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THE MAXIMAL FUNCTION ON VARIABLE L^p SPACES

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Abstract. We give continuity conditions on the exponent function $p(x)$ which are sufficient for the Hardy–Littlewood maximal operator to be bounded on the variable Lebesgue space $L^{p(x)}(\Omega)$, where Ω is any open subset of $\mathbf{R}^n$. Further, our conditions are necessary on $\mathbf{R}$. Our result extends the recent work of Pick and Růžička [20], Diening [3] and Nekvinda [19]. We also show that under much weaker assumptions on $p(x)$, the maximal operator satisfies a weak-type modular inequality.

1. Introduction

Given an open set $\Omega \subset \mathbf{R}^n$, and a measurable function $p\colon \Omega \to [1,\infty)$, let $L^{p(x)}(\Omega)$ denote the Banach function space of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega |f(x)/\lambda|^{p(x)}\,dx < \infty,$$

with norm

$$\|f\|_{p(x),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

These spaces are a special case of the Musielak–Orlicz spaces (cf. Musielak [18]). When $p(x) = p_0$ is constant, $L^{p(x)}(\Omega)$ becomes the standard Lebesgue space $L^{p_0}(\Omega)$.

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Functions in these spaces and the associated Sobolev spaces $W^{k,p(x)}(\Omega)$ have been considered by a number of authors: see, for example, [1], [6]–[9], [11]–[17], [21], [22] and [24]. They appear in the study of variational integrals and partial differential equations with non-standard growth conditions.

Some of the properties of the Lebesgue spaces readily generalize to the spaces $L^{p(x)}(\Omega)$: see, for example, Kováčik and Rákosník [15]. On the other hand, elementary properties, such as the continuity of translation, often fail to hold (see [15] or [10]), and for applications it is an important and open problem to determine which results from harmonic analysis remain true in the variable exponent setting.

In this paper we consider the Hardy–Littlewood maximal operator,

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B\cap\Omega} |f(y)|\,dy, \tag{1.1}$$

where the supremum is taken over all balls B which contain x and for which $|B \cap \Omega| > 0$. It is well known (cf. Duoandikoetxea [5]) that the maximal operator satisfies the following weak and strong-type inequalities:

$$|\{x \in \Omega : Mf(x) > t\}| \le \frac{C}{t^p} \int_\Omega |f(y)|^p\,dy, \qquad 1 \le p < \infty,$$

$$\int_\Omega Mf(y)^p\,dy \le C \int_\Omega |f(y)|^p\,dy, \qquad 1 < p < \infty.$$

We prove analogous inequalities for functions in $L^{p(x)}(\Omega)$.

Strong-type inequalities have been studied recently by several authors. Pick and Růžička [20] constructed examples which showed that the following uniform continuity condition on $p(x)$ is necessary (in some sense) for the maximal operator to be bounded on $L^{p(x)}(\Omega)$:

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|}, \qquad x, y \in \Omega,\ |x-y| < \tfrac{1}{2}. \tag{1.2}$$

This condition appears to be natural in the study of variable L^p spaces; see [1], [20] and the references contained therein.

Diening [3] has shown that this condition is sufficient on bounded domains. To state his result, let $p_* = \inf\{p(y) : y \in \Omega\}$, $p^* = \sup\{p(y) : y \in \Omega\}$.

Theorem 1.1 (Diening). *Let $\Omega \subset \mathbf{R}^n$ be an open, bounded domain, and let $p\colon \Omega \to [1,\infty)$ satisfy (1.2) and be such that $1 < p_* \le p^* < \infty$. Then the maximal operator is bounded on $L^{p(x)}(\Omega)$: $\|Mf\|_{p(x),\Omega} \le C(p(x),\Omega)\|f\|_{p(x),\Omega}$.*

Remark 1.2. As stated by Diening, this result is for the centered maximal operator, i.e., where the supremum in (1.1) is restricted to balls centered at x. However, his proof can be readily adapted to the larger, "uncentered" maximal operator. Also, the assumption that $p^* < \infty$ holds automatically since Ω is bounded and $p(x)$ is uniformly continuous.

Very recently, Diening [4] has extended Theorem 1.1 to all of $\mathbf{R}^n$ with the additional assumption that $p(x)$ is constant outside of a fixed ball. Further, Nekvinda [19] has shown that this hypothesis can be weakened as follows.

Theorem 1.3 (Nekvinda). *Let $p\colon \mathbf{R}^n \to [1,\infty)$ satisfy (1.2) and be such that $1 < p_* \le p^* < \infty$. Suppose further that there is a constant $p_\infty > 1$ such that $p(x) = p_\infty + \phi(x)$, where there exists $R > 0$ such that $\phi(x) \ge 0$ if $|x| > R$, and $\beta > 0$ such that*

$$\int_{\{x\in\mathbf{R}^n:\phi(x)>0\}} \phi(x)\beta^{1/\phi(x)}\,dx < \infty. \tag{1.3}$$

Then the maximal operator is bounded on $L^{p(x)}(\mathbf{R}^n)$.

