Geophysical and Astrophysical Fluid Dynamics
Vol. 97, No. 3, June 2003, pp. 249–274

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THE MEAN ELECTROMOTIVE FORCE FOR MHD TURBULENCE: THE CASE OF A WEAK MEAN MAGNETIC FIELD AND SLOW ROTATION

KARL-HEINZ RÄDLER[a], NATHAN KLEEORIN[b] and
IGOR ROGACHEVSKII[b,*]

[a]*Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482, Potsdam, Germany;*
[b]*Department of Mechanical Engineering, The Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel*

(Received 22 April 2002; In final form 6 January 2003)

The mean electromotive force that occurs in the framework of mean-field magnetohydrodynamics is studied for cases in which magnetic field fluctuations are not only due to the action of velocity fluctuations on the mean magnetic field. The possibility of magnetic field fluctuations independent of a mean magnetic field, as they may occur as a consequence of a small-scale dynamo, is taken into account. Particular attention is paid to the effect of a mean rotation of the fluid on the mean electromotive force, although only small rotation rates are considered. Anisotropies of the turbulence due to gradients of its intensity or its helicity are admitted. The mean magnetic field is considered to be weak enough to exclude quenching effects. A τ-approximation is used in the equation describing the deviation of the cross-helicity tensor from that for zero mean magnetic field, which applies in the limit of large hydrodynamic Reynolds numbers.

For the effects described by the mean electromotive force like α-effect, turbulent diffusion of magnetic fields etc in addition to the contributions determined by the velocity fluctuations also those determined by the magnetic field fluctuations independent of the mean magnetic field are derived. Several old results are confirmed, partially under more general assumptions, and quite a few new ones are given. Provided the kinematic helicity and the current helicity of the fluctuations have the same signs the α-effect is always diminished by the magnetic fluctuations. In the absence of rotation these have, however, no influence on the turbulent diffusion. Besides the diamagnetic effect due to a gradient of the intensity of the velocity fluctuations there is a paramagnetic effect due to a gradient of the intensity of the magnetic fluctuations. In the absence of rotation these two effects compensate each other in the case of equipartition of the kinetic and magnetic energies of the fluctuations of the original turbulence, i.e. that with zero mean magnetic field, but the rotation makes the situation more complex. The $\boldsymbol{\Omega} \times \mathbf{J}$-effect works in the same way with velocity fluctuations and magnetic field fluctuations. A contribution to the electromotive force connected with the symmetric parts of the gradient tensor of the mean magnetic field, which does not occur in the absence of rotation, was found in the case of rotation, resulting from velocity or magnetic fluctuations.

The implications of the results for the mean electromotive force for mean-field dynamo models are discussed with special emphasis to dynamos working without α-effect.

The results for the coefficients defining the mean electromotive force which are determined by the velocity fluctuations in the case of vanishing mean motion agree formally with the results obtained in the kinematic approach, specified by second-order approximation and high-conductivity limit. However, their range of validity is clearly larger.

Keywords: Mean electromotive force; MHD anisotropic rotating turbulence; Magnetic fields

*Corresponding author. E-mail: gary@menix.bgu.ac.il

ISSN 0309-1929 print: ISSN 1029-0419 online © 2003 Taylor & Francis Ltd
DOI:10.1080/0309192031000151212

1. INTRODUCTION

The mean-field approach proved to be very useful in studying dynamo processes in turbulently moving electrically conducting fluids (see, e.g., Moffatt, 1978; Parker, 1979; Krause and Rädler, 1980; Zeldovich *et al.*, 1983). The crucial point of this approach is the mean electromotive force which is determined by the fluctuations of the fluid velocity and the magnetic field. It describes several physical effects like α-effect, turbulent diffusion of the magnetic field or turbulent diamagnetism. In many investigations on the kinematic level the magnetic fluctuations are understood as caused by the action of the velocity fluctuations on the mean magnetic field. This implies that they vanish if the mean magnetic field does so. The mean electromotive force can then be considered as a quantity determined, apart from the mean velocity, by the velocity fluctuations and the mean magnetic field.

There are, however, many realistic cases with magnetic fluctuations which exist independent of the action of the velocity fluctuations on the mean magnetic field and do not vanish if it does so. We recall here the numerous investigations showing the possibility of small-scale dynamos with a zero mean magnetic field (see, e.g., Kazantsev, 1968; Meneguzzi *et al.*, 1981; Zeldovich *et al.*, 1990; Nordlund *et al.*, 1992; Childress and Gilbert 1995; Brandenburg *et al.*, 1996; Rogachevskii and Kleeorin, 1997; Kleeorin *et al.* 2002a). As pointed out already by Pouquet *et al.* (1976) in such cases the mean electromotive force has in addition to the contributions mentioned above, which can be ascribed to the velocity fluctuations, also others due to those magnetic fluctuations which exist independent of the mean magnetic field.

Several investigations on the mean electromotive force comprising both kinds of contributions have been carried out so far. We mention in particular those by Kichatinov (1982) and by Vainshtein and Kichatinov (1983), in which an isotropic turbulence with a scale-independent correlation time was assumed. Results concerning the α-effect, the turbulent magnetic diffusivity and the turbulent diamagnetism or paramagnetism for an originally isotropic turbulence subject to mean rotation have been derived using a modified second-order correlation approximation by Kichatinov (1991), Rüdiger and Kichatinov (1993) and Kichatinov *et al.* (1994).

In this article we present an approach to the mean electromotive force which reproduces such results under more general assumptions and reveals new ones, which are of particular importance for astrophysical applications. We exclude mean motions of the fluid other than a rigid body rotation with some small rotation rate. We further consider only a weak mean magnetic field so that its energy density is small compared to the kinetic energy density. In this way we do not consider quenching effects, i.e. reductions of the coefficients defining the mean electromotive force with growing mean magnetic field, which are important in view of the nonlinear behavior of dynamos and have been discussed in a number of articles (see, e.g., Gruzinov and Diamond, 1994; Cattaneo and Hughes, 1996; Seehafer, 1996; Field *et al.*, 1999; Kulsrud, 1999; Kleeorin *et al.*, 2002b; Rogachevskii and Kleeorin, 2001; Blackman and Brandenburg, 2002). We also do not deal with the evolution of the magnetic helicity, which implies a constraint to the magnetic part of the α-effect (see, e.g., Kleeorin *et al.*, 1995; Kleeorin and Rogachevskii, 1999; Kleeorin *et al.*, 2000, 2002b; Blackman and Brandenburg, 2002). Finally we use a τ-approximation in the equations describing the deviation of the cross-helicity tensor from that for zero magnetic field. In contrast to the often used second-order correlation approximation it does not totally ignore higher than second-order correlations but considers their influence in some summary way. The τ-approximation in that sense applies in the limit of high hydrodynamic Reynolds numbers.

