

HISTORICAL PAPER

The Maximum Principle, Bellman's Equation, and Carathéodory's Work^{1,2}

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Abstract. One of the most important and deep results in optimal control theory is the maximum principle attributed to Hestenes (1950) and in particular to Boltyanskii, Gamkrelidze, and Pontryagin (1956). Another prominent result is known as the Bellman equation, which is associated with Isaacs' and Bellman's work (later than 1951). However, precursors of both the maximum principle and the Bellman equation can already be found in Carathéodory's book of 1935 (Ref. 1a), the first even in his earlier work of 1926 which is given in Ref. 2. This is not a widely acknowledged fact. The present tutorial paper traces Carathéodory's approach to the calculus of variations, once called the "royal road in the calculus of variations," and shows that famous results in optimal control theory, including the maximum principle and the Bellman equation, are consequences of Carathéodory's earlier results.

Key Words. Maximum principle, Bellman equation, Carathéodory's work, calculus of variations, optimal control theory, history of the calculus of variations, history of optimal control theory.

¹This paper is in honor of the seventieth birthday of Professor Angelo Miele. It was exactly 70 years ago, in the months of August and September 1922, around the time of the birth of Professor Miele, when Constantin Carathéodory wrote a paper in Italian entitled "Sui Campi di Estremali Uscenti da un Punto e Riempienti Tutto lo Spazio—On Extremal Fields Emanating from a Point and Covering All the Space" (Ref. 3). This paper was inspired by the great Italian mathematician Leonida Tonelli, one of Angelo Miele's academic teachers.

²The authors would like to thank Professors Sandra Hayes-Widmann and Donald Smith for checking the English of their manuscript, Dr. Uwe Dubielzig for providing them with the entire classical literature on the bull-hide story of the foundation of Carthage and for drawing their attention to Ref. 4, Dr. Ingeborg Neske for directing them to the codex of the City Library of Nuremberg, and the City Library of Nuremberg for permission to publish the page of the codex given in Fig. 1.

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1. Introduction

Constantin Carathéodory, one of the brightest mathematicians of this century in Germany, was born of Greek parents in Berlin on the 13th of September in 1873. Carathéodory was a scion of an old Greek family that brought forth many eminent men for several generations. His great-uncle Alexander Carathéodory Pascha, for example, was the prime delegate of the great power Turkey at the Congress in Berlin in 1878 and later became the Turkish secretary of state. His grandfather Constantin exerted a great influence on Sultan Mahmoud II and on his son and successor Abd-ul-Medjid as their personal physician. His father Stephanos, too, was in the diplomatic service of the Sublime Port. He was an attaché of the Turkish embassy in Berlin when Constantin Carathéodory was born, and became sometime later the ambassador of Turkey in Brussels. So, Constantin Carathéodory grew up in Brussels in a parental home that was characterized by frequent contacts with many distinguished and important persons of diplomacy, science, music, and art from many different countries of wide cultural diversity. This highly intellectual background shaped Carathéodory into a multilingual cosmopolitan of an extraordinarily extensive education.

Carathéodory graduated from the *École militaire de Belgique* in 1895 as engineer officer. He took his first job in his cousin's engineering office and helped with the planning of the road network of Samos. From 1898 until 1900, Carathéodory worked as assistant engineer on the regulation of the Nile in Assiout, Egypt. During this time, his love of mathematics predominated: "I could not, however, resist the obsessive idea, that only the unrestrained study of mathematics would give my life its *raison d'être*."⁵

To the astonishment of his family and friends, Constantin Carathéodory gave up his secure position to follow a romantic inclination. In 1900, he began

⁵Original: "Ich konnte aber der Zwangsvorstellung nicht widerstehen, daß erst die hemmungslose Beschäftigung mit Mathematik meinem Leben seinen Inhalt geben würde."

⁶Indeed, the first isoperimetric problem the solution of which has come down to us in written form by Theon Alexandreus (see Ref. 4, p. 33) is due to Zenodoros' treatise *On Isoperimetric Figures*, of about the 2nd century B.C.: "Just as well, since those of different figures which have the same contour are larger which have more angles, the circle is larger than the (other) plane figures and the sphere than the (other) solids. We are going to present the proof for this in an extract of the arguments as has been given by Zenodoros in his work *On Isoperimetric Figures*." (Translation from the ancient Greek original, compare Figs. 1 and 2.)

Figure 1 shows Zenodoros' theorem in a fourteenth century manuscript of the City Library of Nuremberg. This codex was possessed by the mathematician and astronomer Johannes Müller known as Regiomontanus, who received it as a gift from his patron Cardinal Johannes Bessarion, titular patriarch of Constantinople. The codex serves as the original printing copy for the *editio princeps* of 1538 published in Basel.

his extraordinarily successful study of mathematics in Berlin. His decision to prefer Berlin to Paris might have been influenced by a steel engraving showing Alexander von Humboldt, the sovereign of scholars. This picture contained a personal dedication to Constantin's father and reminded Carathéodory of the friendships his father made in Berlin and which were later of great value to him. This picture always decorated Carathéodory's office.

In 1904, Carathéodory received a doctorate in Göttingen with a sensational dissertation on discontinuous solutions of variational problems. Only a year later, Carathéodory completed his *Habilitationschrift*; he was encouraged by Felix Klein and David Hilbert. Then, he accepted professorships in Bonn, Hannover, Breslau, Göttingen (as successor of F. Klein), Berlin, Smyrna (where he founded the Greek university sponsored by the Greek government and where he stayed until the expelling of the entire Greek population when Turkey reconquered Izmir), Athens, and finally Munich in 1924. He also accepted visiting professorships at several American universities during this period.

During World War II, Carathéodory lived a very secluded life in Munich. Oskar Perron in his obituary to Constantin Carathéodory: "He looked at the Third Reich through the eyes of an historian who is always drawing parallels to dictatorships of bygone times, and also through the eyes of a foreigner (Carathéodory possessed both Greek and German nationality) whose attention is attracted by many strange cultic customs which he however can simply accept without having to be ashamed. . . . Because of his worldwide relations, he also succeeded in finding opportunities of existence through emigration for quite a few of his Jewish colleagues."

Constantin Carathéodory died in Munich on the 2nd of February in 1950.

Carathéodory continued the tradition of the classical Hellenic culture^{6,7}

⁷The engineering conversion of Zenodoros' result was performed by the legendary Phoenician princess Dido of Tyros after the story that the Phoenicians, whom the Libyans allowed only so much land as the hide of a bull (Greek: *βύρσα*) would cover, cut the hide into thin strips and thus encircled a large area on which they erected the citadel Byrsa of Carthage. This tale was sung by Publius Vergilius Maro (70–19 B.C.) in his famous *Aeneid* (book one, verses 365–368):

devenere locos, ubi nunc ingentia cernis
moenia surgentemque novae Karthaginis arcem,
mercatique solum, facti de nomine Byrsam,
taurino quantum possent circumdare tergo.

and in English verses from the translation of the famous English poet John Dryden, a contemporary of the Bernoulli brothers:

At last they landed, where from far your Eyes
May view the Turrets of new Carthage rise:
There bought a space of Ground, which Byrsa call'd
From the Bulls hide, they first inclos'd, and wall'd.