Remark 1.4 (Added in proof). We have learned that Nekvinda has improved this result by removing the requirement that ϕ be nonnegative.

Note that together, conditions (1.2) and (1.3) imply $\phi(x) \to 0$ as $|x| \to \infty$.

Our first result is the following theorem; it is similar to Theorem 1.3 since it is for exponent functions $p(x)$ of the same form (though ϕ need not be positive). Further, it gives a pointwise characterization of how quickly $\phi(x)$ must converge to zero at infinity.

Theorem 1.5. *Given an open set $\Omega \subset \mathbf{R}^n$, let $p\colon \Omega \to [1,\infty)$ be such that $1 < p_* \le p^* < \infty$. Suppose that $p(x)$ satisfies (1.2) and*

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}, \qquad x, y \in \Omega,\ |y| \ge |x|. \tag{1.4}$$

Then the Hardy–Littlewood maximal operator is bounded on $L^{p(x)}(\Omega)$.

Condition (1.4) is the natural analogue of (1.2) at infinity. It implies that there is some number p_∞ such that $p(x) \to p_\infty$ as $|x| \to \infty$, and this limit holds uniformly in all directions. It is also necessary (in some sense) on $\mathbf{R}$, as the next example shows.

Theorem 1.6. *Fix p_∞, $1 < p_\infty < \infty$, and let $\phi\colon [0,\infty) \to [0, p_\infty - 1)$ be such that $\phi(0) = 0$, ϕ is decreasing on $[1,\infty)$, $\phi(x) \to 0$ as $x \to \infty$, and*

$$\lim_{x\to\infty} \phi(x)\log(x) = \infty. \tag{1.5}$$

Define the function $p\colon \mathbf{R} \to [1,\infty)$ by

$$p(x) = \begin{cases} p_\infty, & x \le 0,\\ p_\infty - \phi(x), & x > 0; \end{cases}$$

then the maximal operator is not bounded on $L^{p(x)}(\mathbf{R})$.

The assumption in Theorem 1.5 that $p^* < \infty$ again holds automatically: it follows from (1.4). However, the assumption that $p_* > 1$ is necessary, as the following example shows.

Theorem 1.7. *Let $\Omega \subset \mathbf{R}^n$ be open, and let $p\colon \Omega \to [1,\infty)$ be upper semicontinuous. If $p_* = 1$ then the maximal operator is not bounded on $L^{p(x)}(\Omega)$.*

In passing, we note that an immediate application of Theorem 1.5 has been given by Diening [4]: he has shown that if $\partial\Omega$ is Lipschitz, and the maximal operator is bounded on $L^{p(x)}(\Omega)$, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$.

Unlike the case of the strong-type inequalities, we appear to be the first authors to prove an analogue of the weak (p,p) inequality for the maximal operator. Our weak-type result is somewhat surprising, since it requires no continuity assumptions on $p(x)$, and it is satisfied by unbounded functions. To state it, we need a definition. Given a non-negative, locally integrable function u on $\mathbf{R}^n$, we say that $u \in RH_\infty$ if there exists a constant C such that for every ball B,

$$u(x) \le \frac{C}{|B|}\int_B u(y)\,dy \qquad \text{a.e. } x \in B.$$

Denote the smallest constant C such that this inequality holds by $RH_\infty(u)$. The RH_∞ condition is satisfied by a variety of functions u: for instance, if there exist positive constants A and B such that $A \le u(x) \le B$ for all x. More generally, $u \in RH_\infty$ if $u(x) = |x|^a$, $a > 0$, or if there exists $r > 0$ such that u^{-r} is in the Muckenhoupt class A_1. For further information about RH_∞, see Cruz-Uribe and Neugebauer [2].

Theorem 1.8. *Given an open set Ω, suppose the function $p\colon \Omega \to [1,\infty)$ can be extended to $\mathbf{R}^n$ in such a way that $1/p \in RH_\infty$. Then for all $f \in L^{p(x)}(\Omega)$ and $t > 0$,*

$$|\{x \in \Omega : Mf(x) > t\}| \le C\int_\Omega \left(\frac{|f(y)|}{t}\right)^{p(y)} dy. \tag{1.6}$$

Remark 1.9. Since $p_* \ge 1$, $1/p$ is automatically in RH_∞ if $p^* < \infty$. But, as the above remarks show, this condition is not necessary.

Remark 1.10. We can give an alternative version of Theorem 1.8 which does not require extending $p(x)$ to all of $\mathbf{R}^n$, but to do so we must replace the assumption that $1/p \in RH_\infty$ with the following condition: given any ball B, $|B \cap \Omega| > 0$, and $x \in B \cap \Omega$,

$$1/p(x) \le \frac{C}{|B|}\int_{B\cap\Omega} \frac{dy}{p(y)}.$$

Note, however, that this condition need not hold if $p(x)$ is constant, and so we do not recapture the classical result. We leave the details of the proof to the reader.

