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The Maximum Idempotent Separating Congruence on E-inversive E-semigroups

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Abstract

A semigroup S is an E-inversive E-semigroup if for every $a \in S$, there exists an element $x \in S$ such that ax is idempotent and the set of all idempotents of S forms a subsemigroup. The aim of this paper is to investigate the maximum idempotent separating congruence on E-inversive E-semigroups by using a full and weakly self-conjugate subsemigroup.

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1 Introduction and Preliminaries

Let S be a semigroup and $E(S)$ denote the set of all idempotents of S. An element a in a semigroup S is called *E-inversive* [1] if there exists $x \in S$ such that ax is idempotent of S. A semigroup S is called an *E-inversive* if every element of S is E-inversive. A semigroup S is called an *E-semigroup* if $E(S)$ forms a subsemigroup of S. A semigroup S is said to be a *band* if every element of S is idempotent, and a band S is *rectangular* if for all $x, y \in S, x = xyx$ [3, page 10]. For a semigroup S and $a \in S$, $V(a) := \{x \in S \mid a = axa, x = xax\}$ is the set of all *inverses* of a and $W(a) := \{x \in S \mid x = xax\}$ is the set of all *weak inverses* of a. A congruence ρ on a semigroup S is called an *idempotent separating congruence* if every ρ-class contains at most one idempotent.

In an E-inversive E-semigroup S, one important thing to note here is that the maximum idempotent separating congruence on S in general may not exist, see [1]. Weipoltshammer [1] described the idempotent separating congruence on an E-inversive E-semigroup [1, Theorem 6.1] and described the maximum idempotent separating congruence on an E-inversive semigroup such that $E(S)$ forms a rectangular band [1, Corollary 6.2]. Basic properties and results of E-inversive E-semigroups were given by Mitsch [3] and Weipoltshammer [1].

In this paper, we investigated characterizations of the maximum idempotent separating congruence on an E-inversive E-semigroup S by using a full and weakly self-conjugate subsemigroups of S. The last theorem, we described an idempotent separating congruence on S concerning the centralizer $C_S(H)$ of H in S.

To present the main results we first recall some definitions and a relation on a semigroup which is important here.

A subset H of a semigroup S is *full* [4] if $E(S) \subseteq H$. A subsemigroup H of a semigroup S is called *weakly self-conjugate* if for all $a \in S, x \in H, a' \in W(a)$, we have $axa', a'xa \in H$. For any subsets H and B of a semigroup S, let

$$H_{\omega_B} := \{a \in S \mid ba \in H \text{ for some } b \in B\}.$$

If $B = H$ then H_{ω_H} will be denoted by H_ω and it is called the *closure* of H. If H is a subsemigroup of a semigroup S, then $H \subseteq H_\omega$. H is called a *closed subsemigroup* [4] of S if $H = H_\omega$.

For any nonempty subset H of a semigroup S, we define a relation δ on S as follows :

$$\begin{aligned}\delta \ := \ &\{(a,b) \in S \times S \mid \text{ for all } a' \in W(a) \text{ there exists } b' \in W(b) \\ &\text{such that } axa' = bxb', a'xa = b'xb \text{ for all } x \in H, \text{ and for} \\ &\text{all } b' \in W(b) \text{ there exists } a' \in W(a) \text{ such that } axa' = bxb', \\ &a'xa = b'xb \text{ for all } x \in H\}.\end{aligned}$$

Note that δ may be an empty set. If S is an E-inversive semigroup, then $(a,a) \in \delta$ for all $a \in S$, so δ is not an empty set.
For basic concepts in semigroup theory, see[2] and [5] and for examples of E-inversive E-semigroups, see [1].

The following results are used in this research.

Lemma 1.1. [1] *A semigroup S is E-inversive if and only if $W(a) \neq \emptyset$ for all $a \in S$.*

Proposition 1.2. [1] *For any semigroup S, the following statements are equivalent:*

(i) *S is an E-semigroup.*
(ii) *$W(ab) = W(b)W(a)$ for all $a, b \in S$.*

Proposition 1.3. [1] *Let S be an E-semigroup. Then*

(i) *for all $a \in S, a' \in W(a), e, f \in E(S), ea', a'f, fa'e \in W(a)$,*
(ii) *for all $a \in S, a' \in W(a), e \in E(S), a'ea, aea' \in E(S)$,*
(iii) *for all $e \in E(S), W(e) \subseteq E(S)$,*
(iv) *for all $e, f \in E(S), W(ef) = W(fe)$.*