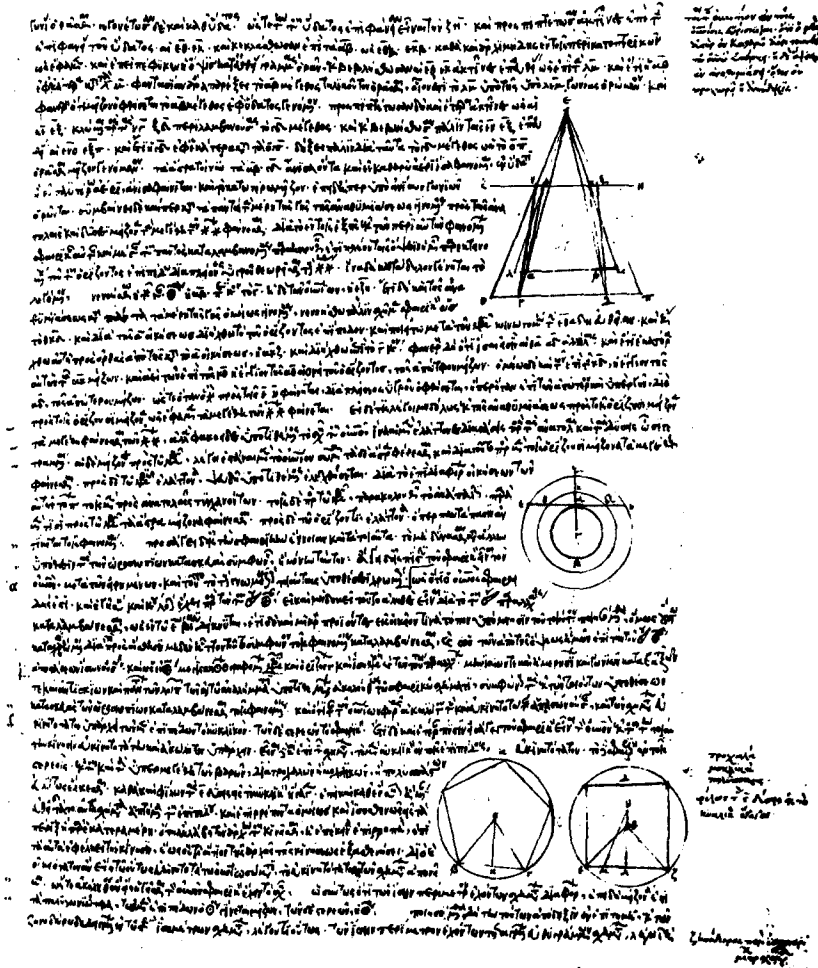


Fig. 1. Zeondoros' theorem in a 14th century manuscript of the City Library of Nuremberg (Cod. Nür. Cent. V App. 8, p. 58').

through his high intellect and by his untiring striving for knowledge. Among his many fields of research, the calculus of variations was surely the one where Carathéodory achieved his most outstanding results. In his book (Ref. 1), he depicted the calculus of variations in an entirely new aspect by converting the historical modus operandi. Making use of the close relation of variational problems to first-order partial differential equations, Carathéodory was able to derive all the classical results of the calculus of variations from the Hilbert integral up to the Euler equations in a very elegant way which was later called the "royal road of the calculus of

From Zenodoros' treatise "Περὶ ἰσομέτρων σχημάτων":

Ἐσαύτως δ' ὅτι τῶν ἴσην περίμετρον ἔχόντων
 σχημάτων διαφόρων, ἐπειδὴ μείζονά ἐστι τὰ
 πολυγωνιώτερα, τῶν μὲν ἐπιπέδων ὁ κύκλος
 γίνεται μείζων, τῶν δὲ στερεῶν ἡ σφαῖρα.
 Ποιησόμεθα δὴ τὴν τούτων ἀπόδειξιν ἐν ἐπιτομῇ
 ἐκ τῶν Ζηνοδώρου δεδειγμένων ἐν τῷ Ἐπιτομῆ
 ἰσοπεριμέτρων σχημάτων'.

Fig. 2. Again Zenodoros' theorem, now printed. Compare last three lines at the bottom of the page of the codex given in Fig. 1.

variations."⁸ From the beginning, his objective was therefore directed toward sufficient conditions for weak minima of variational problems.

The purpose of the present paper is to show that the most prominent results in optimal control theory, the maximum principle and the Bellman equation, are consequences of Carathéodory's results published more than a decade before optimal control theory had begun to develop from the calculus of variations.

2. Carathéodory's Royal Road in the Calculus of Variations

This section of the paper traces Carathéodory's approach to the solution of variational problems and follows, with slight modifications of the notation, the lines of his book; see Ref. 1, Chapter 12 entitled "Simple Variational Problems in the Small." See also Ref. 5.

2.1. Carathéodory's Derivation of the Legendre–Clebsch Condition. Consider continuously differentiable curves represented as

$$x = x(t) = (x_1(t), \dots, x_n(t))^T, \quad t' \leq t \leq t'', \quad (1)$$

in an $(n + 1)$ -dimensional Euclidian space \mathcal{R}_{n+1} with coordinates (t, x_1, \dots, x_n) . The line elements (t, x, \dot{x}) of the curves are regarded as elements of a $(2n + 1)$ -dimensional Euclidian space, say \mathcal{S}_{2n+1} .

In addition to the curve x , a second curve

$$\bar{x} = \bar{x}(t) = (\bar{x}_1(t), \dots, \bar{x}_n(t))^T, \quad t' \leq t \leq t'', \quad (2)$$

⁸See Ref. 5: "Königsweg der Variationsrechnung."

is considered, which will be called a variation of x or a comparison curve of x . Here, it is assumed that \bar{x} is continuous with a piecewise-continuous derivative. Such a comparison curve \bar{x} is said to be in a close (ϵ, η) -neighborhood of x , if there holds

$$|\bar{x}(t) - x(t)| < \epsilon \quad \text{and} \quad |d\bar{x}(t)/dt - \dot{x}(t)| < \eta, \quad \text{for all } t \text{ with } t' \leq t \leq t''. \quad (3)$$

Carathéodory then considered a real-valued C^2 -function $L = L(t, x, \dot{x})$ defined in a domain $A \subset \mathcal{S}_{2n+1}$.⁹ The integrals

$$I = \int_{t_1}^{t_2} L(t, x, \dot{x}) dt \quad \text{and} \quad \bar{I} = \int_{t_1}^{t_2} L(t, \bar{x}, d\bar{x}/dt) dt \quad (4)$$

are then well defined, if the line elements of the curves x and \bar{x} lie in A for all t with $t_1 < t < t_2$.

Now, Carathéodory's definition of an extremal is as follows.¹⁰ A curve x is called an extremal belonging to the basic function L , if the following conditions are satisfied:

- (i) for all $t_0 \in (t', t'')$ in which the extremal is defined, there exists a line element (t_0, x_0, \dot{x}_0) lying in the domain A where $x_0 := x(t_0)$ and $\dot{x}_0 := \dot{x}(t_0)$;
- (ii) for all $t_0 \in (t', t'')$, there exists a quadruple of real numbers $t_1, t_2, \epsilon > 0$, and $\eta > 0$ such that there holds

$$t' < t_1 < t_0 < t_2 < t'', \quad (5)$$

and the integral \bar{I} is well defined for all comparison curves \bar{x} which are in a close (ϵ, η) -neighborhood of x in the interval (t_1, t_2) , and moreover satisfy

$$\bar{x}(t_1) = x(t_1), \quad \bar{x}(t_2) = x(t_2), \quad (6)$$

and finally either $I \leq \bar{I}$ or $I \geq \bar{I}$.