Remark 1.11. In the case of the Lebesgue spaces, the strong-type inequality is deduced from the weak-type inequality via the Marcinkiewicz interpolation theorem. It would be interesting to generalize this approach to use Theorem 1.8 to prove Theorem 1.5.

We prove Theorem 1.5 in Section 2, Theorems 1.6 and 1.7 in Section 3, and Theorem 1.8 in Section 4. Throughout the proofs, notation will be standard or defined as needed. In order to emphasize that we are dealing with variable exponents, we will always write $p(x)$ instead of p to denote an exponent function. Given an open set Ω and function $p(x)$, $1 \le p(x) \le \infty$, define the conjugate function $q(x)$ to satisfy $1/p(x) + 1/q(x) = 1$, where we take $1/\infty = 0$. Given a set E, let $|E|$ denote its Lebesgue measure, and let $p_*(E) = \inf\{p(y) : y \in E\}$ and $p^*(E) = \sup\{p(y) : y \in E\}$. For brevity, let $p_* = p_*(\Omega)$ and $p^* = p^*(\Omega)$. Given a function f, let

$$|f|_{p(x),\Omega} = \int_\Omega |f(y)|^{p(y)}\, dy.$$

Finally, C and c will denote positive constants which will depend only on the dimension n, the underlying set Ω and the exponent function $p(x)$, but whose value may change at each appearance.

2. Proof of Theorem 1.5

The proof of Theorem 1.5 requires a series of lemmas. Throughout this section, let $\alpha(x) = (e + |x|)^{-n}$.

The first lemma is due to Diening [3, Lemma 3.1]. For completeness we include its short proof.

Lemma 2.1. *Given an open set Ω and a function $p\colon \Omega \to [1,\infty)$ which satisfies* (1.2), *then for any ball B such that $|B \cap \Omega| > 0$,*

$$|B|^{p_*(B\cap\Omega)-p^*(B\cap\Omega)} \le C.$$

Proof. Since $p_*(B\cap\Omega) - p^*(B\cap\Omega) \le 0$, we may assume that if r is the radius of B, then $r < \frac{1}{4}$. But in that case, (1.2) implies that

$$p^*(B\cap\Omega) - p_*(B\cap\Omega) \le \frac{C}{\log(1/2r)}.$$

Therefore,

$$|B|^{p_*(B\cap\Omega)-p^*(B\cap\Omega)} \le cr^{-n(p^*(B\cap\Omega)-p_*(B\cap\Omega))} \le cr^{-nC/\log(1/2r)} \le C. \ \square$$

Though our proof of the following lemma is not directly dependent on Nekvinda [19], our understanding of it was greatly enhanced by a close reading of his work, and we are grateful to him for sharing it with us.

Lemma 2.2. *Given a set G and two non-negative functions $r(x)$ and $s(x)$, suppose that for each $x \in G$,*

$$0 \leq s(x) - r(x) \leq \frac{C}{\log(e+|x|)}.$$

Then for every function f,

$$\int_G |f(x)|^{r(x)}\,dx \leq C\int_G |f(x)|^{s(x)}\,dx + \int_G \alpha(x)^{r_*(G)}\,dx.$$

Proof. Let $G^\alpha = \{x \in G : |f(x)| \geq \alpha(x)\}$. Then

$$\int_G |f(x)|^{r(x)}\,dx = \int_{G^\alpha} |f(x)|^{r(x)}\,dx + \int_{G\setminus G^\alpha} |f(x)|^{r(x)}\,dx,$$

and we estimate each integral separately. First, since $\alpha(x) \leq 1$,

$$\int_{G\setminus G^\alpha} |f(x)|^{r(x)}\,dx \leq \int_{G\setminus G^\alpha} \alpha(x)^{r(x)}\,dx \leq \int_G \alpha(x)^{r_*(G)}\,dx.$$

On the other hand, if $x \in G^\alpha$, then

$$|f(x)|^{r(x)} = |f(x)|^{s(x)}|f(x)|^{r(x)-s(x)} \leq |f(x)|^{s(x)}\alpha(x)^{-C/\log(e+|x|)} \leq C|f(x)|^{s(x)}.$$

The desired inequality now follows immediately. □

The next two lemmas generalize the key step in Diening's proof of Theorem 1.1 (see [3, Lemma 3.2]).

Lemma 2.3. *Given Ω and p as in the statement of Theorem 1.5, suppose that $|f|_{p(x),\Omega} \leq 1$, and $|f(x)| \geq 1$ or $f(x) = 0$, $x \in \Omega$. Then for all $x \in \Omega$,*

$$Mf(x)^{p(x)} \leq CM(|f(\cdot)|^{p(\cdot)/p_*})(x)^{p_*} + C\alpha(x)^{p_*}, \tag{2.1}$$

where $\alpha(x) = (e+|x|)^{-n}$.