In Section 2 we will explain the concept of mean-field magnetohydrodynamics for a homogeneous incompressible fluid. In Section 3 we introduce a Fourier representation of the velocity and magnetic fluctuations, define correlation tensors and express the mean electromotive force by the cross-helicity tensor. After giving an equation for this tensor in Section 4 we introduce in Section 5 the mentioned τ-approximation, which leads to closed equations for that part of the cross-helicity tensor which determines the mean electromotive force. In Section 6 general relations for this mean electromotive force are given, and in Section 7 more specific relations for the case for small rotation rates. In Section 8 we restrict ourselves to the limit of weak mean magnetic fields so that the mean electromotive force can be considered as linear in the mean magnetic field and specify the correlation tensors for the "original" turbulence, that is, the turbulence for zero mean magnetic field. For zero rotation they are determined by simple assumptions concerning the deviations from a homogeneous isotropic turbulence, and the influence of a slow rotation on the velocity fluctuations is calculated by a perturbation procedure, again based on a τ-approximation. We further introduce Kolmogorov-type spectra of the relevant quantities. On this basis we deliver results for the coefficients defining the mean electromotive force and discuss them with special attention to the different effects of velocity and magnetic field fluctuations. Finally, in Section 9 our results are compared with results of the kinematic approach in the second-order correlation approximation, some remarks concerning their range of validity are made, and some prospects are mentioned concerning the extension of the approach of this article to related questions of mean-field magnetohydrodynamics.

2. FORMULATION OF THE PROBLEM

We consider a turbulent motion of an electrically conducting incompressible fluid in interaction with a magnetic field. Let us assume that the fluid velocity $\mathbf{U}$ and the magnetic field $\mathbf{B}$ are governed by the equations

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{\nabla P}{\rho} + \frac{1}{\mu\rho}(\nabla\times\mathbf{B})\times\mathbf{B} + \nu\nabla^2\mathbf{U} + 2\mathbf{U}\times\mathbf{\Omega} + \mathbf{F}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, \tag{2}$$

$$\nabla\cdot\mathbf{U} = \nabla\cdot\mathbf{B} = 0, \tag{3}$$

where P is a modified pressure and $\mathbf{F}$ an external force. We refer to a rotating frame with $\mathbf{\Omega}$ being the angular velocity responsible for the Coriolis force. As usual ν and η are the kinematic and magnetic viscosities, ρ is the mass density and μ is the magnetic permeability of the fluid, all considered as constants.

We further assume that there is an averaging procedure which defines for each quantity Q an average $\langle Q \rangle$ and satisfies the Reynolds rules. In the spirit of the mean-field concept we split the fluid velocity **U** and the magnetic field **B** according to

$$\mathbf{U} = \overline{\mathbf{U}} + \mathbf{u}, \quad \mathbf{B} = \overline{\mathbf{B}} + \mathbf{b} \tag{4}$$

into mean parts, $\overline{\mathbf{U}} = \langle \mathbf{U} \rangle$ and $\overline{\mathbf{B}} = \langle \mathbf{B} \rangle$, and fluctuations, **u** and **b**. Analogously we split P into $\overline{P}$ and p, and **F** into $\overline{\mathbf{F}}$ and **f**, etc. The mean fields $\overline{\mathbf{U}}$ and $\overline{\mathbf{B}}$ satisfy the equations

$$\frac{\partial \overline{\mathbf{U}}}{\partial t} + (\overline{\mathbf{U}} \cdot \nabla)\overline{\mathbf{U}} = -\frac{\nabla \overline{P}}{\rho} + \frac{1}{\mu\rho}(\nabla \times \overline{\mathbf{B}}) \times \overline{\mathbf{B}} + \nu \nabla^2 \overline{\mathbf{U}} + 2\overline{\mathbf{U}} \times \mathbf{\Omega} + \mathcal{F} + \overline{\mathbf{F}}, \tag{5}$$

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \mathcal{E}) + \eta \nabla^2 \overline{\mathbf{B}}, \tag{6}$$

$$\nabla \cdot \overline{\mathbf{U}} = \nabla \cdot \overline{\mathbf{B}} = 0, \tag{7}$$

with $\mathcal{F}$ and $\mathcal{E}$ being a pondermotive and an electromotive force due to fluctuations

$$\mathcal{F} = -\langle (\mathbf{u} \cdot \nabla)\mathbf{u} \rangle + \frac{1}{\mu\rho} \langle (\nabla \times \mathbf{b}) \times \mathbf{b} \rangle, \tag{8}$$

$$\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle. \tag{9}$$

For the determination of $\mathcal{F}$ and $\mathcal{E}$ we need information on the fluctuations **u** and **b**. These have to obey the equations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} = {} & -(\overline{\mathbf{U}} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\overline{\mathbf{U}} - \frac{\nabla p}{\rho} + \frac{1}{\mu\rho}[(\nabla \times \overline{\mathbf{B}}) \times \mathbf{b} \\ & + (\nabla \times \mathbf{b}) \times \overline{\mathbf{B}}] + \mathbf{T} + \nu \nabla^2 \mathbf{u} + 2\mathbf{u} \times \mathbf{\Omega} + \mathbf{f}, \end{aligned} \tag{10}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \overline{\mathbf{B}} + \overline{\mathbf{U}} \times \mathbf{b}) + \eta \nabla^2 \mathbf{b} + \mathbf{G}, \tag{11}$$

where **T** and **G** summarize terms nonlinear in **u** and **b**;

$$\mathbf{T} = \langle (\mathbf{u} \cdot \nabla)\mathbf{u} \rangle - (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\mu\rho}[\langle \mathbf{b} \times (\nabla \times \mathbf{b}) \rangle - \mathbf{b} \times (\nabla \times \mathbf{b})], \tag{12}$$

$$\mathbf{G} = \nabla \times (\mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle). \tag{13}$$