The well-known smoothing lemma now says that a restriction of the class of comparison curves will not lead to new extremals, even if analytical

⁹For Sections 2.1 and 2.2, it is sufficient to postulate the existence and continuity of the derivatives $L_x, L_{\dot{x}}, L_{\dot{x}\dot{x}}$ only.

¹⁰Note that Carathéodory's definition differs from the usual definition where an extremal is any solution of the Euler-Lagrange equation. Carathéodory's extremal is either a so-called minimal or a so-called maximal, i.e., the integral

$$I = \int_{t_1}^{t_2} L(t, x, \dot{x}) dt$$

has, in the small, at least a weak extremum.

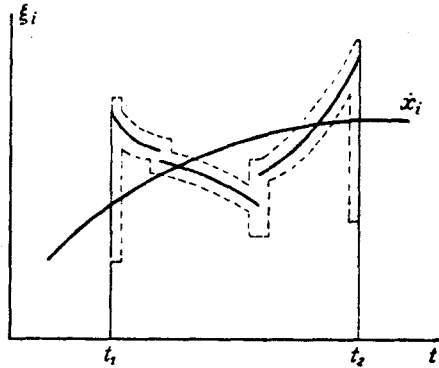


Fig. 3. Carathéodory's figure for the proof of the lemma of the smoothing of corners.

comparison curves are admitted only. Corners can be round-out. Figure 3 is used for the proof of this lemma in Ref. 1.

According to Carathéodory, the necessary condition of Legendre–Clebsch can now be proven as follows. Consider a continuously differentiable curve $x = x(t)$ in an interval $t_0 - h \leq t \leq t_0 + h$ and define a comparison curve $\bar{x} = \bar{x}(t)$ by

$$\bar{x}(t) := x(t) + (t - t_0 + h)\xi, \quad t_0 - h \leq t \leq t_0, \tag{7a}$$

$$\bar{x}(t) := x(t) + (h + t_0 - t)\xi, \quad t_0 \leq t \leq t_0 + h, \tag{7b}$$

where $\xi \in \mathbb{R}^n$ is an arbitrary constant vector. This gives rise to the integrals

$$I(h) = \int_{t_0-h}^{t_0+h} L(t, x, \dot{x}) dt \quad \text{and} \quad \bar{I}(h) = \int_{t_0-h}^{t_0+h} L(t, \bar{x}, d\bar{x}/dt) dt. \tag{8}$$

An estimation of the difference $\bar{I}(h) - I(h)$ for values of h converging to zero shows that a curve x defined in $t' \leq t \leq t''$ and containing the line element (t_0, x_0, \dot{x}_0) cannot be an extremal if the quadratic form

$$Q = \xi^T L_{\dot{x}\dot{x}}(t_0, x_0, \dot{x}_0)\xi \tag{9}$$

is indefinite. For details see Ref. 1a, pp. 193–196. Carathéodory called line elements with a positive (negative) definite quadratic form Q positive (negative) regular and those with a positive (negative) semidefinite quadratic form Q positive (negative) singular.

Hence, the necessary condition of Legendre–Clebsch can be formulated as follows. For a minimal (maximal), i.e., an extremal where $\bar{I} \geq I$ ($\bar{I} \leq I$) always holds, to pass through a line element (t_0, x_0, \dot{x}_0) , it is necessary that the line element be positive (negative) regular or at least positive (negative) singular.

It can be shown that the condition is also sufficient, in the sense of a weak extremum, if the line element is regular.

2.2. Carathéodory's Fundamental Equations of the Calculus of Variations. In the following, only regular line elements are considered. Without loss of generality, one can assume that the line element (t_0, x_0, \dot{x}_0) to be considered is positive regular. Hence, the determinant

$$L_1(t, x, \dot{x}) := \det L_{\dot{x}\dot{x}} > 0 \quad (10)$$

in a certain neighborhood $A \subset \mathcal{S}_{2n+1}$ of (t_0, x_0, \dot{x}_0) . Moreover, one can assume the set A to be convex with respect to \dot{x} .

At this point, Carathéodory introduced his important definition of equivalent variational problems. Let $\alpha(t, x)$ be an arbitrary C^2 -function that is considered along a continuous curve $x(t)$, $t^{(1)} \leq t \leq t^{(2)}$, which has a piecewise-continuous derivative. Then, the following identity always holds:

$$\alpha^{(2)} - \alpha^{(1)} = \int_{t^{(1)}}^{t^{(2)}} (\alpha_t + \alpha_x \dot{x}) dt, \quad (11)$$

with $\alpha^{(i)} := \alpha(t^{(i)}, x(t^{(i)}))$, $i = 1, 2$.

Introducing the function

$$L^*(t, x, \dot{x}) := L(t, x, \dot{x}) + (\alpha_t + \alpha_x \dot{x}), \quad (12)$$

the line integrals of L^* along two curves c and γ which have the same endpoints are considered and compared with the line integrals of the function L along these curves. It can be easily seen that, because of Eq. (11), there always holds

$$I^* - J^* = I - J, \quad (13)$$

where I and J denote the line integrals of L along the two curves c and γ and I^* and J^* denote the corresponding line integrals of L^* . Thus, each extremal of the variational problem with the basic function L is also an extremal of the variational problem with the basic function L^* , and conversely. The two associated variational problems are said to be equivalent. Note that, for equivalent variational problems, one has

$$L_{\dot{x}\dot{x}} = L^*_{\dot{x}\dot{x}}, \quad (14)$$

and the function L_1 likewise has the same value for both problems. It should be mentioned here that the existence of the second derivatives of L^* with respect to x and t cannot be guaranteed. However, this will in no way encroach upon the further conclusions; compare Footnote 9.

The ultimate objective is now to show that exactly one curve which has the extremal property can pass through each regular line element of a

variational problem. For this purpose, one must show that extremals exist at all. Therefore, one considers at first a variational problem with a special basic function L^* for which the proof can be easily obtained. This variational problem is characterized by the following two properties of L^* :

- (i) there exists a continuously differentiable n -vector function $\psi(t, x)$ in a domain $B \subset \mathcal{R}_{n+1}$ with

$$L^*(t, x, \psi) = 0; \tag{15}$$

- (ii) there exists a positive real number η such that, for all $(t, x) \in B$ and for all line elements (t, x, x') passing through these points and satisfying

$$0 \neq |x' - \psi(t, x)| < 2\eta, \tag{16}$$

there always holds

$$L_1^*(t, x, x') > 0. \tag{17}$$

Under these conditions, one can prove that the solutions of the differential equations

$$\dot{x} = \psi(t, x) \tag{18}$$

are extremals of the variational problem with the basic function L^* . Moreover, one can show that, for any comparison curve γ having the same endpoints as the extremal curve e and being in a close neighborhood of e , there always holds $J > I$, if γ does not coincide with e . Here, I and J denote the line integrals of L^* along e and γ , respectively. For details, see again Ref. 1a, pp. 199–200.

