

The mass of the cosmos

Charles Hellaby[★]

Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa

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ABSTRACT

We point out that the mass of the cosmos on gigaparsec-scales can be measured, owing to the unique geometric role of the maximum in the areal radius. Unlike all other points on the past null cone, this maximum has an associated mass, which can be calculated with very few assumptions about the cosmological model, providing a measurable characteristic of our cosmos. In combination with luminosities and source counts, it gives the bulk mass to light ratio. The maximum is particularly sensitive to the values of the bulk cosmological parameters. In addition, it provides a key reference point in attempts to connect cosmic geometry with observations. We recommend the determination of the distance and redshift of this maximum be explicitly included in the scientific goals of the next generation of redshift surveys. The maximum in the redshift space density provides a secondary large-scale characteristic of the cosmos.

Key words: cosmological parameters.

1 INTRODUCTION

The emergence of automated large-scale redshift surveys (<http://www.sdss.org/>; <http://www.aao.gov.au/2df/>) is opening up new possibilities for measuring the content and dynamics of the cosmos on very large scales. We point out a significant characterization of the cosmos on gigaparsec-scales that will become measurable with the next generation of surveys.

It is well known in observational cosmology that our past null cone (pnc) has a maximum in its areal radius, \hat{R}_m , where the angular size of sources of a given size is minimum. Beyond this point, more distant images, though dimmer, subtend larger angular sizes (McCrea 1934¹; Hoyle 1961²). It is also known in relativistic cosmology that this maximum occurs where the observer's pnc crosses the apparent horizon. What hasn't been realized is that a measurement of this maximum is equivalent to a measurement of the mass within a sphere of areal radius \hat{R}_m , and that this relationship is quite general, not requiring the assumption of homogeneity for example. Less well known is that the number density of sources in redshift space also has a maximum. This latter maximum, however, does not have such a deep significance as the former.

It is best to use the Lemaître–Tolman (LT) model, as both R and M are primary functions, so the relationships we are interested in are

[★]E-mail: cwh@maths.uct.ac.za

¹Although the paper doesn't state the diameter distance has a maximum, equation (55) is the standard result for the diameter distance in an Einstein–de Sitter model.

²This is apparently the first article to explicitly state the angular diameter corresponding to a fixed physical diameter has a minimum as redshift increases.

particularly clear. Also, the equation of state – pressure-free matter plus Λ – is very suitable for the post-recombination universe, and the spherical symmetry of the model about the origin is entirely natural in the context of observations, since isotropy is fairly well established, and of course our own past light cone is centred on ourselves.

The LT metric (Lemaître 1933; Tolman 1934) is

$$ds^2 = -dt^2 + \frac{(R')^2}{1+2E} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $R = R(t, r)$ is the areal radius, $R' = \partial R / \partial r$ and $E = E(r)$ is an arbitrary function of coordinate radius r . We use geometric units in all equations. This metric describes pressure-free matter in comoving coordinates. From the Einstein field equations,

$$\dot{R}^2 = \frac{2M}{R} + 2E + \frac{\Lambda R^2}{3}, \quad (2)$$

$$\kappa\rho = \frac{2M'}{R^2 R'}, \quad (3)$$

where $\kappa = 8\pi G/c^4 = 8\pi$, $\rho(t, r)$ is the density, $\dot{R} = \partial R / \partial t$ and $M = M(r)$ is also arbitrary. The solutions to (2) are best found numerically for the general case. The density is divergent at the bang and crunch, $R = 0$, and also at shell crossings, where $R' = 0$ but $M' \neq 0$. The latter singularity is avoidable (Hellaby & Lake 1985), but the former isn't.

As can be seen from (1), the function E determines the local geometry. In addition, (2) shows that it is a kind of energy parameter. Function M is the gravitational mass within a comoving sphere of radius r , so it includes any putative dark matter component(s), but does not include the 'density' associated with Λ . This M is not the

same as the integrated proper density \mathcal{M} ,

$$\mathcal{M} = \int_0^r \rho d^3V = \int_0^r (M'/\sqrt{1+2E}) dr. \quad (4)$$

This latter is the mass one would obtain by summing the masses of individual galaxies, gas clouds, dark matter concentrations etc. See Krasiński (1997) for further discussion of the LT model and supporting references.