3. FOURIER REPRESENTATION, CORRELATION TENSORS

Let us now represent quantities like $\mathbf{u}$ and $\mathbf{b}$, or $\overline{\mathbf{U}}$ and $\overline{\mathbf{B}}$, by Fourier integrals defined according to

$$Q(\mathbf{x}, t) = \int \hat{Q}(\mathbf{k}, t) \exp(i\,\mathbf{k}\cdot\mathbf{x})\, d^3k. \tag{14}$$

We may rewrite the Eqs. (10) and (11) for $\mathbf{u}$ and $\mathbf{b}$ into equations for their components u_i and b_i with respect to a Cartesian coordinate system, derive equations for their Fourier transforms $\hat{u}_i$ and $\hat{b}_i$ and subject the equation for $\hat{u}_i$ to a projection operator $P_{ij} = \delta_{ij} - k_i k_j / k^2$, with δ_{ij} being the Kronecker tensor, in order to eliminate the pressure p. In this way we obtain

$$\begin{aligned} \partial \hat{u}_i(\mathbf{k})/\partial t = {} & -P_{ij}[\hat{L}_j(\mathbf{u}, \overline{\mathbf{U}};\, \mathbf{k}) + \hat{S}_j(\mathbf{u}, \overline{\mathbf{U}};\, \mathbf{k})] + (2P_{ij} - \delta_{ij})\hat{L}_j(\mathbf{b}, \overline{\mathbf{B}};\, \mathbf{k})/\mu\rho \\ & + \hat{S}_i(\mathbf{b}, \overline{\mathbf{B}};\, \mathbf{k})/\mu\rho + D_{ij}(\mathbf{k})\hat{u}_j(\mathbf{k}) - \nu k^2 \hat{u}_i(\mathbf{k}) - P_{ij}(\hat{T}_j(\mathbf{k}) - \hat{f}_j(\mathbf{k}))/\rho, \end{aligned} \tag{15}$$

$$\begin{aligned} \partial \hat{b}_i(\mathbf{k}, t)/\partial t = {} & \hat{S}_i(\mathbf{u}, \overline{\mathbf{B}};\, \mathbf{k}) - \hat{S}_i(\mathbf{b}, \overline{\mathbf{U}};\, \mathbf{k}) - \hat{L}_i(\mathbf{u}, \overline{\mathbf{B}};\, \mathbf{k}) + \hat{L}_i(\mathbf{b}, \overline{\mathbf{U}};\, \mathbf{k}) \\ & - \eta k^2 \hat{b}_i(\mathbf{k}) + \hat{G}_i(\mathbf{k}), \end{aligned} \tag{16}$$

where

$$\hat{L}_i(\mathbf{a}, \mathbf{A};\, \mathbf{k}) = i \int a_j(\mathbf{k} - \mathbf{K}) K_j A_i(\mathbf{K})\, d^3K, \tag{17}$$

$$\hat{S}_i(\mathbf{a}, \mathbf{A};\, \mathbf{k}) = i k_j \int a_i(\mathbf{k} - \mathbf{K}) A_j(\mathbf{K})\, d^3K, \tag{18}$$

$$D_{ij}(\mathbf{k}) = 2\varepsilon_{ijm} k_m (\mathbf{k}\cdot\boldsymbol{\Omega})/k^2, \tag{19}$$

and ε_{ijk} is the Levi–Civita tensor. For the sake of simplicity the argument t is dropped everywhere. Concerning the derivation of (19) we refer to (A1).

We will use these equations for calculating the two-point correlation functions. Let us consider, e.g., the correlation tensor $\langle v_i(\mathbf{x}_1) w_j(\mathbf{x}_2)\rangle$ of two vector fields $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ denote two points in space but both fields are taken at the same time. Using the definition (14) of the Fourier transformation and following a pattern introduced by Roberts and Soward (1975) we write

$$\begin{aligned} \langle v_i(\mathbf{x}_1) w_j(\mathbf{x}_2)\rangle &= \int\!\!\int \langle \hat{v}_i(\mathbf{k}_1)\hat{w}_j(\mathbf{k}_2)\rangle \exp i(\mathbf{k}_1\cdot\mathbf{x}_1 + \mathbf{k}_2\cdot\mathbf{x}_2)\, d^3k_1\, d^3k_2 \\ &= \int \tilde{\varphi}_{ij}(\mathbf{r}, \mathbf{K}) \exp(i\mathbf{K}\cdot\mathbf{R})\, d^3K \\ &= \int \varphi_{ij}(\mathbf{k}, \mathbf{R}) \exp(i\mathbf{k}\cdot\mathbf{r})\, d^3k, \end{aligned} \tag{20}$$

where

$$\tilde{\varphi}_{ij}(\mathbf{r},\mathbf{K}) = \int \langle \hat{v}_i(\mathbf{k}+\mathbf{K}/2)\hat{w}_j(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{k}\cdot\mathbf{r})\, d^3k, \tag{21}$$

$$\varphi_{ij}(\mathbf{k},\mathbf{R}) = \int \langle \hat{v}_i(\mathbf{k}+\mathbf{K}/2)\hat{w}_j(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{K}\cdot\mathbf{R})\, d^3K, \tag{22}$$

and $\mathbf{R} = (\mathbf{x}_1 + \mathbf{x}_2)/2$, $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$, $\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2)/2$. We relate later $\mathbf{r}$ and $\mathbf{k}$ to small scales and $\mathbf{R}$ and $\mathbf{K}$ to large scales in the physical space.