Proof. Without loss of generality, we may assume that f is non-negative. Fix $x \in \Omega$, and fix a ball B of radius $r > 0$ containing x such that $|B \cap \Omega| > 0$. Let $B_\Omega = B \cap \Omega$. It will suffice to show that (2.1) holds with the left-hand side replaced by

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)},$$

and with a constant independent of B. We will consider three cases.

Case 1: $r < |x|/4$. Define $\bar{p}(x) = p(x)/p_*$. Then $\bar{p}(x) \geq 1$, and (1.4) holds with p replaced by $\bar{p}$. In particular, by our assumption on r, if $y \in B_\Omega$,

$$0 \leq \bar{p}(y) - \bar{p}_*(B_\Omega) \leq \frac{C}{\log(e+|y|)}. \tag{2.2}$$

Therefore, by Hölder's inequality and by Lemma 2.2 with $r(x)$ replaced by the constant $\bar{p}_*(B_\Omega)$ and $s(x)$ by $\bar{p}(y)$, we have that

$$\begin{aligned}\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} &\leq \left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}_*(B_\Omega)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)}\\ &\leq \left(\frac{C}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy + \frac{1}{|B|}\int_{B_\Omega} \alpha(y)^{\bar{p}_*(B_\Omega)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)};\end{aligned}$$

since $r < |x|/4$ and $p(x)/\bar{p}_*(B_\Omega) \leq p^* < \infty$,

$$\begin{aligned}&\leq \left(\frac{C}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy + C\alpha(x)^{\bar{p}_*(B_\Omega)}\right)^{p(x)/\bar{p}_*(B_\Omega)}\\ &\leq 2^{p^*}C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)} + 2^{p^*}C\alpha(x)^{p(x)}.\end{aligned}$$

If $|B| \geq 1$, then by Hölder's inequality and since $|f|_{p(x),\Omega} \leq 1$,

$$\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy \leq \left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{1/p_*} \leq \left(\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{1/p_*} \leq 1.$$

Hence, since $p(x)/\bar{p}_*(B_\Omega) \geq p_*$ and $\alpha(x) \leq 1$, we have that

$$\begin{aligned}\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} &\leq C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p_*} + C\alpha(x)^{p_*}\\ &\leq CM(f(\,\cdot\,)^{\bar{p}(\,\cdot\,)})(x)^{p_*} + C\alpha(x)^{p_*}.\end{aligned}$$

If, on the other hand, $|B| \leq 1$, then, again since $|f|_{p(x),\Omega} \leq 1$,

$$\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy \leq |B_\Omega|^{1/p_*'}\left(\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{1/p_*} \leq 1.$$

Therefore,

$$\begin{aligned}\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} &\leq C|B|^{-p(x)/\bar{p}_*(B_\Omega)}\left(\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)} + C\alpha(x)^{p_*}\\ &\leq C|B|^{-p(x)/\bar{p}_*(B_\Omega)+p_*}\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p_*} + C\alpha(x)^{p_*}.\end{aligned}$$

Since $|B| \leq 1$, and since

$$\begin{aligned} -p(x)/\bar{p}_*(B_\Omega) + p_* &= \big(p_*/p_*(B_\Omega)\big)\big(p_*(B_\Omega) - p(x)\big) \\ &\geq \big(p_*/p_*(B_\Omega)\big)\big(p_*(B_\Omega) - p^*(B_\Omega)\big), \end{aligned}$$

by Lemma 2.1,

$$\leq C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p_*} + C\alpha(x)^{p_*} \leq CM\big(f(\,\cdot\,)^{\bar{p}(\,\cdot\,)}\big)(x)^{p_*} + C\alpha(x)^{p_*}.$$

This is precisely what we wanted to prove.

Case 2: $|x| \leq 1$ and $r \geq |x|/4$. The proof is essentially the same as in the previous case: since $|x| \leq 1$, $\alpha(x) \approx 1$, so inequality (2.2) and the subsequent argument still hold.

Case 3: $|x| \geq 1$ and $r \geq |x|/4$. Since $f(x) \geq 1$, $p_* \geq 1$ and $|f|_{p(x),\Omega} \leq 1$,

$$\begin{aligned} \left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} &\leq |B|^{-p(x)}\left(\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{p(x)} \\ &\leq Cr^{-np(x)}|f|^{p(x)}_{p(x),\Omega} \leq C|x|^{-np_*} \leq C\alpha(x)^{p_*} \\ &\leq CM\big(f(\,\cdot\,)^{\bar{p}(\,\cdot\,)}\big)(x)^{p_*} + C\alpha(x)^{p_*}. \end{aligned}$$

This completes the proof. □

Definition 2.4. Given a function f on Ω, we define the Hardy operator H by

$$Hf(x) = |B_{|x|}(0)|^{-1}\int_{B_{|x|}(0)\cap\Omega} |f(y)|\,dy.$$

Lemma 2.5. *Given Ω and p as in the statement of Theorem* 1.5, *suppose that $|f|_{p(x),\Omega} \leq 1$, and $|f(x)| \leq 1$, $x \in \Omega$. Then for all $x \in \Omega$,*

$$Mf(x)^{p(x)} \leq CM\big(|f(\,\cdot\,)|^{p(\,\cdot\,)/p_*}\big)(x)^{p_*} + C\alpha(x)^{p_*} + CHf(x)^{p(x)}, \tag{2.3}$$

where $\alpha(x) = (e + |x|)^{-n}$.

