The Max k-Cut Game and its Strong Equilibria[⋆]

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Abstract. An instance of the MAX $k-$CUT game is an edge weighted graph. Every vertex is controlled by an autonomous agent with strategy space $[1..k]$. Given a player i, his payoff is defined as the total weight of the edges $[i,j]$ such that player j's strategy is different from player i's strategy. The social welfare is defined as the weight of the cut, i.e. half the sum of the players payoff. It is known that this game always has a pure strategy Nash equilibrium, a state from which no single player can deviate. Instead we focus on strong equilibria, a robust refinement of the pure Nash equilibrium which is resilient to deviations by coalitions of any size. We study the strong equilibria of the MAX $k-$CUT game under two perspectives: existence and worst case social welfare compared to a social optimum.

1 Introduction

Given a graph $G=(V,E)$ and a weight function $w:E\rightarrow\mathbb{R}_+$, the MAX $k-$CUT problem is to partition V into k sets V_1, V_2 $\dots V_k$ such that the sum of the weight of the edges having their endpoints not in the same part of the partition is maximum. In this paper we study a strategic game defined upon MAX $k-$CUT. Each vertex is controlled by a player with strategy set $\{1,2,\dots,k\}$. A player's utility is the total weight of the edges incident to her and such that her neighbor has a different strategy.

The game models a large class of situations where there are k available facilities and every agent must choose one. The facilities are inherently similar but their number is typically smaller than the number of agents (e.g. compartments in a train). Then the agents must share the facilities. In this game every agent is "hindered" by the other agents but solely by those who chose the same facility. So every agent makes his choice according to the agents that he wants to avoid. In the MAX $k-$CUT game, the weight of an edge $[i,j]$ represents the strength of interference that agents i and j exert on each others if they choose the same facility. The social welfare for a given state is defined as the total weight of the edges with corresponding endpoint agents making distinct choices of a facility (i.e. half the sum of the player's utility).

This paper is devoted to the existence and the quality of pure[1] equilibria in the MAX $k-$CUT game. Our work is motivated by the study of large scale distributed systems which usually lack a central control authority. Instead these systems are operated by self interested entities. Though the uncoordinated decisions made by the entities often end up in a stable configuration (an equilibrium), these configurations are rarely socially

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[1] We only consider pure strategies so we often omit the word pure.

optimal. Two main questions naturally arise in this context: For which instances an equilibrium exists? How far from the social optimum these equilibria can be?

When the focus is on pure Nash equilibria in the MAX $k-$CUT game, the answer to these two questions is known. For every instance (and every k) an optimal cut is a pure Nash equilibrium. Furthermore, the *price of anarchy* (PoA in short) [1], defined as the worst case ratio between the social welfare of a Nash equilibrium and the optimal social welfare, is $\frac{k-1}{k}$ [2]. This paper is devoted to the (even more) appealing concept of *strong equilibrium* (SE in short) [3]. This notion refines the NE because it considers deviations by coalitions of any size whereas NE are restricted to deviations by a single player. When it exists, a SE is a very robust state of the game, it is also more sustainable than a NE. Strong equilibria are the topic of many recent articles including [4–8].

We are interested in the existence of SE in the MAX $k-$CUT game and their quality with respect to socially optimal configurations. In particular, we resort to the *strong price of anarchy* (SPoA in short) [4] which is the price of anarchy restricted to strong equilibria.

Previous related work and Contribution The MAX $k-$CUT game or similar games like the party affiliation game, the interference game or the consensus game have been studied in [9, 10, 5, 6, 2] from different perspectives: existence of a pure equilibrium, convergence time to an equilibrium, complexity for computing an equilibrium and worst case quality of an equilibrium. In this paper we only deal with the existence and the worst case quality of a pure equilibrium.

For the MAX $2-$CUT game, the picture is complete. A SE always exists because the state corresponding to an optimal cut is a SE. The PoA is $1/2$ by a well known result from local search theory and the SPoA is $2/3$ [5]. From now on we consider that $k \geq 3$. A NE always exists because the state corresponding to an optimal cut is a NE. In [2] it is shown that the PoA of the *unweighted* MAX $k-$CUT game is $\frac{k-1}{k}$ and one can easily extend the result to the weighted case. In [5] it is shown that an optimal cut is not necessarily a SE but the instance presented admits another optimal cut which turns out to be a SE. The state corresponding to an optimal cut is a 3-strong equilibrium (a state immune to deviations by coalitions of at most 3 players) but not necessarily a 4-strong equilibrium [5].

In Section 2 we provide some useful definitions and notations. The results presented in this paper deal with the existence of a SE (Section 3) and if a SE exists, we bound its quality compared to a social optimum (Section 4). In Section 3 we do not prove or disprove that every instance of the MAX $k-$CUT game admits a SE. Instead we give both negative and positive results related to this question. In Section 4 we give an upper bound of $\frac{2k-2}{2k-1}$ and a matching lower bound on the SPoA. It is noteworthy that the upper bound is derived without any assumption on the instance so it applies every time a SE exists. We conclude in Section 5. We conjecture that a SE always exists for the MAX $k-$CUT game, but we are not able to prove this for the moment.

