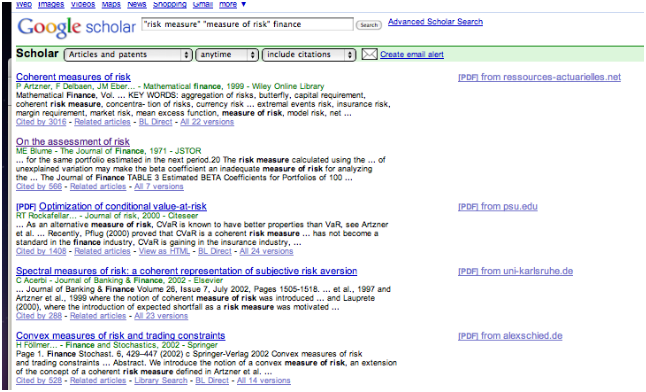

The Marginal Cost of Risk, Risk Measures, and Capital Allocation

Department of Risk Management and Insurance ROBINSON

Daniel Bauer & George Zanjani

(both Georgia State University)

Introduction
- Risk Measures and Capital Allocation
- Preview of Results

Profit Maximization and Capital Allocation

Capital Allocation and Risk Measures

Application

Conclusion

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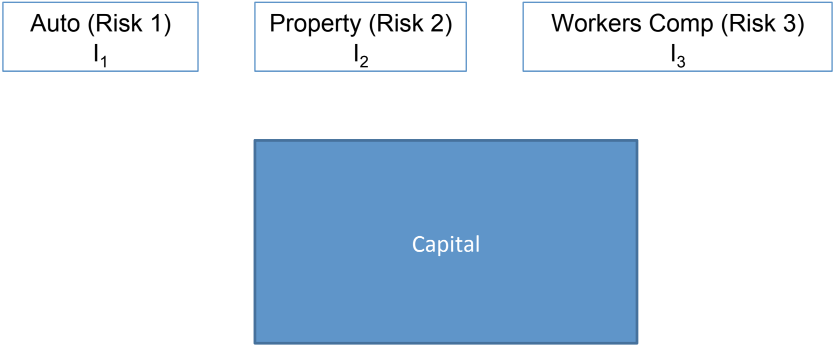
The Capital Allocation Problem

Auto (Risk 1) I_1	Property (Risk 2) I_2	Workers Comp (Risk 3) I_3

The Capital Allocation Problem



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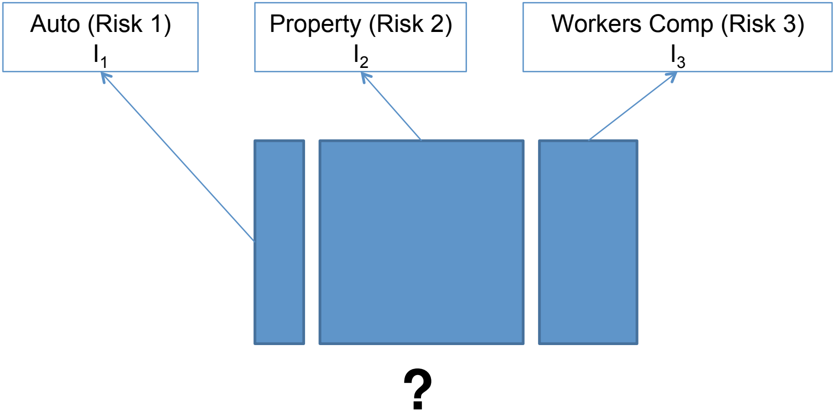

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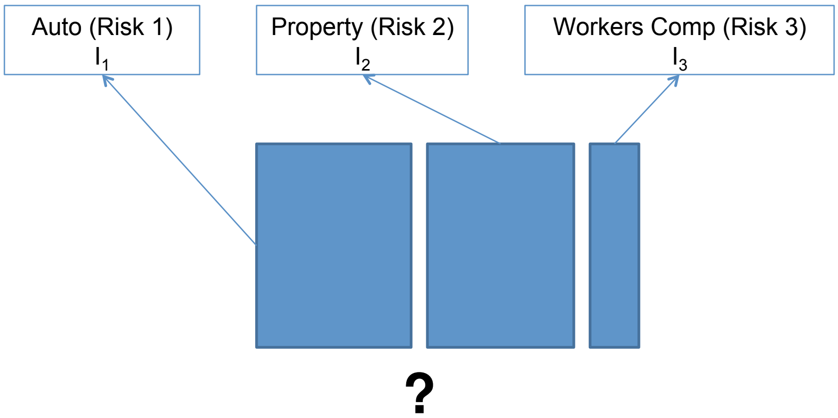