2 OBSERVABLES

The light rays making up the pnc are the incoming radial null geodesics arriving at the central observer at a given moment, in particular,

$$\frac{dt}{dr} = \frac{-R'}{\sqrt{1+2E}}, \quad t = t_0 \text{ at } r = 0, \quad (5)$$

with solution $t = \hat{t}(r)$. Any quantity $Q(t, r)$ evaluated on our current pnc will be indicated with a hat: $\hat{Q} = \hat{Q}(r) = Q(\hat{t}(r), r)$, and for expressions a square bracket with subscript 'n' will be used. The redshift of an observed object, located at r_e , is given by

$$\ln(1+z) = \int_0^{r_e} \frac{\hat{R}'}{\sqrt{1+2E}} dr, \quad (6)$$

where \hat{R}' may be found from (2). Along a ray, the areal radius is $\hat{R} = R(\hat{t}(r), r)$, and its rate of change along the ray, using (2) and (5), is

$$\hat{R}' = [\hat{R}' + R']_n = \left[\left(-\frac{\sqrt{(2M/R) + 2E + (\Lambda R^2/3)}}{\sqrt{1+2E}} + 1 \right) R' \right]_n. \quad (7)$$

Since the coordinate r is not observable or physically meaningful, we calculate

$$\frac{d\hat{R}}{dz} = \frac{\hat{R}'}{z'} = \left[\frac{R'}{\hat{R}'(1+z)} \left(\sqrt{1+2E} - \sqrt{\frac{2M}{R} + 2E + \frac{\Lambda R^2}{3}} \right) \right]_n. \quad (8)$$

Near the origin we have $M \rightarrow 0$, $E \rightarrow 0$ and $z \rightarrow 0$, so

$$\frac{d\hat{R}}{dz} \rightarrow [R'/\hat{R}']_{n0}. \quad (9)$$

See Mustapha, Hellaby & Ellis (1997) and Mustapha et al. (1998) for further details.

If a source such as a galaxy has measured angular diameter δ and known actual diameter D , then the diameter distance is identically the areal radius on the pnc and is defined by

$$\hat{R} = d_D = D/\delta. \quad (10)$$

If the apparent luminosity of an observed source is ℓ , and its absolute luminosity is known to be L , then the luminosity distance is defined by

$$d_L = \sqrt{L/\ell} d_a = 10^{(m-\tilde{m})/5} d_a, \quad (11)$$

where $d_a = 10$ pc, m is the apparent magnitude and \tilde{m} the absolute magnitude. For a general curved space-time and arbitrary motion, the reciprocity theorem tells us

$$d_L = d_D(1+z)^2. \quad (12)$$

If $d_D = \hat{R}$ is calculated from m and z measurements using (12), the fractional error is comparable with that of d_L , assuming the redshifts are fairly accurate:

$$\delta \hat{R}/\hat{R} = 0.2 \ln 10 \delta(m - \tilde{m}) - 2\delta z/(1+z). \quad (13)$$

For each of these distances, it is essential to know an intrinsic source property, D or L (or \tilde{m}). While their values for nearby sources can be determined using other distance measures, for distant sources there is the extra difficulty of determining how much they have evolved with time, or even what type of nearby object the source corresponds to.

If n is the number density of sources in redshift space/(steradian/unit redshift interval), and μ is the mass per source, then the relation between n and ρ is

$$\mu n = \left[\frac{\rho R^2 R'}{\sqrt{1+2E}} \right]_n \frac{dr}{dz} = \left[\frac{2M'}{\kappa \hat{R}'(1+z)} \right]_n = \left[\frac{2 d\mathcal{M}}{\kappa dz} \right]_n. \quad (14)$$

See Hellaby (2001) for a discussion of multiple source types and multicolour observations.