2 Definitions and notations

A *strategic game* is a tuple $\langle N, (\Sigma_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where N is the set of players (we suppose that $|N| = n$), Σ_i is the set of strategies of player i and $u_i : \times_i \Sigma_i \to \mathbb{R}$ is player i's utility function. A *pure state* or *pure strategy profile* of the game is an element of $\Sigma := \times_i \Sigma_i$. Although players may choose a probability distribution over their strategy set, we only consider pure strategy profiles in this paper. Players are supposed to be rational, i.e. each of them plays in order to *maximize* his utility.

Given a state $a \in \Sigma$, (a_{-i}, b_i) denotes the state where a_i is replaced by b_i in a while the strategy of the other players remains unchanged. A state a is a *Nash equilibrium* (NE) if there no player $i \in N$ and a strategy $b_i \in A_i$ such that $u_i\big((a_{-i}, b_i)\big) > u_i(a)$.

Given two states a, a' and a coalition $C \subseteq N$, (a_{-C}, a') denotes the state where a_i is replaced by a'_i in a for all $i \in C$. A state a is a *strong equilibrium* (SE) if there is no non-empty coalition $C \subseteq N$ and a profile $a' \in A$ such that $u_i\big((a_{-C}, a')\big) > u_i(a)$ for all $i \in C$. A state a is an r*-strong equilibrium* (r-SE) if there is no non-empty coalition $C \subseteq N$ of size at most r and a profile $a' \in A$ such that $u_i\big((a_{-C}, a')\big) > u_i(a)$ for all $i \in C$. Therefore a SE is a NE, a NE is a 1-SE and a n-SE is SE (n is the number of players).

The *price of anarchy* (PoA) measures the performance of decentralized systems [1] via its Nash equilibria. More formally, let Γ be a family of strategic games, let γ be an instance of Γ, let A_γ be the strategy space of γ, let $\mathcal{Q} : A_\gamma \to \mathbb{R}_+$ be the social welfare, let $\mathcal{E}(\gamma)$ be the set of all pure Nash equilibria of γ and let o_γ be a social optimum for γ (i.e. $o_\gamma = \mathrm{argmax}_{a \in A_\gamma} \mathcal{Q}(a)$). The *pure* price of anarchy of Γ is $\min_{\gamma \in \Gamma} \min_{a \in \mathcal{E}(\gamma)} \mathcal{Q}(a)/\mathcal{Q}(o_\gamma)$. If $\mathcal{SE}(\gamma)$ denotes the set of all strong equilibria of γ then the strong price of anarchy (SPoA) [4] is $\min_{\gamma \in \Gamma} \min_{a \in \mathcal{SE}(\gamma)} \mathcal{Q}(a)/\mathcal{Q}(o_\gamma)$.

3 On the existence of strong equilibria

This section contains both negative and positive results on the existence of a SE in the MAX $k-$CUT game. The negative results are (often non trivial) observations that all proof techniques that we are aware of, to show the existence of a SE, fail. The positive results are (often tight) sufficient conditions for the existence of a SE, and the existence of a good approximation of it in every instance.

Negative results The strategy profiles which correspond to optimal cuts play an important role because they are often stable states. When $k = 2$ and $k \geq 3$, optimal cuts are respectively strong equilibria and 3-strong equilibria [5]. An instance presented in [5] admits two optimal cuts: one is a SE while the other is not a 4-SE. It shows that an optimal cut is not necessarily a SE but it does not prevent (at least) one optimal cut to be a SE. In this paper we propose a new and simpler instance in which the unique optimal cut is not a SE. Consider the instance given on the left part of Figure 1. An exhaustive search can show that the given 3-cut is the only optimal solution. However, it is not a SE as nodes a, b, c and f can modify their strategy and benefit (see the right part of Figure 1).

A second way to prove the existence of a SE is to exhibit a strong potential function Φ_S and an order $\prec$ such that $\Phi_S(\sigma) \prec \Phi_S(\sigma')$ holds for every improving pair of strategy profiles (σ, σ')[2] [6]. This technique captures the fact that the players naturally converge

[2] The cost (resp. the payoff) in σ of every player having a distinct strategy in σ' strictly decreases (resp. strictly increases) when switching to σ'.

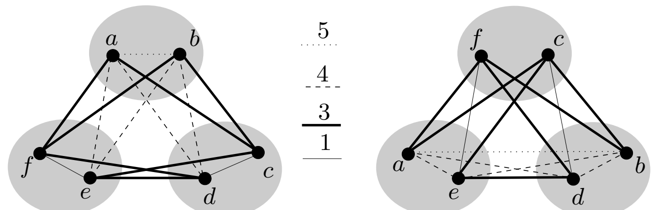

Fig. 1. Left: An optimal 3-cut with value 37. Right: Starting from the left configuration, vertices a, b, c and f can move and benefit but the value of the cut is 36.

to a SE (a state σ^* such that $\Phi_S(\sigma^*)$ is locally maximum for $\prec$) since every sequence of improvements is finite. However no pair $(\Phi_S, \prec)$ can exist for the MAX $k-$CUT game since the dynamics can cycle. We are given an instance of the MAX $3-$CUT with 4 nodes and three strategy profiles (see Figure 2). At each deviation by a coalition, the utility of every member strictly increases but the three configurations form a cycle. It is noteworthy that the *interference game* studied by Harks, Klimm and Möhring [6], and for which they prove the existence of a SE by the strong potential function, is slightly different to the MAX k-CUT game. The slight difference makes both results (existence of a strong potential function for the interference game, and non existence of a strong potential function for the MAX $k-$CUT game) consistent.

