THE MAXIMUM DEGREE OF THE BARABÁSI–ALBERT RANDOM TREE

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In a one-parameter model for evolution of random trees, which also includes the Barabási–Albert random graph [1], law of large numbers and central limit theorem are proved for the maximal degree. In the proofs martingale methods are applied.

1. Introduction

In the classical Erdős–Rényi model of random graphs, when the number of edges is proportional to the number of vertices, the degree distribution is approximately Poisson with a tail decreasing even faster than exponentially. However, in many real life networks power law degree distributions were observed with different exponents. To introduce a more realistic model for the evolution of random networks, Barabási and Albert [1] proposed the following one, which they called scale free.

In the beginning, at the first step, we only have a single edge. At every further step we start a new (undirected) edge from one of the vertices created so far. The other endpoint of the edge is a new vertex, while the starting point is chosen from the existing vertices at random, in such a way that each vertex is selected with probability proportional to its degree (in other words, to choose an existing vertex we first choose one of the edges with equal probability, then one of the endpoints of that edge). In this model the asymptotic proportion of vertices with degree k decreases as k^{-3}, which is the same power law that was observed in the World Wide Web. A couple of papers has recently been devoted to the study of this random graph as well as to other similar models, all different from the classical Erdős–Rényi construction. Here we only mention [2].

A generalization of this model was investigated in [9]. There, at the n-th step, a vertex of degree k was chosen with probability proportional to $k+\beta$, where β was a fixed parameter of the model, $\beta > -1$. Thus, a vertex of degree k was selected with probability $\dfrac{k+\beta}{S_n}$, where S_n denoted the sum of weights over all vertices of the random tree with n edges and $n+1$ vertices; that is, $S_n = 2n+(n+1)\beta = (2+\beta)n+\beta$. In [9] the proportion of vertices of degree k was shown to converge almost surely

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to a limit c_k, which, as a function of k, decreased at the rate $k^{-(3+\beta)}$. It turned out, in addition, that the number of degree k vertices had an asymptotic normal distribution around nc_k with variance of order n.

The aim of the present note is to show that maximal degree M_n of the random tree, divided by $n^{\frac{1}{2+\beta}}$, converges a.s. to a positive random variable, as n tends to infinity. In Erdős-Rényi graphs of n vertices and cn edges M_n is asymptotically equal to $\log n/\log\log n$.

2. Basic martingales

Our starting point is common with [2] to a certain extent: we first consider the degree sequences of individual vertices. Let the vertices of the only edge existing at start be labelled by 0 and 1, then every new vertex is labelled by step number when it is born. Let $X[n,j]$ denote the weight ($=$ degree $+\,\beta$) of vertex j after the n-th step, with initial values $X[n,j]=0$ for $n<j$, $X[j,j]=1+\beta$ for $j>0$, and $X[1,0]=1+\beta$. Let $\Delta[n+1,j]$ denote the increment $X[n+1,j]-X[n,j]$.

Let us introduce a double sequence of normalizing constants by

$$c[n,k]=\frac{\Gamma\Bigl(n+\frac{\beta}{2+\beta}\Bigr)}{\Gamma\Bigl(n+\frac{k+\beta}{2+\beta}\Bigr)},\quad n\ge 1,\ k\ge 0. \tag{2.1}$$

Then we clearly have

$$\frac{c[n+1,k]}{c[n,k]}=\frac{S_n}{S_n+k};$$

and $c[n,k]=n^{-\frac{k}{2+\beta}}\bigl(1+O\bigl(n^{-1}\bigr)\bigr)$, for fixed k, as $n\to\infty$.

Finally, let $\mathcal{F}_n$ denote the σ-field generated by the first n (random) steps.

The degree process is driven by the basic dynamics

$$P\bigl(\Delta[n+1,j]=1\big|\mathcal{F}_n\bigr)=1-P\bigl(\Delta[n+1,j]=0\big|\mathcal{F}_n\bigr)=\frac{X[n,j]}{S_n}. \tag{2.2}$$

Hence one immediately gets that

$$E\bigl(X[n+1,j]\big|\mathcal{F}_n\bigr)=X[n,j]\Bigl(1+\frac{1}{S_n}\Bigr)=X[n,j]\,\frac{c[n,1]}{c[n+1,1]}.$$

Thus

$$\Bigl(c[n,1]X[n,j],\ \mathcal{F}_n\Bigr),\quad n\ge\max\{j,1\} \tag{2.3}$$

is a positive martingale, which is known to converge a.s. to a random variable ζ_j.

