization*, 27(4):2301–2331, 2017. arXiv:1410.6852, 31 pages, to appear.

[49] D. Drusvyatskiy, G. Pataki, and H. Wolkowicz. Coordinate shadows of semidefinite and Euclidean distance matrices. *SIAM J. Optim.*, 25(2):1160–1178, 2015.

[50] H. Dym and I. Gohberg. Extensions of band matrices with band inverses. *Linear Algebra Appl.*, 36:1–24, 1981.

[51] C.S. Edwards. The derivation of a greedy approximator for the koopmans-beckmann quadratic assignment problem. *Proc. CP77 Combinatorial Prog. Conf.*, pages 55–86, 1977.

[52] M. Fazel. *Matrix Rank Minimization with Applications.* PhD thesis, Stanford University, Stanford, CA, 2001.

[53] M. Fazel, H. Hindi, and S.P. Boyd. A rank minimization heuristic with application to minimum order system approximation. In *Proceedings American Control Conference*, pages 4734–4739, 2001.

[54] G. Finke, R.E. Burkard, and F. Rendl. Quadratic assignment problems. *Ann. Discrete Math.*, 31:61–82, 1987.

[55] G. Finke, R.E. Burkard, and F. Rendl. Quadratic assignment problems. In *Surveys in combinatorial optimization (Rio de Janeiro, 1985)*, volume 132 of *North-Holland Math. Stud.*, pages 61–82. North-Holland, Amsterdam, 1987.

[56] J. Gauvin and J.W. Tolle. Differential stability in nonlinear programming. *SIAM J. Control Optimization*, 15(2):294–311, 1977.

[57] S.J. Gortler and D.P. Thurston. Characterizing the universal rigidity of generic frameworks. *Discrete Comput Geom*, 51:1017–1036, 2014.

[58] S.J. Gortler and D.P. Thurston. Characterizing the universal rigidity of generic frameworks. *Discrete Comput. Geom.*, 51(4):1017–1036, 2014.

[59] F.J. Gould and J.W. Tolle. Geometry of optimality conditions and constraint qualifications. *Math. Programming*, 2(1):1–18, 1972.

[60] J.C. Gower. Euclidean distance geometry. *Math. Sci.*, 7(1):1–14, 1982.

[61] J.C. Gower. Properties of Euclidean and non-Euclidean distance matrices. *Linear Algebra Appl.*, 67:81–97, 1985.

[62] H. Grassmann. *Extension theory*, volume 19 of *History of Mathematics*. American Mathematical Society, Providence, RI, 2000. Translated from the 1896 German original and with a foreword, editorial notes and supplementary notes by Lloyd C. Kannenberg.

[63] B. Grone, C.R. Johnson, E. Marques de Sa, and H. Wolkowicz. Positive definite completions of partial Hermitian matrices. *Linear Algebra Appl.*, 58:109–124, 1984.

[64] M. Guignard. Generalized Kuhn-Tucker conditions for mathematical programming problems in a Banach space. *SIAM J. of Control*, 7(2):232–241, 1969.

[65] K.J. Harrison. Matrix completions and chordal graphs. *Acta Math. Sin. (Engl. Ser.)*, 19(3):577–590, 2003. International Workshop on Operator Algebra and Operator Theory (Linfen, 2001).

[66] T.F. Havel, I.D. Kuntz, and B. Crippen. Errata: "The theory and practice of distance geometry" [Bull. Math. Biol. **45** (1983), no. 5, 665–720; MR0718540 (84k:92037)]. *Bull. Math. Biol.*, 47(1):157, 1985.

[67] C. Helmberg and F. Rendl. A spectral bundle method for semidefinite programming. *SIAM J. Optim.*, 10(3):673 – 696, 2000.

[68] S. Huang and H. Wolkowicz. Low-rank matrix completion using nuclear norm with facial reduction. *J. Global Optim.*, 23 pages, to appear, 2017.

[69] X. Huang. *Preprocessing and postprocessing in linear optimization*. PhD thesis, McMaster University, 2004.

[70] C.R. Johnson. Matrix completion problems: a survey. In *Matrix theory and applications (Phoenix, AZ, 1989)*, pages 171–198. Amer. Math. Soc., Providence, RI, 1990.

[71] C.R. Johnson. Matrix completion problems: a survey. *Proceedings of Symposium in Applied Mathematics*, 40:171–198, 1990.

[72] C.R. Johnson, editor. *Matrix theory and applications*, volume 40 of *Proceedings of Symposia in Applied Mathematics*. American Mathematical Society, Providence, RI, 1990. Lecture notes prepared for the American Mathematical Society Short Course held in Phoenix, Arizona, January 10–11, 1989, AMS Short Course Lecture Notes.