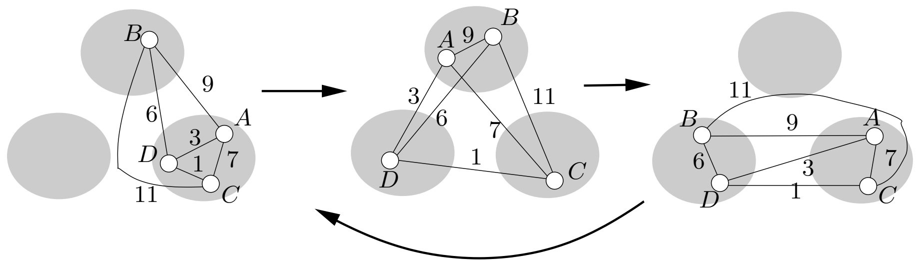

Fig. 2. A cycle disproving the existence of a strong potential function.

A last attempt is to observe that the MAX $k-$CUT game is a *congestion game* [11, 12]. Congestion games are extensively studied for two reasons, they always admit a pure strategy Nash equilibrium and they are general models for resource sharing in networks. Sufficient conditions on the strategy space to show the existence of a SE were derived [7, 8]. These works are based on the notion of *bad* configurations in the strategy space. An instance of a congestion game without any bad configuration admits a SE (but a bad configuration does not prevent some instances to admit a SE). If we turn the MAX $k-$CUT game into a congestion game and consider the clique on 3 nodes then we get a bad configuration.

Positive results A direct observation is that every k-colorable graph admits a SE: a k coloration induces a state σ^*, where every vertex with color c plays c, such that $u_i(\sigma^*) = \sum_{[i,j]\in E} w([i,j])$ for all $i \in V$. However this condition is not necessary: take a clique of size 3 for the MAX 2-CUT game. In every state σ there is a node i which satisfies $u_i(\sigma) < \sum_{[i,j]\in E} w([i,j])$ but this instance admits a SE.

Another direction to prove the existence of a SE is to limit the number players.

Proposition 1. *If $|V| \leq k+2$ then an optimal state of the* MAX k-CUT *game is a SE.*

Proof. Let σ be an optimal state. Hence σ is a NE. Suppose there is $j \in \{1,\dots,k\}$ and no player i such that $\sigma(i) = j$. For every pair of nodes i, i' such that $\sigma(i) = \sigma(i')$, it must be $w([i,i']) = 0$ since otherwise σ is not optimal. Thus G is $k-1$ colorable and σ must be a SE.

Now suppose that for every $j \in \{1,\dots,k\}$, there is at least one player i such that $\sigma(i) = j$. By contradiction, suppose that there is a coalition $C \subseteq V$ of players who can modify their strategy and benefit. Let σ' be the resulting strategy profile. A result of [5] states that an optimal state is a 3-SE, i.e. $|C| > 3$.

Let $V_1,\dots,V_k$ (resp. $V'_1,\dots,V'_k$) be the k partition corresponding to σ (resp. σ'). By hypothesis $|V_j| \geq 1$ for $j = 1..k$. Since $|V| \leq k+2$, we can consider two cases:

- $|V_1| = 3$ and $|V_j| = 1$ for $j = 2,\dots,k$. For every $i \in V_2 \cup \ldots \cup V_k$, we know that $i \notin C$ because $u_i(\sigma)$ is maximum. Then $C \subset V_1$, $|C| \leq |V_1| = 3$, contradiction with $|C| > 3$.
- $|V_1| = |V_2| = 2$ and $|V_j| = 1$ for $j = 3..k$. For every $i \in V_3 \cup \ldots \cup V_k$, we know that $i \notin C$ because $u_i(\sigma)$ is maximum. If $V_1 \cap \big(V'_3 \cup \ldots \cup V'_k\big) \neq \emptyset$ or $V_2 \cap \big(V'_3 \cup \ldots \cup V'_k\big) \neq \emptyset$ then it contradicts the fact that σ is a NE. Indeed, if a player i belongs to $V_1 \cap \big(V'_3 \cup \ldots \cup V'_k\big)$, then it means that it can deviate unilaterally and improve its utility, contradiction with the fact that σ is a NE. Thus $\sigma(i) \in \{1,2\} \Rightarrow \sigma'(i) \in \{1,2\}$. If $V_1 \subseteq C$ then $u_i(\sigma) \geq u_i(\sigma')$ holds for every $i \in V_1$. We deduce that $V_1 \not\subseteq C$. It follows that $|C| \leq |V_1| - 1 + |V_2| = 3$, contradiction with $|C| > 3$. □

One can observe that Proposition 1 is tight when $k = 3$. Every instance with 5 nodes and $k = 3$ admits a SE by the proposition, but one cannot go beyond since for the 6 nodes instance of Figure 1, the optimal cut is not a SE.

Our last positive result is about the existence of an approximate strong equilibrium in every instance. Given a real $\epsilon \geq 0$, a state a is an *ϵ-approximate strong equilibrium* if there is no non-empty coalition $C \subseteq N$ and a profile $a' \in A$ such that $u_i\big((a_{-C}, a')\big) > (1+\epsilon)u_i(a)$ for all $i \in C$. Therefore a 0-approximate SE is a SE. Approximate equilibria are appealing concepts in game theory. They capture the fact a player does not deviate if his gain is negligible. Approximate equilibria are the topic of many recent articles including [13–15].