3 CHARACTERISTIC COSMIC MASS AND DENSITY

In an expanding, non-inflating universe, the diameter distance $\hat{R}(r)$ necessarily has a maximum. The locus of such points, where different rays are momentarily at constant R , is the apparent horizon (AH). We find it by putting $\hat{R}' = 0$ in (7), giving

$$\Lambda \hat{R}_m^3 - 3\hat{R}_m + 6M_m = 0, \quad (15)$$

so thus if R_m is on the AH, then $2M_m = R_m - \Lambda R_m^3/3$, and if $\Lambda = 0$ this is the familiar $R = 2M$. In general the locus of the pnc, and the variation of measurables down it, are strongly affected by the details of the cosmological model. This point, however, where the pnc crosses the AH, has a unique meaning:

- (a) there is a simple direct relationship between R and M that is independent of any inhomogeneities in E , t_B and M ,
- (b) since R is measurable, the mass M within that radius is immediately determined,
- (c) this point flags a major causal feature of model,
- (d) the maximum in R is a distinctive feature of the R - z plot.

No other point on the pnc has such a simple, generic M - R relationship, and this holds true whether not the model is homogeneous. Therefore, R_m and the redshift z_m where it occurs are distinctive characteristics of the cosmological model.

For $M_m \geq 0$ and $R_m \geq 0$, equation (15) has solutions if

$$M_m \leq 1/(3\sqrt{\Lambda}) = (M_m)_{\max}, \quad (16)$$

and this maximum possible value of M_m occurs at

$$(R_m)_{\max} = 1/\sqrt{\Lambda}. \quad (17)$$

Thus if an incoming ray reaches $R > (R_m)_{\max}$, then it has no maximum in R . World lines with $M > (M_m)_{\max}$ never meet the AH, and those with $M < (M_m)_{\max}$ cross it twice.

Condition (15) will always hold somewhere for any cosmology with a non-zero matter density, because $M = 0$ at the observer, and increases outwards. In all LT models with a bang, the AH goes as $\hat{R} \approx 2M$ near the bang. In ever-expanding models, the AH asymptotically approaches the de Sitter horizon, $R = \sqrt{3/\Lambda}$, as $t \rightarrow \infty$. The 'incoming' ray that is tangential to $R = \sqrt{3/\Lambda}$ at $t = \infty$ divides rays that have a maximum and reach the origin, from those

that have no maximum and have ever-increasing \hat{R} . In closed recollapsing models, putting $\dot{R} = 0$ and $E = -1/2$ in (2) reproduces (15), thus showing that the moment of maximum expansion at the maximum in the spatial sections lies on the AH. (In fact it is where the past and future AHs cross.) Thus all pncs in (physically well behaved) closed recollapsing LT models have maxima in their areal radii. For further discussion of the apparent horizon, see Hellaby (1987) and Krasinski & Hellaby (2004).

It is evident from (14) that the redshift space density also has a maximum, because, although $M' \geq 0$ and increases from zero at the origin,³ both \hat{R}' and $(1+z)$ are finite at the observer, and diverge towards the bang. For the maximum in μn , we put

$$0 = \frac{d(\mu n)}{dz} = \left[\frac{2}{\kappa} \frac{d^2 \mathcal{M}}{dz^2} \right]_n = \left[\frac{2}{\kappa (\hat{R}')^3 (1+z)^2} \right. \\ \left. \times \left\{ \sqrt{1+2E} (M'' \hat{R}' - M' \hat{R}'') + M' (\hat{R}' \hat{R}' - (\hat{R}')^2) \right\} \right]_n, \quad (18)$$

and solve for $\tilde{\mu} n_m$ and \tilde{z}_m . The locus of this maximum doesn't have a deep geometric or physical meaning, but depends on the redshift behaviour down the pnc, which depends strongly on the details of the model. Nevertheless, one can say that the maximum in μn is a large-scale characteristic density of the model.