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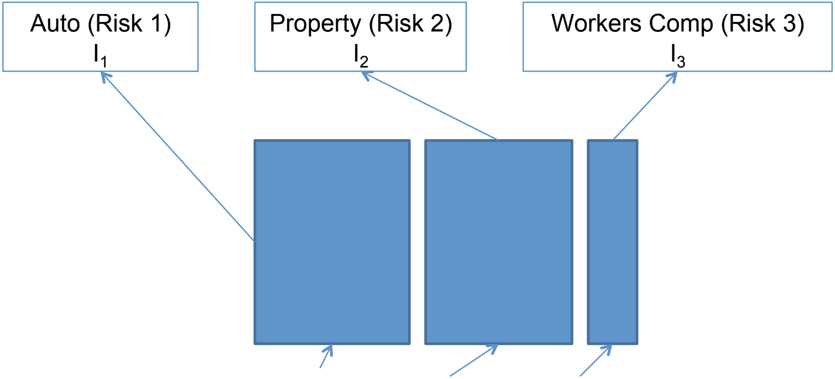

$$\frac{\partial \rho(I)}{\partial q_1} \times q_1 + \frac{\partial \rho(I)}{\partial q_2} \times q_2 + \frac{\partial \rho(I)}{\partial q_3} \times q_3 = a,$$

... where $a = \rho(I)$, $I = I_1 + I_2 + I_2$ and $I_i = q_i \times L_i$

$\rightarrow$ Easy to implement, billed as economic (connection to marginal cost)

$\rightarrow$ So: [(1) Choose $\rho \Rightarrow$ (2) Allocate Capital] – but how to choose ρ?

What we do: The opposite

Our approach

We start with a primitive economic model of profit maximizing insurer, calculate marginal cost and the implied capital allocation, and then figure out what risk measure would yield the correct allocation

Economic Model $\Rightarrow$ **Marginal Cost** $\Rightarrow$ **Capital Allocation** $\Rightarrow$ **Risk Measure**

Preview of Results

- Study profit maximizing insurer with risk averse counterparties, facing a (possibly non-binding) regulatory capital constraint.
  Thus, there are three sources of “discipline" – (1) the regulator (via risk measure s), (2) shareholders’ access to future profits, and (3) counterparties that determine capital allocation:

$$\lambda_1 \times \left[\frac{\partial s(I)}{\partial q_i}\right] \quad + \quad \lambda_2 \times \left[\tilde{\theta}_i\right] \times \quad + \quad (1-(\lambda_1+\lambda_2)) \times \left[\tilde{\phi}_i\right]$$

$$\swarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \searrow$$

$$(1) \qquad\qquad\qquad (2) \qquad\qquad\qquad (3)$$

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$$\tilde{\theta}_i = \frac{\partial}{\partial q_i} \mathrm{VaR}_{\alpha^*}(I)$$

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- "Going concern" allocation $\tilde{\theta}_i$ is determined via gradient of Value-at-Risk:

$$\tilde{\theta}_i = \frac{\partial}{\partial q_i} \mathrm{VaR}_{\alpha^*}(I)$$

- "Counterparty-driven" allocation $\tilde{\phi}_i$ is determined via gradient of "new" risk measure $\tilde{\rho}$:

$$\tilde{\phi}_i = \frac{\partial \tilde{\rho}(I)}{\partial q_i} \text{ where } \boxed{\tilde{\rho}(X) = \exp\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\log\{X\}\right]\right\}}$$

Preview of Results (2)

- $\tilde{\rho}$ is **neither convex nor coherent** due to embedded log-transformation
  - $\rightarrow$ Stems from **"limited liability"** – extreme states less important since there is not much left to share
- But includes an absolutely continuous measure transformation $\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}}$ that
  - keeps the **focus on the default states**, i.e. $\tilde{\mathbb{P}}(I \geq a) = 1$
  - **depends on the consumers' marginal utilities in loss states**, which are higher in extreme states
- In a setting with security markets, result pertains in the "branch" where market becomes incomplete
  - In limiting case of completeness, results in Ibragimov et al. (2010) allocation
- For $X = I$, we can represent $\bar{\rho}(I) = \exp\left\{\mathbb{E}\left[\psi(I)\,\log\{I\}|I \geq a\right]\right\}$
  - Relationship to *Spectral Risk Measure* (Acerbi, 2002)
  - For homogeneous exponential losses and CARA utility (ARA a, N risks):
    $$\psi(I) = \text{const} \times \mathbf{1}_{\{I \geq a\}} \times \sum_{k=0}^{\infty} \frac{(k+1)(\alpha(I-a))^k}{(N+k)!}$$
  - In comparison to other allocation methods (here CTE), results may be more or less conservative, depending on e.g. expected loss and risk aversion