This is surely not new; in the case of $\beta=0$ this martingale was also considered in [2], but even that was not the first place where it appeared. When β is an integer number, $X[n,j]$ can be considered as the number of white balls in a generalized Pólya–Eggenberger urn model after $n-j$ draws, where the urn initially contains $2j-1+j\beta$ black balls and $1+\beta$ white ones; having drawn a white ball we return it into the urn together with one white and $1+\beta$ black balls, while in case of drawing a black ball we only put into the urn $2+\beta$ additional black balls and no white ones. Generalized Pólya–Eggenberger urns have been studied exhaustively in the last decades. Section 4.3 of the monograph [6] is devoted to certain particular cases of the general model, but it still does not cover our case. [3] applies martingale methods for a.s. limit results for the proportion of white balls as the number of draws goes to infinity. In our case the limit of the proportion is clearly 0; the real question is the proper order of magnitude. That can be found e.g. in [4] and [5]. Martingale techniques are quite common in all these papers.

The martingale property in (2.3) is just a particular case of the following, more general result.

Theorem 2.1. *Let r be a positive integer, $k_1, \dots k_r$ and $0 \le j_1 < \dots < j_r$ arbitrary nonnegative integers. Introduce*

$$Z[n;\ j_1, \dots, j_r,\ k_1, \dots, k_r] = c[n, k_1 + \dots + k_r] \prod_{i=1}^{r} \binom{X[n, j_i] + k_i - 1}{k_i}.$$

Then

$$\Big(Z[n;\ j_1, \dots, j_r,\ k_1, \dots, k_r],\ \mathcal{F}_n \Big), \qquad n \ge \max\{j_r, 1\}$$

is a martingale.

Proof. It can be assumed that each k_i is positive. Clearly,

$$\begin{aligned}
\binom{X[n+1, j] + k - 1}{k} &= \binom{X[n, j] + k - 1}{k} + \Delta[n, j] \binom{X[n, j] + k - 1}{k - 1} \\
&= \binom{X[n, j] + k - 1}{k} \left(1 + \frac{k \Delta[n, j]}{X[n, j]} \right),
\end{aligned}$$

hence

$$\prod_{i=1}^{r} \binom{X[n+1, j_i] + k_i - 1}{k_i} = \prod_{i=1}^{r} \binom{X[n, j_i] + k_i - 1}{k_i} \left(1 + \sum_{i=1}^{r} \frac{\Delta[n, j_i] k_i}{X[n, j_i]} \right),$$

because at most one of the increments $\Delta[n, j_i]$ can differ from 0 at the same time. Consequently,

$$\begin{aligned}
E\bigg(Z[n+1;\ j_1, \dots, j_r,\ k_1, \dots, k_r] \,\bigg|\, \mathcal{F}_n \bigg) &= \\
&= Z[n;\ j_1, \dots, j_r,\ k_1, \dots, k_r] \, \frac{c[n+1, k_1 + \dots + k_r]}{c[n, k_1 + \dots + k_r]} \left(1 + \frac{k_1 + \dots + k_r}{S_n} \right) \\
&= Z[n;\ j_1, \dots, j_r,\ k_1, \dots, k_r],
\end{aligned}$$

as claimed.