4 ROBERTSON-WALKER CASE

The key results are those given above in Section 3, which apply to a fairly general class of realistic post-recombination cosmologies. However, it is useful to look at the characteristic cosmic mass and density in the standard homogeneous model. The above results are specialized to the homogeneous case in Appendix A.

The Robertson–Walker (RW) version of the AH equation (A12) seems quite complex, and hides the key relationship that is evident in (15).

The dependence of \hat{R}_m , M_m , \mathcal{M}_m and $\tilde{\mu} n$ on H_0 , Ω_Λ and Ω_m is shown in Fig. 1, and for reference the $\hat{R}(z)$ and $\mu n(z)$ curves for a range of values of these parameters are given in the appendix.⁴

5 DISCUSSION

The properties outlined in Section 3 below equation (15) make the maximum in \hat{R} a very significant feature of a cosmological model, both theoretically and observationally.

Both \hat{R}_m and z_m may be measured, and calculating M_m from (15) requires only the value of Λ , and is independent of the details of the cosmology, or even whether large-scale homogeneity exists. Note also that M_m is not very sensitive to uncertainties in Ω_Λ ; for a given measured R_m , an increase from 0 to 0.7 decreases the calculated M_m by 12 per cent.⁵

Current galaxy redshift surveys only extend to $z = 0.3$, while quasar redshift surveys extend to $z = 6$. However, the data is not complete enough, and the quasar population too diverse to be useful. The next generation of galaxy surveys will extend past the redshift

³Here we use a well-behaved r coordinate, such as one with $M \propto r^3$ or $R(t = \text{const}, r) \propto r$.

⁴For a related earlier treatment with $\Lambda = 0$, see Ellis & Tivon (1985).

⁵This is the uncertainty in M_m once R_m has been measured, rather than the variation in the M_m predicted by a variety of models with different Ω_Λ values, in which R_m also varies.

