THE MASS-PARTICLE IN AN EXPANDING UNIVERSE.

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I. *Introduction and Summary*

In the astronomical applications of General Relativity two types of metric for the universe are used. For discussing the motion of planets round the Sun, the statical Schwarzschild metric is employed, which may be written in isotropic co-ordinates * as

$$ds^2 = \left(\frac{1 - m/2r_1}{1 + m/2r_1}\right)^2 dt^2 - \frac{1}{c^2}\left(1 + \frac{m}{2r_1}\right)^4 \{dr_1{}^2 + r_1{}^2(d\theta^2 + \sin^2\theta d\phi^2)\}; \qquad (1)$$

on the other hand, for dealing with the phenomenon of the recession of the spiral nebulæ non-statical metrics are used, which can be subdivided into two classes : the Lemaître class, in which

$$ds^2 = dt^2 - \frac{e^{\beta(t)}}{c^2}\left\{\frac{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)}{(1 + r^2/4R^2)^2}\right\}, \qquad (2)$$

and the de Sitter class, in which

$$ds^2 = dt^2 - \frac{e^{\beta(t)}}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \qquad (3)$$

In (2) the constant $1/R^2$, which may be positive or negative, gives the curvature of space as a whole, local irregularities being disregarded. In (3) the curvature of space is zero. A particular case of (3) is the original de Sitter universe, for which $\beta(t) = t/a$ where a is a constant.†

One important respect in which the metric (1) differs from (2) and (3) is that in the former the co-ordinate r_1 is what we shall call an "observer's" co-ordinate, *i.e.* it is one based on the assumption that the distance between two points at relative rest is independent of the time. In (2) and (3) the co-ordinate r is one which will be called "cosmical." It is used when the system of nebulæ is taken as the basis of reference. An observer in the field uses a co-ordinate $r_1 = re^{\frac{\beta(t_1)}{2}}$ at the instant t_1; so that the "observer's" co-ordinate for a fixed value of r is not independent of the time.

When we consider that we are compelled to observe the universe from the near neighbourhood of a mass-particle (the Sun) it becomes of some interest to find a form of metric which will reduce to the statical metric (1) in terms of observer's co-ordinates, but which can also be expressed, approximately at least, in one of the forms (2) and (3) when cosmical co-ordinates are used. Some solutions of the problem have already been proposed. G. Lemaître ‡

* A. S. Eddington, *Math. Th. of Relativity*, 1924, § 43.
† H. P. Robertson, *Nat. Acad. Sci. Proc.*, **15**, 822–829, 1929.
‡ G. Lemaître, *M.N.*, **91**, 490–501, 1931.

has put forward one type of metric. Unfortunately his solution appears to depend on the assumption that the pressure of the matter outside the mass-particle is negative if the density is positive or *vice versa*. This condition is true whatever the co-ordinate system used. It is difficult to see how such a case could be applied to the actual universe unless, indeed, both pressure and density were zero. But this reduces Lemaître's result to the well-known one of a mass-particle in an, otherwise empty, de Sitter universe.* An alternative type of metric put forward by Dr. W. H. McCrea and the author † is also open to criticism on the ground that it implies that the matter in the universe outside the Sun is flowing toward it with a high velocity which is certainly not observed.‡

Hitherto it has been assumed that the problem can be solved by the choice of any set of "polar" co-ordinates, with origin at the mass-particle, that happen to yield some mathematical simplification of the equations involved. It is therefore hardly surprising that the resulting metrics do not apply to the actual universe. In the present investigation we have started from the point of view that an observer is necessarily constrained, by the methods he uses for setting up a co-ordinate system, to those systems in which the metric of space-time is expressed in an isotropic and orthogonal form. The predictions of any theory must, therefore, be calculated with respect to such a system before they can be compared with the observer's measurements.

We shall suppose that our observer takes into account the gravitational effect of the matter in the universe, but outside the mass-particle, by assuming that it is evenly spread out through space as if it were a gas. It will thus be characterised by (1) its density, T_4^4; (2) its pressure-components, T_1^1, T_2^2, T_3^3; and (3) its momentum, T_{14}, if it is flowing towards or away from the mass-particle. These are the only non-zero components of the energy-tensor when spherical symmetry is assumed. The pressure is, of course, due to the random velocities of the particles constituting the gas. Both density and pressure may, however, be in part due to the presence of radiation. It appears to us axiomatic that, in any space-time model applicable to the actual universe, the density and pressure cannot be negative but may, of course, be zero in a first approximation.

The solution of the problem now follows if the observer makes the general assumption that the mass-particle does not occupy a peculiar point in the distribution of matter in the universe. This leads him to conclude that, firstly, the pressure is everywhere isotropic, and secondly, that the matter is "at rest" with respect to his co-ordinate system. By this is meant that it has, on the whole, zero co-ordinate velocity, and therefore zero momentum, in his system. It might be thought that, in the actual universe, this could not be true because of the phenomenon of the "recession" of the nebulæ. But it must be emphasized that we do not *observe* a velocity in this connection,

* A. S. Eddington, *loc. cit.*, § 45.

† W. H. McCrea and G. C. McVittie, *M.N.*, **91**, pp. 128–133, 1930; *ibid.*, **92**, 7–12, 1931.