Proposition 1.4. [1] *For any E- inversive semigroup S, the following are equivalent.*

(i) *$E(S)$ is a rectangular band.*
(ii) *For all $a, b \in S, W(a) \cap W(b) \neq \emptyset$ implies $W(a) = W(b)$.*

Proposition 1.5. *Let S be an E-semigroup. If $a' \in V(a)$ for all $a \in S$ then $W(a) = W(a'a)a'W(aa')$.*

Proof. Let $a \in S$ and $a' \in V(a)$. Let $x \in W(a'a)$ and $y \in W(aa')$. By Proposition 1.3(iii) and (i), we have $W(a'a) \subseteq E(S)$, $W(aa') \subseteq E(S)$ and $xa'y \in W(a)$, respectively. So $W(a'a)a'W(aa') \subseteq W(a)$.

Let $z \in W(a)$ and $a' \in V(a), a = aa'a$. Consider $z = zaz = z(aa'a)z = (za)a'(az)$ and $(za)(a'a)(za) = z(aa'a)za = zaza = za$. Then $za \in W(a'a)$.

Similarly, we have $az = (az)(aa')az$ and so $az \in W(aa')$. Therefore $z \in W(a'a)a'W(aa')$, so $W(a) \subseteq W(a'a)a'W(aa')$ and $W(a) = W(a'a)a'W(aa')$. □

A subset H of a semigroup S is called *unitary* if for all $a \in S$, and for all $h \in H, ha \in H$or $ah \in H$ implies $a \in H$.

We have the following properties :

Proposition 1.6. *Let S be an E-inversive semigroup with a full subset H of S. Then H is unitary if and only if H is closed.*

Proof. Suppose that H is unitary. Let $x \in H_\omega$. Then there exists $h \in H$ such that $hx \in H$, which implies that $x \in H$, and so $H_\omega \subseteq H$. Since H is a subsemigroup of S, we have $H \subseteq H_\omega$. Hence $H = H_\omega$.

Conversely, let $hx, h \in H$. Then $x \in H_\omega$. Since H is closed, we have $x \in H$. If $h, xh \in H$ and $x' \in W(x)$, then $(x'xh)x \in H$ since H is full. It follows that $x \in H_\omega = H$. Then H is unitary. □

Proposition 1.7. *Every full and closed subsemigroup of an E-inverse semigroup is E-inversive.*

Proof. Let H be a full and closed subsemigroup of an E-inversive semigroup S. Let $h \in H$ and $h' \in W(h)$. Then $hh' \in E(S) \subseteq H$, so $h' \in H_\omega$. Since H is closed, $h' \in H$. This shows that H is an E-inversive subsemigroup of S. □

By Propositions 1.6 and 1.7, we have

Proposition 1.8. *Every full and unitary subsemigroup of an E-inversive semigroup contains all the weak inverses of its elements.*

Proof. Let H be a full and unitary subsemigroup of an E-inversive semigroup S. We shall show that for all $a \in H, a' \in W(a)$ implies $W(a) \subseteq H$. Let $a \in H$ and $a' \in W(a)$. Then $a'a, aa' \in E(S) \subseteq H$. Since H is unitary, it follows that $a' \in H$ and $W(a) \subseteq H$. This shows that H contains all the weak inverses of its elements. □

Proposition 1.9. *Let S be an E-inversive semigroup. If H is a full and closed subsemigroup of S, then $E \subseteq H_{\omega_E} = H_\omega$ where $E = E(S)$.*

Proof. Clearly, $E(S) \subseteq H_{\omega_E}$. Let $x \in H_{\omega_E}$. Then there exists $e \in E(S)$ such that $ex \in H$. Since H is full, we have $e \in H$ and so $x \in H_\omega$. Therefore $H_{\omega_E} \subseteq H_\omega$. Let $y \in H_\omega$. By Proposition 1.7, there exists $h' \in W(h) \cap H$ such that $(h'h)y = h'(hy) \in H$. Since $h'h \in E(S)$, we have $y \in H_{\omega_E}$. It follows that $H_{\omega_E} = H_\omega$.