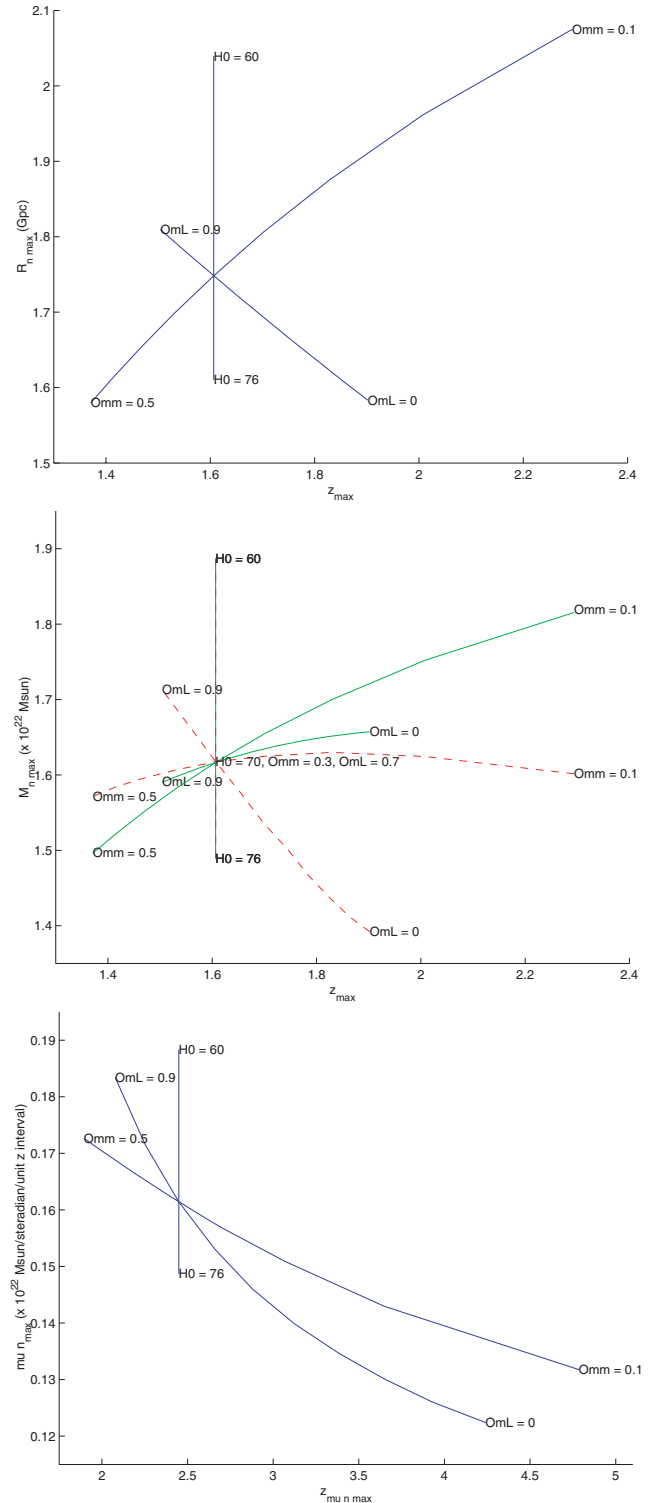


Figure 1. (a) The dependence of \hat{R}_m and z_m on the RW parameters; (b) the dependence of M_m (solid lines) and \mathcal{M}_m (dashed lines) on the RW parameters; (c) the dependence of $\tilde{\mu} n_m$ and \tilde{z}_m on the RW parameters.

of the maximum, and recent supernova observations (Strolger et al. 2004) are already approaching it. The $\hat{R}-z$ plot may be obtained from direct measurements of redshifts and angular diameters, or derived from $m-z$ measurements. While the former would be ideal, the latter is acceptable, since the reciprocity theorem is very general.

Of course, knowledge (or assumptions) about true diameters and absolute luminosities of sources, and their z evolution, is essential, and at large z this is a significant uncertainty.

While locating \hat{R}_m reliably requires a good sample of data points in a range near z_m , determination of $n(z)$ is not so easy, since one must be sure of detecting or reliably estimating all masses. Nevertheless, a sufficiently deep, complete survey should provide an indication of $\widetilde{\mu n}_m$ and \widetilde{z}_m .

An issue for future consideration is the effect of inhomogeneities on the uncertainty in \hat{R}_m and z_m determinations. For example, Mustapha et al. (1998) showed inhomogeneity can cause loops in the $\hat{R}-z$ curve near the maximum.

The mass-to-light ratio is difficult to determine outside gravitationally bound systems, and direct estimates of (gravitational) mass from galaxy and cluster dynamics, and from gravitational lensing only extend up to cluster or supercluster size, whereas the determination of M_m reaches several orders of magnitude larger.

The possibility of determining the metric of the cosmos from observations of redshifts, luminosities (or angular diameters) and the number density of sources, combined with the evolution of absolute luminosities, true diameters and mass per source, was considered in Kristian & Sachs (1966), Ellis & Tivon (1985), Stoeger et al. (1992a), Stoeger, Ellis & Nel (1992b), Stoeger, Nel & Ellis (1992c,d), Maartens & Matravets (1994), Mustapha et al. (1997), Araújo & Stoeger (1999), Celerier (2000), Araújo et al. (2001a), Araújo, Roveda & Stoeger (2001b), Hellaby (2001) and Ribeiro & Stoeger (2003). A project to begin implementing this is now underway (Lu & Hellaby, in preparation). The cosmic mass M_m provides an important cross-check on the summed mass at that radius. In fact the theorem of Mustapha et al. (1997) – that, given any reasonable set of observations and any reasonable source evolutions functions, an LT model can be found that fits them – needs to be qualified, as the combination of observations and evolution functions must mesh correctly close to \hat{R}_m .