‡ G. C. McVittie, *M.N.*, **92**, 512 *et seq.*, 1932.

but only a shift to the red of the lines of the spectrum in the light emitted by distant objects. It is the whole object of the "expanding universe" theory to show that *even if* an observer assigns zero velocity to the nebulæ at each instant, yet this red-shift will be observed owing to the properties of space.

In arriving at the generalized Schwarzschild field we do not employ observer's co-ordinates, in the first instance. Instead we obtain the metric, using cosmical co-ordinates analogous to those used in (2) and (3). We then show that the metric can be transformed into the statical field (1) independently of the moment in time at which the transformation is made. This possibility turns out to be closely bound up with the condition that the matter in the universe has zero velocity, on the average, in the observer's system. If the mass-particle has a planetary system we show that it appears to be permanent and unchanging to the observer. The distant nebulæ, meanwhile, exhibit the recession phenomenon. We also point out that the use of cosmical co-ordinates, for dealing with this phenomenon, in no way implies that there is a "cosmical" time common to all observers. Lastly, we give reasons for believing that the observer must conclude that the cosmical constant λ cannot be zero or negative, and we discuss the present position of the theory of the disturbance of the equilibrium of the Einstein universe by the formation of mass-particles.

I wish to express my thanks to Sir Arthur S. Eddington for some helpful criticisms and suggestions.

II. *Equations Determining the Metric of Space-time*

We consider an observer engaged in setting up a co-ordinate system in the neighbourhood of a mass-particle (*e.g.* the Sun), which he takes as his origin of spatial co-ordinates. He is provided with rigid measuring rods and makes use of light-triangulations for dealing with points he cannot reach with his measuring rods. He works under the assumptions made by terrestrial observers, viz. :

(*a*) The length of a measuring rod is constant in time and independent of orientation around a given point.

(*b*) The backwards and forwards velocity of light between any two points is the same.

(*c*) The velocity of light is the same in every direction around a given point.

Under these circumstances he sets up an "observer's" co-ordinate system of the orthogonal and isotropic type,* in terms of which he expresses the metric applicable to the whole universe. In order to determine the coefficients of this metric he will have to make some assumptions regarding the distribution of matter in the universe. If he believes that his part of the universe is similar to every other part (except for the "singularity" corresponding to the mass-particle), he will be entitled to assume the following:—

(*d*) The matter in the universe is distributed with spherical symmetry around the origin where there is a mass-particle.

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* A. S. Eddington, *loc. cit.*, § 43.

(*e*) There is no flow of the matter as a whole either towards or away from the origin—otherwise it would be necessary to postulate that, at some time or other, the neighbourhood of the origin had been the scene of an explosion (for instance) great enough to set the matter in the whole universe in motion away from that point. Until some physical process, capable of producing an upheaval of such magnitude, is discovered, our observer will be constrained to postulate (*e*).

(*f*) At any point in the universe the pressure is isotropic. This seems a natural consequence of (*e*) since there is now no preferential direction towards which the velocities of the particles, or the flow of radiation, might be directed.

Our object is now to find, by means of Einstein's equations, the metric which the observer assigns to the universe in this way. We shall not, however, determine it in terms of observer's co-ordinates directly, but instead find a metric which satisfies the requirements (*d*) to (*f*) in terms of isotropic cosmical co-ordinates. We shall then show that, on transforming this metric into observer's co-ordinates, the properties (*a*) to (*f*) are all found to hold. We deduce that this metric is the one actually used by our observer.

Consider, then, the most general form of metric which is orthogonal, isotropic in the space co-ordinates and which expresses the condition for spherical symmetry around the origin. Using cosmical co-ordinates, it can be written as

$$ds^2 = e^{\zeta(r,t)}dt^2 - \frac{e^{\nu(r,t)}}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \tag{4}$$

The distribution of density and pressure is given by Einstein's equations

$$G_{ik} - \lambda g_{ik} = -\kappa c(T_{ik} - \tfrac{1}{2}g_{ik}T), \tag{5}$$

where G_{ik} is the Riemann-Christoffel tensor; λ the cosmical constant; T_{ik} the energy-tensor; T the proper density, and κ is a constant related to the gravitational constant, γ, and the velocity of light, c, by

$$\frac{\kappa}{c} = \frac{8\pi\gamma}{c^4} = 2 \times 10^{-48} \text{ c.g.s units.}$$