Of course, if the function L is prescribed, one cannot expect to find, in general, a function $\psi(t, x)$ which fulfills the conditions (15) and (17). It is, however, sufficient to assume that, among the equivalent variational problems, there exists one problem with the basic function

$$L^*(t, x, \dot{x}) = L(t, x, \dot{x}) - S_t - S_x \dot{x}, \tag{19}$$

so that the conditions (15) and (17) are satisfied. This immediately leads to the following central theorem in Carathéodory's approach.

Theorem 2.1. Carathéodory's Sufficient Condition. If a continuously differentiable function $\psi(t, x)$ and an at least twice continuously differentiable function $S(t, x)$ can be determined for which, on the one hand, there always holds

$$L(t, x, \psi) - S_x \psi \equiv S_t, \tag{20}$$

and on the other hand,

$$L(t, x, x') - S_x x' > S_t, \tag{21}$$

for all x' with $|x' - \psi|$ sufficiently small and $|x' - \psi| \neq 0$, then the solutions of the differential equations

$$\dot{x} = \psi(t, x) \tag{22}$$

are minimal of the variational problem with basic function L .

Carathéodory states: "According to this last result, we must in particular try to determine the functions $\psi_i(t, x_j)$ and $S(t, x_j)$ so that the expression

$$L^*(t, x_j, x'_j) = L(t, x_j, x'_j) - S_t - S_{x_i} x'_i,$$

considered as a function of x'_i , possesses a minimum for $x'_i = \psi_i(t, x_j)$, $i = 1, \dots, n$, which also has the value zero."¹¹ In other words,

$$S_t = \min_{x'} \{L(t, x, x') - S_x x'\}. \tag{23}$$

This equation is well known as the Bellman equation.¹² It was, however, first published by Carathéodory in 1935; see Ref. 1a. The results of Bellman concerning equations of this type go back to the year 1954 (see Refs. 6 and 7 and the 1954 Rand Corporation reports of Bellman cited in Ref. 6). Such equations play an important role in the method of dynamic programming as developed by Bellman and, in more general form, in the theory of differential games as developed by Isaacs at the beginning of the 50's; see Refs. 8 and 9 and the 1954 Rand Corporation reports of Isaacs cited in the last-mentioned reference. Both authors obtained their results

¹¹See the English translation (Ref. 1b, p. 208) of the German original (Ref. 1a, p. 201): "Nach dem letzten Resultat müssen wir insbesondere danach trachten, die Funktionen $\psi_i(t, x_j)$ und $S(t, x_j)$ so zu bestimmen, daß der Ausdruck

$$L^*(t, x_j, x'_j) = L(t, x_j, x'_j) - S_t - S_{x_i} x'_i,$$

als Funktion der x'_i aufgefaßt, für $x'_i = \psi_i(t, x_j)$, $i = 1, \dots, n$, ein Minimum besitze, das überdies den Wert Null habe."

¹²A different form, for example,

$$- \tilde{S}_t = \min_{x'} \{L(t, x, x') + \tilde{S}_x x'\},$$

is obtained when replacing the merit function S by the optimal return function \tilde{S} defined by $S(t, x) = S^{(2)} - S(t, x)$, where $S^{(2)} := S(t^{(2)}, x(t^{(2)}))$.

Note that

$$S^{(2)} - S(t, x) = \int_t^{t^{(2)}} L(\tau, x(\tau), \psi(\tau, x(\tau))) d\tau.$$

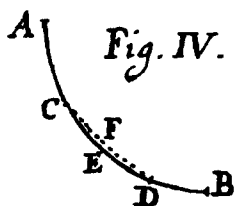


Fig. 4. Jacob Bernoulli's figure for the proof of the principle of optimality.

directly from the principle of optimality^{13,14} under the same differentiability assumptions for the so-called merit function S , which corresponds to the value of a pursuit-evasion differential game problem.

¹³Without doubt, the principle of optimality goes back to Jacob Bernoulli. In his reply (Ref. 10) to the famous brachistochrone problem by which his brother Johann founded the calculus of variations in 1669 (Ref. 11), Jacob Bernoulli wrote: "If $ACEDB$ is the required curve, along which a heavy particle descends under the action of the downward directing gravity from A to B in shortest time, and if C and D are two arbitrarily close points of the curve, the part CED of the curve is, among all other parts having endpoints C and D , that part which a particle falling from A under the action of gravity traverses in shortest time. Viz., if a different part CFD of the curve would be traversed in a shorter time, the particle would traverse $ACFDB$ in a shorter time as $ACEDB$, in contrast to the hypothesis." See Fig. 4.

Original: "Si curva $ACEDB$ talis sit, quae requiritur, h.e. per quam descendendo grave brevissimo tempore ex A ad B perveniat, atque in illa assumantur duo puncta quantumlibet propinqua C & D : Dico, proportionem Curvae CED omnium aliarum punctis C & D terminatorum Curvarum illam esse, quam grave post lapsum ex A brevissimo quoque tempore emetiatur. Si dicas enim, breviori tempore emetiri aliam CFD , breviori ergo emetietur $ACFDB$, quam $ACEDB$, contra hypoth."

¹⁴In Ref. 12, Jacob Bernoulli's result was later formulated by Euler (Carathéodory: "in one of the most wonderful books that has ever been written about a mathematical subject") as a theorem; see the *Propositio II* in Ref. 12. Indeed, Jacob Bernoulli's methods were so powerful and general that they have inspired all his illustrious successors in the field of the calculus of variations, and he himself was conscious of his outstanding results which is testified to in one of his most important papers (Carathéodory: "eine Leistung allerersten Ranges") not only by the dedication to the four mathematical heroes Marquis de l'Hôpital, Leibniz, Newton, and Fatio de Duillier, but also by the very unusual and dignified closing of this paper (Ref. 13): "Verily be everlasting praise, honor and glory to eternal God for the grace accorded man in granting mortals the goal of introspection, by faint (or vain) lines, into the mysterious depths of His boundless knowledge and of discovery of it up to a certain point."

Original: "Deo autem immortalis, qui imperscrutabilem inexhaustae suae sapientiae abyssum leviusculis radiis intraspicere, & aliquousque rimari concessit mortalibus, pro praestita nobis gratia sit laus, honos & gloria in sempiterna secula."

This prayer contains a nice play upon words: "radius" means "ray" or "line" as well as "drawing pencil" or also the slat with which the antique mathematicians have drawn their figures into the green powdered glass on the plates of their drawing tables.

For an excellent review of the early history of the calculus of variations, see Carathéodory's paper in Ref. 14.

Carathéodory's equation (23) leads immediately to the fundamental equations of the calculus of variations,

$$S_x = L_{\dot{x}}(t, x, \psi), \quad (24a)$$

$$S_t = L(t, x, \psi) - L_{\dot{x}}(t, x, \psi)\psi. \quad (24b)$$

Carathéodory's equations are the starting point on his royal road from the Hilbert integral to the Euler-Lagrange equations, reversing the historical *modus operandi*.