Being a nonnegative martingale, $Z[n;\ j_1, \dots, j_r,\ k_1, \dots, k_r]$ is bounded in L_1. It converges a.s. to

$$\frac{\zeta_{j_1}^{k_1} \dots \zeta_{j_r}^{k_r}}{k_1! \dots k_r!},$$

whatever the exponents k_i be. Since

$$Z[n;\ j_1,\dots,j_r,\ k_1,\dots,k_r]^2 \le C\, Z[n;\ j_1,\dots,j_r,\ 2k_1,\dots,2k_r],$$

where the constant C does not depend on n, this martingale is bounded in L_2 as well. This implies that it converges in L_1, too. Using that fact one can easily write down all moments and mixed moments of the random variables $\zeta_0,\zeta_1,\dots$, together with their joint Laplace transform (or characteristic function).

$$\frac{E\zeta_j^k}{k!} = \lim_{n\to\infty} EZ[n;\ j,k] = EZ[j;\ j,k] = c[j,k]\binom{k+\beta}{k},\ j\ge 1, \tag{2.4}$$

particularly

$$E\zeta_j = (1+\beta)c[j,1], \tag{2.5}$$

$$\begin{aligned}\operatorname{var}\zeta_j &= (1+\beta)(2+\beta)c[j,2]-(1+\beta)^2c[j,1]^2\\ &= (E\zeta_j)^2\Big(\frac{c[j,2]}{c[j,1]^2}-1\Big)+(1+\beta)c[j,2],\end{aligned} \tag{2.6}$$

$$E\exp(-t\zeta_j) = \sum_{k=0}^{\infty} c[j,k]\binom{k+\beta}{k}(-t)^k = \sum_{k=0}^{\infty} c[j,k]\binom{-\beta}{k}t^k. \tag{2.7}$$

Note that ζ_0 and ζ_1 have the same distribution (and, in fact, the two random variables are interchangeable in the sequence $\zeta_0,\zeta_1,\zeta_2,\dots$).

The mixed moments can be computed by iteration.

$$\begin{aligned}\frac{E\zeta_0^{k_0}\zeta_1^{k_1}\dots\zeta_r^{k_r}}{k_0!k_1!\dots k_r!} &= EZ[r;\ 0,\dots,r,k_0\dots,k_r]\\ &= \frac{c[r;\ k_0+\dots+k_r]}{c[r;\ k_0+\dots+k_{r-1}]}\binom{k_r+\beta}{k_r}EZ[r;\ 0,\dots,r-1,k_0,\dots k_{r-1}]\\ &= \frac{c[r;\ k_0+\dots+k_r]}{c[r;\ k_0+\dots+k_{r-1}]}\binom{k_r+\beta}{k_r}EZ[r-1;\ 0,\dots,r-1,k_0,\dots k_{r-1}]\\ &= \ \cdots\ = \prod_{i=0}^{r}\binom{k_i+\beta}{k_i}\prod_{i=1}^{r}\frac{c[i;\ k_0+\dots+k_i]}{c[i;\ k_0+\dots+k_{i-1}]}\,c[1,k_0].\end{aligned} \tag{2.8}$$

Hence the covariances are

$$\operatorname{cov}(\zeta_i,\zeta_j) = E\zeta_iE\zeta_j\Big(\frac{c[j,2]}{c[j,1]^2}-1\Big),\quad 0\le i<j. \tag{2.9}$$

By the convexity of the function $\psi(z)=\log\Gamma(z)$ it follows that the random variables $\zeta_0,\zeta_1,\zeta_2,\dots$ are negatively correlated, as expected.

3. Almost sure convergence of the maximal degree

Let M_n denote the maximal degree appearing in our random tree after n steps, and for $n\ge j$ let $M[n,j]=\max\{Z[n;\ i,1]: 0\le i\le j\}$. In these terms $M[n,n]=c[n,1](M_n+\beta)$. Define $\mu(j)=\max\{\zeta_i: 0\le i\le j\}$, and $\mu=\mu(\infty)=\sup_{j\ge0}\zeta_j$. First we show that

Theorem 3.1. *With probability* 1 *we have*

$$\lim_{n\to\infty} n^{-\frac{1}{2+\beta}} M_n = \mu;$$

the limit is a.s. positive and finite, and it has an absolutely continuous distribution. The convergence also holds in L_p*, for all* p*,* $1 \le p < \infty$*.*

Proof. Being the maximum of martingales, $\big(M[n,n],\ \mathcal{F}_n\big)$ is a (non-negative) submartingale, for which

$$EM[n,n]^k \le \sum_{j=0}^{n} EZ[n;\ j,1]^k \le \sum_{j=0}^{\infty} E\zeta_j^k = k!\binom{k+\beta}{k}\sum_{j=0}^{\infty} c[j,k] < \infty,$$

if k is large enough ($k > 2+\beta$). Here $c[0,k]$ is defined to be equal to $c[1,k]$. Thus, our submartingale is bounded in L_k for every positive integer k, which implies not only almost sure convergence but convergence in L_p, $p \ge 1$, as well.