Theorem 1. *Every NE of the* MAX k-CUT *game is a $\frac{1}{k-1}$-approximate SE.*

Proof. Let σ be a NE. Take a player p and suppose w.l.o.g. that $\sigma(p) = k$. Let $E(p,\sigma,i)$ be the set of edges $[p,q]$ such that $\sigma(q) = i$. Let $W(p,\sigma,i) = \sum_{e \in E(p,\sigma,i)} w(e)$ when $E(p,\sigma,i) \neq \emptyset$ and $W(p,\sigma,i) = 0$ otherwise. The utility of p under σ is equal to $\sum_{i=1}^{k-1} W(p,\sigma,i)$. If p unilaterally replaces his strategy by j then his utility becomes $\sum_{\substack{i=1 \\ i\neq j}}^{k} W(p,\sigma,i)$.

Since σ is a NE, $\sum_{i=1}^{k-1} W(p,\sigma,i) \geq \sum_{\substack{i=1\\ i\neq j}}^{k} W(p,\sigma,i)$, which is equivalent to $W(p,\sigma,j) \geq W(p,\sigma,k)$ for every $j \in \{1,\ldots,k-1\}$. Sum up this inequality for every $j \in \{1,\ldots,k-1\}$ to get that $\frac{1}{k-1}\sum_{i=1}^{k-1} W(p,\sigma,i) \geq W(p,\sigma,k)$. The utility of p in any state σ' is at most $\sum_{i=1}^{k} W(p,\sigma,i)$. We deduce that

$$u_p(\sigma') \leq \sum_{i=1}^{k-1} W(p,\sigma,i) + W(p,\sigma,k) \leq (1+\frac{1}{k-1})\sum_{i=1}^{k-1} W(p,\sigma,i) = (1+\frac{1}{k-1})u_p(\sigma).$$

It follows that σ must be a $\frac{1}{k-1}$-approximate SE. □

Since the MAX $k-$CUT game always possesses a NE, the existence of a $\frac{1}{k-1}$-approximate SE is guaranteed. Interestingly Theorem 1 is tight because there are instances where a NE is a $\frac{1}{k-1}$-approximate SE but not an ϵ-approximate SE for some $\epsilon < \frac{1}{k-1}$.

4 On the quality of strong equilibria

In the previous section we identified some cases where a SE exists. Here we bound the strong price of anarchy but we do not make any assumption on the instance so the result applies for every instance admitting a SE.

Theorem 2. *When $k \geq 3$, the SPoA of the* MAX $k-$CUT *game is at least* $\frac{2k-2}{2k-1}$.

Proof. Let $k \geq 3$ and $G = (V,E)$ be an instance of the MAX $k-$CUT game. Let σ be a SE of G. Let σ^* be an optimal state of G.

Let E_{OS} (resp. E_{OO}) be the set of edges which are only in cut induced by σ (resp. σ^*). Let E_{COM} be the set of edges which are in common. Let OS, OO and COM be the weight of E_{OS}, E_{OO} and E_{COM} respectively. Suppose that the following inequality holds.

$$COM + kOS \geq (2k-2)OO \quad (1)$$

Add $(2k-2)COM$ on both sides to get $(2k-1)COM + kOS \geq (2k-2)(OO + COM)$. Since $k \geq 1 \Leftrightarrow k \leq 2k-1$, we deduce that $(2k-1)(COM+OS) \geq (2k-2)(OO+COM)$. Using $COM + OS = \mathcal{Q}(\sigma)$ and $COM + OO = \mathcal{Q}(\sigma^*)$, the result follows.

Now let us prove inequality (1). We partition V into k^2 sets $X_{i,j} := \{v \in V : \sigma(v) = i \text{ and } \sigma^*(v) = j\}$ for $i,j = 1,\ldots,k$ (see Figure 3 for an illustration). Given two disjoint sets $X \subseteq V$ and $Y \subseteq V$, $w(X,Y)$ denotes $\sum_{x\in X}\sum_{y\in Y} w([x,y])$. Similarly, $w(x,Y)$ denotes $\sum_{y\in Y} w([x,y])$ where $Y \subset V$ and $x \in V \setminus Y$.

Let π be a permutation of $\{1,\cdots,k\}$. One can list $k!$ optimal cuts representing the same state σ^*, one per permutation, if $\sigma^*(v)$ is replaced by $\pi(\sigma^*(v))$ for all $v \in V$. Let us denote by $\pi(\sigma^*)$ the optimum state associated with π. Let V_π be the nodes of V which are misplaced according to $\pi(\sigma^*)$, i.e. $v \in V_\pi$ if $\sigma(v) \neq \pi(\sigma^*(v))$. Let $r_\pi = |V_\pi|$. We are going to rename the nodes of V_π so that $V_\pi = \{v_1,\cdots,v_{r_\pi}\}$.