We write

$$T_4^4 = \rho; \qquad T_1^1 = -p_1; \qquad T_2^2 = T_3^3 = -p_2;$$

so that

$$T = \rho - p_1 - 2p_2.$$

The equations (5) reduce to

$$-\frac{\kappa c}{2}(\rho + p_1 + 2p_2) + \lambda = e^{-\zeta}\{\tfrac{3}{2}\ddot{\nu} + \tfrac{3}{4}(\dot{\nu})^2 - \tfrac{3}{4}\dot{\zeta}\dot{\nu}\} - c^2e^{-\nu}\left\{\tfrac{1}{2}\zeta'' + \tfrac{1}{4}(\zeta')^2 + \frac{1}{r}\zeta' + \tfrac{1}{4}\zeta'\nu'\right\}, \tag{6}$$

$$-\frac{\kappa c}{2}(2p_2 - p_1 - \rho) + \lambda = e^{-\zeta}\{\tfrac{1}{2}\ddot{\nu} + \tfrac{3}{4}(\dot{\nu})^2 - \tfrac{1}{4}\dot{\nu}\dot{\zeta}\} - c^2e^{-\nu}\left\{\tfrac{1}{2}\zeta'' + \tfrac{1}{4}(\zeta')^2 - \tfrac{1}{4}\zeta'\nu' + \nu'' + \frac{1}{r}\nu'\right\}, \tag{7}$$

$$-\frac{\kappa c}{2}(p_1-\rho)+\lambda=e^{-\zeta}\{\tfrac{1}{2}\ddot{\nu}+\tfrac{3}{4}(\dot{\nu})^2-\tfrac{1}{4}\dot{\nu}\dot{\zeta}\}$$
$$-c^2e^{-\nu}\left\{\tfrac{1}{2}\nu''+\frac{3}{2r}\nu'+\tfrac{1}{4}(\nu')^2+\frac{1}{2r}\zeta'+\tfrac{1}{4}\nu'\zeta'\right\}, \qquad (8)$$
$$-\kappa cT_{14}=\dot{\nu}'-\tfrac{1}{2}\dot{\nu}\zeta', \qquad (9)$$

where a dot denotes differentiation with respect to t and a dash differentiation with respect to r.

The two conditions (e) and (f) are now expressed by

$$T_{14}=0 \quad\text{and}\quad p_1=p_2.$$

These lead, together with (7) and (8), to our two fundamental equations for determining the coefficients of the metric (4), viz.:

$$\dot{\nu}'-\tfrac{1}{2}\dot{\nu}\zeta'=0, \qquad (10)$$
$$\zeta''+\nu''-\frac{1}{r}(\nu'+\zeta')-\nu'\zeta'-\tfrac{1}{2}(\nu')^2+\tfrac{1}{2}(\zeta')^2=0. \qquad (11)$$

III. *Solution of the Equations*

We shall now show that ν and ζ can be determined by the use of (10) and (11) alone.

The equation (10) can be solved immediately. Dividing throughout by $\dot{\nu}$, we can write it as

$$\frac{\partial(\log\dot{\nu})}{\partial r}=\frac{1}{2}\frac{\partial\zeta}{\partial r},$$

whence

$$\dot{\nu}=\alpha(t)e^{\frac{1}{2}\zeta(r,t)}.$$

Hence finally

$$\nu=\int\alpha(t)e^{\frac{1}{2}\zeta(r,t)}dt+a(r), \qquad (12)$$

where, in the integral, r is treated as a constant and $a(r)$ is a function of r alone.

We shall now show that the generalized Schwarzschild field we are seeking cannot be included amongst those solutions for which ζ is a function of r alone. For, in such a case,

$$\nu=\beta(t)e^{\frac{1}{2}\zeta(r)}+a(r),$$

where

$$\beta=\int\alpha(t)dt.$$

Substitution into (11) shows that we can have two possibilities: (1) β is constant; in which case we can only arrive at the statical Schwarzschild case (1); or (2) β is arbitrary, ζ is constant, and $a(r)$ is given by

$$\frac{d^2a}{dr^2}-\frac{1}{r}\frac{da}{dr}-\frac{1}{2}\left(\frac{da}{dr}\right)^2=0,$$

whence

$$a=-2\log\left(1+\frac{r^2}{4R^2}\right). \qquad (13)$$

This merely brings us back to the cases (2) and (3). It therefore appears that there is no generalization of the Schwarzschild metric (in terms of cosmical co-ordinates) in which the mass of the Sun enters as a constant independent of the time.

Turning to solutions of (10) in which ζ is a function of both r and t, we consider first cases in which $a(r)=0$. This function is evidently dependent on the curvature of space as a whole, so that putting it equal to zero is equivalent to dealing with metrics analogous to (3). We shall show later that it is easy to extend the results we obtain to cases when $a(r)$ is of the form (13).

The solution we require must have a singularity at the origin similar to that possessed by (1). ζ and ν must therefore be expressible as power series in $1/r$. We assume

$$e^{\frac{1}{2}\zeta}=\gamma=1+\alpha_1 u^{m_1}+\alpha_2 u^{m_2}+\alpha_3 u^{m_3}+\ldots, \tag{14}$$

where $u=\dfrac{1}{r}$, the α_s are functions of t and the powers of u are arranged in ascending order. Substitution into (12) yields

$$\nu=\beta(t)+\sum_{s=1}^{\infty}u^{m_s}\beta_s(t), \tag{15}$$

where

$$\beta(t)=\int\alpha(t)dt, \tag{16}$$

$$\beta_s(t)=\int\alpha(t)\alpha_s(t)dt. \tag{17}$$

The equation (11) becomes, on changing the independent variable from r to u and substituting for ζ in terms of γ,

$$\gamma\left(u\frac{\partial^2\nu}{\partial u^2}+3\frac{\partial\nu}{\partial u}\right)+2u\frac{\partial^2\gamma}{\partial u^2}+\left(6-2u\frac{\partial\nu}{\partial u}\right)\frac{\partial\gamma}{\partial u}-\tfrac{1}{2}u\gamma\left(\frac{\partial\nu}{\partial u}\right)^2=0. \tag{18}$$