Let again $k > 2+\beta$, fixed. Then clearly

$$E\big(M[n,n] - M[n,j]\big)^k \le \sum_{i=j+1}^{n} EZ[n;i,1]^k,$$

for $\ge k$. The limit as $n \to \infty$ of the left-hand side is equal to

$$E\Big(\lim_{n\to\infty} n^{-\frac{1}{2+\beta}} M_n - \mu(j)\Big)^k;$$

while the right-hand side converges increasingly to

$$\sum_{i=j+1}^{\infty} E\zeta_i^k = k!\binom{k+\beta}{k}\sum_{i=j+1}^{\infty} c[i,k],$$

which can be arbitrarily small if j is large enough. Hence the limit of $n^{-\frac{1}{2+\beta}} M_n$ is just μ, as claimed.

The positivity of μ can be proved, e.g., by showing that any of the Laplacians (2.7) tends to 0 as $t \to \infty$. Instead, we choose another, more direct way, which also provides a lower estimation of the probabilities $P(\zeta_j > t)$. By the equivalence of ζ_0 and ζ_1 we can assume $j > 0$. We are going to prove the following estimation.

Lemma 3.1. *Let the random variable* ξ *have a positive stable distribution with characteristic exponent* $1/(2+\beta)$*; more precisely, let the Laplace transform of* ξ *be of the form* $\exp\big(-\kappa t^{1/(2+\beta)}\big)$*,* $t \ge 0$*, where*

$$\kappa = \int_0^{\infty} \Big(1 - \exp\big(-y^{-2-\beta}\big)\Big).$$

Case (i): $\beta \ge 0$*. Then* ζ_j *is minorized stochastically by* $(j\xi)^{-\frac{1}{2+\beta}}$*.*

Case (ii): $\beta < 0$*. Then* ζ_j *is minorized stochastically by* $\varrho\big((j+1)\xi\big)^{-\frac{1}{2+\beta}}$*, where* ϱ *is independent of* ξ*, and* $P(\varrho \le t) = t^{1+\beta}$*,* $0 \le t \le 1$*.*

Recall that random variable Y_1 is said to minorize Y_2 stochastically (or equivalently, Y_2 majorizes Y_1), if for every real t we have $P(Y_1 > t) \le P(Y_2 > t)$.

Proof of Lemma 3.1. Let $T[j,k]$ denote the number of steps needed until the degree of vertex j reaches k. Then $T[j,1]=j$; and for arbitrary $\beta>-1$ and $t>n$ we have

$$P\left(T[j,k+1]>t \,\middle|\, T[j,k]=n\right)=\prod_{i=n}^{\lfloor t\rfloor-1}\Bigl(1-\frac{k+\beta}{S_i}\Bigr).$$

On the right-hand side we have

$$1-\frac{k+\beta}{S_i}\le\left(\exp\Bigl(\frac{2+\beta}{S_i}\Bigr)\right)^{-\frac{k+\beta}{2+\beta}}\le\left(1+\frac{2+\beta}{S_i}\right)^{-\frac{k+\beta}{2+\beta}}=\left(\frac{S_i}{S_{i+1}}\right)^{\frac{k+\beta}{2+\beta}},$$

hence

$$P\left(T[j,k+1]>t \,\middle|\, T[j,k]=n\right)\le\left(\frac{S_n}{S_{\lfloor t\rfloor}}\right)^{\frac{k+\beta}{2+\beta}}\le\left(\frac{(2+\beta)n+\beta}{(2+\beta)(t-1)+\beta}\right)^{\frac{k+\beta}{2+\beta}}.$$

