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THE MAXIMUM NUMBERS OF FACES OF A CONVEX POLYTOPE

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Abstract. In this paper we give a proof of the long-standing Upper-bound Conjecture for convex polytopes, which states that, for $1 \leqslant j < d < v$, the maximum possible number of j-faces of a d-polytope with v vertices is achieved by a cyclic polytope $C(v, d)$.

1. *Introduction.* Let P be a d-polytope, and for $0 \leqslant j \leqslant d-1$, let $f_j(P)$ denote the number of its j-faces. (In matters of terminology, we shall follow Grünbaum [1967] throughout.) Of some practical, as well as considerable theoretical importance is the problem of determining

$$\mu_j(v, d) = \max\left\{f_j(P) \mid P \text{ a } d\text{-polytope, with } f_0(P) = v\right\}.$$

The convex hull of v distinct points on the moment curve

$$M_d = \left\{(\tau, \tau^2, \ldots, \tau^d) \in E^d \mid -\infty < \tau < \infty\right\},$$

is called a cyclic polytope $C(v, d)$. The combinatorial type of $C(v, d)$ is independent of the particular choice of the v vertices on M_d (see Grünbaum [1967, §4.7]); we write

$$f_j(v, d) = f_j\big(C(v, d)\big).$$

T. S. Motzkin [1957] formulated what has come to be known as

THE UPPER-BOUND CONJECTURE. *For all* $1 \leqslant j < d < v$,

$$\mu_j(v, d) = f_j(v, d).$$

In fact, Motzkin's formulation was categorical; however, no proof was subsequently published, and so it seems more reasonable to call the statement a conjecture.

Since 1957, considerable efforts have been devoted to proving various cases of the Upper-bound Conjecture. Of particular importance were the contributions of Fieldhouse [1961], Klee [1964b] and Gale [1964]; it should also be noted that the cases $d \leqslant 4$ were already known to Brückner [1893]. For detailed discussions of the history of these investigations, the reader should consult Grünbaum [1967, §10.1, 1970].

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In this paper, we shall give a complete proof of the Upper-bound Conjecture. Before stating the form in which we shall prove the conjecture, we remark upon two important properties of the cyclic polytope $C(v, d)$. Firstly, $C(v, d)$ is *simplicial*; that is, all its faces are simplices. Secondly, $C(v, d)$ is *neighbourly*; that is, for each $1 \leqslant r \leqslant n = [\frac{1}{2}d]$, every subset of r vertices of $C(v, d)$ is the set of vertices of a $(r-1)$-face of $C(v, d)$. (For proofs of these assertions, see, for example, Grünbaum [1967, §4.7].) Then we shall prove

THEOREM. *Let P be a d-polytope with v vertices. Then for $1 \leqslant j \leqslant d-1$,*

$$f_j(P) \leqslant f_j(v, d).$$

Moreover, if equality is attained for some $[\frac{1}{2}(d-1)] \leqslant j \leqslant d-1$, then P is simplicial and neighbourly, and equality is attained for each j.

2. *The Dehn–Sommerville Equations.* Klee [1964b] has shown that if P is any d-polytope, then there is some simplicial d-polytope Q with the same number of vertices such that, for $1 \leqslant j \leqslant d-1$,

$$f_j(Q) \geqslant f_j(P).$$

In fact, McMullen [1970a] has shown further that, if P is non-simplicial, then the inequality is strict whenever $[\frac{1}{2}(d-1)] \leqslant j \leqslant d-1$. It is thus clear that if we are interested in the problem of maximizing $f_j(P)$, we may without loss of generality restrict our attention to simplicial polytopes P.

If P is a simplicial d-polytope, then the numbers of its faces satisfy various linear relations, called

THE DEHN–SOMMERVILLE EQUATIONS. *For each $-1 \leqslant k \leqslant d-2$,*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(P) = (-1)^{d-1} f_k(P),$$

where, conventionally, we write $f_{-1}(P) = 1$.

For proofs of this result, see Dehn [1905] (in case $d = 4, 5$), Sommerville [1927], and, more recently, Klee [1964a] and Grünbaum [1967].