[73] C.R. Johnson. Positive definite completions: a guide to selected literature. In *Signal processing, Part I*, volume 22 of *IMA Vol. Math. Appl.*, pages 169–188. Springer, New York, 1990.

[74] C.R. Johnson, B. Kroschel, and H. Wolkowicz. An interior-point method for approximate positive semidefinite completions. *Comput. Optim. Appl.*, 9(2):175–190, 1998.

[75] C.R. Johnson and P. Tarazaga. Connections between the real positive semidefinite and distance matrix completion problems. *Linear Algebra Appl.*, 223/224:375–391, 1995. Special issue honoring Miroslav Fiedler and Vlastimil Pták.

[76] M. Kojima, S. Kim, and H. Waki. Sparsity in sums of squares of polynomials. *Math. Program.*, 103(1, Ser. A):45–62, 2005.

[77] N. Krislock. *Semidefinite Facial Reduction for Low-Rank Euclidean Distance Matrix Completion*. PhD thesis, University of Waterloo, 2010.

[78] N. Krislock and H. Wolkowicz. Explicit sensor network localization using semidefinite representations and facial reductions. *SIAM Journal on Optimization*, 20(5):2679–2708, 2010.

[79] N. Krislock and H. Wolkowicz. Euclidean distance matrices and applications. In *Handbook on Semidefinite, Cone and Polynomial Optimization*, number 2009-06 in International Series in Operations Research & Management Science, pages 879–914. Springer-Verlag, 2011.

[80] J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.*, 11(3):796–817 (electronic), 2000/01.

[81] J.B. Lasserre. *Moments, positive polynomials and their applications*, volume 1 of *Imperial College Press Optimization Series*. Imperial College Press, London, 2010.

[82] M. Laurent. A connection between positive semidefinite and Euclidean distance matrix completion problems. *Linear Algebra Appl.*, 273:9–22, 1998.

[83] M. Laurent. A tour d'horizon on positive semidefinite and Euclidean distance matrix completion problems. In *Topics in semidefinite and interior-point methods (Toronto, ON, 1996)*, volume 18 of *Fields Inst. Commun.*, pages 51–76. Amer. Math. Soc., Providence, RI, 1998.

[84] M. Laurent. Polynomial instances of the positive semidefinite and Euclidean distance matrix completion problems. *SIAM J. Matrix Anal. Appl.*, 22:874–894, 2000.

[85] M. Laurent. Matrix completion problems. In *Encyclopedia of Optimization*, pages 1311–1319. Springer US, 2001.

[86] M. Laurent and F. Vallentin. Semidefinite Optimization. `http://homepages.cwi.nl/~monique/master_SDP_2016.pdf`, 2016. [Online; accessed 25-April-2016].

[87] L. Liberti and C. Lavor. Open research areas in distance geometry. *ArXiv e-prints*, October 2016.

[88] L. Liberti, C. Lavor, N. Maculan, and A. Mucherino. Euclidean distance geometry and applications. *SIAM Review*, 56(1):3–69, 2014.

[89] M. Liu and G. Pataki. Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming. Technical report, Department of Statistics and Operations Research, University of North Carolina at Chapel Hill, 2015.

[90] M.S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret. Applications of second-order cone programming. *Linear Algebra Appl.*, 284(1-3):193–228, 1998. ILAS Symposium on Fast Algorithms for Control, Signals and Image Processing (Winnipeg, MB, 1997).

[91] E.M. Loiola, Nair M. Maia d A., P.O. Boaventura-Netto, P. Hahn, and T. Querido. A survey for the quadratic assignment problem. *European J. Oper. Res.*, 176(2):657–690, 2007.

[92] Z-Q. Luo, J.F. Sturm, and S. Zhang. Duality results for conic convex programming. Technical Report Report 9719/A, April, Erasmus University Rotterdam, Econometric Institute EUR, P.O. Box 1738, 3000 DR, The Netherlands, 1997.

[93] O. L. Mangasarian and S. Fromovitz. The Fritz John necessary optimality conditions in the presence of equality and inequality constraints. *J. Math. Anal. Appl.*, 17:37–47, 1967.

[94] O.L. Mangasarian. *Nonlinear Programming*. McGraw-Hill, New York, NY, 1969.

[95] H. Massam. Optimality conditions for a cone-convex programming problem. *J. Austral. Math. Soc. Ser. A*, 27(2):141–162, 1979.

[96] H. Massam. Optimality conditions using sum-convex approximations. *J. Optim. Theory Appl.*, 35(4):475–495, 1981.

[97] J. Maybee and J. Quirk. Qualitative problems in matrix theory. *SIAM Rev.*, 11:30–51, 1969.