Introduce $U[j,k]=(2+\beta)T[j,k]+\beta$. Then, for $t>1$ we can write

$$P\left(U[j,k+1]-2-\beta>tU[j,k]\,\middle|\,U[j,k]\right)\le t^{-\frac{k+\beta}{2+\beta}}=P\left(\exp(\eta_k)>t\right),$$

where η_k denotes an exponentially distributed random variable with parameter $\frac{k+\beta}{2+\beta}$. In other words, $U[j,k+1]$ is majorized stochastically by

$$2+\beta+U[j,k]\exp(\eta_k),$$

where η_k is independent of $U[j,k]$. After some iteration we obtain that $U[j,k+1]$ is majorized stochastically by

$$((2+\beta)j+\beta)\exp\left(\eta_1+\dots+\eta_k\right)+(2+\beta)\sum_{i=1}^{k}\exp\left(\eta_{i+1}+\dots+\eta_k\right), \tag{3.1}$$

with independent variables η_i.

In Case (i) (3.1) can be majorized by decreasing the parameter of each η_i to $\frac{i}{2+\beta}$. From (3.1) one easily obtains that $T[j,k+1]$ is majorized stochastically by

$$\frac{2}{2+\beta}+j\sum_{i=1}^{k}\exp\left(\eta_i+\dots+\eta_k\right), \tag{3.2(i)}$$

where $\eta_1,\dots,\eta_k$ are independent, exponentially distributed random variables, and this time η_i with parameter $\frac{i}{2+\beta}$. It is well known that the joint distribution of

$$\eta_1+\dots+\eta_k,\ \eta_2+\dots+\eta_k,\ \dots,\ \eta_k$$

coincides with that of a (decreasingly) ordered sample of size k from the exponential distribution with parameter $1/(2+\beta)$. Hence the second term of (3.2(i)) is, in distribution, just j times the sum of k i.i.d. random variables with the same distribution as $\exp\eta_1$ (namely, Pareto$(2+\beta)$). That distribution belongs to the domain of attraction of the distribution of ξ; more precisely, (3.2(i)), divided by $n^{2+\beta}$, converges in distribution to $j\xi$.

Let us turn to ζ_j. Clearly, $P\left(X[n,j] < k+\beta\right) = P\left(T[j,k] > n\right)$, hence, with the notation $k = \lceil n^{1/(2+\beta)}t - \beta\rceil$ we can write

$$\begin{aligned}
P\left(\zeta_j < t\right) &= P\Big(\lim_{n\to\infty} n^{-1/(2+\beta)} X[n,j] < t\Big) \\
&\le \liminf_{n\to\infty} P\Big(X[n,j] < n^{1/(2+\beta)} t\Big) \\
&= \liminf_{n\to\infty} P\big(T[j,k] > n\big) \\
&= \liminf_{k\to\infty} P\big(k^{-2-\beta} T[j,k] > t^{-2-\beta}(1+o(1))\big) \\
&\le P\big(j\xi \ge t^{-2-\beta}\big)\,,
\end{aligned}$$

and the proof is completed.

In case (ii) (3.1) can be majorized by decreasing the parameter of $\eta_i,\ i\ge 2$, to $\frac{i-1}{2+\beta}$, but this time η_1 does not change. From (3.1) we obtain that $T[j,k+1]$ is majorized stochastically by the following modification of (3.2(i)).

$$\frac{2}{2+\beta} + (j+1)\exp(\eta_1)\sum_{i=2}^{k}\exp\left(\eta_i + \cdots + \eta_k\right), \tag{3.2(ii)}$$

From this point the proof can be completed in the same way as in Case(i). Note that $\varrho = \big(\exp(\eta_1)\big)^{-\frac{1}{2+\beta}}$.

Finally, the absolute continuity of μ is a corollary of the following assertions.

Lemma 3.2. $\mu = \max\{\zeta_j : j \ge 0\}$ *with probability* 1.