2.3. The Weierstrass Excess Function and the Hamilton Function. If ψ is replaced by \dot{x} in the right-hand sides of the fundamental equations (24) and if these values of S_t and S_x are introduced into Eq. (19) with \dot{x} replaced by x' , one obtains the Weierstrass \mathcal{E} -function

$$\mathcal{E}(t, x, \dot{x}, x') = L(t, x, x') - L(t, x, \dot{x}) - L_{\dot{x}}(t, x, \dot{x})(x' - \dot{x}). \quad (25)$$

By Taylor expansion, one has

$$\mathcal{E}(t, x, \dot{x}, x') = (1/2)(x' - \dot{x})^\top \tilde{L}_{\dot{x}\dot{x}}(x' - \dot{x}), \quad (26)$$

with

$$\tilde{L}_{\dot{x}\dot{x}} = L_{\dot{x}\dot{x}}(t, x, \dot{x} + \vartheta(x' - \dot{x}))$$

and an appropriate $\vartheta \in (0, 1)$. For line elements (t, x, \dot{x}) and (t, x, x') in the interior of A , the preceding result yields

$$\mathcal{E}(t, x, \dot{x}, x') \geq 0, \quad (27)$$

because of Eq. (10) and the convexity of A with respect to \dot{x} . In addition, there holds

$$\mathcal{E}(t, x, \dot{x}, x') = 0, \quad \text{if and only if } x' - \dot{x} = 0.$$

This is the well-known necessary condition of Weierstrass.

Using the relation

$$L(t, x, x') = S_t + S_x x' + \mathcal{E}(t, x, \psi, x'), \quad (28)$$

there immediately follows that

$$\int_{\gamma} L(t, x(t), x'(t)) dt > S^{(2)} - S^{(1)}, \quad (29)$$

with $S^{(i)} := S(t^{(i)}, x(t^{(i)}))$, $i = 1, 2$, for all piecewise-continuous curves γ which satisfy the condition (16) at all continuity points of its derivative γ' and do not coincide with one of the curves of the field defined by Eq. (22).

For curves e satisfying Eq. (22), there always holds

$$\int_e L(t, x(t), x'(t)) dt = S^{(2)} - S^{(1)}. \tag{30}$$

This is the well-known independent Hilbert integral. Hence, the solutions of the differential equation (22) are minimals of the variational problem with the basic function L . Equation (30) explains the name “merit function” for S .

The extremals can therefore be determined by the integration of the fundamental equations which take a simple form if canonical coordinates are introduced. For this purpose, let

$$y := L_{\dot{x}}(t, x, \dot{x})^\top. \tag{31}$$

Because of Eq. (10), one can solve the preceding equation for \dot{x} ,

$$\dot{x} = \varphi(t, x, y), \tag{32}$$

where φ is a C^1 -function since L is assumed to be a C^2 -function. Hence, the Hamiltonian defined by

$$H(t, x, y) := -L(t, x, \varphi) + y^\top \varphi(t, x, y) \tag{33}$$

is at least a C^1 -function. Because of the formulas of the Legendre transformation,

$$H_t(t, x, y) = -L_t(t, x, \varphi), \tag{34a}$$

$$H_x(t, x, y) = -L_x(t, x, \varphi), \tag{34b}$$

$$H_y^\top(t, x, y) = \varphi(t, x, y), \tag{34c}$$

the Hamiltonian is at least twice continuously differentiable. Moreover, the Legendre–Clebsch condition implies

$$H_{yy} \geq 0. \tag{35}$$

This follows immediately from differentiating the identity

$$\dot{x} \equiv H_y^\top(t, x, L_{\dot{x}}(t, x, \dot{x})) \tag{36}$$

with respect to \dot{x} . The components of the triple (t, x, y) are then called the canonical coordinates of the line elements (t, x, \dot{x}) .

The Weierstrass \mathcal{E} -function can now be expressed in canonical coordinates. Denoting the canonical coordinates of the line elements (t, x, \dot{x}) and (t, x, x') by (t, x, y) and (t, x, y') , respectively, one obtains, from Eqs. (25),

(31), and (33) together with the relations

$$L(t, x, x') = -H(t, x, y') + H_y(t, x, y')y', \quad (37a)$$

$$L(t, x, \dot{x}) = -H(t, x, y) + H_y(t, x, y)y, \quad (37b)$$

$$L_x(t, x, \dot{x}) = y^\top, \quad \dot{x} = H_y^\top(t, x, y), \quad x' = H_y^\top(t, x, y'), \quad (37c)$$

the important representation of the \mathcal{E} -function by the Hamiltonian H ,

$$\mathcal{E} = H(t, x, y) - H(t, x, y') - H_y(t, x, y')(y - y'). \quad (38)$$

In Section 4, we will see that the Eqs. (38) and (27) provide a precursor of the well-known maximum principle.

However, before turning to this point, Carathéodory's royal road should be terminated.

2.4. The Hamilton–Jacobi Equation and the Euler Equation. As mentioned above, the fundamental equations (24) have a simple form when using canonical coordinates. From Eqs. (24a), (31) and (24b), (33), one obtains the well-known Hamilton–Jacobi equation which necessarily must be satisfied by the merit function S ,

$$S_t + H(t, x, S_x) = 0. \quad (39)$$

Vice versa, any at least twice continuously differentiable function S satisfying the partial differential equation (39) yields via

$$\dot{x} = H_y^\top(t, x, S_x) \quad (40)$$

a field of extremals of the variational problem if all line elements are regular.¹⁵ Moreover, one obtains in this way all regular extremals of the variational problem, i.e., extremals having only regular line elements.

Finally, the characteristic equations of the Hamilton–Jacobi equation (39) yield the Euler equations in canonical form,

$$\dot{x} = H_y^\top(t, x, y), \quad \dot{y} = -H_x^\top(t, x, y). \quad (41)$$

¹⁵The Italian mathematician Eugenio Beltrami was the first who found, in 1868, the fundamental relation between variational problems and first-order partial differential equations: "Indeed, if the derivative y' is eliminated from the two preceding equations, one obtains a result of the form

$$\Phi(x, y, \partial F/\partial x, \partial F/\partial y) = 0,$$

i.e., a nonlinear first-order partial differential equation which must be satisfied by the function F and which may therefore serve to determine it." See Ref. 15b, p. 368.

Original: "Infatti, eliminando dalle due precedenti equazioni . . . , cioè un'equazione a derivate parziali del primo ordine non lineare, cui deve soddisfare la funzione F e che può quindi servire a determinarla."

For that, the Hamiltonian H needs to be a C^2 -function. Then, y is eliminated from Eqs. (34b) and (41) by means of Eq. (31), and one obtains the well-known Euler equations,

$$(d/dt)L_{\dot{x}} - L_x = 0. \quad (42)$$

Note that the approach via the canonical coordinates (31) only requires L to be twice continuously differentiable when Eq. (31) is solved for \dot{x} , contrary to the smoothness requirements for the unique solvability of the Euler equation; see Ref. 16.¹⁶

Carathéodory then shows that exactly one extremal can pass through each regular line element of a variational problem. This extremal must necessarily be a solution of the Euler equation. For the details, see Ref. 1a, pp. 208–211, where it is also shown by examples that this theorem is not correct when singular line elements occur; see pp. 309–314.

This completes Carathéodory's royal road in the calculus of variations.

