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The Maximum Number of Odd Integral Distances Between Points in the Plane

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Dedicated to Professor Dr. H.-J. Kanold on the occasion of his eightieth birthday

Abstract. Four points in the plane with pairwise odd integral distances do not exist. The maximum number of odd distances between n points in the plane is proved to be $n^2/3 + r(r-3)/6$ for all n, where $r = 1, 2, 3$ and $n \equiv r \pmod 3$. This solves a recently stated problem of Erdős.

For every dimension d and for every natural number n there exist n points in the Euclidean space E^d with pairwise integral distances, since n equidistributed points on a line determine by an appropriate scaling only integral distances. However, the maximum number of points with pairwise odd integral distances in E^d is $d+2$ if $d+2 \equiv 0 \pmod{16}$ and $d+1$ in the remaining cases (see [2]; an independent proof was given in [4]). Thus four points in the plane with mutual odd distances do not exist.

Erdős [1] asked for the maximum number $f(n)$ of odd integral distances occurring between n points in the plane. This maximum $f(n)$ is determined in Theorem 1. The proof is based on a construction given in [3]. Moreover, we make use of Turán's theorem from graph theory (see p. 30 of [5]).

Turán's Theorem. *The maximum number of edges in a graph with n vertices that does not contain a complete subgraph with four vertices is*

$$\frac{n^2}{3} + \frac{r(r-3)}{6} \qquad \textit{with} \quad r = 1, 2, \textit{ or } 3, \textit{ and} \quad n \equiv r \pmod 3.$$

The upper bound is attained if and only if the graph is isomorphic to the complete 3-partite graph T_n with three vertex classes that have almost the same size.

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Theorem 1. *The maximum number $f(n)$ of odd integral distances between n points in the plane is*

$$f(n) = \frac{n^2}{3} + \frac{r(r-3)}{6} \qquad \textit{with} \quad r = 1, 2, \textit{ or } 3, \textit{ and } \quad n \equiv r \pmod 3.$$

Proof. The upper bound $f(n) \le n^2/3 + r(r-3)/6$ follows immediately from Turán's theorem if we consider a set $\mathcal{S}$ of n points in the plane with the maximum number of odd integral distances together with the graph $G(\mathcal{S})$ induced by the points and the odd distances of $\mathcal{S}$. Since $G(\mathcal{S})$ does not contain a complete graph with four vertices the number of edges in $G(\mathcal{S})$ and hence the upper bound for the number of odd integral distances in $\mathcal{S}$ follows by Turán's theorem.

The lower bound $f(n) \ge n^2/3 + r(r-3)/6$ follows by construction of a set $\mathcal{P}_m = \{p_0, \dots, p_{3m-1}\}$ of $n = 3m$ points in the complex plane with $3m^2$ odd integral distances for each natural number m. Since $G(\mathcal{P}_m)$ is isomorphic to T_{3m} by Turán's theorem, the deletion of one or two appropriate vertices leaves a T_{3m-1} or a T_{3m-2} so that the corresponding point sets prove the lower bound for $n = 3m - 1$ or $n = 3m - 2$.

For the third root of unity $\rho = (-1 + \sqrt{3}i)/2$ with $i^2 = -1$, and for $\omega = 3 + \rho$ let $\eta_k = 7^{m - \lfloor k/3 \rfloor} \rho^k \omega^{2\lfloor k/3 \rfloor}$, and $p_k = \sqrt{3}\eta_k^2/(3 \cdot 7^m)$ with $0 \le k < 3m$. This is the special case $R = 7^m = (\omega\bar{\omega})^m$ of a construction given in [3] (see Fig. 1 for the case $m = 2$). The points of the construction consist of the vertices of a regular triangle of side lengths 7^m that is rotated $m - 1$ times about its center through an angle $\theta = \arccos 71/98$.

In [3] it is shown that $\bar{\eta}_s \eta_t = x + \rho 7^m |p_s - p_t|$ for $0 \le s < t < 3m$ and with a real number x. Comparing the imaginary parts implies

$$|p_s - p_t| = \frac{\operatorname{Im}(\bar{\eta}_s \eta_t)}{7^m \operatorname{Im}(\rho)} = \frac{(2\sqrt{3}/3) \operatorname{Im}(\bar{\eta}_s \eta_t)}{7^m}.$$

With $\bar{\rho} = \rho^2$ and $\omega\bar{\omega} = 7$ it follows that

$$\bar{\eta}_s \eta_t = 7^{2m + \lfloor s/3 \rfloor - \lfloor t/3 \rfloor} \rho^{t+2s} \omega^{2\lfloor t/3 \rfloor - 2\lfloor s/3 \rfloor}.$$

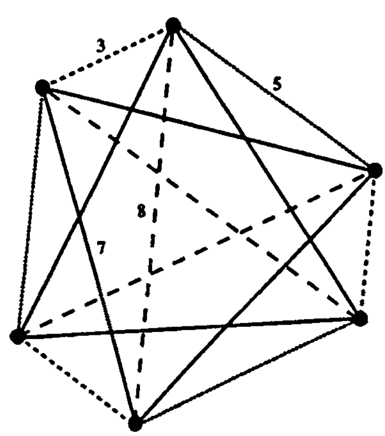

Fig. 1. Six points with 12 odd distances.

Since all occurring distances $|p_s - p_t|$ are integers as shown in [3] the number of odd distances equals the number of odd numbers $(2\sqrt{3}/3)\ \mathrm{Im}(\rho^{t+2s}\omega^{2\lfloor t/3\rfloor - 2\lfloor s/3\rfloor})$, with $0 \le s < t < 3m$.

With $\omega^2 = 8 + 5\rho$, $\rho^2 = -1 - \rho$, and $\rho^3 = 1$, and with

$$(8+5\rho)^k = \sum_{\nu\equiv 0\ (\mathrm{mod}\ 3)} 8^{k-\nu}5^{\nu}\binom{k}{\nu} + \rho \sum_{\nu\equiv 1\ (\mathrm{mod}\ 3)} 8^{k-\nu}5^{\nu}\binom{k}{\nu} - (1+\rho)\sum_{\nu\equiv 2\ (\mathrm{mod}\ 3)} 8^{k-\nu}5^{\nu}\binom{k}{\nu}$$

it follows that ω^{2k} and ρ^k are both of the form $a + b\rho$ with integers a, b satisfying $a \equiv 1, 0, 1\ (\mathrm{mod}\ 2)$ and $b \equiv 0, 1, 1\ (\mathrm{mod}\ 2)$ for $k \equiv 0, 1, 2\ (\mathrm{mod}\ 3)$, respectively. Thus $(2\sqrt{3}/3)\ \mathrm{Im}(\omega^{2k}\rho^l)$ for fixed k is an odd number in exactly two of the three cases $1 = 0, 1, 2$. Now it follows for all pairs s, t with $0 \le \lfloor s/3\rfloor < \lfloor t/3\rfloor < m$ that exactly six of all nine distances $|p_s - p_t|$ are odd. In the remaining m cases where $\lfloor s/3\rfloor = \lfloor t/3\rfloor$ all three distances $|p_s - p_t|$ are odd. Altogether there are $6\binom{m}{2} + 3m = 3m^2$ odd integral distances in $\mathcal{P}_m$ and the proof is complete. □

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