[98] C. Mészáros and U.H. Suhl. Advanced preprocessing techniques for linear and quadratic programming. *OR Spectrum*, 25:575–595, 2003. 10.1007/s00291-003-0130-x.

[99] A. Mucherino, C. Lavor, L. Liberti, and N. Maculan, editors. *Distance geometry*. Springer, New York, 2013. Theory, methods, and applications.

[100] Y.E. Nesterov and A.S. Nemirovski. *Interior-point polynomial algorithms in convex programming*, volume 13 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[101] C.E. Nugent, T.E. Vollman, and J. Ruml. An experimental comparison of techniques for the assignment of facilities to locations. *Operations Research*, 16:150–173, 1968.

[102] D.E. Oliveira, H. Wolkowicz, and Y. Xu. ADMM for the SDP relaxation of the QAP. Technical report, University of Waterloo, Waterloo, Ontario, 2015. arXiv:1512.05448, under revision Oct. 2016, 12 pages.

[103] P. Pardalos, F. Rendl, and H. Wolkowicz. The quadratic assignment problem: a survey and recent developments. In P.M. Pardalos and H. Wolkowicz, editors, *Quadratic assignment and related problems (New Brunswick, NJ, 1993)*, pages 1–42. Amer. Math. Soc., Providence, RI, 1994.

[104] P. Pardalos and H. Wolkowicz, editors. *Quadratic assignment and related problems*. American Mathematical Society, Providence, RI, 1994. Papers from the workshop held at Rutgers University, New Brunswick, New Jersey, May 20–21, 1993.

[105] P.A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Math. Program.*, 96(2, Ser. B):293–320, 2003. Algebraic and geometric methods in discrete optimization.

[106] G. Pataki. A simple derivation of a facial reduction algorithm and extended dual systems. Technical report, Columbia University, New York, 2000.

[107] G. Pataki. Bad semidefinite programs: they all look the same. *ArXiv e-prints*, December 2011.

[108] G. Pataki. On the connection of facially exposed and nice cones. *J. Math. Anal. Appl.*, 400(1):211–221, 2013.

[109] G. Pataki. Strong duality in conic linear programming: facial reduction and extended duals. In David Bailey, Heinz H. Bauschke, Frank Garvan, Michel Thera, Jon D. Vanderwerff, and Henry Wolkowicz, editors, *Computational and analytical mathematics*, volume 50 of *Springer Proc. Math. Stat.*, pages 613–634. Springer, New York, 2013.

[110] J. Peña and J. Renegar. Computing approximate solutions for convex conic systems of constraints. *Math. Program.*, 87(3, Ser. A):351–383, 2000.

[111] F. Permenter, H. Friberg, and E. Andersen. Solving conic optimization problems via self-dual embedding and facial reduction: a unified approach. Technical report, MIT, Boston, MA, 2015.

[112] F. Permenter and P. Parrilo. Partial facial reduction: simplified, equivalent SDPs via approximations of the PSD cone. Technical Report Preprint arXiv:1408.4685, MIT, Boston, MA, 2014.

[113] F. Permenter and P. A. Parrilo. Basis selection for sos programs via facial reduction and polyhedral approximations. In *53rd IEEE Conference on Decision and Control*, pages 6615–6620, Dec 2014.

[114] F. Permenter and P. A. Parrilo. Dimension reduction for semidefinite programs via Jordan algebras. *ArXiv e-prints*, August 2016.

[115] T.K. Pong, H. Sun, N. Wang, and H. Wolkowicz. Eigenvalue, quadratic programming, and semidefinite programming relaxations for a cut minimization problem. *Comput. Optim. Appl.*, 63(2):333–364, 2016.

[116] M.V. Ramana. *An Algorithmic Analysis of Multiquadratic and Semidefinite Programming Problems*. PhD thesis, Johns Hopkins University, Baltimore, Md, 1993.

[117] M.V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Math. Programming*, 77(2):129–162, 1997.

[118] M.V. Ramana, L. Tunçel, and H. Wolkowicz. Strong duality for semidefinite programming. *SIAM J. Optim.*, 7(3):641–662, 1997.

[119] B. Recht, M. Fazel, and P. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Rev.*, 52(3):471–501, 2010.

[120] J. Renegar. Some perturbation theory for linear programming. *Math. Programming*, 65(1, Ser. A):73–91, 1994.

[121] J. Renegar. Incorporating condition measures into the complexity theory of linear programming. *SIAM J. Optim.*, 5(3):506–524, 1995.

[122] J. Renegar. Linear programming, complexity theory and elementary functional analysis. *Math. Programming*, 70(3, Ser. A):279–351, 1995.

[123] S.M. Robinson. Stability theorems for systems of inequalities, part i: linear systems. *SIAM J. Numerical Analysis*, 12:754–769, 1975.