3. The Problem of Lagrange

In this section, we consider variational problems belonging to a basic function L with side conditions given by differential equations,

$$G(t, x, \dot{x}) = 0. \quad (43)$$

Here, G denotes a p -vector-valued function with $p < n$. Problems of this type are usually called Lagrange problems although the ansatz named after Lagrange for variational problems with different types of side conditions can be implicitly found also in Euler's work. The material of this section, too, has already been developed by Carathéodory; compare both Ref. 2 of 1926 and his book, Ref. 1, Chapter 18 entitled "The Problem of Lagrange".

The Jacobian of G with respect to \dot{x} is assumed to have full rank,

$$\text{rank}(\partial G_k / \partial \dot{x}_j)_{\substack{k=1,\dots,p \\ j=1,\dots,p}} = p. \quad (44)$$

¹⁶Carathéodory: "The result we have obtained seems to be very remarkable to me. . . . For the general problem one also could restrict our assumptions by applying the methods which Mr. L. Tonelli has developed with such great skill."

Original: "Das Resultat, zu dem wir gelangt sind, scheint mir sehr bemerkenswert zu sein. . . . Auch für das allgemeine Problem könnte man durch Übertragung der Methoden, die Herr L. Tonelli mit so großer Kunst entwickelt hat, unsere Voraussetzungen einschränken." Carathéodory refers here to Tonelli's fundamental contribution to the calculus of variations from 1923; see Ref. 17.

Applying the same ideas leading to Theorem 2.1, one can proceed as follows. Firstly, a family of curves is considered which is assumed to cover simply a certain domain of \mathcal{S}_{n+1} and to be defined by the differential equations

$$\dot{x} = \psi(t, x). \quad (45)$$

The function ψ is assumed to be continuously differentiable, and the curves are assumed to satisfy the constraints

$$G(t, x, \psi) \equiv 0. \quad (46)$$

Secondly, it is assumed that an at least twice continuously differentiable function $S(t, x)$ can be determined for which, on the one hand, there always holds

$$L(t, x, \psi) - S_x \psi \equiv S_t, \quad (47)$$

and on the other hand,

$$L(t, x, x') - S_x x' > S_t, \quad (48)$$

for all x' which satisfy the constraints

$$G(t, x, x') = 0, \quad (49)$$

and where $|x' - \psi|$ is sufficiently small and $|x' - \psi| \neq 0$ for the associated line elements (t, x, x') .

The solutions of the differential equations are then minimals of the variational problem with the basic function L subject to the differential equations (43). Hence, Carathéodory's necessary condition (23) reads for the Lagrange problem as follows:

$$S_t = \min_{\substack{x' \text{ such that} \\ G(t, x, x') = 0}} \{L(t, x, x') - S_x x'\}. \quad (50)$$

Introducing the function

$$M(t, x, \dot{x}, \mu) := L(t, x, \dot{x}) + \mu^T G(t, x, \dot{x}), \quad (51)$$

where μ denotes a p -vector Lagrange multiplier, the fundamental equations (24) take the form

$$S_x = M_{\dot{x}}(t, x, \psi, \mu), \quad (52a)$$

$$S_t = M(t, x, \psi, \mu) - M_{\dot{x}}(t, x, \psi, \mu)\psi, \quad (52b)$$

$$G(t, x, \psi) = 0. \quad (52c)$$

These equations can already be found in Carathéodory's paper given in Ref. 2.

The Weierstrass \mathcal{E} -function (25) for the constrained variational problem is then given by

$$\mathcal{E}(t, x, \dot{x}, x', \mu) = M(t, x, x', \mu) - M(t, x, \dot{x}, \mu) - M_{\dot{x}}(t, x, \dot{x}, \mu)(x' - \dot{x}), \quad (53)$$

where line elements (t, x, \dot{x}) and (t, x, x') are considered for which the side conditions (43) and (49) are fulfilled. A sufficient condition for

$$\mathcal{E}(t, x, \dot{x}, x', \mu) \geq 0 \quad (54)$$

in a certain neighborhood of the line element (t, x, \dot{x}) is the Legendre–Clebsch condition, which here can be formulated as follows: The minimum of the quadratic form

$$Q = \xi^T M_{\dot{x}\dot{x}}(t, x, \dot{x}, \mu)\xi, \quad (55)$$

subject to the constraints

$$(\partial G / \partial \dot{x})\xi = 0 \quad (56)$$

on the sphere $|\xi|_2 = 1$, must be positive. This immediately implies that

$$\begin{vmatrix} M_{\dot{x}\dot{x}} & \partial G^T / \partial \dot{x} \\ \partial G / \partial \dot{x} & 0 \end{vmatrix} \neq 0, \quad (57)$$

which is the analogue to Eq. (10). For details, see Refs. 1 or 2. As we will see, Eq. (57) will play an important role when canonical coordinates are introduced.

Of course, results analogous to Eqs. (29) and (30) also hold for the variational problem subject to the side conditions (43). In order to show this, an arbitrary solution $x = x(t)$ of the differential equation (49) is considered. The endpoints of this curve are denoted by $(t^{(1)}, x(t^{(1)}))$ and $(t^{(2)}, x(t^{(2)}))$. Because of the relation

$$S^{(2)} - S^{(1)} = \int_{t_1}^{t_2} (dS/dt)(x, t) dt, \quad (58)$$

the Hilbert integral can be obtained using the fundamental equations (52),

$$S^{(2)} - S^{(1)} = \int_{t_1}^{t_2} (M(t, x, \dot{x}, \mu) + M_{\dot{x}}(t, x, \dot{x}, \mu)(x' - \dot{x})) dt. \quad (59)$$

Its value depends only on the fixed endpoints $(t^{(1)}, x(t^{(1)}))$ and $(t^{(2)}, x(t^{(2)}))$ of the curve $x = x(t)$. On the other hand, because of Eq. (49), one has the relation

$$M(t, x, x', \mu) = L(t, x, x'). \quad (60)$$

Hence, the line integral of L along the curve $x = x(t)$ can be written as

$$J = \int_{t_1}^{t_2} M(t, x, x', \mu) dt. \tag{61}$$

By subtracting Eq. (59) from Eq. (61) and using Eq. (53), one has

$$J - (S^{(2)} - S^{(1)}) = \int_{t_1}^{t_2} \mathcal{E}(t, x, \dot{x}, x', \mu) dt. \tag{62}$$

Therefore, Eq. (54) yields

$$J \geq I := S^{(2)} - S^{(1)}, \tag{63}$$

where I denotes the line integral along a minimal e of the constrained variational problem with the endpoints $(t^{(1)}, x(t^{(1)}))$ and $(t^{(2)}, x(t^{(2)}))$ prescribed. For any other curve γ having the same endpoints as e , there always holds that $J > I$ if γ lies in an appropriate close neighborhood of e and if the line elements of γ satisfy the constraints (49).

As we will see in the following, the representation (38) of the Weierstrass \mathcal{E} -function in canonical coordinates remains valid for the Lagrange problems, too. Similarly as in Section 2.3, new variables are firstly introduced,

$$y := M_{\dot{x}}^T(t, x, \dot{x}, \mu), \tag{64a}$$

$$z := G(t, x, \dot{x}). \tag{64b}$$

Because of (57), these equations can be solved for \dot{x} and μ ,

$$\dot{x} = \Phi(t, x, y, z), \tag{65a}$$

$$\mu = X(t, x, y, z). \tag{65b}$$

Since the side conditions can be expressed as $z = 0$, the so-called complete line elements $(t, x, \dot{x}_{p+1}, \dots, \dot{x}_n, \mu_1, \dots, \mu_p)$ are entirely characterized by its canonical coordinates (t, x, y) .