In the following we consider in particular the correlation tensors for the velocity and magnetic fluctuations, $\langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle$ and $\langle b_i(\mathbf{x}_1)b_j(\mathbf{x}_2)\rangle$, and the cross-helicity tensor, $\langle u_i(\mathbf{x}_1)b_j(\mathbf{x}_2)\rangle$, and we use the definitions

$$v_{ij}(\mathbf{k},\mathbf{R}) = \Phi(\hat{u}_i, \hat{u}_j; \mathbf{k},\mathbf{R}), \quad m_{ij}(\mathbf{k},\mathbf{R}) = \Phi(\hat{b}_i, \hat{b}_j; \mathbf{k},\mathbf{R})/\mu\rho, \tag{23}$$

$$\chi_{ij}(\mathbf{k},\mathbf{R}) = \Phi(\hat{u}_i, \hat{b}_j; \mathbf{k},\mathbf{R}), \tag{24}$$

where

$$\Phi(v, w; \mathbf{k},\mathbf{R}) = \int \langle v(\mathbf{k}+\mathbf{K}/2) w(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{K}\cdot\mathbf{R})\, d^3K. \tag{25}$$

The definition of v_{ij} implies

$$v_{ij}(\mathbf{k},\mathbf{R}) = v_{ji}(-\mathbf{k},\mathbf{R}) \tag{26}$$

and

$$v_{ij}(\mathbf{k},\mathbf{R})k_i = \frac{i}{2}\nabla_i v_{ij}(\mathbf{k},\mathbf{R}), \quad v_{ij}(\mathbf{k},\mathbf{R})k_j = -\frac{i}{2}\nabla_j v_{ij}(\mathbf{k},\mathbf{R}). \tag{27}$$

Here and in the following ∇ stands for $\partial/\partial\mathbf{R}$. A relation analogous to (26) applies to m_{ij}, too, and relations analogous to (27) apply to m_{ij} and even to χ_{ij}.

If we know v_{ij}, m_{ij} and χ_{ij} we may calculate the pondermotive force $\mathcal{F}$ and the electromotive force $\mathcal{E}$. In this article, however, we will focus attention on $\mathcal{E}$ only. Since

$$\mathcal{E}_i = \varepsilon_{ijk} \int \chi_{jk}(\mathbf{k},\mathbf{R})\, d^3k, \tag{28}$$

we first deal with the cross-correlation tensor χ_{ij}.

4. THE EQUATION FOR THE CROSS-HELICITY TENSOR

According to the definition (25) of χ_{ij} its time derivative is given by

$$\frac{\partial \chi_{ij}(\mathbf{k},\mathbf{R})}{\partial t} = \Phi\left(\frac{\partial \hat{u}_i}{\partial t}, \hat{b}_j; \mathbf{k},\mathbf{R}\right) + \Phi\left(\hat{u}_i, \frac{\partial \hat{b}_j}{\partial t}; \mathbf{k},\mathbf{R}\right). \tag{29}$$

We evaluate this using Eqs. (15) and (16) and neglecting all contributions containing higher than first order terms in the operator ∇, independent on whether it acts on v_{ij}, m_{ij}, χ_{ij}, $\overline{\mathbf{U}}$ or $\overline{\mathbf{B}}$. With manipulations explained in Appendix B we obtain

$$\frac{\partial \chi_{ij}}{\partial t} + L_{ijklm}\frac{\partial \chi_{kl}}{\partial k_m} + M_{ijklm}\nabla_m \chi_{kl} + N_{ijkl}\chi_{kl} + C_{ij} = I_{ij}\,, \tag{30}$$

where

$$L_{ijklm} = -\delta_{ik}\delta_{jl}\overline{U}_{p,m}k_p, \tag{31}$$

$$M_{ijklm} = -i\,[D_{ik}\delta_{jl}k_m - \varepsilon_{ikm}\delta_{jl}(\mathbf{k}\cdot\mathbf{\Omega}) - \varepsilon_{ikp}k_p\delta_{jl}\Omega_m]/k^2 + i(\eta - \nu)\delta_{ik}\delta_{jl}k_m, \tag{32}$$

$$N_{ijkl} = \overline{U}_{i,k}\delta_{jl} - \delta_{ik}\overline{U}_{j,l} - 2k_{ip}\overline{U}_{p,k}\delta_{jl} - D_{ik}\delta_{jl} + (\nu + \eta)k^2\delta_{ik}\delta_{jl}, \tag{33}$$

$$C_{ij} = \Phi[P_{ik}(\hat{T}_k - \hat{f}_k), \hat{b}_j] - \Phi(\hat{u}_i, \hat{G}_j), \tag{34}$$

$$I_{ij} = -i(\mathbf{k}\cdot\overline{\mathbf{B}})(v_{ij} - m_{ij}) + \frac{1}{2}\overline{B}_p\nabla_p(v_{ij} + m_{ij}) - \overline{B}_{j,p}v_{ip} + \overline{B}_{i,p}m_{pj} - 2\frac{k_ik_p}{k^2}\overline{B}_{p,q}m_{qj} - \overline{B}_{p,q}k_p(v_{ijq} + m_{ijq}), \tag{35}$$

and $\overline{B}_{i,j} = \nabla_j\overline{B}_i$, $\overline{U}_{i,j} = \nabla_j\overline{U}_i$, furthermore $k_{ij} = k_ik_j/k^2$, $v_{ijq} = (1/2)\partial v_{ij}/\partial k_q$ and $m_{ijq} = (1/2)\partial m_{ij}/\partial k_q$.

5. THE τ-APPROXIMATION

When dealing with Eq. (30) for χ_{ij} we are confronted with the difficulty that we know nothing about the term C_{ij} which stands for couplings of χ_{ij} with higher-order correlation tensors. We assume now that we know the solution χ_{ij} of Eq. (30) for vanishing mean magnetic field, that is $I_{ij} = 0$, denote it by $\chi_{ij}^{(0)}$, and the corresponding C_{ij} by $C_{ij}^{(0)}$. For the case with nonvanishing mean magnetic field we put then $\chi_{ij} = \chi_{ij}^{(0)} + \chi_{ij}^{(B)}$ and $C_{ij} = C_{ij}^{(0)} + C_{ij}^{(B)}$. Then we have

$$\frac{\partial \chi_{ij}^{(B)}}{\partial t} + L_{ijklm}\frac{\partial \chi_{kl}^{(B)}}{\partial k_m} + M_{ijklm}\nabla_m\chi_{kl}^{(B)} + N_{ijkl}\chi_{kl}^{(B)} + C_{ij}^{(B)} = I_{ij}. \tag{36}$$

At this level we introduce the τ-approximation

$$C_{ij}^{(B)} = \chi_{ij}^{(B)}/\tau(k) \tag{37}$$

with some relaxation time $\tau(k)$ (see, e.g., Orszag, 1970; Monin and Yaglom, 1975; Pouquet *et al.*, 1976; McComb, 1990, Kleeorin *et al.*, 1990, 1996). This approximation applies for large hydrodynamic Reynolds numbers, i.e. for fully developed turbulence. In this case the relaxation time $\tau(k)$ is determined by the correlation time of the turbulent velocity field only. The limit $\tau \to \infty$ corresponds to cancelling all higher-order correlations, that is, to some kind of second-order correlation approximation (see also Section 9). Note that we do not introduce any closure assumption for the original turbulence; Eq. (37) concerns only deviations from the original turbulence.