We now substitute from (14) and (15) into (17). The lowest power of u turns out to be u^{m_1-1}. On equating its coefficient to zero we obtain

$$m_1(m_1+2)(2\alpha_1+\beta_1)=0. \tag{19}$$

The next two powers of u are u^{2m_1-1} and u^{m_2-1}. Hence we have $m_2=2m_1$ and, in general, we find $m_s=sm_1$. It therefore follows from the indicial equation (19) that the only way in which we can obtain a solution as a power series in u which includes the first power of u, is by taking $m_1=1$ and

$$2\alpha_1+\beta_1=0. \tag{20}$$

This relation, together with (16) and (17), gives

$$\frac{\dot{\alpha}_1}{\alpha_1}=-\tfrac{1}{2}\dot{\beta}. \tag{21}$$

The coefficient of u^{2m_1-1} when equated to zero gives, on using (20),

$$2\alpha_2+\beta_2=\tfrac{1}{2}\alpha_1{}^2=c_2\alpha_1{}^2.$$

Differentiating this relation with respect to t and using (16), (17) again, it follows that

$$\alpha_2 = c_2\alpha_1^2 \quad \text{and} \quad \beta_2 = -c_2\alpha_1^2.$$

Proceeding in this manner it soon becomes apparent that

$$\alpha_n = c_n\alpha_1^n \quad \text{and} \quad \beta_n = -\frac{2c_n}{n}\alpha_1^n,$$

where the c_n are constants. Hence, if the solution we are seeking exists at all, it must be of the form

$$\gamma = 1 + \sum_{s=1}^{\infty} c_s\left(\frac{\mu(t)}{r}\right)^s, \tag{22}$$

$$\nu = \beta(t) - 2\sum_{s=1}^{\infty}\frac{c_s}{s}\left(\frac{\mu(t)}{r}\right)^s, \tag{23}$$

with

$$\tfrac{1}{2}\dot{\beta} = -\frac{\dot{\mu}}{\mu}.$$

In the above we have returned to the variable r and have written μ for α_1.

We can now show that γ, ν given by (22) and (23) can be expressed in finite form. For these series imply

$$\frac{\partial\nu}{\partial r} = \frac{2(\gamma-1)}{r} \quad \text{and} \quad \frac{\partial\nu}{\partial t} = -\frac{2\dot{\mu}}{\mu}\gamma\,; \tag{24}$$

hence (11) becomes

$$r^2\frac{\partial^2\gamma}{\partial r^2} - r(\gamma-1)\frac{\partial\gamma}{\partial r} - \gamma(\gamma^2-1) = 0. \tag{25}$$

We solve this equation as if γ were a function of r alone and then treat the "constants of integration" as functions of t. If we put

$$r = e^x$$

we obtain

$$\frac{d^2\gamma}{dx^2} - \gamma\frac{d\gamma}{dx} - \gamma(\gamma^2-1) = 0. \tag{26}$$

A particular first integral is obviously

$$\frac{d\gamma}{dx} = \gamma^2 - 1. \tag{27}$$

The general first integral is found as follows. Put $z = \gamma^2 - 1$ and assume $\frac{d\gamma}{dx} = z(1+w)$. Then (26) reduces to

$$z\frac{dw}{dz} = -\frac{w(2w+3)}{2(1+w)},$$

the integral of which is

$$w^2(2w+3) = \frac{A^3}{z^3}$$

or

$$\left(\frac{dy}{dx}-y^2+1\right)^2\left(2\frac{dy}{dx}+y^2-1\right)=A^3. \tag{28}$$

The expression (22) for y involves one "arbitrary constant," $\mu(t)$, with respect to integrations by r. It must, therefore, be obtained by means of a particular first integral of (26). We notice from (28) that the particular first integral (27) is the singular solution of that equation. The particular first integral we require is, therefore, the alternative one provided by (28), viz.:

$$2\frac{dy}{dx}+y^2-1=0,$$

which gives

$$y=\frac{1-\frac{\mu}{2}e^{-x}}{1+\frac{\mu}{2}e^{-x}},$$

$\frac{\mu}{2}$ being the "constant" of integration.

Reverting to the variable r we have

$$y=\frac{1-\frac{\mu(t)}{2r}}{1+\frac{\mu(t)}{2r}},$$

where μ is now regarded as an arbitrary function of t. The first of the relations (24) then gives, on integration,

$$\nu=\beta(t)+4\log\left(1+\frac{\mu(t)}{2r}\right),$$

whilst the second is satisfied if

$$\tfrac{1}{2}\dot{\beta}=-\frac{\dot{\mu}}{\mu}. \tag{28a}$$

We can therefore say that, in terms of cosmical co-ordinates, the Schwarzschild field has the form

$$ds^2=\left(\frac{1-\mu(t)/2r}{1+\mu(t)/2r}\right)^2dt^2-\frac{1}{c^2}\left(1+\frac{\mu(t)}{2r}\right)^4e^{\beta(t)}\{dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\}, \tag{29}$$

where $\frac{1}{2}\dot{\beta}=-\frac{\dot{\mu}}{\mu}$, and μ is identified with the mass of the particle at the origin. The curvature of space is here supposed to be zero.