Next, one defines

$$\tilde{H}(t, x, y, z) := -M(t, x, \Phi, X) + y^T \Phi + z^T X, \tag{66}$$

$$H(t, x, y) := \tilde{H}(t, x, y, 0). \tag{67}$$

Note that \tilde{H} is the Legendre transformation of M . Hence, one has

$$\tilde{H}_t = -M_t, \quad \tilde{H}_x = -M_x, \quad \tilde{H}_y^T = \dot{x}, \quad \tilde{H}_z^T = \mu, \tag{68}$$

and

$$H_t(t, x, y) = \tilde{H}_t(t, x, y, 0), \tag{69a}$$

$$H_x(t, x, y) = \tilde{H}_x(t, x, y, 0), \tag{69b}$$

$$H_y(t, x, y) = \tilde{H}_y(t, x, y, 0). \tag{69c}$$

Equations (68) and (69) imply that

$$H_t = -M_t, \quad H_x = -M_x, \quad H_y^T = \dot{x}, \tag{70}$$

for all line elements which satisfy the side conditions (43). The function H can also be obtained directly, if the equations

$$y := M_x^T(t, x, \dot{x}, \mu), \tag{71a}$$

$$0 = G(t, x, \dot{x}) \tag{71b}$$

are solved for \dot{x} and μ ,

$$\dot{x} = \varphi(t, x, y) = \Phi(t, x, y, 0), \tag{72a}$$

$$\mu = \chi(t, x, y) = X(t, x, y, 0). \tag{72b}$$

In this case, one defines

$$H(t, x, y) = -M(t, x, \varphi, \chi) + y^T \varphi. \tag{73}$$

The function H is called the Hamiltonian of the Lagrange problem.

The Weierstrass \mathcal{E} -function in canonical coordinates follows as in Section 2.3. Let (t, x, y) and (t, x, y') be the canonical coordinates of two complete line elements passing through the same point. Because of Eqs. (64), (70), and (73), one obtains the relations

$$M(t, x, \dot{x}, \mu) = -H(t, x, y) + H_y(t, x, y)y, \tag{74a}$$

$$M(t, x, \dot{x}', \mu) = -H(t, x, y') + H_y(t, x, y')y', \tag{74b}$$

$$M_x(t, x, \dot{x}, \mu)(x' - \dot{x}) = (H_y(t, x, y') - H_y(t, x, y))y. \tag{74c}$$

The definition of the \mathcal{E} -function (53) finally gives

$$\mathcal{E} = H(t, x, y) - H(t, x, y') - H_y(t, x, y')(y - y'). \tag{75}$$

From this equation with the sign condition (54), Carathéodory's precursor of the maximum principle will follow.

However, before we attend to this, it should be mentioned that all solutions of the Euler equations (41) in canonical form with Hamiltonian defined by Eq. (73) are extremals of the Lagrange problem, if these extremals can be varied in the sense of Section 2.1. For conditions under which this is possible, see Ref. 1.

4. Carathéodory's Precursor of the Maximum Principle

In this section, we continue presenting evidence that Carathéodory's results can be interpreted as a precursor of the well-known maximum

principle. In particular, we consider Carathéodory's necessary condition given by Eqs. (54) and (75),

$$H(t, x, y) - H(t, x, y') - H_y(t, x, y')(y - y') \geq 0, \quad (76)$$

with the Hamiltonian as defined by Eq. (73). The only small step required to obtain a special case of the maximum principle is to substitute the variational form of an optimal control problem into Eq. (76).

As it is known, the simple optimal control problem for a p -dimensional state vector z and a k -dimensional control vector u ,

$$\int_{t_1}^{t_2} L(t, z, u) dt \stackrel{!}{=} \min, \quad (77)$$

subject to the side conditions

$$\dot{z} = g(t, z, u), \quad (78)$$

can be written as

$$\int_{t_1}^{t_2} L(t, x_1, \dots, x_p, \dot{x}_{p+1}, \dots, \dot{x}_n) dt \stackrel{!}{=} \min, \quad (79)$$

subject to

$$G_i(t, x, \dot{x}) := \dot{x}_i - g_i(t, x_1, \dots, x_p, \dot{x}_{p+1}, \dots, \dot{x}_n), \quad i = 1, \dots, p. \quad (80)$$

Here, we define

$$n := p + k,$$

$$(x_1, \dots, x_p) := (z_1, \dots, z_p),$$

$$(\dot{x}_1, \dots, \dot{x}_n) := (\dot{z}_1, \dots, \dot{z}_p, u_1, \dots, u_k).$$

Using the definition (51),

$$M(t, x, \dot{x}, \mu) := L(t, x_1, \dots, x_p, \dot{x}_{p+1}, \dots, \dot{x}_n) + \mu^T G(t, x_1, \dots, x_p, \dot{x}_1, \dots, \dot{x}_n), \quad (81)$$

Eqs. (71) and (72) yield

$$\dot{x}_i = \begin{cases} \varphi_i(t, x_1, \dots, x_p, \varphi_{p+1}, \dots, \varphi_n) \\ = g_i(t, x_1, \dots, x_p, \varphi_{p+1}, \dots, \varphi_n), & i = 1, \dots, p, \\ \varphi_i(t, x_1, \dots, x_p, y_1, \dots, y_n), & i = p + 1, \dots, n, \end{cases} \quad (82a)$$

$$\mu_i = \chi_i(t, x_1, \dots, x_p, y_1, \dots, y_n) = y_i, \quad i = 1, \dots, p. \quad (82b)$$

Equations (82a) for $i = p + 1, \dots, n$ are obtained when solving the k equations

$$y_i = L_{\dot{x}_i} - (\mu^T g)_{\dot{x}_i}, \quad i = p + 1, \dots, n, \quad (83)$$

in Eq. (71) for the k variables $(\dot{x}_{p+1}, \dots, \dot{x}_n)$. This is possible because of Eq. (57). A simple calculation then shows that the second of the Euler equations (41) implies

$$y_i \equiv 0, \quad i = p + 1, \dots, n. \tag{84}$$

Because of

$$H_{y_i} = \begin{cases} g_i, & i = 1, \dots, p, \\ \varphi_i, & i = p + 1, \dots, n, \end{cases} \tag{85}$$

Eq. (76) finally leads to

$$-L + \sum_{i=1}^p y_i g_i + L' - \sum_{i=1}^p y_i g'_i \geq 0, \tag{86}$$

with

$$L' := L(t, x_1, \dots, x_p, \varphi'_{p+1}, \dots, \varphi'_n), \tag{87a}$$

$$g'_i := g_i(t, x_1, \dots, x_p, \varphi'_{p+1}, \dots, \varphi'_n), \quad i = 1, \dots, p, \tag{87b}$$

$$\varphi'_i := \varphi_i(t, x_1, \dots, x_p, y'_1, \dots, y'_n), \quad i = p + 1, \dots, n. \tag{87c}$$

Hence, the maximum principle reads

$$\mathcal{H}(t, z, y, u) \geq \mathcal{H}(t, z, y, u'), \tag{88}$$

with the Hamiltonian \mathcal{H} of the control problem defined by

$$\mathcal{H} := -L(t, z, u) + \sum_{i=1}^p y_i g_i(t, z, u). \tag{89}$$

5. Some Historical Remarks on the Maximum Principle

The above derivation of the maximum principle (and also its equivalence to the Weierstrass condition) is correct only in the special case in which the control functions are assumed to be continuous and to have values in an open control domain and all line elements are assumed to be regular, which is a severe restriction when practical applications are considered. In this sense, optimal control problems with piecewise-continuous control functions and bounded control domains, which are of utmost importance for many practical applications, are nonclassical variational problems or, in other words, generalized Lagrange problems.