6. RELATIONS FOR THE MEAN ELECTROMOTIVE FORCE

Returning now to the electromotive force $\boldsymbol{\mathcal{E}}$ we first put in the above sense $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(B)}$. We assume however that $\boldsymbol{\mathcal{E}}^{(0)}$, if not at all equal to zero, becomes unimportant in comparison to $\boldsymbol{\mathcal{E}}^{(B)}$ when $\overline{\mathbf{B}}$ grows and put therefore $\boldsymbol{\mathcal{E}}^{(0)} = 0$. Thus the electromotive force $\boldsymbol{\mathcal{E}}$ is given by

$$\mathcal{E}_i = \varepsilon_{ijk} \int \chi_{jk}^{(B)}(\mathbf{k}, \mathbf{R})\, d^3k. \tag{38}$$

Let us assume that the time variations of I_{ij}, that is, of $\overline{B}_i$, $\overline{B}_{i,j}$, v_{ij}, and m_{ij}, are sufficiently weak, so that $\chi_{ij}^{(B)}$, apart from some initial time, can be considered as a solution of the steady version of Eq. (36), that is, with $\partial\chi_{ij}^{(B)}/\partial t$ being neglected. Thus the dependence of $\chi_{ij}^{(B)}$ on $\overline{B}_i$ and $\overline{B}_{i,j}$ is linear and homogeneous and, in addition, instantaneous. So we may conclude that

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\overline{B}_{j,k} \tag{39}$$

with tensors a_{ij} and b_{ijk} determined by v_{ij} and m_{ij}.

This relation can be rewritten in another form, which might be more suitable for some discussions. To derive it we split a_{ij} in to a symmetric and an antisymmetric part and express the latter by a vector. We further split the gradient tensor of $\overline{\mathbf{B}}$ into a symmetric part $\partial\overline{\mathbf{B}}$, defined by $(\partial\overline{B})_{ij} = (1/2)(\overline{B}_{i,j} + \overline{B}_{j,i})$, and an antisymmetric one, which can be expressed by $\nabla \times \overline{\mathbf{B}}$. We have then $\overline{B}_{i,j} = (\partial\overline{B})_{ij} - (1/2)\varepsilon_{ijk}(\nabla \times \overline{\mathbf{B}})_k$. Finally we express the tensorial coefficient occurring then with $\nabla \times \overline{\mathbf{B}}$ again by a symmetric tensor and a vector. In this way we arrive at

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\alpha}\overline{\mathbf{B}} - \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta}(\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\delta} \times (\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\kappa}\partial\overline{\mathbf{B}} \tag{40}$$

(see Rädler, 1980, 1983), where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are symmetric tensors of the second rank, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ vectors, and $\boldsymbol{\kappa}$ is a tensor of the third rank; $\boldsymbol{\kappa}$ can be considered to be symmetric in the indices connecting it with $\partial\overline{\mathbf{B}}$, and contributions can be dropped which would produce $\nabla \cdot \overline{\mathbf{B}}$. We have

$$\alpha_{ij} = -\tfrac{1}{2}(a_{ij} + a_{ji}), \quad \beta_{ij} = \tfrac{1}{4}(\varepsilon_{ikl}b_{jkl} + \varepsilon_{jkl}b_{ikl}), \tag{41}$$

$$\gamma_i = \tfrac{1}{2}\varepsilon_{ijk}a_{jk}, \quad \delta_i = \tfrac{1}{4}(b_{jji} - b_{jij}), \quad \kappa_{ijk} = -\tfrac{1}{2}(b_{ijk} + b_{ikj}). \tag{42}$$

For the sake of simplicity we restrict ourselves on the case $\overline{U}_i = \text{const}$. Thus $\chi_{ij}^{(B)}$ is governed by the steady version of Eq. (36) with $L_{ijklm} = 0$ and $N_{ijkl} = -D_{ik}\delta_{jl} + (\nu + \eta)\ k^2\delta_{ik}\delta_{jl}$. Using (37) the equation for $\chi_{ij}^{(B)}$ can be written in the form

$$\tilde{D}_{ik}\chi_{kj}^{(B)} + M_{ijklm}\nabla_m\chi_{kl}^{(B)}\tau_* = I_{ij}\tau_*, \tag{43}$$

where

$$\tilde{D}_{ij} = \delta_{ij} - D_{ij}\tau_*, \quad \tau_*^{-1} = \tau^{-1} + (\nu + \eta)k^2. \tag{44}$$

In deriving Eq. (30) we have neglected all contributions to I_{ij} of higher than first order in $\mathbf{\nabla}$. In the same sense we may replace $\nabla_m \chi_{kl}^{(B)}$ in Eqs. (30), (36) and (43) by $\nabla_m \chi_{kl}^{o(B)}$, where $\chi_{ij}^{o(B)}$ summarizes the contributions to $\chi_{ij}^{(B)}$ which are of zero order in $\mathbf{\nabla}$. We have then

$$\tilde{D}_{ik}\chi_{kj}^{o(B)} = I_{ij}^{o}\tau_*, \quad I_{ij}^{o} = -i(\overline{\mathbf{B}}\cdot\mathbf{k})(v_{ij} - m_{ij}), \tag{45}$$

and consequently

$$\chi_{ij}^{o\,(B)} = \tilde{D}_{il}^{-1} I_{lj}^{o}\tau_*, \tag{46}$$

where $\tilde{\mathbf{D}}^{-1}$ is the inverse of $\tilde{\mathbf{D}}$, satisfying $\tilde{D}_{ik}^{-1}\tilde{D}_{kj} = \delta_{ij}$, that is,

$$\tilde{D}_{ij}^{-1} = (1 + \omega^2 k^2)^{-1}(\delta_{ij} + \omega\varepsilon_{ijk}k_k + \omega^2 k_i k_j), \tag{47}$$

where $\omega = 2\tau_*(\mathbf{\Omega}\cdot\mathbf{k})/k^2$. Proceeding as described above we finally obtain the solution of Eq. (43) in the form

$$\chi_{ij}^{(B)} = \tilde{D}_{ik}^{-1} I_{kj}\tau_* - \tilde{D}_{ip}^{-1} M_{pjklm}\tilde{D}_{kq}^{-1}\nabla_m I_{ql}^{o}\tau_*^2. \tag{48}$$