Since σ is a SE, there is at least one node $v \in V_\pi$ such that $u_v(\sigma) \geq u_v(\sigma_{-V_\pi},\pi(\sigma^*))$. In other words, v does not benefit if all nodes of V_π replace their strategy in σ by their strategy in $\pi(\sigma^*)$. Rename v by v_1. Again, since σ is a SE, there is at least one node

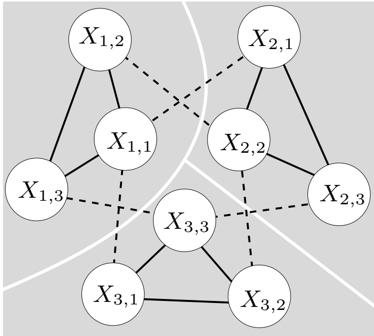

Fig. 3. Partition of V into k^2 sets according to σ and σ^*, case $k = 3$. Dashed edges belong to the cut induced by σ but they are not in the cut induced by σ^*. Solid edges belong to the cut induced by σ^* but they are not in the cut induced by σ. Non represented edges are in the intersection.

$v \in V_\pi \setminus \{v_1\}$ such that $u_v(\sigma) \geq u_v(\sigma_{-V_\pi \setminus \{v_1\}}, \pi(\sigma^*))$. Rename v by v_2. The procedure is run until all nodes of V_π are renamed, that is $V_\pi = \{v_1, \cdots, v_{r_\pi}\}$.

Let us define V_π^ℓ as $\{v_\ell, v_{\ell+1}, \cdots, v_{r_\pi}\}$ for $1 \leq \ell \leq r_\pi$. For every $v_\ell \in V_\pi$, one has

$$u_{v_\ell}(\sigma) \geq u_{v_\ell}(\sigma_{-V_\pi^\ell}, \pi(\sigma^*)) \tag{2}$$

Take a vertex $v_\ell \in V_\pi$ and suppose that $v_\ell \in X_{i_\ell, j_\ell}$. There are three cases where the weight of an edge $[v_\ell, y]$ is present in $u_{v_\ell}(\sigma)$ but not in $u_{v_\ell}(\sigma_{-V_\pi^\ell}, \pi(\sigma^*))$:

- **CASE 1.** y is not misplaced and he plays in σ the strategy that v_ℓ plays in $\pi(\sigma^*)$. In other words, $y \in X_{i',j'}$ where $i' = \pi(j')$ and $i' = \pi(j_\ell)$; thus, $j' = j_\ell$ since π is a permutation. In this case $[v_\ell, y] \in E_{OS}$.
- **CASE 2.** y is misplaced, he was renamed *after* v_ℓ, he does not play the same strategy as v_ℓ in σ but he plays the same strategy as v_ℓ in $\pi(\sigma^*)$. In other words, $y \in X_{i',j'} \cap V_\pi^{\ell+1}$ where $i' \neq \pi(j')$, $i' \neq i_\ell$ and $j' = j_\ell$. In this case $[v_\ell, y] \in E_{OS}$.
- **CASE 3.** y is misplaced, he was renamed *before* v_ℓ, he plays in σ the strategy that v_ℓ plays in $\pi(\sigma^*)$. In other words, $y \in X_{i',j'} \cap (V_\pi \setminus V_\pi^\ell)$ where $i' \neq \pi(j')$, $i' \neq i_\ell$ and $i' = \pi(j_\ell)$ (hence, $j' \neq j_\ell$). In this case $[v_\ell, y] \in E_{COM}$.

There are three cases where the weight of an edge $[v_\ell, y]$ is present in $u_{v_\ell}(\sigma_{-V_\pi^\ell}, \pi(\sigma^*))$ but not in $u_{v_\ell}(\sigma)$:

- **CASE 4.** y is not misplaced and he plays the same strategy as v_ℓ in σ. In other words, $y \in X_{i',j'}$ where $\pi(j') = i'$ and $i' = i_\ell$. In this case $[v_\ell, y] \in E_{OO}$.
- **CASE 5.** y is misplaced, he was renamed *after* v_ℓ, he plays the same strategy as v_ℓ in σ but he does not play the same strategy as v_ℓ in $\pi(\sigma^*)$. In other words, $y \in X_{i',j'} \cap V_\pi^{\ell+1}$ where $i' = i_\ell$, $j' \neq j_\ell$ and $i' \neq \pi(j')$. In this case $[v_\ell, y] \in E_{OO}$.
- **CASE 6.** y is misplaced, he was renamed *before* v_ℓ and he plays the same strategy as v_ℓ in σ. In other words, $y \in X_{i',j'} \cap (V_\pi \setminus V_\pi^\ell)$ where $i' \neq \pi(j')$ and $i' = i_\ell$. In this case $[v_\ell, y] \notin E_{OO} \cup E_{COM} \cup E_{OS}$ if y plays the same strategy as v_ℓ in $\pi(\sigma^*)$, i.e. $j' = j_\ell$, otherwise $[v_\ell, y] \in E_{OO}$.

Using cases 1 to 6, one can rewrite inequality (2) as follows.

$$w(v_\ell, X_{\pi(j_\ell),j_\ell}) + \sum_{i'=1, i'\neq i_\ell, i'\neq \pi(j_\ell)}^{k} w(v_\ell, X_{i',j_\ell} \cap V_\pi^{\ell+1}) + \sum_{j'=1, j'\neq j_\ell}^{k} w(v_\ell, X_{\pi(j_\ell),j'} \cap (V_\pi \setminus V_\pi^\ell)) \geq \sum_{j'=1}^{k} w(v_\ell, X_{i_\ell,\pi^{-1}(i_\ell)}) + \sum_{j'=1, j'\neq j_\ell, j'\neq \pi^{-1}(i_\ell)}^{k} w(v_\ell, X_{i_\ell,j'} \cap V_\pi^{\ell+1}) + \sum_{j'=1, j'\neq \pi^{-1}(i_\ell)}^{k} w(v_\ell, X_{i_\ell,j'} \cap (V_\pi \setminus V_\pi^\ell)) \quad (3)$$