Proof. We may assume without loss of generality that f is non-negative. We argue almost exactly as we did in the proof of Lemma 2.3. In that proof we only used the fact that $f(x) \geq 1$ in Case 3, so it will suffice to fix $x \in \Omega$, $|x| \geq 1$, and a ball B containing x with radius $r > |x|/4$, and prove that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} \leq CM\big(|f(\,\cdot\,)|^{p(\,\cdot\,)/p_*}\big)(x)^{p_*} + C\alpha(x)^{p_*} + CHf(x)^{p(x)}.$$

Since $p^* < \infty$, we have that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} \le 2^{p^*}\left(\frac{1}{|B|}\int_{B_\Omega\cap B_{|x|}(0)} f(y)\,dy\right)^{p(x)} + 2^{p^*}\left(\frac{1}{|B|}\int_{B_\Omega\setminus B_{|x|}(0)} f(y)\,dy\right)^{p(x)};$$

since $r > |x|/4$,

$$\le C\left(|B_{|x|}(0)|^{-1}\int_{B_{|x|}(0)\cap\Omega} |f(y)|\,dy\right)^{p(x)} + C\left(\frac{1}{|B|}\int_{B_\Omega\setminus B_{|x|}(0)} f(y)\,dy\right)^{p(x)}$$
$$= CHf(x)^{p(x)} + C\left(\frac{1}{|B|}\int_{B_\Omega\setminus B_{|x|}(0)} f(y)\,dy\right)^{p(x)}.$$

To estimate the last term, note that if $y \in B_\Omega \setminus B_{|x|}(0)$ then (2.2) holds and $\alpha(y) \le \alpha(x)$, so the argument in Case 1 of the proof of Lemma 2.3 goes through. This shows that

$$\left(\frac{1}{|B|}\int_{B_\Omega\setminus B_{|x|}(0)} f(y)\,dy\right)^{p(x)} \le CM\big(|f(\cdot)|^{p(\cdot)/p_*}\big)(x)^{p_*} + C\alpha(x)^{p_*},$$

and this completes the proof. □

Lemma 2.6. *If $i(x)$ is a radial, increasing function, $i_* > 1$, and if $|f(x)| \le 1$, then*

$$\int_\Omega Hf(y)^{i(y)}\,dy \le C\big(n, i(x)\big)\int_\Omega |f(y)|^{i(y)}\,dy.$$

Proof. Without loss of generality we may assume that f is non-negative. Also, for clarity of notation, we extend f to all of $\mathbf{R}^n$ by setting it equal to zero on $\mathbf{R}^n \setminus \Omega$.

We first assume only that $i_* \ge 1$. Recall that $|B_{|x|}(0)| = |B_1(0)|\,|x|^n$. Let S denote the unit sphere in $\mathbf{R}^n$. Then by switching to polar coordinates and making a change of variables, we get that

$$Hf(x)^{i(x)} = \left(|B_1(0)|^{-1}|x|^{-n}\int_{B_{|x|}(0)} f(y)\,dy\right)^{i(x)}$$
$$= \left(|B_1(0)|^{-1}|x|^{-n}\int_S\int_0^{|x|} f(r\theta)r^{n-1}\,dr\,d\theta\right)^{i(x)}$$
$$= \left(|B_1(0)|^{-1}\int_S\int_0^1 f(|x|r\theta)r^{n-1}\,dr\,d\theta\right)^{i(x)}$$

$$
\begin{aligned}
&= \left(|B_1(0)|^{-1} \int_{B_1(0)} f(|x|y)\,dy\right)^{i(x)} \\
&\le |B_1(0)|^{-1} \int_{B_1(0)} f(|x|y)^{i(x)}\,dy,
\end{aligned}
$$

by Hölder's inequality.

Now let $r > 1$; the exact value of r will be chosen below. By Minkowski's integral inequality, and again by switching to polar coordinates,

$$
\begin{aligned}
\|Hf(\,\cdot\,)^{i(\,\cdot\,)}\|_{r,\mathbf{R}^n} &\le C\left(\int_{\mathbf{R}^n}\left(\int_{B_1(0)} f(|x|y)^{i(x)}\,dy\right)^r dx\right)^{1/r} \\
&\le C\int_{B_1(0)}\left(\int_{\mathbf{R}^n} f(|x|y)^{ri(x)}\,dx\right)^{1/r} dy \\
&= C\int_S\int_0^1\left(\int_{\mathbf{R}^n} f(|x|s\theta)^{ri(x)}\,dx\right)^{1/r} s^{n-1}\,ds\,d\theta \\
&= C\int_S\int_0^1 s^{-n/r}\left(\int_{\mathbf{R}^n} f(|x|\theta)^{ri(x/s)}\,dx\right)^{1/r} s^{n-1}\,ds\,d\theta,
\end{aligned}
$$