7. THE MEAN ELECTROMOTIVE FORCE AT SLOW ROTATION

We split the electromotive force $\boldsymbol{\mathcal{E}}^{(B)}$ according to

$$\boldsymbol{\mathcal{E}}^{(B)} = \boldsymbol{\mathcal{E}}^{(B0)} + \boldsymbol{\mathcal{E}}^{(B\Omega)} \tag{49}$$

into parts $\boldsymbol{\mathcal{E}}^{(B0)}$ and $\boldsymbol{\mathcal{E}}^{(B\Omega)}$, the first of which does not depend on the rotation while the second one does but vanishes with vanishing rotation. For $\mathcal{E}_i^{(B0)}$ we have simply

$$\mathcal{E}_i^{(B0)} = \varepsilon_{ijk}\int I_{jk}(\mathbf{k})\tau_*(k)\,d^3k. \tag{50}$$

As for $\boldsymbol{\mathcal{E}}^{(B\Omega)}$ we restrict ourselves to the case of slow rotation. More precisely, we neglect terms of third and higher order in $\Omega\tau_*$ in comparison to unity. So we find

$$\begin{aligned}\mathcal{E}_i^{(B\Omega)} = \int\Bigg(&2(\mathbf{k}\cdot\mathbf{\Omega})(k_i I_{jj} - k_j I_{ij})\\ &+ i\left\{\left[2\frac{(\mathbf{k}\cdot\mathbf{\Omega})}{k^2}(\mathbf{k}\cdot\mathbf{\nabla}) - (\mathbf{\Omega}\cdot\mathbf{\nabla})\right]k_i I_{jj}^{(o)} - (\mathbf{k}\cdot\mathbf{\Omega})(\nabla_i I_{jj}^{(o)} - \nabla_j I_{ij})\right\}\\ &- 4(\mathbf{k}\cdot\mathbf{\Omega})^2\left(\varepsilon_{ijk} + \varepsilon_{ikl}\frac{k_l k_j}{k^2}\right)I_{jk}\tau_* - 2i(\mathbf{k}\cdot\mathbf{\Omega})\left\{2\left[\frac{(\mathbf{k}\cdot\mathbf{\Omega})}{k^2}(\mathbf{k}\cdot\mathbf{\nabla}) - (\mathbf{\Omega}\cdot\mathbf{\nabla})\right]\varepsilon_{ijk}I_{jk}^{(o)}\right.\\ &\left.+\frac{(\mathbf{k}\cdot\mathbf{\Omega})}{k^2}\varepsilon_{ijk}k_j\nabla_l I_{lk}^{(o)}\right\}\tau_*\Bigg)\frac{\tau_*^2}{k^2}d^3k.\end{aligned} \tag{51}$$

In the integrand we have dropped terms of the structures $\nabla_j k_k I^{(o)}_{kl}$ or $\nabla_j k_k I^{(o)}_{lk}$, which would, by reason connected with (27), lead to contributions of the second order in ∇. Moreover, for the sake of simplicity in both (50) and (51) we ignored terms containing the factor $(\eta - \nu)k^2\tau_*$. We will later point out the consequences of that for our final result.

In relation (39) for $\boldsymbol{\mathcal{E}}$ we split a_{ij} and b_{ijk} into parts corresponding to their dependence on v_{ij} or m_{ij}, e.g.,

$$a_{ij} = a^{(v)}_{ij} + a^{(m)}_{ij}, \tag{52}$$

and each of them are splitted into one which is independent of Ω and a remaining one, e.g.,

$$a^{(v)}_{ij} = a^{(v0)}_{ij} + a^{(v\Omega)}_{ij}, \quad a^{(m)}_{ij} = a^{(m0)}_{ij} + a^{(m\Omega)}_{ij}. \tag{53}$$

We note that the splitting of $\boldsymbol{\mathcal{E}}^{(B)}$ according to (49) and the splittings of $a^{(v)}_{ij}$, $a^{(m)}_{ij}, \ldots$ according to (53) refer only to the dependencies on $\boldsymbol{\Omega}$ which occur explicitly in this stage of our derivations. In general v_{ij} will depend on $\boldsymbol{\Omega}$, too, and then $\boldsymbol{\mathcal{E}}^{(B0)}$ and $a^{(v0)}_{ij}$, $a^{(m0)}_{ij}, \ldots$ will have also contributions with $\boldsymbol{\Omega}$, and the dependencies of $\boldsymbol{\mathcal{E}}^{(B\Omega)}$ and $a^{(v\Omega)}_{ij}$, $a^{(m\Omega)}_{ij}, \ldots$ on $\boldsymbol{\Omega}$ will be more complex.

A straightforward calculation yields

$$a^{(v0)}_{ij} = \varepsilon_{ilm} v^{(1)}_{lmj}, \quad a^{(m0)}_{ij} = -\varepsilon_{ilm} m^{(1)}_{lmj}, \tag{54, 55}$$

$$b^{(v0)}_{ijk} = \varepsilon_{ijl} v^{(1)}_{lk}, \quad b^{(m0)}_{ijk} = \varepsilon_{ijl} m^{(1)}_{lk} + 2\varepsilon_{ilm} m^{(1)}_{kljm}, \tag{56, 57}$$

$$a^{(v\Omega)}_{ij} = \Big[2v^{(2)}_{ppijl} - \nabla_i v^{(2)}_{ppjl} + \nabla_j v^{(2)}_{ppil} + 2\nabla_p\Big(v^{(2)}_{qqijlp} + v^{(2)}_{ipjl}\Big) - \nabla_l v^{(2)}_{ppij}\Big]\Omega_l - 4\varepsilon_{ipq} v^{(3)}_{pqjlm}\Omega_l\Omega_m, \tag{58}$$

$$a^{(m\Omega)}_{ij} = \Big[-2m^{(2)}_{ppijl} + \nabla_i m^{(2)}_{ppjl} + \nabla_j m^{(2)}_{ppil} - 2\nabla_p\Big(m^{(2)}_{qqijlp} + m^{(2)}_{ipjl}\Big) + \nabla_l m^{(2)}_{ppij}\Big]\Omega_l + 4\varepsilon_{ipq} m^{(3)}_{pqjlm}\Omega_l\Omega_m, \tag{59}$$