Under the same restrictions as above, Hestenes seems to be the first who obtained the relation (88); see his Ref. 18 from 1950.¹⁷ His derivation

¹⁷Hestenes: "Thus, H has a maximum value with respect to u_i along a minimizing curve C_0 ."

is very similar to the one based on Carathéodory's results and takes its starting point directly from the Weierstrass necessary condition, Eqs. (25) and (27). Hestenes showed the equivalence of the control problem (77)–(78) with the variational problem for the basic function M as defined by Eq. (81). By means of the Euler equation, the necessary condition of Weierstrass can then be rewritten in terms of the Hamiltonian of the control problem. In addition, Hestenes has also considered control problems with control variable equality and inequality constraints and has applied his results to the minimum-time flight path control problem for an aircraft under a constraint to the angle of attack.

Decidedly, the achievement of Boltyanskii, Gamkrelidze, and Pontryagin, who coined the term "maximum principle" in their 1956 paper given in Ref. 19,¹⁸ lies in the fact that they later gave a rigorous proof for the general case of an arbitrary (for example, closed) control domain, and for bounded measurable control functions; see the pioneering book of Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko from 1961, Ref. 20. Indeed, the new ideas in this book led to the cutting of the umbilical cord between the calculus of variations and optimal control theory. The first papers on the maximum principle¹⁹ at an early stage are the papers of Gamkrelidze from 1957 and 1958 for linear control systems. The first proof was given by Boltyanskii in 1958 and later improved by several other authors. All these references are cited in Ref. 20 and, for example, also in Ref. 21 where the more recent proofs of the maximum principle, which are based on new ideas, can be found, too.

For the sake of completeness, it should be mentioned that also Isaacs can be regarded as one of the more intuitive discoverers of the maximum principle. His "tenet of transition" in his 1951 RAND report (Ref. 8) can be considered as a generalized form of the maximum principle, which becomes a minimax principle in differential game theory.²⁰

These illustrious researchers and their successors, inspired by the diversity of so many attractive fields of application of the calculus of variations and their offsprings, optimal control and differential game theory, have continued and are still continuing Carathéodory's royal road

¹⁸Boltyanskii, Gamkrelidze, Pontryagin: "This fact is a special case of the following general principle which we call maximum principle."

¹⁹For the maximum principle, Pontryagin was conferred the highest ranking order of the USSR, the Lenin order.

²⁰Isaacs in Ref. 22: "Once I felt that here was the heart of the subject . . . Later I felt that it . . . was a mere truism. Thus in my book '*Differential Games*', it is mentioned only by title. This I regret. I had no idea that Pontryagin's principle and Bellman's maximal principle (a special case of the tenet . . .) would enjoy such widespread citation."

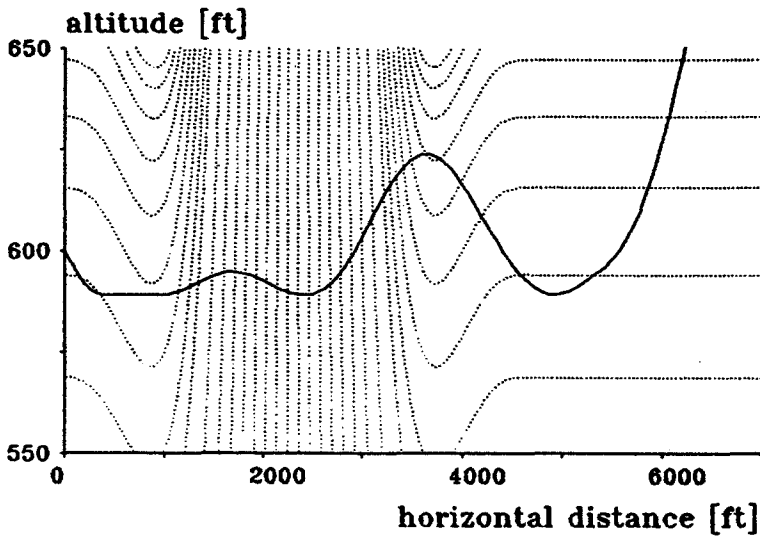


Fig. 5. Abort landing under a windshear including an upwind.

into various branches. Carathéodory in Ref. 23: "One can affirm that the charm, exerted all along by the calculus of variations on so many first rate greatneses of mind, is chiefly traceable to the role which particular problems have played and are playing even today in the development of this theory."²¹

One of these particular problems Carathéodory had in mind surely is the abort landing of a passenger aircraft in the presence of windshear. Angelo Miele has made many contributions to this field, which is of utmost importance for aviation safety. See, for example, Refs. 24 and 25, to cite only two of his many papers. Miele's papers have inspired many researchers, including the authors (Ref. 26). Therefore, we close the present paper with a new result from nonhistorical research: Figure 5 shows a wind flow field including an upwind and the associated altitude versus horizontal distance. This optimal control problem is of the minimax type—the minimum altitude is to be maximized—and includes one control variable and two state variable inequality constraints; one is of order one, the other is of order three. The optimal trajectory exhibits one boundary arc and two

²¹Original: "Et l'on peut affirmer que le charme exercé de tout temps par le Calcul des Variations, sur tant d'esprits de premier ordre, est dû en grande partie au rôle qu'ont joué et que jouent jusqu'à ce jour les problèmes particuliers pour le développement de cette théorie."

touch points due to the third-order state constraint, two boundary arcs due to the first-order state constraint, six bang-bang and three singular subarcs. This complicated switching structure can only be computed by carefully applying new necessary conditions for optimality; details can be found in Ref. 27.

6. Conclusions

We conclude with a statement of Constantin Carathéodory addressing American mathematicians on August 31, 1936:²² "I will be glad if I have succeeded in impressing the idea that it is not only pleasant and entertaining to read at times the works of the old mathematical authors, but that this may occasionally be of use for the actual advancement of science.

Besides this there is a great lesson we can derive from the facts which I have just referred to. We have seen that even under conditions which seem most favorable very important results can be discarded for a long time and whirled away from the main stream which is carrying the vessel science. Sometimes it is of no use even if such results are published in very conspicuous places. It may happen that the work of most celebrated men may be overlooked.

If their ideas are too far in advance of their time, and if the general public is not prepared to accept them, these ideas may sleep for centuries on the shelves of our libraries. Occasionally, as we have tried to do to-day, some of them may be awakened to life. But I can imagine that the greater part of them is still sleeping and is awaiting the arrival of the prince charming who will take them home."

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²²Meeting of the Mathematical Association of America in Cambridge, Massachusetts, during the tercentenary celebration of Harvard University; see Ref. 28.

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