by a change of variables in the inner integral. Since i is a radial increasing function, $i(x/s) \ge i(x)$; since $f(|x|\theta) \le 1$,

$$
\begin{aligned}
&\le C\int_S\int_0^1 s^{-n/r}\left(\int_{\mathbf{R}^n} f(|x|\theta)^{ri(x)}\,dx\right)^{1/r} s^{n-1}\,ds\,d\theta \\
&\le C\int_S\left(\int_{\mathbf{R}^n} f(|x|\theta)^{ri(x)}\,dx\right)^{1/r} d\theta.
\end{aligned}
$$

Since S has constant, finite measure, by Hölder's inequality,

$$
\le C\left(\int_S\int_{\mathbf{R}^n} f(|x|\theta)^{ri(x)}\,dx\,d\theta\right)^{1/r}.
$$

Since i is a radial function, if we rewrite the inner integral in polar coordinates, we get that

$$
\begin{aligned}
&= C\left(\int_S\int_S\int_0^\infty f(u\theta)^{ri(u)}u^{n-1}\,du\,d\phi\,d\theta\right)^{1/r} \\
&= C\left(\int_S\int_0^\infty f(u\theta)^{ri(u)}u^{n-1}\,du\,d\theta\right)^{1/r} = C\left(\int_{\mathbf{R}^n} f(y)^{ri(y)}\,dy\right)^{1/r}.
\end{aligned}
$$

To complete the proof, we repeat the above argument with $i(x)$ replaced by $\bar{i}(x) = i(x)/i_*$ and with $r = i_*$, since $i_* > 1$. □

Remark 2.7. While Theorem 1.5 shows that we must have $p^* < \infty$ for the norm inequality to be true in general, we do not need this assumption in restricted cases. If f is a bounded, radial, decreasing function, then $Mf(x) \approx Hf(x)$, and so it follows from Lemma 2.6 that if p is a radial increasing function, $\|Mf\|_{p(x),\Omega} \leq C\|f\|_{p(x),\Omega}$.

Proof of Theorem 1.5. Without loss of generality we may assume that f is non-negative. We first show there exists a constant C such that if $|f|_{p(x),\Omega} \leq 1$, then $|Mf|_{p(x),\Omega} \leq C$. Fix f, $|f|_{p(x),\Omega} \leq 1$. Let $f = f_1 + f_2$, where

$$f_1(x) = f(x)\chi_{\{x: f(x)\geq 1\}}(x).$$

Then for $i = 1, 2$, $|f_i|_{p(x),\Omega} \leq 1$. Since $p^* < \infty$,

$$\int_\Omega Mf(y)^{p(y)}\,dy \leq 2^{p^*}\int_\Omega Mf_1(y)^{p(y)}\,dy + 2^{p^*}\int_\Omega Mf_2(y)^{p(y)}\,dy.$$

We will show that each integral on the right-hand side is bounded by a constant. Since $|f_2(x)| \leq 1$, by Lemma 2.5, f_2 satisfies inequality (2.3). Therefore, if we integrate over Ω we get that

$$\begin{aligned}\int_\Omega Mf_2(y)^{p(y)}\,dy &\leq C\int_\Omega M\big(f_2(\,\cdot\,)^{p(\,\cdot\,)/p_*}\big)(y)^{p_*}\,dy + C\int_\Omega \alpha(y)^{p_*}\,dy\\ &\quad + C\int_\Omega Hf_2(y)^{p(y)}\,dy.\end{aligned}$$

Since $p_* > 1$, M is bounded on $L^{p_*}(\Omega)$ and $\alpha(x) \in L^{p_*}(\Omega)$, so

$$\leq C\int_\Omega f_2(y)^{p(y)}\,dy + C + C\int_\Omega Hf_2(y)^{p(y)}\,dy \leq C + C\int_\Omega Hf_2(y)^{p(y)}\,dy.$$

Given a function p, define its increasing, radial minorant i_p to be the function

$$i_p(x) = \inf_{|y|\geq|x|} p(y).$$

Clearly, i_p is a radial, increasing function. Further, (1.4) implies that for all $x \in \Omega$,

$$0 \leq p(x) - i_p(x) \leq \frac{C}{\log(e + |x|)}.$$

Therefore, since $f_2(x) \leq 1$ and $(i_p)_* = p_*$, by Lemmas 2.6 and 2.2,

$$\begin{aligned}\int_\Omega Hf_2(y)^{p(y)}\,dy &\leq C\int_\Omega Hf_2(y)^{i_p(y)}\,dy \leq C\int_\Omega f_2(y)^{i_p(y)}\,dy\\ &\leq C\int_\Omega f_2(y)^{p(y)}\,dy + C\int_\Omega \alpha(y)^{p_*}\,dy \leq C.\end{aligned}$$

Hence, $|Mf_2|_{p(x),\Omega} \le C$.