This result can be generalized to take account of the curvature of space when this is different from zero. For it is evident that in a small region near the origin in which the curvature of space is negligible, the field must be given approximately by (29), whilst in distant regions it must be (2). Hence y, ν must be of the forms

$$e^{\frac{1}{2}\zeta} = \gamma = \frac{1-y}{1+y}, \qquad \nu = \beta + 4\log(1+y) + a(r), \tag{30}$$

where y is a function of (r, t) and $a(r) = -2\log(1 + r^2/4R^2)$.

Substitution into (11) gives

$$\frac{\partial^2 y}{\partial r^2} - \frac{3}{y}\left(\frac{\partial y}{\partial r}\right)^2 - \frac{\partial y}{\partial r}\left(\frac{1}{r} + \frac{da}{dr}\right) = 0,$$

whence

$$\frac{1}{y^3}\left(\frac{\partial y}{\partial r}\right) = \frac{B(t)r}{(1 + r^2/4R^2)^2}$$

and

$$\frac{1}{y^2} = C(t) + \frac{4R^2 B(t)}{(1 + r^2/4R^2)},$$

where B, C are constants of integration with respect to r and so may be functions of t.

If we choose

$$C = 16R^2/\mu^2 \quad \text{and} \quad B = -4/\mu^2,$$

where μ is an arbitrary function of t, we obtain

$$y = \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right), \tag{31}$$

and this is easily seen to be the only value of y which reduces to $\mu/2r$ in regions where r/R is negligible. Corresponding to (31) we have, by (30),

$$\nu = \beta(t) + 4\log\left\{1 + \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right)^{\frac{1}{2}}\right\} - 2\log\left(1 + \frac{r^2}{4R^2}\right), \tag{32}$$

$$\zeta = 2\log\left\{\frac{1 - \dfrac{\mu(t)}{2r}\left(1 + \dfrac{r^2}{4R^2}\right)^{\frac{1}{2}}}{1 + \dfrac{\mu(t)}{2r}\left(1 + \dfrac{r^2}{4R^2}\right)^{\frac{1}{2}}}\right\}. \tag{33}$$

Direct substitution of these values into (10) gives, as before,

$$\frac{\dot{\mu}}{\mu} = -\tfrac{1}{2}\dot{\beta}.$$

Thus finally: when the curvature of space is not zero, the Schwarzschild field in cosmical co-ordinates has the form

$${}^2 = \left\{\frac{1 - \dfrac{\mu(t)}{2r}\left(1 + \dfrac{r^2}{4R^2}\right)^{\frac{1}{2}}}{1 + \dfrac{\mu(t)}{2r}\left(1 + \dfrac{r^2}{4R^2}\right)^{\frac{1}{2}}}\right\}^2 dt^2 - \frac{\left\{1 + \dfrac{\mu(t)}{2r}\left(1 + \dfrac{r^2}{4R^2}\right)^{\frac{1}{2}}\right\}^4}{\left(1 + \dfrac{r^2}{4R^2}\right)^2} \frac{e^{\beta(t)}}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \tag{34}$$

with

$$\frac{\dot{\mu}}{\mu} = -\tfrac{1}{2}\dot{\beta}.$$

It should be noted that (34) gives the field of a particle which has neither a similar particle nor a "hole" at its antipodes.* For it is obvious that as $r \to \infty$, the metric (34) tends to the form

$$ds^2 = \left(\frac{1 - \frac{\mu}{4R}}{1 + \frac{\mu}{4R}}\right)^2 dt^2 - \frac{\left(1 + \frac{\mu}{4R}\right)^4}{\left(1 + \frac{r^2}{4R^2}\right)^2} \frac{e^\beta}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\},$$

which differs from the Lemaître solution (2) only in terms of μ/R, which are negligibly small when μ is the mass of a star similar to the Sun and R is of the order of the radius of space. Hence there is no singularity as $r \to \infty$.

IV. *Planetary Systems*

We consider the reduction of the metrics found in the previous paragraphs to observer's co-ordinates. From our present point of view, we can neglect the curvature of space and take as the expression of the Schwarzschild field in cosmical co-ordinates the result (29) viz.:

$$ds^2 = \left(\frac{1 - \mu(t)/2r}{1 + \mu(t)/2r}\right)^2 dt^2 - \left(1 + \frac{\mu(t)}{2r}\right)^4 \frac{e^{\beta(t)}}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \tag{35}$$

At any instant of time t_1 the observer's co-ordinate for measuring distance from the origin is

$$r_1 = e^{\frac{\beta(t_1)}{2}} r.$$

If we write

$$m = \mu(t_1) e^{\frac{\beta(t_1)}{2}},$$

the above metric becomes

$$ds^2 = \left(\frac{1 - m/2r_1}{1 + m/2r_1}\right)^2 dt^2 - \frac{1}{c^2}\left(1 + \frac{m}{2r_1}\right)^4 \{dr_1^2 + r_1^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \tag{36}$$

and thus appears to the observer, at the instant t_1, as the usual statical field (1). But, in virtue of (28*a*), the derivative of m as a function of t is zero. Hence the reduction of (35) to (36) is independent of the instant in time at which the transformation is made and the field of the mass-particle will therefore be considered as strictly statical by the observer.