The Dehn–Sommerville equations can be reformulated in many ways. For example, we can write $f_n(P), \ldots, f_{d-1}(P)$ (we shall always let $n = [\frac{1}{2}d]$) in terms of $f_0(P), \ldots, f_{n-1}(P)$ (for proofs see, among others, Fieldhouse [1961], Klee [1964a, 1964b], McMullen–Shephard [1970] and McMullen [1970b]). Here we shall find more useful a reformulation due to Sommerville [1927] (see also Grünbaum [1967, §9.2], McMullen–Walkup [1970]). For $-1 \leqslant k \leqslant d-1$, we write

$$g_k^{(d)}(P) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} f_j(P),$$

where, as usual, $f_{-1}(P) = 1$. Then we have

LEMMA 1. *The Dehn–Sommerville equations are equivalent to the following relations. For $-1 \leqslant k \leqslant n-1$ ($n = [\frac{1}{2}d]$),*

$$g_k^{(d)}(P) = g_{d-k-2}^{(d)}(P).$$

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The relationship between the numbers $f_j(P)$ and $g_k^{(d)}(p)$ can also be expressed as follows. For $-1 \leqslant j \leqslant d-1$,

$$f_j(P) = \sum_{k=-1}^{j} \binom{d-k-1}{d-j-1} g_k^{(d)}(P).$$

In view of the relations of Lemma 1, we can rewrite these as

$$f_j(P) = \sum_{k=-1}^{n-1} \left\{ \binom{d-k-1}{d-j-1} + (1 - \delta_{k,\, d-n-1}) \binom{k+1}{d-j-1} \right\} g_k^{(d)}(P),$$

where $\delta_{k,\, d-n-1}$ is Kronecker's delta function. It should be particularly noted that the coefficients are non-negative, and positive if $n-1 \leqslant j \leqslant d-1$.

If P is a neighbourly d-polytope with v vertices, it is clear that, for $-1 \leqslant j \leqslant n-1$,

$$f_j(P) = \binom{v}{j+1}.$$

This implies that, for $-1 \leqslant k \leqslant n-1$,

$$g_k^{(d)}(P) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} \binom{v}{j+1} = \binom{v-d+k}{k+1}.$$

Thus, to prove the Upper-bound Conjecture, in the formulation of the theorem of this paper, it is enough to prove

LEMMA 2. *Let P be a simplicial d-polytope with v vertices. Then for $1 \leqslant k \leqslant n-1$,*

$$g_k^{(d)}(P) \leqslant \binom{v-d+k}{k+1}.$$

Notice that equality in all these relations will characterize P as neighbourly.

3. *Shelling the boundary complex.* The set of faces of a d-polytope P, together with the empty face $\varnothing$ (whose dimension is conventionally taken to be -1), forms a cell complex ∂P, the *boundary complex* of P, whose underlying polyhedron (in the topological sense) is the boundary bd P of P. Bruggesser–Mani [1970] have proved that ∂P is *shellable*, in the sense that the facets (*i.e.* the $(d-1)$-faces) of P may be labelled $F_1, \ldots, F_m$ $(m = f_{d-1}(P))$, in such a way that, for $1 \leqslant s \leqslant m-1$,

$$M_s = \bigcup_{t=1}^{s} F_t,$$

is a (topological) $(d-1)$-ball, with the property that, for $2 \leqslant s \leqslant m-1$, $M_{s-1} \cap F_s$ is a (topological) $(d-2)$-ball.

The boundary complex of a polytope can be shelled in many ways; since we shall later need to choose a shelling with particular properties, we shall give a brief description of the method of Bruggesser–Mani [1970]. Let L be a line through an interior point of the polytope P, in general position with respect to the supporting hyperplanes corresponding to the facets of P. We label the facets $F_1, \ldots, F_m$ in such a way that, as we proceed along L from the interior of P, we meet successively

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the corresponding support hyperplanes $H_1, \ldots, H_m$. We must regard order along L in the projective sense, so that we pass through infinity along L. Then if y is any point on L between H_s and H_{s+1}, before passing through infinity, M_s is the set of points of bd P visible from y, and after passing through infinity, M_s is the set of points of bd P not visible from y.

If P is a simplicial polytope, for $1 \leqslant s \leqslant m$, we write

$$g_k^{(d)}(M_s) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} f_j(M_s),$$

where $f_j(M_s)$ denotes the number of j-faces of P in M_s, with the usual convention $f_{-1}(M_s) = 1$. We shall also let $M_0 = \varnothing$, with $f_j(M_0) = 0$ for every j. Thus, in particular, $g_k^{(d)}(M_m) = g_k^{(d)}(P)$ for each k.