Basic Model Setup (one period model without security market)

- Consumer i faces loss L_i (non-negative random variable)
- Firm determines optimal **asset** level a, optimal coverage **indemnification level**, which is given by $I_i = I_i(L_i, q_i)$ with choice parameter q_i, $I = \sum I_i$, and optimal **premium** level p_i
- In non-default states, consumer gets full indemnification amount. In default states, all claimants are paid at the same rate per dollar of coverage

→ Recovery $R_i = \min\left\{I_i, \frac{a}{I} I_i\right\}$ with expected value

$$e_i = \mathbb{E}[R_i] = \underbrace{\mathbb{E}[R_i \, \mathbf{1}_{\{I<a\}}]}_{e_i^Z} + \underbrace{\mathbb{E}[R_i \, \mathbf{1}_{\{I\geq a\}}]}_{e_i^D}$$

- Tax on assets $(\tau \times a)$
- Consumer i with wealth level w_i has utility function U_i with

$$v_i = \mathbb{E}\left[U_i\left(\mathrm{w}_i - p_i - L_i + R_i\right)\right]$$

- Firm solves

$$\begin{cases} \max_{a,\{p_i\},\{q_i\}} \sum p_i - \sum e_i - \tau \times a \\ \text{s.th.} \\ v_i \geq \gamma_i, i = 1, \ldots, N \quad \text{(participation constraint)} \\ s(q_1, \ldots, q_n) \leq a \quad \text{(external solvency constraint)} \end{cases}$$

$\Rightarrow$ Under certain assumptions, optimal solution can be implemented by a monotonic premium schedule $p^*(\cdot)$ that satisfies

$$\frac{\partial p_i^*}{\partial q_i} = \underbrace{\frac{\partial e_i^Z}{\partial q_i}}_{\text{Extra claims cost to consumer } i} + \underbrace{\frac{\partial s}{\partial q_i}\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right]}_{\text{Cost relating to regulatory constraint}}$$

$$\underbrace{+ \underbrace{\frac{\mathbb{E}\left[\frac{\frac{\partial I_i}{\partial q_i}}{I} \sum_k \frac{U_k'}{v_k'} \frac{I_k}{I} \middle| I \geq a\right]}{\mathbb{E}\left[\sum_k \frac{U_k'}{v_k'} \frac{I_k}{I} \middle| I \geq a\right]}}_{\tilde{\phi}_i} \times \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \times a}_{\textbf{Cost relating to externalities on other consumers}}$$

From Marginal Cost to Capital Allocation

- Marginal cost implies allocation of capital:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \underbrace{\frac{\partial s}{\partial q_i}\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right]}_{\text{Regulator driven allocation}} + \underbrace{\tilde{\phi}_i \times \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \times a}_{\text{Counterparty driven allocation}}$$

- Why are we calling it an allocation?

$$\sum_i \frac{\partial p_i^*}{\partial q_i} \times q_i = e_i^Z + [\mathbb{P}(I \geq a)\, a + \tau\, a]$$

- Additional terms in multi-period model:

$$\underbrace{\frac{\partial s}{\partial q_i}\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} - V\, f_I(a)\right]}_{\text{Regulator driven allocation}} + \underbrace{\tilde{\phi}_i \times \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \times a}_{\text{Counterparty driven allocation}} \underbrace{+\,\mathbb{E}\left[\frac{\partial I_i}{\partial q_i} \,|I = a\right] \times V\, f_I(a) \times a}_{\text{Shareholder driven allocation } (\tilde{\theta}_i)}$$

- State prices enter when considering security market, but result pertains in "branch" where market becomes incomplete

Special Cases:

$$\underbrace{\frac{\partial s}{\partial q_i}\overbrace{\left[1-\frac{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}}{\mathbb{P}(I\geq a)+\tau}-\frac{V\,f_I(a)}{\mathbb{P}(I\geq a)+\tau}\right]}^{1-(\lambda_1+\lambda_2)}}_{\text{Regulator driven allocation}}\underbrace{+\tilde{\phi}_i\times a\times\overbrace{\left[\frac{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}}{\mathbb{P}(I\geq a)+\tau}\right]}^{\lambda_1}}_{\text{Counterparty driven allocation}}\underbrace{+\tilde{\theta}_i\times a\times\overbrace{\left[\frac{V\,f_I(a)}{\mathbb{P}(I\geq a)+\tau}\right]}^{\lambda_2}}_{\text{Shareholder driven allocation}}$$