Again, if we compute the components, $(T_{ik})^*$, of the energy tensor for the matter outside the mass-particle in terms of the observer's system of co-ordinates (r_1, t, θ, ϕ) we find

$$\left.\begin{aligned} (T_4^4)^* = T_4^4 = \rho;\ (T_1^1)^* = T_1^1 = -p_1;\ (T_2^2)^* = (T_3^3)^* = T_2^2 = T_3^3 = -p_2 \\ (T_{14})^* = e^{-\frac{\beta(t_1)}{2}} T_{14}. \end{aligned}\right\} \tag{37}$$

* This is in contrast to all the cases previously studied by Dr. W. H. McCrea and the author. See *M.N.*, **92,** 7–12, 1931.

Hence, since $p_1 = p_2$ and $T_{14} = 0$, the conditions (*a*) to (*f*) of paragraph II all hold in the observer's system. He concludes that the system of nebulæ is "at rest" in his system in the sense explained in the Introduction. He is therefore justified in using a set of cosmical co-ordinates, as we have done, when it is mathematically convenient. The use of such a system, it should be noticed, does not imply the existence of a "cosmical time" common to all observers in the universe.* For we can suppose the observer to set up first of all the system (36) in which he chooses the "time" to suit his local circumstances without reference to the "times" of other observers. It is then this purely local time that he employs if he decides to introduce the cosmical co-ordinates in which the metric takes the form (35).

In the case of our Sun, the terms m/r_1 in (36) are already negligibly small at the distance of the Earth. If our observer is near a mass-particle of stellar mass, the metric (35) rapidly becomes indistinguishable from (2) as we recede from the origin. The observer will thus be able to account for the "recession" of a distant nebulæ by assigning a fixed value to the cosmical co-ordinate r of the nebula and interpreting the red-shift of the lines of the spectrum as due to the change in the co-ordinate r_1 during the time the light travels to him from the nebula. In this way he finds that, at the present moment,

$$\dot{\beta} = 2 \times 10^{-17} \text{ sec.}^{-1}.$$

We shall now consider planetary orbits in terms of the cosmical co-ordinates in which the metric is (35). There are, of course, the closed geodesics around the origin. The equations of any geodesic are

$$\frac{d^2x^\alpha}{ds^2} + \{\mu\nu, \alpha\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \qquad (\alpha = 1, 2, 3, 4),$$

where x^α stands for any one of (r, t, θ, ϕ) and the $\{\mu, \nu, \alpha\}$ are the Christoffel symbols with respect to (35). Direct calculation of the equation for θ shows that if, initially,

$$\frac{d\theta}{ds} = 0, \qquad \frac{d^2\theta}{ds^2} = 0, \qquad \theta = \tfrac{1}{2}\pi,$$

then the geodesic will always be in the plane $\theta = \frac{1}{2}\pi$. The problem of determining the orbit of the planet thus reduces to finding the (r, ϕ) equation of the geodesics in a plane whose metric is

$$ds^2 = e^{\zeta(r,t)}dt^2 - \frac{e^{\nu(r,t)}}{c^2}\{dr^2 + r^2d\phi^2\}, \tag{38}$$

where ν, ζ are given by

$$\zeta = 2 \log\left(\frac{1 - \mu/2r}{1 + \mu/2r}\right), \qquad \nu = \beta(t) + 4 \log(1 + \mu/2r). \tag{39}$$

To do this we write

$$\sigma = is, \qquad \tau = it,$$

* This has been contended by E. A. Milne, *Nature*, **130**, 9–10, 1932 July 2, and pp. 507–508, 1932 October 1.

so that (38) can be written

$$1 = e^{\zeta(r,\tau)}\left(\frac{d\tau}{d\sigma}\right)^2 + \frac{e^{\nu(r,\tau)}}{c^2}\left\{\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2\right\}. \tag{40}$$

We put

$$L = \tfrac{1}{2}\left[e^{\zeta}\left(\frac{d\tau}{d\sigma}\right)^2 + \frac{e^{\nu}}{c^2}\left\{\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2\right\}\right].$$