A very similar argument using Lemma 2.3 shows that $|Mf_1|_{p(x),\Omega} \le C$.

Therefore, we have shown that if $|f|_{p(x),\Omega} \le 1$, then $|Mf|_{p(x),\Omega} \le C$. Since $C > 1$, it follows that

$$\int_\Omega \big(C^{-1}Mf(x)\big)^{p(x)}\,dx \le 1,$$

which in turn implies that

$$\|Mf\|_{p(x),\Omega} \le C.$$

To complete the proof we fix a function $g \in L^{p(x)}(\Omega)$, and let $f(x) = g(x)/\|g\|_{p(x),\Omega}$. Then $\|f\|_{p(x),\Omega} \le 1$, so $|f|_{p(x),\Omega} \le 1$. Hence,

$$\|Mg\|_{p(x),\Omega} = \|g\|_{p(x),\Omega}\|Mf\|_{p(x),\Omega} \le C\|g\|_{p(x),\Omega}. \ \square$$

3. Proofs of Theorems 1.6 and 1.7

Proof of Theorem 1.6. Our proof is closely modeled on the construction given by Pick and Růžička in [20].

By inequality (1.5), we have that

$$\lim_{x\to\infty} \left(1 - \frac{p_\infty}{p(2x)}\right)\log(x) = -\infty,$$

which in turn implies that

$$\lim_{x\to\infty} x^{1-p_\infty/p(2x)} = 0.$$

Therefore, we can form a sequence $\{c_n\}_{n=1}^\infty$, $c_{n+1} < 2c_n \le -1$, such that

$$|c_n|^{1-p_\infty/p(2|c_n|)} \le 2^{-n}.$$

Let $d_n = 2c_n < c_n$, and define the function f on $\mathbf{R}$ by

$$f(x) = \sum_{n=1}^\infty |c_n|^{-1/p(|d_n|)}\chi_{(d_n,c_n)}(x).$$

We claim that $|f|_{p(x),\mathbf{R}} \le 1$ and $|Mf|_{p(x),\mathbf{R}} = \infty$; it follows immediately from this that $\|f\|_{p(x),\mathbf{R}} \le 1$ and $\|Mf\|_{p(x),\mathbf{R}} = \infty$, so the maximal operator is not bounded on $L^{p(x)}(\mathbf{R})$. First, we have that

$$\begin{aligned}|f|_{p(x),\mathbf{R}} &= \sum_{n=1}^\infty \int_{d_n}^{c_n} |c_n|^{-p(x)/p(|d_n|)}\,dx = \sum_{n=1}^\infty \int_{d_n}^{c_n} |c_n|^{-p_\infty/p(|d_n|)}\,dx \\ &= \sum_{n=1}^\infty |c_n|^{1-p_\infty/p(|d_n|)} \le \sum_{n=1}^\infty 2^{-n} = 1.\end{aligned}$$

On the other hand, if $x \in (|c_n|, |d_n|)$, then

$$\begin{aligned} Mf(x) &\geq \frac{1}{2|d_n|} \int_{d_n}^{|d_n|} f(y)\,dy \geq \frac{1}{2|d_n|} \int_{d_n}^{c_n} f(y)\,dy \\ &= \frac{|c_n|^{1-1/p(|d_n|)}}{2|d_n|} = \frac{1}{4}|c_n|^{-1/p(|d_n|)}. \end{aligned}$$

Therefore, since $p(x)$ is an increasing function and $|c_n| \geq 1$,

$$\begin{aligned} |Mf|_{p(x),\mathbf{R}} &\geq \frac{1}{4} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(x)/p(|d_n|)} \geq \frac{1}{4} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(|d_n|)/p(|d_n|)} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} 1 = \infty. \ \square \end{aligned}$$

Proof of Theorem 1.7. Fix $k \geq 1$. Since $p_* = 1$, Ω is open and p is upper semi-continuous, there exists $x_k \in \Omega$ and $\varepsilon_k > 0$ such that $B_k = B_{\varepsilon_k}(x_k) \subset \Omega$, and such that if $x \in B_k$, $p(x) < 1 + 1/k$. We define the function $f_k(x) = |x_k - x|^{-nk/(k+1)} \chi_{B_k}(x)$. Then $f_k \in L^{p(x)}(\Omega)$. On the other hand, for $x \in B_k$, let $r = |x - x_k|$; then

$$Mf_k(x) \geq \frac{c}{|B_r(x_k)|} \int_{B_r(x_k)} f_k(y)\,dy = c(k+1) f_k(x).$$

Hence, $\|Mf_k\|_{p(x),\Omega} \geq c(k+1)\|f_k\|_{p(x),\Omega}$; since we may take k arbitrarily large, the maximal operator is not bounded on $L^{p(x)}(\Omega)$. $\square$

4. Proof of Theorem 1.8

We begin with a lemma which, intuitively, plays the role that Hölder's inequality does in the standard proof that the maximal operator is weak (p,p).