We consider the quantity

$$g_k^{(d)}(M_s) - g_k^{(d)}(M_{s-1}),$$

for $1 \leqslant s \leqslant m$. If $2 \leqslant s \leqslant m-1$, $M_{s-1} \cap F_s$ is a $(d-2)$-ball composed of faces of the $(d-1)$-simplex F_s; thus it is the union of all the $(d-2)$-faces of F_s which contain some particular face of F_s. If this face of F_s has dimension $d-r-2$ (say), then, in going from M_{s-1} to M_s, the new faces of P added are precisely the opposite r-face of F_s and the j-faces $(r < j \leqslant d-1)$ of F_s which contain it (including F_s itself). There are $\binom{d-r-1}{d-j-1}$ such j-faces, and so

$$\begin{aligned} g_k^{(d)}(M_s) - g_k^{(d)}(M_{s-1}) &= \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} \binom{d-r-1}{d-j-1} \\ &= \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-r-1}{d-k-1} \binom{k-r}{j-r} \\ &= \delta_{k,r}, \end{aligned}$$

the Kronecker delta. It is clear that this equation also holds in the extreme cases $s = 1$ (with $r = -1$) and $s = m$ (with $r = d-1$).

It incidentally follows from this argument that $g_k^{(d)}(P) \geqslant 0$ for each k. Further, since if we consider the facets of P in the reverse order $F_m, \ldots, F_1$, we again obtain a shelling of ∂P, with the rôles of the r-face and $(d-r-2)$-face of F_s interchanged, we also deduce that $g_k^{(d)}(P) = g_{d-k-2}^{(d)}(P)$; that is, in view of Lemma 1, we have a new proof of the Dehn–Sommerville equations.

Since $g_0^{(d)}(P) = f_0(P) - d$, the crucial Lemma 2 is clearly an immediate consequence of

LEMMA 3. *Let P be a simplicial d-polytope with v vertices. Then for $1 \leqslant k \leqslant d-1$,*

$$(k+1)\, g_k^{(d)}(P) \leqslant (v-d+k)\, g_{k-1}^{(d)}(P).$$

Let x be a vertex of P, and let P_x be the vertex-figure of P at x. That is, P_x is a simplicial $(d-1)$-polytope obtained as the intersection of P with any hyperplane which strictly separates x from the remaining vertices of P. We shall prove Lemma 3 by comparing the values of $g_k^{(d-1)}(P_x)$ and $g_k^{(d)}(P)$ in two ways.

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Firstly, we obtain the exact relation

$$\sum_{x \in \operatorname{vert} P} g_{k-1}^{(d-1)}(P_x) = (k+1)\, g_k^{(d)}(P) + (d-k)\, g_{k-1}^{(d)}(P),$$

where vert P denotes the set of vertices of P. This can be proved by writing down the expression for $g_{k-1}^{(d-1)}(P_x)$ in terms of $f_{j-1}(P_x)$, and using the relationship

$$\sum_{x \in \operatorname{vert} P} f_{j-1}(P_x) = (j+1) f_j(P).$$

However, we can give a more geometrical proof as follows. Let $F_1, \ldots, F_m$ be a shelling of ∂P, as described above. This clearly gives rise to a shelling of the boundary complex of each vertex-figure of P. Again as above, let us suppose that in adding the simplex F_s to M_{s-1} to obtain M_s, the faces of P added consist of an r-face of F_s and the faces of F_s which contain it; that is, we just increase $g_r^{(d)}$ by 1. Clearly, d vertex-figures of P are affected by adding F_s; in $r+1$ of them we increase $g_{r-1}^{(d-1)}$ by 1, and in the remaining $d-r-1$, we increase $g_r^{(d-1)}$ by 1. From this, the expression above follows at once.

The second relation is

$$\sum_{x \in \operatorname{vert} P} g_{k-1}^{(d-1)}(P_x) \leqslant v\, g_{k-1}^{(d)}(P).$$

For, consider any vertex-figure P_x. The method of Bruggesser–Mani [1970] clearly allows us to choose a shelling of ∂P so that, for some s, $F_1, \ldots, F_s$ are the facets of P which contain x. Then if at any of the first s steps of the shelling we add 1 to $g_r^{(d)}$ for P, we also add 1 to $g_r^{(d-1)}$ for P_x. We conclude that

$$g_r^{(d-1)}(P_x) \leqslant g_r^{(d)}(P),$$

from which the second relation follows at once.