Lemma 3.3. *For* $j = 1, 2, \ldots$ *the distribution of the random variable*

$$\tau_j = \frac{\zeta_0 + \cdots + \zeta_{j-1}}{\zeta_0 + \cdots + \zeta_j} \tag{3.3}$$

is $\mathrm{Beta}\big(j(2+\beta)-1, 1+\beta\big)$. *In addition,* $\tau_1, \tau_2 \ldots, \tau_j,\ \zeta_0+\zeta_1\cdots+\zeta_j$ *are independent.*

Indeed, by Lemma 3.3 and the positivity of the sums $\zeta_0 + \cdots + \zeta_j$ it follows that $\zeta_0, \zeta_1, \ldots, \zeta_j$ have absolutely continuous (joint) distribution for every $j = 1, 2, \ldots$. It is easy to see that μ, being the *maximum* of (a countable number of) absolutely continuous random variables by Lemma 3.2, is absolutely continuous. Let A_j, $j = 0, 1, \ldots$ be the event that $\min\{i : \mu(i) = \mu\} = j$; they are pairwise disjoint, and $P(\cup A_j) = 1$. Since $\mu = \zeta_j$ on A_j, it has a density equal to the density of ζ_j a.e. on A_j.

Proof of Lemma 3.2. For arbitrary real $t > 0$, and integers $j > 0, k > 2+\beta$ we can write

$$\begin{aligned}
P(\mu \ne \mu(j)) &\le P(\zeta_0 \le t) + \sum_{i>j} P(\zeta_i > t) \le P(\zeta_0 \le t) + t^{-k}\sum_{i>j} E\zeta_i^k \\
&\le P(\zeta_0 \le t) + t^{-k} k! \binom{k+\beta}{k} \sum_{i=j+1}^{\infty} c[i,k].
\end{aligned}$$

Here on the right-hand side the first probability can be arbitrarily small if t is small enough, and for fixed t the infinite sum can be arbitrarily small with a sufficiently large j.

Proof of Lemma 3.3. Let us look at the process only at steps where one of the vertices $0, 1, \ldots, j$ is selected. Then it can be interpreted as a Pólya–Eggenberger urn process in the following way. Suppose we have S_j balls in an urn, $1 + \beta$ of them are white, the others are black (this has meaning only for integer values of β in this context, but the results we want to refer to are also valid for arbitrary real $\beta > -1$). Balls are drawn from the urn one after another, the ball drawn is always returned to the urn together with an additional ball of the same color. Drawing of a white ball corresponds to the selection of vertex j, while drawings of black balls are interpreted as selections of any of vertices $0, 1, \ldots, j-1$. The proportion of black balls in the urn after n drawings is equal to the ratio

$$\frac{Z[m;\ 0,1] + \cdots + Z[m;\ j-1,1]}{Z[m;\ 0,1] + \cdots + Z[m;\ j,1]} \tag{3.4}$$

at the (random) moment m when it occurs for the nth time that a vertex from $0, 1, \ldots, j$ is selected. This proportion is known to converge a.s., and the limit is $\mathrm{Beta}\big(S_j - 1 - \beta, 1 + \beta\big)$ distributed (see e.g. Section 6.3.3 of [6]); while (3.4) converges to (3.3) as $n \to \infty$. The embedded process is transparently independent of the moments of embedding.

4. Central limit theorem for the maximal degree

In this section we will prove the following limit theorem, which enhances the description, given in Theorem 3.1, of the asymptotic behaviour of the maximal degree.

Theorem 4.1.

(i) The normalized maximal degree

$$n^{\frac{1}{2(2+\beta)}}\Big(n^{-\frac{1}{2+\beta}} M_n - \mu\Big)$$

converges in distribution to the normal mixture $N(0, \mu)$ defined as the distribution of the product $\sqrt{\mu}\mathcal{N}$, where $\mathcal{N}$ is a standard normal random variable, independent of μ.

(ii) Furthermore,

$$n^{\frac{1}{2(2+\beta)}}\mu^{-1/2}\Big(n^{-\frac{1}{2+\beta}} M_n - \mu\Big)$$

converges in distribution to the standard normal law $N(0,1)$, as $n \to \infty$.

Proof. Consider the Doob–Meyer decomposition of the submartingale $M[n, n]$ into a convergent martingale and a predictable increasing process. We shall see that the martingale obeys the central limit theorem, and the increasing process will turn out to be negligible.