Lemma 4.1. *Given an open set Ω, a function p: $\mathbf{R}^n \to [1,\infty)$ such that $1/p$ is locally integrable, f in $L^{p(x)}(\Omega)$ and $t > 0$, suppose that B is a ball such that*

$$\frac{1}{|B|} \int_{B\cap\Omega} |f(y)|\,dy > t.$$

Then

$$\int_B \frac{dx}{p(x)} \leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy.$$

Proof. Fix a sequence of simple functions $\{s_n(x)\}$ on B, such that $s_n(x) \geq p_*(B)$ and such that the sequence increases monotonically to $p(x)$ on B. For each n we have that

$$s_n(x) = \sum_{j=1}^{k_n} \alpha_{n,j} \chi_{A_{n,j}}(x),$$

where the $A_{n,j}$'s are disjoint sets whose union is B. Let $t_n(x)$ be the conjugate function associated to $s_n(x)$; then $t_n(x)$ decreases to $q(x)$, the conjugate function of $p(x)$.

By Hölder's and Young's inequalities,

$$\begin{aligned}
\int_{B\cap\Omega} \frac{|f(y)|}{t}\,dy &= \sum_{j=1}^{k_n} \int_{A_{n,j}\cap\Omega} \frac{|f(y)|}{t}\,dy \\
&\leq \sum_{j=1}^{k_n} \left(\int_{A_{n,j}\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} dy \right)^{1/\alpha_{n,j}} |A_{n,j}|^{1/\alpha'_{n,j}} \\
&\leq \sum_{j=1}^{k_n} \left(\frac{1}{\alpha_{n,j}} \int_{A_{n,j}\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} dy + \frac{|A_{n,j}|}{\alpha'_{n,j}} \right) \\
&\leq \sum_{j=1}^{k_n} \left(\frac{1}{p_*(B)} \int_{A_{n,j}\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{s_n(y)} dy + \int_{A_{n,j}} \frac{dy}{t_n(y)} \right) \\
&\leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{s_n(y)} dy + \int_B \frac{dy}{t_n(y)}.
\end{aligned}$$

Since this is true for all n, by the monotone convergence theorem,

$$\int_B \frac{|f(y)|}{t}\,dy \leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy + \int_B \frac{dy}{q(y)}.$$

Therefore,

$$\begin{aligned}
\int_B \frac{dy}{p(y)} = |B| - \int_B \frac{dy}{q(y)} &< \int_{B\cap\Omega} \frac{|f(y)|}{t}\,dy - \int_B \frac{dy}{q(y)} \\
&\leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy. \quad \square
\end{aligned}$$

Proof of Theorem 1.8. For each $N > 0$, define the operator M_N by

$$M_N f(x) = \sup \frac{1}{|B|} \int_{B\cap\Omega} |f(y)|\,dy,$$

where the supremum is taken over all balls containing x such that $|B| \le N$. The sequence $\{M_N f(x)\}$ is increasing and converges to $Mf(x)$ for each $x \in \Omega$. Thus, by the monotone convergence theorem, for each $t > 0$,

$$|\{x \in \Omega : Mf(x) > t\}| = \lim_{N\to\infty} |\{x \in \Omega : M_N f(x) > t\}|.$$

Therefore, it will suffice to prove (1.6) with M replaced by M_N, and with a constant independent of N.

Fix $t > 0$ and let $E_N = \{x \in \Omega : M_N f(x) > t\}$. Then for each $x \in E_N$, there exists a ball B_x containing x, $|B_x| \le N$, such that

$$\frac{1}{|B_x|}\int_{B_x\cap\Omega} |f(y)|\,dy > t.$$

By a weak variant of the Vitali covering lemma (cf. Stein [23, p. 9]), there exists a collection of disjoint balls, $\{B_k\}$, contained in $\{B_x : x \in E_N\}$, and a constant C depending only on the dimension n, such that

$$|E_n| \le C\sum_k |B_k|.$$

Therefore, by Lemma 4.1,

$$\begin{aligned}|E_N| &\le C\sum_k |B_k| \le \sum_k |B_k|\left(\int_{B_k}\frac{dy}{p(y)}\right)^{-1}\int_{B_k}\frac{dy}{p(y)}\\ &\le \sum_k\left(\frac{1}{|B_k|}\int_{B_k}\frac{dy}{p(y)}\right)^{-1}\frac{1}{p_*(B_k)}\int_{B_k\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{p(y)}dy;\end{aligned}$$

since $p_*(B_k)^{-1} = (1/p)^*(B_k)$, by the definition of RH_∞,

$$\le RH_\infty(1/p)\sum_k\int_{B_k\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{p(y)}dy \le C\int_\Omega\left(\frac{|f(y)|}{t}\right)^{p(y)}dy. \ \square$$

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