Using $w(v_\ell, X_{i_\ell,j'} \cap (V_\pi \setminus V_\pi^\ell)) \geq 0$ (the weight of an edge is always non negative) and the previous inequality, we get that:

$$w(v_\ell, X_{\pi(j_\ell),j_\ell}) + \sum_{i'=1, i'\neq i_\ell, i'\neq \pi(j_\ell)}^{k} w(v_\ell, X_{i',j_\ell} \cap V_\pi^{\ell+1}) + \sum_{j'=1, j'\neq j_\ell}^{k} w(v_\ell, X_{\pi(j_\ell),j'} \cap (V_\pi \setminus V_\pi^\ell)) \geq \sum_{j'=1}^{k} w(v_\ell, X_{i_\ell,\pi^{-1}(i_\ell)}) + \sum_{j'=1, j'\neq j_\ell, j'\neq \pi^{-1}(i_\ell)}^{k} w(v_\ell, X_{i_\ell,j'} \cap V_\pi^{\ell+1}) + \sum_{j'=1, j'\neq \pi^{-1}(i_\ell), j'\neq j_\ell}^{k} w(v_\ell, X_{i_\ell,j'} \cap (V_\pi \setminus V_\pi^\ell)) \quad (4)$$

Actually, we have "removed" the weight of edges $[v_\ell, y] \notin E_{OO} \cup E_{COM} \cup E_{OS}$ which appear on Case 6. Observe that the last two terms of inequality (4) can be grouped as follows:

$$w(v_\ell, X_{\pi(j_\ell),j_\ell}) + \sum_{i'=1, i'\neq i_\ell, i'\neq \pi(j_\ell)}^{k} w(v_\ell, X_{i',j_\ell} \cap V_\pi^{\ell+1}) + \sum_{j'=1, j'\neq j_\ell}^{k} w(v_\ell, X_{\pi(j_\ell),j'} \cap (V_\pi \setminus V_\pi^\ell)) \geq \sum_{j'=1}^{k} w(v_\ell, X_{i_\ell,\pi^{-1}(i_\ell)}) + \sum_{j'=1, j'\neq j_\ell, j'\neq \pi^{-1}(i_\ell)}^{k} w(v_\ell, X_{i_\ell,j'} \cap V_\pi) \quad (5)$$

Summing inequality (5) for $\ell = 1, \cdots, r_\pi$, i.e. for each $v_\ell \in V_\pi$, we get that

$$\sum_{\ell=1}^{r_\pi} \Big(w(v_\ell, X_{\pi(j_\ell),j_\ell}) + \sum_{i'=1, i'\neq i_\ell, i'\neq \pi(j_\ell)}^{k} w(v_\ell, X_{i',j_\ell} \cap V_\pi^{\ell+1}) + \sum_{j'=1, j'\neq j_\ell}^{k} w(v_\ell, X_{\pi(j_\ell),j'} \cap (V_\pi \setminus V_\pi^\ell)) \Big) \geq \sum_{\ell=1}^{r_\pi} \Big(\sum_{j'=1}^{k} w(v_\ell, X_{i_\ell,\pi^{-1}(i_\ell)}) + \sum_{j'=1, j'\neq j_\ell, j'\neq \pi^{-1}(i_\ell)}^{k} w(v_\ell, X_{i_\ell,j'} \cap V_\pi) \Big) \quad (6)$$

Now we give an intermediate property (proof in the appendix).

Property 1. For any edge $[x, y]$, if $w([x, y])$ appears in the left-hand part of inequality (6) then it appears once.

Cases 1 and 2 state that if $w[x, y]$ appears in the left-hand part of inequality (6) then $[x, y] \in E_{OS}$. Using the fact that $w[x, y]$ appears at most once (by Property 1), we deduce that

$$\sum_{\ell=1}^{r_\pi} \big(w(v_\ell, X_{\pi(j_\ell),j_\ell}) + \sum_{i'=1, i' \neq i_\ell, i' \neq \pi(j_\ell)}^{k} w(v_\ell, X_{i',j_\ell} \cap V_\pi^{\ell+1}) \big) \leq OS \tag{7}$$

Case 3 states that if $w[x, y]$ appears in the left-hand part of inequality (6) then $[x, y] \in E_{COM}$ and $\{x, y\} \subseteq V_\pi$. Moreover $w([x, y])$ appears at most once, we deduce that

$$\sum_{\ell=1}^{r_\pi} (\sum_{j'=1, j' \neq j_\ell}^{k} w(v_\ell, X_{\pi(j_\ell),j'} \cap (V_\pi \setminus V_\pi^\ell))) \leq \sum_{i=1}^{k} \sum_{j=1, j \neq \pi^{-1}(i)}^{k} \sum_{j'=1, j' \neq j}^{k} w(X_{i,j}, X_{\pi(j),j'}) \tag{8}$$

Using inequalities (7) and (8), we obtain the following upper bound on the left-hand part of inequality (6):

$$OS + \sum_{i=1}^{k} \sum_{j=1, j \neq \pi^{-1}(i)}^{k} \sum_{j'=1, j' \neq j}^{k} w(X_{i,j}, X_{\pi(j),j'}) \tag{9}$$

Now we give a second intermediate property (proof in the appendix).