- Full deposit insurance and perfect competition: $\boxed{\lambda_1=\lambda_2=0}$ and allocation solely determined by externally specified risk measure
  - World of Myers and Read (2001), Tasche (2004) etc.
- Full deposit insurance, no or non-binding regulation, and monopolistic competition: $\boxed{\lambda_1=0,\lambda_2=1}$ – only $\tilde{\theta}_i$ matters, which derives as the gradient of Value-at-Risk
  - May explain popularity of VaR (deposit insurance prevalent)
- Competition and no regulation: $\boxed{\lambda_1=1,\lambda_2=0}$ – only $\tilde{\phi}_i$ matters, which is driven by counterparty risk aversion

A Novel Risk Measure

- Regulator-driven allocation based on external risk measure, shareholder-driven allocation based on VaR
- **But what about counterparty-driven allocation?**

A Novel Risk Measure

- Regulator-driven allocation based on external risk measure, shareholder-driven allocation based on VaR
- **But what about counterparty-driven allocation?**

1. Define the probability measure $\tilde{\mathbb{P}}$ by the Radon-Nikodym derivative

$$\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} = \frac{\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k'}{v_k'} \frac{I_k}{I}}{\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k'}{v_k'} \frac{I_k}{I}\right]}$$

   - $\tilde{\mathbb{P}}$ absolutely continuous with respect to $\mathbb{P}$ with $\tilde{\mathbb{P}}(I \geq a) = 1$

2. On $L_+^2 = \left\{X \in (\Omega, \mathcal{F}, \tilde{\mathbb{P}}) | X > 0\right\}$, define the risk measure

$$\tilde{\rho}(X) = \exp\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\log\{X\}\right]\right\}$$

   - While $\tilde{\rho}$ is monotonic, homogenous, and satisfies constancy, it is **not translation-invariant** and **not sub-additive**, and therefore **not coherent** and **not convex**
   - However, it is correct for internal allocation according to the Euler principle...

The Euler Principle Revisited: No Deposit Insurance/No Regulation/One-Period

- Define $\tilde{\chi}_\rho = \frac{a^*}{\tilde{\rho}(I^*)}$ as the "exchange rate" between capital and risk. Then

$$\begin{cases} \pi(q_1, \ldots, q_N, a) \to \max \\ \tilde{\rho}(q_1, \ldots, q_N)\, \tilde{\chi}_\rho \le a \end{cases}$$

yields allocation

$$\begin{aligned} \sum_k \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_k} q_k^* \underbrace{\left(-\frac{\partial \pi}{\partial a} \right)}_{\mathbb{P}(I \geq a^*) + \tau} &= \sum_k \tilde{\phi}_k\, q_k^*\, a^* \, [\mathbb{P}(I \geq a^*) + \tau] \\ &= a^* \, [\mathbb{P}(I \geq a^*) + \tau] \\ \Longrightarrow \sum_k \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_k} q_k^* &= a^* \end{aligned}$$

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$\Rightarrow$ Capital allocation can be implemented by differentiating **novel risk measure** at current portfolio

The Euler Principle Revisited: General Case

- Here we have two restrictions: $(\alpha^* = \mathbb{P}(I^* \leq a^*))$

$$\begin{cases} \pi^{\text{1per.}}(q_1, \ldots, q_N, a) \to \max \\ s(q_1, \ldots, q_N) \leq a \\ \tilde{\rho}(q_1, \ldots, q_N)\, \tilde{\chi}_\rho \leq a \\ \text{VaR}_{\alpha^*}(I) \leq a \end{cases}$$

so in addition to partial derivatives, the Lagrange multipliers matter:

$$\sum_i \Bigg[\underbrace{\frac{\partial s}{\partial q_i} \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} - V\, f_I(a) \right]}_{\text{Regulator driven allocation}} + \underbrace{\frac{\partial \tilde{\rho}}{\partial q_i} \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \right]}_{\text{Counterparty driven allocation}} + \underbrace{\frac{\partial \text{VaR}_{\alpha^*}(I^*)}{\partial q_i} \times [V\, f_I(a)]}_{\text{Shareholder driven allocation}} \Bigg]$$

$$= \quad a^* \times [\mathbb{P}(I \geq a) + \tau]$$

$$\Rightarrow a^* \quad = \quad \sum_j q_j^* \frac{\partial}{\partial q_j} \left([1 - (\lambda_1 + \lambda_2)]\, s + \lambda_1\, \tilde{\chi}_\rho\, \tilde{\rho} + \lambda_2\, \text{VaR}_{\alpha^*} \right)(I^*)$$