Hence the proof is completed. □

2 Main Results

The idempotent separating congruence μ on an E-inversive E-semigroups can be found by Weipoltshammer [1] as follows :

$$\begin{aligned} \mu \ := \ & \{(a,b) \in S \times S \mid \text{ for all } a' \in W(a) \text{ there exists} \\ & b' \in W(b), aea' = beb', a'ea = b'eb \text{ for all } e \in E(S) \\ & \text{and for all } b' \in W(b) \text{ there exists } a' \in W(a), \\ & aea' = beb', a'ea = b'eb \text{ for all } e \in E(S)\}. \end{aligned} \qquad (*)$$

Let $\mathcal{C}$ be the class of all full and weakly self-conjugate subsemigroups of a semigroup S. For $H \in \mathcal{C}$, we replace $E(S)$ in $(*)$ by H. Then we have

Theorem 2.1. *If S is an E-inversive E-semigroup and $H \in \mathcal{C}$, then a binary relation*

$$\begin{aligned} \delta \ := \ & \{(a,b) \in S \times S \mid \ for\ each\ a' \in W(a)\ there\ exists \\ & b' \in W(b), a'xa = b'xb,\ axa' = bxb'\ for\ all\ x \in H \\ & and\ for\ all\ b' \in W(b)\ there\ exists\ a' \in W(a), \\ & a'xa = b'xb, axa' = bxb'\ for\ all\ x \in H\} \end{aligned}$$

is an idempotent separating congruence on S. Moreover, if $E(S)$ is a rectangular band with $E(S) = H_{\omega_E}$, then δ is the maximum idempotent separating congruence on S.

Proof. Obviously, δ is reflexive and symmetric. Let a, b, c be elements in S such that $a\delta b$ and $b\delta c$ and let $a' \in W(a)$. Then there exists $b' \in W(b)$ such that $a'xa = b'xb, axa' = bxb'$ for all $x \in H$. Since $b\delta c$ and $b' \in W(b)$, there is $c' \in W(c)$ such that $b'xb = c'xc, bxb' = cxc'$ for all $x \in H$. Thus $a'xa = c'xc$ and $axa' = cxc'$ for all $x \in H$.

Similarly, we can show that for all $c' \in W(c)$, there exists $a' \in W(a)$ such that $a'xa = c'xc$ and $axa' = cxc'$ for all $x \in H$. Hence $a\delta c$.

Let $a, b, c \in S$ with $a\delta b$, and let $a' \in W(a)$. Then there exists $b' \in W(b)$ such that $a'xa = b'xb, axa' = bxb'$ for all $x \in H$. Let $c' \in W(c)$. By $c'a' \in W(ac)$ and $c'b' \in W(bc)$, we get that for all $x \in H$,

$$\begin{aligned} (ac)x(c'a') &= a(cxc')a' \\ &= b(cxc')b' \\ &= (bc)x(c'b') \quad \text{since } cxc' \in H, \end{aligned}$$

and

$$\begin{aligned} (c'a')x(ac) &= c'(a'xa)c \\ &= c'(b'xb)c \\ &= (c'b')x(bc) \quad \text{since } b'xb \in H. \end{aligned}$$

Hence $ac\delta bc$, so δ is a right compatible. Similarly, we can show that δ is a left compatible. Therefore δ is a congruence on S.

Let e, f be elements in S such that $e\delta f$. Since $e \in W(e)$, there exists $f' \in W(f)$ such that $exe = f'xf = fxf'$ for all $x \in H$.

Since H is full, $e \in H$ and $e = eee = f'ef = fef'$, we have $ef = (f'ef)f = f'ef = e$. Now $f \in W(f)$. There exists $e' \in W(e)$ such that $fxf = e'xe = exe'$ for all $x \in H$. Since $f \in H$, $f = fff = e'fe = efe'$, it follows that $ef = e(efe') = efe' = f$. Therefore $e = f$ and so δ is an idempotent separating congruence on S.