We can then regard L as the kinetic potential of a dynamical system moving under no forces, in which σ plays the part of the time and (40) is the integral of energy. Our problem is then equivalent to finding the equations of motion of this system with ϕ instead of σ as independent variable, using the integral of energy (40). The details of the method will be found elsewhere.* The result is that the equations of motion are expressible in terms of a function

$$L' = \left[e^{\zeta}\left(\frac{d\tau}{d\phi}\right)^2 + \frac{e^{\nu}}{c^2}\left\{\left(\frac{dr}{d\phi}\right)^2 + r^2\right\}\right]^{\frac{1}{2}} \tag{41}$$

in the usual form

$$\frac{d}{d\phi}\left(\frac{\partial L'}{\partial r'}\right) - \frac{\partial L'}{\partial_r} = 0; \qquad \frac{d}{d\phi}\left(\frac{\partial L'}{\partial \tau'}\right) - \frac{\partial L'}{\partial \tau} = 0, \tag{42}$$

where

$$r' = \frac{dr}{d\phi}, \qquad \tau' = \frac{d\tau}{d\phi}.$$

An integral of these equations is

$$\tau'\frac{\partial L'}{\partial \tau'} + r'\frac{\partial L'}{\partial r'} - L' = -\frac{ih}{c}, \tag{43}$$

where h is a (real) constant. This equation becomes, on substituting from (41) and squaring both sides,

$$e^{\zeta}\left(\frac{d\tau}{d\phi}\right)^2 + \frac{e^{\nu}}{c^2}\left\{\left(\frac{dr}{d\phi}\right)^2 + r^2\right\} = -\frac{r^4 e^{2\nu}}{h^2 c^2}. \tag{44}$$

On using (40) and returning to the variables t, s we obtain

$$r^2 e^{\nu(r,t)}\frac{d\phi}{ds} = hc, \tag{45}$$

so that (43) is the "integral of angular momentum" for the motion of the planet in the field (35).

The equation of the orbit is the first of the equations (42). If we put $u = 1/r$ and use (44) we get, after some reduction,

$$\frac{d^2u}{d\phi^2} + u = -\frac{e^{\nu(r,t)}\zeta_u}{2h^2} + \frac{u^2(\nu_u - \zeta_u)}{2} - \frac{(\zeta_u - \nu_u)}{2}\left(\frac{du}{d\phi}\right)^2, \tag{46}$$

where ν_u, ζ_u denote derivatives with respect to u.

* E. T. Whittaker, *Analytical Dynamics*, § 42, 3rd ed., 1927.

We consider the "Newtonian" approximation to (46) when ζ, ν are given by (39). The result is

$$\frac{d^2u}{d\phi^2}+u=\frac{e^{\beta(t)}\mu(t)}{h^2}. \tag{47}$$

If now the planet concerned has a period short compared with the rate of change of β and moves with a velocity small compared with that of light, we can take the integral of (47) to be

$$u=\frac{e^{\beta(t)}\mu(t)}{h^2}(1+\epsilon\cos\phi), \tag{48}$$

where ϵ is the eccentricity of the orbit. Suppose ϵ is fixed; then the value of r, for a given ϕ, will change at a rate

$$\frac{1}{r^2}\frac{dr}{dt}=-\frac{e^{\beta(t)}\dot{\beta}\mu}{2h^2}(1+\epsilon\cos\phi)$$

since $\dot{\beta}=-2\dot{\mu}/\mu$, and h is an absolute constant by (45). Also $\dot{\beta}$ is positive. Thus in terms of cosmical co-ordinates the orbit of the planet continually shrinks and, from (28*a*), it follows that the mass of the central particle continually decreases. Hence, from the point of view of cosmical co-ordinates, the configuration of the nebulæ is unchanging, but the mass-particle and its planetary system perpetually shrink in size.*

On the other hand, from the observer's point of view, the orbit of the planet is (at the instant t_1)

$$\frac{1}{r_1}=\frac{m}{h^2}(1+\epsilon\cos\phi)$$

and the mass, m, of the central particle is constant. Hence, for the observer, the orbit of the planet and the mass of the central particle remain fixed, whilst the system of the nebulæ increases in size.

It should be remarked that this result turns on the relation (28*a*) which itself follows from the assumption that the system of the nebulæ has, on the whole, zero momentum in the observer's system. If, as has been suggested by E. A. Milne,† the "recession" were due to the nebulæ having true co-ordinate velocities directed away from the observer, the component (T_{14})* of the energy-tensor could certainly not be zero in the observer's co-ordinate system. It is difficult to see how the "recession" could then fail to have observable repercussions on planetary systems,‡ even if the serious difficulty of endowing the nebulæ with sufficient momentum to produce velocities corresponding to the observed enormous red-shifts is overlooked.

* A. S. Eddington, *Phys. Soc. Proc.*, 44, 1–17, 1932 January 1.

† E. A. Milne, *loc cit.*

‡ See in this connection G. C. McVittie, *M.N.*, 92, 500–518, 1932, where the theory of a particle with constant mass in the cosmical co-ordinate system is worked out.

V. *The Pressure and Density and the Cosmical Constant*

Returning, for a moment, to cosmical co-ordinates we calculate the density and pressure outside the mass-particle in the field (34). Since the pressure is now isotropic, we put $p_1 = p_2 = p$. Solving the equations (6) and (7) or (8) for ρ and p, we obtain

$$\kappa c\rho = -\lambda + \tfrac{3}{4}e^{-\zeta}(\dot{\nu})^2 - c^2 e^{-\nu}\left\{\nu'' + \frac{2}{r}\nu' + \tfrac{1}{4}(\nu')^2\right\}, \tag{49}$$

$$\kappa cp = \lambda - e^{-\zeta}\{\ddot{\nu} + \tfrac{3}{4}(\dot{\nu})^2 - \tfrac{1}{2}\dot{\zeta}\dot{\nu}\} + \frac{c^2 e^{-\nu}}{4}\left\{\zeta'' + \nu'' + \frac{3}{r}(\zeta' + \nu') + \tfrac{1}{2}(\zeta' + \nu')^2\right\}, \tag{50}$$

where ζ, ν are given by (33) and (32) respectively.