Within the dust- Λ -RW model, it is noteworthy that the maximum is where the $\hat{R}(z)$ curve is most sensitive to variations in Ω_Λ and H_0 , and nearly so for Ω_m , and that variations in these three parameters move the (z_m, \hat{R}_m) and $(\widetilde{\mu n}_m, \widetilde{z}_m)$ loci in very different directions. Thus a determination of \hat{R}_m and z_m would provide rather generic limits on H_0 , Ω_m and Ω_Λ , and combined with the initial slope of the $\hat{R}-z$ graph, or with measurements of $\widetilde{\mu n}_m$ and \widetilde{z}_m , would fix all three values. Determining \hat{R}_m and z_m to within 10 per cent would by itself provide confirmation (or otherwise) of current parameter estimations, while 5 per cent accuracy would put new constraints on the possible values.

The problem of averaging in general relativity means that identifying the RW model that best fits the observations is not a well-defined exercise. Therefore, measurements of bulk effects are particularly important, and since R_m , unlike any other point on the pnc, gives the total mass on that scale, the R_m and z_m values provide a natural definition of the best fit RW model.

Although particular models always have a $\hat{R}-M$ relation, such as (A1) and (A2) would give for RW models, this relation is very model dependent, whereas M_m is not. It is more general even than the LT model, as any cosmology will have a locus where the pnc crosses the apparent horizon, and an associated mass is naturally defined there.

6 CONCLUSION

In summary, the maximum in the diameter distance is the only point on the pnc that corresponds to a model-independent mass,

thus allowing direct measurement of a characteristic cosmic mass on gigaparsec-scales. Therefore, it also provides a very large-scale check on the mass to light ratio, as well as a reference point for determining geometry from observations. Since it is a point on the apparent horizon, a measurement of this maximum may actually be the first detection of a relativistic horizon.

For RW models, the region near the maximum is where $R(z)$ is most sensitive to the values of the RW parameters.

We advocate that, with the next generation of surveys, direct measurements of angular sizes and hence diameter distances be compiled and calculated independently of luminosity distances, and that, apart from fitting an Friedmann–Lemaître–Robertson–Walker model to the available data, a separate determination of \hat{R}_m be done with a limited data set near z_m , thus giving a model-independent value for M_m .

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APPENDIX A: SPECIALIZING THE LT EQUATIONS TO RW

The dust- Λ -RW model, in its most common coordinate system, is obtained if we put