Property 2. Let x and y be two vertices of $X_{i,j}$ and $X_{i',j'}$ respectively. Among the $k!$ possible permutations π of $\{1, \cdots, k\}$, exactly $(k-1)!$ of them satisfy simultaneously: $\pi(j) \neq i$, $\pi(j') \neq i'$ and $i' = \pi(j)$.

Now sum up inequality (9) for all permutations π of $\{1, \ldots, k\}$. We can give the following upper bound of the result: $k!OS + (k-1)!COM$. Indeed every edge in E_{OS} appears exactly once for each permutation π. Concerning the edges $[x, y] \in E_{COM}$, each one appears at most $(k-1)!$ times by Property 2.

Now we focus on the right part of inequality (6). Take an edge $[x, y]$ such that $x \in X_{i,j}$ and $y \in X_{i',j'}$. The weight of $[x, y]$ does not appear in the right part of inequality (6) if $i \neq i'$. $w([x, y])$ appears once in the right part of inequality (6) if $\pi(j) = i = i'$ and $\pi(j') \neq i'$; in this case $x \notin V_\pi$ whereas $y \in V_\pi$. $w([x, y])$ appears twice in the right part of inequality (6) if $i \neq i'$, $\pi(j) \neq i$ and $\pi(j') \neq i'$; in this case $x, y \in V_\pi$.

By definition a vertex which is not in V_π must be in $X_{i,\pi^{-1}(i)}$ for some $i \in \{1, \cdots, k\}$. Then inequality (6) is equal to

$$\sum_{i=1}^{k} (\sum_{j=1, j \neq \pi^{-1}(i)}^{k} w(X_{i,\pi^{-1}(i)}, X_{i,j}) + \sum_{j=1, j \neq \pi^{-1}(i)}^{k} \sum_{j'=1,, j' \neq \pi^{-1}(i)}^{k} w(X_{i,j}, X_{i,j'}) \big). \tag{10}$$

If we sum up inequality (10) over the $k!$ permutations of $\{1, \cdots, k\}$ then every term $w(X_{i,j}, X_{i,j'})$, for $j \neq j'$, appears exactly $(k-1)!(2k-2)$ times. Indeed, a set $X_{i,j}$

satisfies $\pi(j) = i$ exactly $(k-1)!$ times, whereas $\pi(j) \neq i$ holds $k! - (k-1)!$ times. In addition it can not be $\pi(j) = \pi(j') = i$ because $j \neq j'$: $2(k-1)! + 2(k! - 2(k-1)!) = (k-1)!(2k-2)$. Therefore summing up inequality (10) over the $k!$ permutations of $\{1, \cdots, k\}$ gives $(k-1)!(2k-2)OO$.

Finally, summing up inequality (6) over all possible permutations of $\{1, \cdots, k\}$, we get that $k!OS + (k-1)!COM \geq (k-1)!(2k-2)OO$ which is equivalent to inequality (1). □

The following matching upper bound on the SPoA of the MAX $k-$CUT game can be derived.

Proposition 2. *The SPoA of the* MAX $k-$CUT *game is at most* $(2k-2)/(2k-1)$.

Proof. Consider an instance with $2k$ vertices $\{v_1, \cdots, v_k\} \cup \{u_1, \cdots, u_k\}$ and the following $2k-1$ edges of weight 1: $[v_1, v_i]$ for $i = 2..k$, $[u_k, u_i]$ for $i = 1..k-1$ and $[v_1, u_k]$. See Figure 4. If every v_i plays i while every u_j plays j then the state is optimal and it has weight $2k-1$. If every v_i plays i while every u_j plays $j+1$ mod k (i.e. u_k plays 1, u_1 plays 2, etc) then the state is a SE of weight $2k-2$. Indeed every node in $\{v_2, \cdots, v_k\} \cup \{u_1, \cdots, u_{k-1}\}$ has the maximum utility that he can expect in this instance so none of them has incentive to deviate. Now if v_1 or u_k moves then one of his incident edge would not be the cut anymore while only $[v_1, u_k]$ can enter the cut. Then v_1 and u_k can not alone or together increase their utility. □

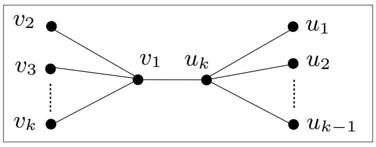

Fig. 4. An instance for the upper bound on the SPoA.

5 Concluding remarks and open questions

The main question remaining open is to prove or disprove that every instance of the MAX $k-$CUT game possesses a strong equilibrium. The technique used by Harks, Klimm and Möhring [6] considers all improving pairs of strategies while Holzman and Law-Yone [7] require minimal improving pairs of strategies (an improving pair of strategies is not minimal if a proper subset of the coalition can also perform an improvement). The cycle presented in Figure 2 is made of three improvements which are not minimal. Then it would be interesting to investigate the existence of a strong potential function restricted to minimal improvements.

A Nash equilibrium of the MAX k-CUT game a $\frac{1}{k-1}$-approximate SE, can we prove the existence of an ϵ-approximate SE for some $\epsilon < \frac{1}{k-1}$? It is known from [5] that every instance of the game possesses a 3-SE, a stronger notion of equilibrium than the NE.