Define L_n as the multiplicity of the maximal degree, that is,

$$L_n = \#\left\{0 \le j \le n : M[n, n] = Z[n;\ j, 1]\right\}.$$

From Lemma 3.2 it follows that $P\big(\zeta_i = \zeta_j \text{ for some } i \neq j\big) = 0$; thus, with probability 1 we eventually have $L_n = 1$. Let

$$d_n = M[n,n] - E\big(M[n,n] \bigm| \mathcal{F}_{n-1}\big) = M[n,n] - M[n-1,n-1]\,\frac{S_{n-1}+L_{n-1}}{S_{n-1}+1};$$

these are martingale differences with respect to the filtration $\big(\mathcal{F}_n\big)$. Further, let

$$a_n = E\big(M[n,n] \bigm| \mathcal{F}_{n-1}\big) - M[n-1,n-1] = M[n-1,n-1]\,\frac{L_{n-1}-1}{S_{n-1}+1};$$

this is non-negative and predictable. Clearly,

$$\mu - M[n,n] = \sum_{i=n+1}^{\infty} d_i + \sum_{i=n+1}^{\infty} a_i. \tag{4.1}$$

The second sum on the right-hand side has only finitely many terms different from 0, hence

$$\lim_{n\to\infty} n^{\frac{1}{2(2+\beta)}} \sum_{i=n+1}^{\infty} a_i = 0. \tag{4.2}$$

Let us apply Corollary 4.2.1 of [7] to the martingale difference array

$$\Big\{\big(n^{\frac{1}{2(2+\beta)}}\, d_i,\ \mathcal{F}_i\big),\ i = n+1, n+2, \dots\Big\}, \quad n = 1, 2, \dots, \tag{4.3}$$

and Theorem 4.2.1 of [7] to the array

$$\Big\{\big(n^{\frac{1}{2(2+\beta)}}\, M[n,n]^{-1/2} d_i,\ \mathcal{F}_i\big),\ i = n+1, n+2, \dots\Big\}, \quad n = 1, 2, \dots. \tag{4.4}$$

For the sake of brevity let c, L, M, and S stand for $c[i,1]$, L_{i-1}, $M_{i-1}+\beta$, and S_{i-1}, resp. Then the conditional distribution of the difference d_i is given by

$$\begin{aligned} &P\Big(d_i = -c\,\tfrac{ML}{S} \Bigm| \mathcal{F}_{i-1}\Big) = 1 - \tfrac{ML}{S},\\ &P\Big(d_i = c\,\big(1 - \tfrac{ML}{S}\big) \Bigm| \mathcal{F}_{i-1}\Big) = \tfrac{ML}{S}. \end{aligned} \tag{4.5}$$

Hence the conditional variance of d_i is

$$E\big(d_i^2 \bigm| \mathcal{F}_{i-1}\big) = c^2\,\tfrac{ML}{S}\big(1 - \tfrac{ML}{S}\big),$$

which is asymptotically equal to $\Big((2+\beta)i^{\frac{1}{2+\beta}+1}\Big)^{-1}\mu$ as $i \to \infty$. Thus the sum of conditional variances in the nth row of the martingale difference array (4.3) converges to μ as $n \to \infty$. As to the Lindeberg condition, that is,

$$\sum_{i=n+1}^{\infty} E\Big(n^{\frac{1}{2+\beta}}\, d_i^2\ I\Big(\Big|n^{\frac{1}{2(2+\beta)}}\, d_i\Big| > \varepsilon\Big) \Bigm| \mathcal{F}_{i-1}\Big) \to 0$$

in probability as $n \to \infty$, for every positive ε, it is obviously implied by the inequality

$$\sup\left\{\left|n^{\frac{1}{2(2+\beta)}} d_i\right| : n < i < \infty\right\} \le n^{\frac{1}{2(2+\beta)}} c[n+1,1] \sim n^{-\frac{1}{2(2+\beta)}}.$$

All conditions of Corollary 4.2.1 of [7] are satisfied, thus

$$n^{\frac{1}{2(2+\beta)}} \sum_{i=n+1}^{\infty} d_i$$

converges in distribution to $N(0,\mu)$. By (4.1) and (4.2) the proof of part (i) is completed.

The proof of part (ii) follows the same lines, therefore all its details will be omitted.

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