$$b^{(v\Omega)}_{ijk} = \big(2v^{(2)}_{ikjl} - 2v^{(2)}_{jkil} + v'_{ppijkl}\big)\Omega_l - 4\varepsilon_{ijp} v^{(3)}_{pklm}\Omega_l\Omega_m, \tag{60}$$

$$\begin{aligned} b^{(m\Omega)}_{ijk} = {} & \big(-2m^{(2)}_{ikjl} + 2m^{(2)}_{jkil} + 2m^{(2)}_{ppij}\delta_{kl} + 2m^{(2)}_{ppjl}\delta_{ik} - 4m^{(2)}_{ppijkl} + m'_{ppijkl}\big)\Omega_l \\ & - 4(\varepsilon_{ijp} m^{(3)}_{kplm} + 2\varepsilon_{ipq} m^{(3)}_{kpjlmq})\Omega_l\Omega_m. \end{aligned} \tag{61}$$

Here we used the definitions

$$v^{(\mu)}_{ijk\dots p} = \int (-ik)^{\lambda} v_{ij}(\mathbf{k}) k_{k\dots p} \tau^{\mu}_{*} \, d^3k, \tag{62}$$

$$v'_{ijklmn} = \int v_{ij}(\mathbf{k}) k_{klmn} (d\tau^2_{*}/dk) \, k d^3k, \tag{63}$$

where $\lambda = 0$ or $\lambda = 1$ for even or odd ν, respectively, and $k_{k\dots p} = k_k \dots k_p / k^{\nu}$, with ν being the rank of this last tensor, and analogous definitions for $m^{(\mu)}_{ijk\dots p}$, and m'_{ijklmn} with v_{ij} replaced by m_{ij}. With the help of (26) we find that the $v^{(\mu)}_{ijk\dots p}$ with $\lambda = 0$ are symmetric and those with $\lambda = 1$ antisymmetric in i and j and moreover in both cases symmetric in every pair of the remaining indices, and that v'_{ijklmn} is symmetric in i and j and again in every pair of the remaining indices. The same applies to $m^{(\mu)}_{ijk\dots p}$ and to m'_{ijklmn}. Furthermore in the case $\lambda = 0$ we may conclude from (27) that $v^{(\mu)}_{pij\dots np} = -(1/2)\nabla_p \tilde{v}^{(\mu)}_{pij\dots n}$ and $v^{(\mu)}_{ipj\dots np} = (1/2)\nabla_p \tilde{v}^{(\mu)}_{ipj\dots n}$, where the $\tilde{v}^{(\mu)}_{ij\dots p}$ (which will never explicitly occur in the following) are defined like the $v^{(\mu)}_{ij\dots p}$ but with $k_{ij\dots p} k^{-2}$ in the integrand instead of $k_{ij\dots p}$. In the case $\lambda = 1$ we have simply $v^{(\mu)}_{pij\dots np} = (1/2)\nabla_p v^{(\mu)}_{pij\dots n}$ and $v^{(\mu)}_{ipj\dots np} = -(1/2)\nabla_p v^{(\mu)}_{ipj\dots n}$. All these statements apply again to $m^{(\mu)}_{pij\dots np}$ and $m^{(\mu)}_{ipj\dots np}$, too. We have used these properties of the $v^{(\mu)}_{ijk\dots p}$, $m^{(\mu)}_{ijk\dots p}, \dots$ in deriving (54)–(61), and we have ignored all terms which would result in contributions to $\boldsymbol{\mathcal{E}}$ of higher than first order in ∇. Likewise we have cancelled contributions to b_{ijk} proportional to δ_{jk}, which cannot contribute to $\boldsymbol{\mathcal{E}}$ since $\nabla \cdot \overline{\mathbf{B}} = 0$.

Let us consider the representation (40) for $\boldsymbol{\mathcal{E}}$ and split each of the quantities $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ after the pattern of (52) and (53) into four parts, e.g.,

$$\alpha_{ij} = \alpha^{(v)}_{ij} + \alpha^{(m)}_{ij}, \tag{64}$$

and

$$\alpha^{(v)}_{ij} = \alpha^{(v0)}_{ij} + \alpha^{(v\Omega)}_{ij}, \quad \alpha^{(m)}_{ij} = \alpha^{(m0)}_{ij} + \alpha^{(m\Omega)}_{ij} \,. \tag{65}$$

The above remarks concerning the dependencies on $\boldsymbol{\Omega}$ apply here analogously. Using (41)–(42) and (54)–(61) all these contributions to $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ can be expressed by the $v^{(\mu)}_{ijk\dots p}$, $m^{(\mu)}_{ijk\dots p}, \dots$. For the sake of simplicity we give here only the contributions which do not depend on $\boldsymbol{\Omega}$. They read

$$\alpha^{(v0)}_{ij} = \tfrac{1}{2}(\varepsilon_{ilk} v^{(1)}_{klj} + \varepsilon_{jlk} v^{(1)}_{kli}), \quad \alpha^{(m0)}_{ij} = -\tfrac{1}{2}(\varepsilon_{ilk} m^{(1)}_{klj} + \varepsilon_{jlk} m^{(1)}_{kli}), \tag{66}$$

$$\beta^{(v0)}_{ij} = \tfrac{1}{2}(v^{(1)}_{pp}\delta_{ij} - v^{(1)}_{ij}), \quad \beta^{(m0)}_{ij} = -\tfrac{1}{2}(m^{(1)}_{pp}\delta_{ij} - m^{(1)}_{ij} - 2m^{(1)}_{ppij}), \tag{67}$$

$$\gamma^{(v0)}_{i} = \tfrac{1}{2}\nabla_i v^{(1)}_{ij}, \quad \gamma^{(m0)}_{i} = -\tfrac{1}{2}\nabla_i m^{(1)}_{ij}, \quad \delta^{(v0)}_{i} = \delta^{(m0)}_{i} = 0, \tag{68}$$

$$\kappa^{(v0)}_{ijk} = -\tfrac{1}{2}(\varepsilon_{ijl} v^{(1)}_{lk} + \varepsilon_{ikl} v^{(1)}_{lj}), \tag{69}$$

$$\kappa^{(m0)}_{ijk} = -\tfrac{1}{2}(\varepsilon_{ijl} m^{(1)}_{lk} + \varepsilon_{ikl} m^{(1)}_{lj}) + \varepsilon_{ilm}(m^{(1)}_{kmjl} + m^{(1)}_{jmkl}). \tag{70}$$

In order to derive (67) we have used (A2).