$$M = M_0 r^3, \quad 2E = -kr^2, \quad t_B = 0, \quad (\text{A1})$$

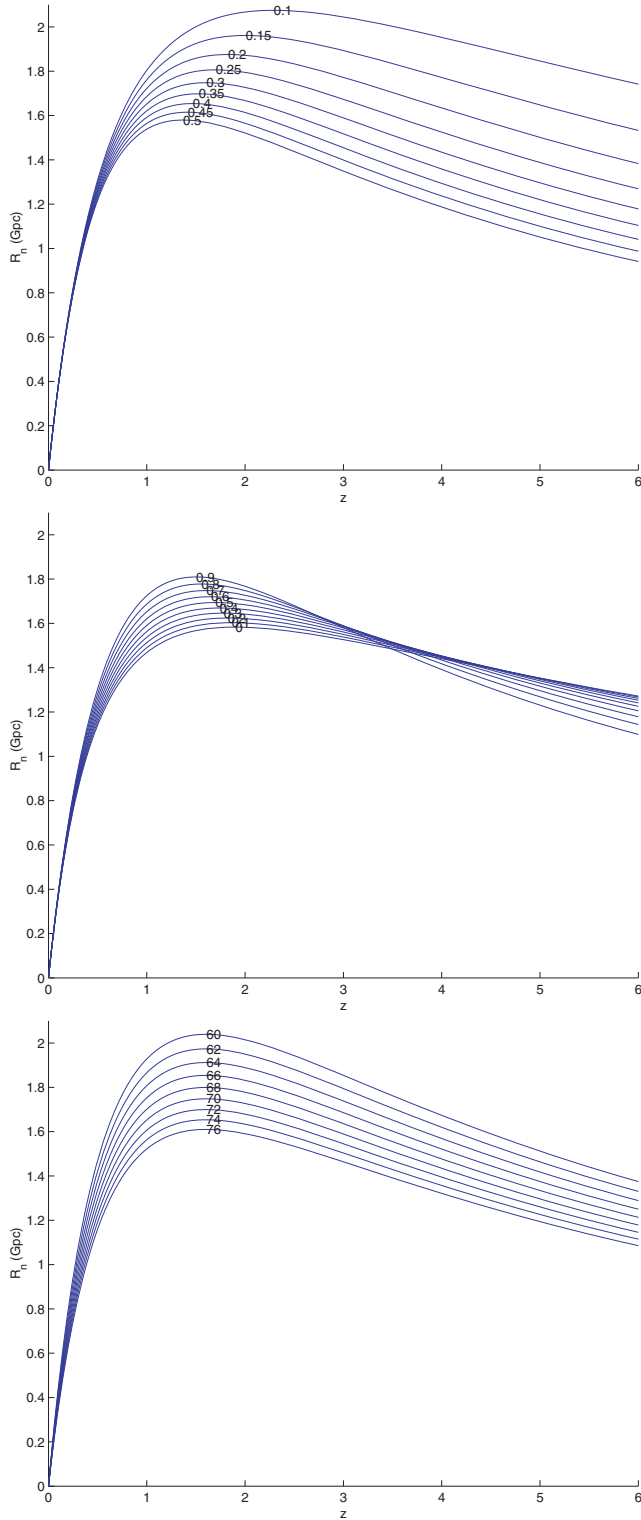


Figure A1. The $\hat{R}(z)$ curves for a range of RW parameters: (a) different Ω_m values ($H_0 = 70$, $\Omega_\Lambda = 0.7$); (b) Ω_Λ values ($H_0 = 70$, $\Omega_m = 0.3$); (c) H_0 values ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$).

from which we have

$$R = rS(t), \quad \dot{S}^2 = 2M_0/S - k + \Lambda S^2/3, \quad (\text{A2})$$

$$H_0 = \dot{S}_0/S_0, \quad \kappa\rho = 6M_0/S^3, \quad \kappa\rho_0 = 6M_0/S_0^3, \quad (\text{A3})$$