Suppose we denote by ρ^* and p^* the values of the density and pressure calculated in the observer's system at the instant t_1. The result (37) shows that ρ^*, p^* are obtained from ρ, p simply by the substitutions $r = r_1 e^{-\frac{\beta(t_1)}{2}}$, $\mu = me^{-\frac{\beta(t_1)}{2}}$. Using (32), (33), (49) and (50) we obtain, after some calculation,

$$\kappa c\rho^* = -\lambda + \tfrac{3}{4}\dot{\beta}_1^2 + \frac{3c^2}{R^2}\left\{1 + \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\right\}^{-4} - \frac{3c^2}{R^2}\frac{m}{r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\left\{1 + \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\right\}^{-5},$$

$$\kappa cp^* = \lambda - \frac{1 + \dfrac{m}{2r_1}\left(1 + \dfrac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}}{1 - \dfrac{m}{2r_1}\left(1 + \dfrac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}}\ddot{\beta}_1 - \tfrac{3}{4}\dot{\beta}_1^2 - \frac{c^2}{R^2}\left\{1 + \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\right\}^{-4},$$

where $\ddot{\beta}_1$, $\dot{\beta}_1$ are the values $\ddot{\beta}$, $\dot{\beta}$ at the instant t_1 and $R_1 = e^{\frac{\beta(t_1)}{2}}R$ is the radius of space at that moment.

It has been suggested by certain investigators * that the cosmical constant, λ, which appears in these formulæ is merely a mathematical device and that its value is indifferent from the physical point of view. They propose putting $\lambda = 0$. This cannot be done, however, without introducing difficulties with regard to the expansion. For suppose $\lambda = 0$. Then it follows, since ρ^*, p^* cannot be negative, that $\ddot{\beta}_1 < 0$ whatever $\dot{\beta}_1$ and $1/R^2$ may be. Moreover, this holds for all instants t_1. But our observer has found from his observations of the nebulæ that $\dot{\beta}_1 > 0$. Thus he must conclude that the expansion is proceeding subject to a retardation. Exactly the same result follows if $\lambda < 0$. Hence, in either case, he must conclude, firstly, that at some time in the past the expansion started instantaneously with a finite velocity; secondly, that there is a "retarding force" slowing up the expansion which, obviously, cannot be the initial cause that started the

* A. Einstein, *Preuss. Akad. Wiss. Berlin, Ber.*, **12**, 235–237, 1931; A. Einstein and W. de Sitter, *Nat. Acad. Sci. Proc.*, **18**, 213–214, 1932.

latter. No attempt has been made to account for these properties of the expansion, nor is it easy to imagine how they could arise. We are driven to the conclusion that our observer would necessarily take λ to be a positive constant. If he does so he is at once presented with a familiar type of mechanism for starting the expansion: the disturbance of the equilibrium (Einstein) state of the universe. As Eddington has shown,* $\ddot{\beta}$ and $\dot{\beta}$ are then always of the same sign and the effect of the disturbance is cumulative, $\ddot{\beta}$ increasing with $\dot{\beta}$ if $\ddot{\beta} > 0$ initially.

VI. *Equilibrium of Einstein Universe*

It is necessary to point out that the new metric (34) for a mass-particle in a spherical universe of initial curvature $1/R^2$ does not throw any new light on the question of the disturbance of the equilibrium of the Einstein universe by the formation of condensations. As in previous cases considered,† the volume of the new universe, for constant proper mass, is identical with that of the corresponding Einstein universe, to a first approximation. No method for dealing with the second approximation has yet been discovered. An alternative method is proposed by Narliker,‡ who suggests that the formation of a condensation is accompanied by an increase in the cosmical constant, λ, from which he deduces that the Einstein universe would begin to expand. A serious objection to this is that the universe after the condensation has formed no longer corresponds to the same Einstein universe as that in which the process of condensation began.§ Because of the increase of λ, the proper-mass of the whole universe has decreased. Narliker's treatment fails to show that it is the formation of the condensation and not this change of proper-mass which actually starts the expansion. It must be remembered, of course, that a decrease of proper-mass, without a corresponding adjustment of the radius, of an Einstein universe will of itself start an expansion.|| The problem, in our opinion, is to determine whether or not an Einstein universe of given proper-mass will expand or contract when a condensation forms, the proper-mass being conserved throughout all changes.

* A. S. Eddington, *M.N.*, **90**, 668–678, 1930.

† W. H. McCrea and G. C. McVittie, *M.N.*, **92**, 7–12, 1931.

‡ V. Narliker, *Phil. Mag.* (7), **14**, 433–436, 1932.

§ The relation between the proper-mass, M, of an Einstein universe and λ is $M = \frac{\pi}{2\sqrt{\lambda}}$. Hence to each value of λ there corresponds an Einstein universe of different proper-mass, radius, etc.

|| A. S. Eddington, *M.N.*, **90**, 668–678, 1930.