Suppose that $E(S)$ is a rectangular band with $E(S) = H_{\omega_E}$. Let ρ be an arbitrary idempotent separating congruence on S. Let $a, b \in S$ with $a\rho b$, and let $a' \in W(a)$. We choose $b^* \in W(b)$. Then $b^*a\rho b^*b$. Let $b' = a'ab^*aa'$. By Proposition 1.3(i), we have $b' \in W(b)$. For any $x \in H, a'xa \ \rho \ a'xb = (a'aa')xb$. Since $a'a, b^*b \in E(S)$ and $E(S)$ is a rectangular band, it follows that $a'a = (a'a)(b^*b)(a'a)$. Thus $a'xa\rho(a'ab^*ba'a)a'xb$. Since $b^*a\rho b^*b$, we have $a'xa\rho a'a(b^*a)a'aa'xb = a'ab^*aa'xb = b'xb$. Thus $a'xa\rho b'xb$. Now $a'xa, b'xb \in H$ because H is weakly self-conjugate. Since $E(S) = H_{\omega_E}$ and $a'xa = a'a(a'xa) \in H$, we have $a'xa \in H_{\omega_E} = E(S)$. Similarly, $b'xb = b'b(b'xb) \in H$. Thus $b'xb \in H_{\omega_E} = E(S)$. Since ρ is an idempotent separating congruence on S, we get $a'xa = b'xb$. The proof of the second part is similar to the proof of the first part. Therefore $\rho \subseteq \delta$.

Hence δ is the maximum idempotent separating congruence on S.

□

The last theorem, we investigated an idempotent separating congruence on an E-inversive E-semigroup S concerning centralizer $C_S(H)$ of H in S where $C_S(H) := \{a \in S \mid ha = ah \text{ for all } h \in H\}$. Then we have

Theorem 2.2. *Let S be an E-inversive E-semigroup with $H \in \mathcal{C}$, and let δ^* be a relation given by*

$$\begin{aligned}\delta^* := \{(a,b) \in S \times S \mid & \textit{ for all } a' \in W(a) \textit{ there exists} \\ & b' \in W(b) \textit{ such that } a'a = b'b, ab' \in C_S(H) \textit{ and for all} \\ & b' \in W(b) \textit{ there exists } a' \in W(a) \textit{ such that } a'a = b'b, \\ & ba' \in C_S(H)\}.\end{aligned}$$

If H is a commutative subsemigroup, then $\delta^* = \delta$, hence δ^* is an idempotent separating congruence on S.

Proof. Let a, b be elements in S such that $a\delta b$, and let $a' \in W(a)$. Then there exists $b' \in W(b)$ such that $axa' = bxb'$ and $a'xa = b'xb$ for all $x \in H$.

Note that $aa' \in E(S) \subseteq H$ and $a'a = a'aa'a = (b'aa')b$. By Proposition 1.3(i), $b'aa' \in W(b)$.
For any $h \in H$,

$$\begin{aligned}(ab'aa')h = ab'a(a'aa')h =& \ ab'aa'(haa') \\ = ab'a(a'ha)a' =& \ ab'b(a'ha)b' \quad \text{(since } a'ha \in H) \\ = h(ab'ba')ab' =& \ hab'aa'aa' \\ = h(ab'aa').&\end{aligned}$$

So $ab'aa' \in C_S(H)$. The proof of the second part is similar to the proof of the first part. Therefore $\delta \subseteq \delta^*$.

Let $a, b \in S$ with $a\delta^* b$, and let $a' \in W(a)$. Then there exists $b' \in W(b)$ such that $a'a = b'b, ab' \in C_S(H)$. We choose $b^* = a'ab'aa'$. By Proposition 1.3(i), we have $b^* \in W(b)$.
For any $x \in H$,

$$\begin{aligned}a'xa &= (a'aa')xa &&= b'b(a'xa) \\ &= a'xa\, b'b &&= a'aa'xab'b \\ &= a'aa'ab'xb &&= (a'ab'aa')xb \\ &= b^*xb.\end{aligned}$$

Consider,

$$\begin{aligned}
axa' &= ax\, a'aa' && = ax\, b'ba' \\
&= ab'bxa' && = ab'bxa'aa' \\
&= ab'bxa'a(a'a)a' && \\
&= ab'bxa'ab'ba' && = (bxa'ab')(ab')(ba') \\
& && \text{(since } ab' \in C_S(H) \text{ and } bxa'ab' \in H) \\
&= bxa'ab'aa'aa' && \text{(since } b'b = a'a) \\
&= bx(a'ab'aa') && = bxb^*.
\end{aligned}$$

Similarly, we can show that for all $b^* \in W(b)$ there exists $a' \in W(a)$ such that $a'xa = b^*xb$ and $axa' = bxb^*$ for all $x \in H$. It follows that $a\delta b$ and hence $\delta^* \subseteq \delta$. Therefore $\delta^* = \delta$ and so δ^* is an idempotent separating congruence on S. □

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