[124] S.M. Robinson. Stability theorems for systems of inequalities, part ii: differentiable nonlinear systems. *SIAM J. Numerical Analysis*, 13:497–513, 1976.

[125] R.T. Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[126] B. Rosgen and L. Stewart. Complexity results on graphs with few cliques. *Discrete Math. Theor. Comput. Sci.*, 9(1):127–135 (electronic), 2007.

[127] S. Sahni and T. Gonzales. P-complete approximation problems. *Journal of ACM*, 23:555–565, 1976.

[128] A. Singer. A remark on global positioning from local distances. *Proc. Natl. Acad. Sci. USA*, 105(28):9507–9511, 2008.

[129] F.J. Sturm. *Primal-dual interior point approach Semidefinite programming.* PhD thesis, Erasmus University, Rotterdam, Netherlands, 1997.

[130] J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optim. Methods Softw.*, 11/12(1-4):625–653, 1999. Interior point methods.

[131] J.F. Sturm. Error bounds for linear matrix inequalities. *SIAM J. Optim.*, 10(4):1228–1248 (electronic), 2000.

[132] H. Sun. ADMM for SDP relaxation of GP. Master's thesis, University of Waterloo, 2016.

[133] Z. Tang and H. Wolkowicz. ADMM for the second lifting SDP relaxation of MC. Technical report, University of Waterloo, Waterloo, Ontario, 2017. in progress.

[134] S. Tanigawa. Singularity degree of the positive semidefinite matrix completion problem. Technical Report arXiv:1603.09586, Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan, 2016.

[135] P. Tarazaga. Faces of the cone of Euclidean distance matrices: characterizations, structure and induced geometry. *Linear Algebra Appl.*, 408:1–13, 2005.

[136] P. Tarazaga, T.L. Hayden, and J. Wells. Circum-Euclidean distance matrices and faces. *Linear Algebra Appl.*, 232:77–96, 1996.

[137] L. Tunçel. On the Slater condition for the SDP relaxations of nonconvex sets. *Oper. Res. Lett.*, 29(4):181–186, 2001.

[138] L. Tunçel. *Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization*, volume 27 of *Fields Institute Monographs.* American Mathematical Society, Providence, RI, 2010.

[139] L. Tunçel. *Polyhedral and semidefinite programming methods in combinatorial optimization*, volume 27 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2010.

[140] L. Tunçel and H. Wolkowicz. Strong duality and minimal representations for cone optimization. *Comput. Optim. Appl.*, 53(2):619–648, 2012.

[141] L. Vandenberghe and M.S. Andersen. Chordal graphs and semidefinite optimization. *Found. Trends Opt.*, 1(4):241–433, 2015.

[142] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Rev.*, 38(1):49–95, 1996.

[143] H. Waki, S. Kim, M. Kojima, and M. Muramatsu. Sums of squares and semidefinite program relaxations for polynomial optimization problems with structured sparsity. *SIAM J. Optim.*, 17(1):218–242, 2006.

[144] H. Waki and M. Muramatsu. A facial reduction algorithm for finding sparse SOS representations. *Oper. Res. Lett.*, 38(5):361–365, 2010.

[145] H. Waki and M. Muramatsu. Facial reduction algorithms for conic optimization problems. *J. Optim. Theory Appl.*, 158(1):188–215, 2013.

[146] H. Whitney. Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.*, 36(1):63–89, 1934.

[147] H. Wolkowicz. Geometry of optimality conditions and constraint qualifications: the convex case. *Math. Programming*, 19(1):32–60, 1980.

[148] H. Wolkowicz. Some applications of optimization in matrix theory. *Linear Algebra Appl.*, 40:101–118, 1981.

[149] H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. *Handbook of semidefinite programming*. International Series in Operations Research & Management Science, 27. Kluwer Academic Publishers, Boston, MA, 2000. Theory, algorithms, and applications.

[150] H. Wolkowicz and Q. Zhao. Semidefinite programming relaxations for the graph partitioning problem. *Discrete Appl. Math.*, 96/97:461–479, 1999. Selected for the special Editors' Choice, Edition 1999.

[151] S. Wright. *Primal-Dual Interior-Point Methods*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, 1996.

[152] Q. Zhao, S.E. Karisch, F. Rendl, and H. Wolkowicz. Semidefinite programming relaxations for the quadratic assignment problem. *J. Comb. Optim.*, 2(1):71–109, 1998. Semidefinite programming and interior-point approaches for combinatorial optimization problems (Fields Institute, Toronto, ON, 1996).

[153] Y-R. Zhu. *Recent Advances and Challenges in Quadratic Assignment and Related Problems*. PhD thesis, University of Pennsylvania, 2007. PhD Thesis.