These two relations immediately imply the assertion of Lemma 3. In view of the remarks before Lemmas 2 and 3, we see that we have proved the theorem; that is, we have proved the Upper-bound Conjecture.

4. *Remarks.* It has also been conjectured that the Upper-bound Conjecture holds for simplicial complexes whose underlying polyhedron is a $(d-1)$-sphere; this question has been discussed in some detail by Grünbaum [1970]. The proof in the case of polytopes cannot be extended to such more general complexes, some of which are not shellable, and certainly not in the special ways demanded by the proof above. It is therefore still a possibility that the Upper-bound Conjecture fails for some triangulated spheres.

Note added in proof (October 1970). An alternative approach to the proof uses the dual formulation. In this, we consider the subcomplexes consisting of all the faces of a simple d-polytope P (possibly including P itself) which are contained in variable closed half-spaces with fixed outward normal vector. For such a complex M, let

$$g_k^*(M) = \sum_{j=k}^{d} (-1)^{j-k} \binom{j}{k} f_j(M),$$

so that

$$f_j(P) = \sum_{k=d-n}^{d} \left\{ \binom{k}{j} + (1-\delta_{kn}) \binom{d-k}{j} \right\} g_k^*(P).$$

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We find that $g_k{}^*(M)$ increases by δ_{kr} as M acquires (say) r edges of P through some vertex x, and from this we obtain the relations

$$(d-k)\, g_k{}^*(P) + (k+1)\, g_{k+1}^*(P) = \textstyle\sum_F g_k{}^*(F) \leqslant f g_{k+1}^*(F),$$

where the sum is taken over the f facets F of P. We deduce that, for $k = d-n, \ldots, d-2$,

$$g_k{}^*(P) \leqslant \binom{f-k-1}{d-k},$$

and equality occurs only when P is the dual of a neighbourly polytope.

References

M. Brückner, 1893, " Die Elemente der vierdimensionalen Geometrie mit besonderer Berücksichtigung der Polytope ", *Jber. Ver. Naturk. Zwickau* (1893).

H. Bruggesser and P. Mani, 1970, " Shellable decompositions of cells and spheres ", *to be published.*

M. Dehn, 1905, " Die Eulersche Formel in Zusammenhang mit dem Inhalt in der nicht-Euklidischen Geometrie ", *Math. Ann.*, 61 (1905), 561–586.

M. Fieldhouse, 1961, *Linear programming*, Ph.D. Thesis, Cambridge Univ. (1961). (Reviewed in *Operations Res.*, 10 (1962), 740.)

D. Gale, 1964, " On the number of faces of a convex polytope ", *Canad. J. Math.*, 16 (1964), 12–17.

B. Grünbaum, 1967, *Convex polytopes* (John Wiley and Sons, London–New York–Sydney, 1967).

———, 1970, *Polytopes, graphs and complexes, to be published.*

V. L. Klee, 1964a, " A combinatorial analogue of Poincaré's duality theorem ", *Canad. J. Math.*, 16 (1964), 517–531.

———, 1964b, " The number of vertices of a convex polytope ", *Canad. J. Math.*, 16 (1964), 701–720.

P. McMullen, 1970a, " On a problem of Klee concerning convex polytopes ", *Israel J. Math.*, 8 (1970), 1–4.

———, 1970b, " On the upper-bound conjecture for convex polytopes ", *J. Combinatorial Theory, to be published.*

——— and G. C. Shephard, 1970, *Convex polytopes and the upper-bound conjecture*, *London Math. Soc. Lecture Notes Series, Volume* 3, *to be published.*

——— and D. W. Walkup, 1970, "A generalized lower-bound conjecture for simplicial polytopes," *to be published.*

T. S. Motzkin, 1957, " Comonotone curves and polyhedra ", *Bull. Amer. Math. Soc.*, 63 (1957), 35.

D. M. Y. Sommerville, 1927, " The relations connecting the angle-sums and volume of a polytope in space of n dimensions ", *Proc. Roy. Soc. London, Ser. A*, 115 (1927), 103–119.

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