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$\Rightarrow$ Euler principe works! Capital allocation can be implemented by differentiating **weighted average of external and internal risk measure** at current portfolio

Properties of $\tilde{\rho}$

Two influences:

1. **log-transform** – driven by limited liability
   - $\rightarrow$ In comparison to linear case ($\rightarrow$Expected Shortfall) **less weight on extreme** loss states
   - $\rightarrow$ Counterparties evaluate changes in risk simply from the perspective of how the expected value of **recoveries** from the firm are affected
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2. **Change of measure** – driven by marginal utility in loss states:
   - $\rightarrow$ If **expected losses large** or **risk aversion high**, relatively **more weight on high loss states** $\rightarrow$ "more conservative"
   - $\rightarrow$ Evaluation for $X = I^*$:

$$\tilde{\rho}(I^*) = \exp\left\{\mathbb{E}\left[\psi(I^*)\,\log(I^*)\,|\,I^* \geq a^*\right]\right\}$$

   - $\rightarrow$ $\psi(\cdot)$ similar as **risk spectrum** within **spectral risk measures** (Acerbi, 2002)
   - $\rightarrow$ Ultimately depends on $\psi(\cdot)$ how this "risk measure" compares and the ensuing allocation compares to other methods

Homogeneous Exponential Losses

- Here, $l_i = q\,L_i$ and $l = q\sum L_i = q\,L$ and (obviously)

$$\frac{a_i}{a} = q\,\tilde{\phi}_i = \frac{1}{N}\mathbb{E}\left[\psi(L)|q\,L \geq a\right] = \frac{1}{N}$$

  with

$$\psi(x) = \text{const} \times \sum_{k=0}^{\infty} \frac{(k+1)\,(\alpha(x-a))^k}{(N+k)!}$$

  and

$$\tilde{\rho}(q\,L) = \exp\left\{\mathbb{E}\left[\psi(L)\,\log\{q\,L\}|\,q\,L \geq a\right]\right\}$$

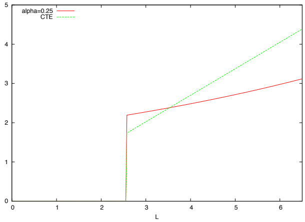
- For Expected Shortfall, we have

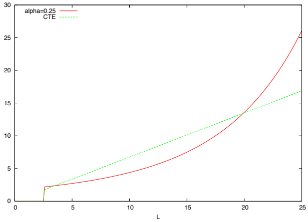
$$\frac{a_i}{a} = \frac{1}{N}\mathbb{E}\left[\text{const} \times L|q\,L \geq a\right] = \frac{1}{N}$$

- Analytical properties:
  - ψ convex for $\alpha > 0$, particularly $\psi(x) = \text{const} \times \exp\{-\alpha(a-x)\}$ for $N = 1$
  - ψ flat for $N \to \infty$ or $\alpha = 0$

Homogeneous Exp. Losses – two possible shapes for ψ:

Low risk aversion / small loss relative to wealth:





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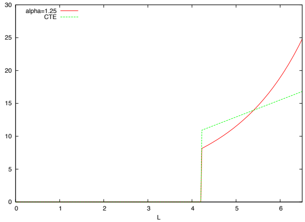

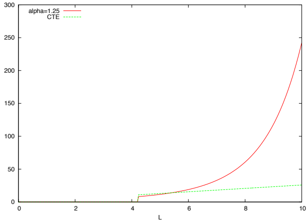

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- We identify the **optimal capital allocation** consistent with the **marginal cost** for a profit-maximizing firm with risk-averse counterparties, and the **supporting risk measure**
- This risk measure is generally **not convex** and **not coherent**, due to limited liability of the firm

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- **Risk measure selection "thorny" issue** that can only be resolved by careful consideration of **institutional context,** particularly when the main purpose is the allocation of risk-based capital
- We identify the **optimal capital allocation** consistent with the **marginal cost** for a profit-maximizing firm with risk-averse counterparties, and the **supporting risk measure**
- This risk measure is generally **not convex** and **not coherent**, due to limited liability of the firm
- However, in includes a measure transform that puts the **focus on default states** and is related to **consumer's marginal utility in default states.** Hence, it may still **penalize high risk states more severely** than coherent risk measures
- Thus, the comparison to Expected Shortfall may result in **qualitative different outcomes**, depending on the size of the losses and risk aversion, among others

Contact

Daniel Bauer
dbauer@gsu.edu

George Zanjani
gzanjani@gsu.edu

Georgia State University
USA

www.rmi.gsu.edu

Thank you!