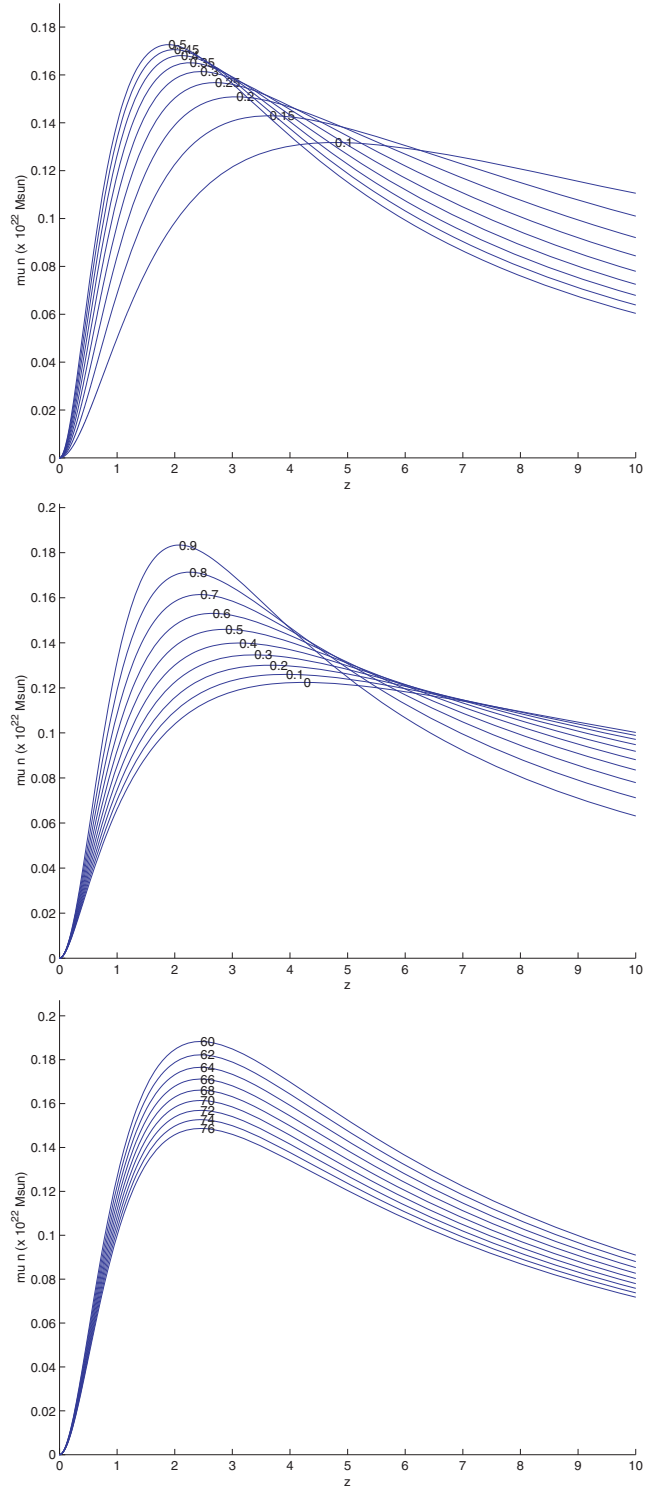


Figure A2. The $\mu n(z)$ curves for a range of RW parameters (note that the z -scale is different from that of Fig. A1: (a) various Ω_m values ($H_0 = 70$, $\Omega_\Lambda = 0.7$); (b) Ω_Λ values ($H_0 = 70$, $\Omega_m = 0.3$); (c) H_0 values ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$).

where S is the scale factor. Given H_0 , Ω_Λ and Ω_m , then

$$t_0 = 2/3H_0, \quad \kappa\rho_0 = 3\Omega_m H_0^2, \quad \Lambda = 3H_0^2\Omega_\Lambda, \quad (\text{A4})$$

$$\Omega_k = 1 - \Omega_\Lambda - \Omega_m, \quad k = -\text{sign}(\Omega_k), \quad (\text{A5})$$

$$S_0 = \begin{cases} \text{arbitrary} & \text{if } k = 0, \\ \frac{1}{H_0} \sqrt{\frac{-k}{\Omega_k}} & \text{if } k \neq 0, \end{cases} \quad (\text{A6})$$

$$M_0 = \Omega_m H_0^2 S_0^3 / 2, \quad (\text{A7})$$

where Ω_m is the baryonic plus dark matter fraction. The integrated density equation is

$$\mathcal{M} = \frac{3M_0 k}{2} \left(\frac{\sin^{-1}(\sqrt{k} r)}{\sqrt{k}} - r \sqrt{1 - kr^2} \right), \quad (\text{A8})$$

so for nearly flat models, where $r \ll 1$,

$$(\mathcal{M} - M)/M \approx 0.3kr^2. \quad (\text{A9})$$

The pnc and AH equations (5), (8), (9), (15) and (18), using $S_m = S(\hat{r}(r_m))$, become

$$\hat{r}' = \frac{-S}{\sqrt{1 - kr^2}}, \quad (\text{A10})$$

$$\frac{dR_n}{dz} = \left[\frac{\sqrt{1 - kr^2} - r\dot{S}}{H(1+z)} \right]_n, \quad \left. \frac{dR_n}{dz} \right|_{z=0} = \frac{1}{H_0}, \quad (\text{A11})$$

$$\Lambda r_m^3 \hat{S}_m^3 - 3r_m \hat{S}_m + 6M_0 r_m^3 = 0, \quad (\text{A12})$$

$$\frac{6M_0}{\kappa \hat{S}^3 (1+z)^2} (2r \sqrt{1 - kr^2} \dot{S} + r^2 (S\ddot{S} - \dot{S}^2)) = 0. \quad (\text{A13})$$

Figs A1 and A2 show $\hat{R}(z)$ and $\mu n(z)$ for a range of H_0 , Ω_Λ and Ω_m values. The primary interest is on how these parameters affect the maxima.

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