However, it is not difficult to build an instance containing a 3-SE σ which is also a $\frac{1}{k-1}$-approximate SE, but σ is not an ϵ'-approximate SE for $\epsilon' < \frac{1}{k-1}$. A promising direction would be to bound the ϵ such that every optimal cut is an ϵ-approximate SE. This ϵ cannot be 0 since an optimal cut is not necessarily a SE and a better lower bound on this ϵ can be derived from the instance of Figure 1.

The *price of stability* (PoS) is a well studied ratio whose definition is close to the price of anarchy [13]. It is the worst case ratio between the social welfare of the best NE and a socially optimal state. Since an optimal cut is a NE in the MAX $k-$CUT game, the PoS is 1. It is natural to restrict this notion to strong equilibria [4]. We know that the price of stability for strong equilibria is 1 when $k = 2$ and strictly less than 1 when $k \geq 3$ (see the instance of Figure 1). It would be interesting to give an explicit lower bound for the case $k \geq 3$.

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Appendix

Property 1. For any edge $[x, y]$, if $w([x, y])$ appears in the left-hand part of inequality (6) then it appears once.

Proof.
- If neither x nor y belong to V_π then $w([x, y])$ can not appear since in this case $[x, y] \in E_{COM}$ or $[x, y] \notin E_{COM} \cup E_{OS} \cup E_{OO}$.
- If only x (or only y) is in V_π then $w([x, y])$ can appear at most once, in the term $w(v_{\ell^*}, X_{\pi(j_{\ell^*}),j_{\ell^*}})$ where $x = v_{\ell^*}$.
- If x and y both belong to V_π then there are ℓ^* and ℓ^{**} such that $x = v_{\ell^*}$ and $y = v_{\ell^{**}}$. Without loss of generality, suppose that $\ell^* < \ell^{**}$, $v_{\ell^*} \in X_{i_{\ell^*},j_{\ell^*}}$ and $v_{\ell^{**}} \in X_{i_{\ell^{**}},j_{\ell^{**}}}$. $w([v_{\ell^*}, v_{\ell^{**}}])$ can appear when $\ell = \ell^*$ and when $\ell = \ell^{**}$, i.e. in the terms $\sum_{i'=1,i'\neq i_{\ell^*},i'\neq\pi(j_{\ell^*})}^{k} w(v_{\ell^*}, X_{i',j_{\ell^*}} \cap V_\pi^{\ell^*+1})$ and $\sum_{j'=1,j'\neq j_{\ell^{**}}}^{k} w(v_{\ell^{**}}, X_{\pi(j_{\ell^{**}}),j'} \cap (V_\pi \setminus V_\pi^{\ell^{**}}))$. However the first term imposes $j_{\ell^{**}} = j_{\ell^*}$ whereas the second one imposes $j_{\ell^{**}} \neq j_{\ell^*}$, contradiction. □

Property 2. Let x and y be two vertices of $X_{i,j}$ and $X_{i',j'}$ respectively. Among the $k!$ possible permutations π of $\{1, \cdots, k\}$, exactly $(k-1)!$ of them satisfy simultaneously: $\pi(j) \neq i$, $\pi(j') \neq i'$ and $i' = \pi(j)$.

Proof. We first observe that $j \neq j'$ by $\pi(j') \neq i'$ and $i' = \pi(j)$. Moreover $i \neq i'$ by $\pi(j) \neq i$ and $i' = \pi(j)$. Then we conduct a case study where $i \neq i'$ and $j \neq j'$. Let a, b, c, d are four distinct elements of $\{1, \cdots, k\}$:

- Case $i = a$, $j = a$, $i' = b$ and $j' = b$. π must satisfy $b = \pi(a)$. Since a permutation is a bijection, it follows that $\pi(b) \neq b$ and $\pi(a) \neq a$. Then π satisfies the three assertions iff $b = \pi(a)$ and there are $(k-1)!$ permutations satisfying $b = \pi(a)$.
- Case $i = a$, $j = b$, $i' = b$ and $j' = a$. π must satisfy $b = \pi(b)$. It follows that $\pi(b) \neq a$ and $\pi(a) \neq b$. Then π satisfies the three assertions iff $b = \pi(b)$.
- Case $i = a$, $j = b$, $i' = b$ and $j' = c$. π must satisfy $b = \pi(b)$. It follows that $\pi(b) \neq a$ and $\pi(c) \neq b$. Then π satisfies the three assertions iff $b = \pi(b)$.
- Case $i = b$, $j = c$, $i' = a$ and $j' = b$. π must satisfy $a = \pi(c)$. It follows that $\pi(b) \neq a$ and $\pi(c) \neq b$. Then π satisfies the three assertions iff $a = \pi(c)$.
- Case $i = a$, $j = b$, $i' = c$ and $j' = c$. π must satisfy $c = \pi(b)$. It follows that $\pi(c) \neq c$ and $\pi(b) \neq a$. Then π satisfies the three assertions iff $c = \pi(b)$.
- Case $i = c$, $j = c$, $i' = a$ and $j' = b$. π must satisfy $a = \pi(c)$. It follows that $\pi(c) \neq c$ and $\pi(b) \neq a$. Then π satisfies the three assertions iff $a = \pi(c)$.
- Case $i = a$, $j = b$, $i' = c$ and $j' = d$. π must satisfy $c = \pi(b)$. It follows that $\pi(d) \neq c$ and $\pi(b) \neq a$. Then π satisfies the three assertions iff $c = \pi(b)$. □