8. THE MEAN ELECTROMOTIVE FORCE IN THE LIMIT OF WEAK MEAN MAGNETIC FIELDS UNDER SPECIFIC ASSUMPTIONS ON THE ORIGINAL TURBULENCE

In general the correlation tensors v_{ij} and m_{ij} depend, of course, on the mean magnetic field $\overline{\mathbf{B}}$. As a consequence the mean electromotive force $\boldsymbol{\mathcal{E}}$ depends in a nonlinear way on $\overline{\mathbf{B}}$. We restrict ourselves now to the approximation in which v_{ij} and m_{ij} are simply replaced by the corresponding tensors $v_{ij}^{(0)}$ and $m_{ij}^{(0)}$ for the "original" turbulence, that is the turbulence for zero mean magnetic field. Then, of course, $\boldsymbol{\mathcal{E}}$ is linear in $\overline{\mathbf{B}}$. This implies that any quenching effects are excluded.

8.1 Nonrotating Turbulence

We first ignore any influence of a rotation of the fluid on the turbulence. As for the correlation tensors $v_{ij}^{(0)}$ and $m_{ij}^{(0)}$ for the original turbulence we assume that, as long it is homogeneous, they are essentially determined by the kinetic and magnetic energy densities, that is by $\langle \mathbf{u}^{(0)2} \rangle$ and $\langle \mathbf{b}^{(0)2} \rangle$, and by the kinematic and current helicities, $\langle \mathbf{u}^{(0)} \cdot (\nabla \times \mathbf{u}^{(0)}) \rangle$ and $\langle \mathbf{b}^{(0)} \cdot (\nabla \times \mathbf{b}^{(0)}) \rangle$, and that inhomogeneities are only due to gradients of these four quantities. With the notations $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$ instead of $\mathbf{u}$ and $\mathbf{b}$ we want to stress that we are dealing with the original turbulence. Note that $\mathbf{b}$ is in general nonzero even if $\mathbf{b}^{(0)} = \mathbf{0}$. Under the assumptions adopted the most general form of the correlation tensor $v_{ij}^{(0)}(\mathbf{k}, \mathbf{R})$ of the velocity fluctuations of the original turbulence is given by

$$v_{ij}^{(0)}(\mathbf{k}, \mathbf{R}) = \frac{1}{8\pi k^2} \Bigg(\Big[P_{ij}(\mathbf{k}) + \frac{i}{2k^2}(k_i \nabla_j - k_j \nabla_i) \Big] W^{(v)}(k, \mathbf{R}) - \frac{1}{2k^2} \Big[\varepsilon_{ijk} k_k \Big[2i + \frac{1}{k^2}(\mathbf{k} \cdot \nabla) \Big] - \frac{1}{k^2}(k_i \varepsilon_{jlm} + k_j \varepsilon_{ilm}) k_l \nabla_m \Big] \mu^{(v)}(k, \mathbf{R}) \Bigg). \quad (71)$$

We note that $v_{ij}^{(0)}$ satisfies the requirements resulting from $\nabla \cdot \mathbf{u}^{(0)} = 0$, and that $W^{(v)}(k, \mathbf{R})$ and $\mu^{(v)}(k, \mathbf{R})$ are spectrum functions depending on $\mathbf{k}$ via k only and possessing the properties $\int_0^\infty W^{(v)}(k, \mathbf{R})\, dk = \langle \mathbf{u}^{(0)2} \rangle$ and $\int_0^\infty \mu^{(v)}(k, \mathbf{R})\, dk = \langle \mathbf{u}^{(0)} \cdot (\nabla \times \mathbf{u}^{(0)}) \rangle$. The definition of the correlation tensor $m_{ij}^{(0)}$ of the magnetic fluctuations of the original turbulence follows from (71) when replacing $v_{ij}^{(0)}$, $W^{(v)}$ and $\mu^{(v)}$ by $m_{ij}^{(0)}$, $W^{(m)}$ and $\mu^{(m)}$, and we have $\int_0^\infty W^{(m)}(k, \mathbf{R})\, dk = \langle \mathbf{b}^{(0)2} \rangle / \mu\rho$ and $\int_0^\infty \mu^{(m)}(k, \mathbf{R})\, dk = \langle \mathbf{b}^{(0)} \cdot (\nabla \times \mathbf{b}^{(0)}) \rangle / \mu\rho$.

Since we ignore here any influence of rotation the $\alpha_{ij}^{(v)}, \alpha_{ij}^{(m)}, \ldots$ coincide with the $\alpha_{ij}^{(v0)}$, $\alpha_{ij}^{(m0)}, \ldots$ given with (66)–(70). Calculating now the $v_{ijk\cdots p}^{(1)}$ with v_{ij} replaced by $v_{ij}^{(0)}$ as specified by (71) we find

$$v_{ij}^{(1)} = \frac{1}{3} \delta_{ij} I^{(v1)} + \cdots, \quad (72)$$

$$v_{ijk}^{(1)} = \frac{1}{12} (\delta_{ik} \nabla_j - \delta_{jk} \nabla_i) I^{(v1)} - \frac{1}{6} \varepsilon_{ijk} J^{(v1)}, \quad (73)$$

$$v_{ijkl}^{(1)} = \frac{1}{6} \left(\delta_{ij} \delta_{kl} - \frac{3}{5} E_{ijkl}^{(4)} \right) I^{(v1)} + \cdots, \quad (74)$$

where

$$I^{(v1)} = \int_0^\infty W^{(v)}(k, \mathbf{R})\tau_* \, dk, \quad J^{(v1)} = \int_0^\infty \mu^{(v)}(k, \mathbf{R})\tau_* \, dk, \tag{75}$$

and $E^{(4)}_{ijkl}$ is a completely symmetric tensor, which we define with a view to the following in a more general frame by

$$E^{(2\nu)}_{ijkl\cdots pq} = \frac{1}{2\nu - 1}\Big(\delta_{ij}E^{(2\nu-2)}_{kl\cdots pq} + \delta_{ik}E^{(2\nu-2)}_{jl\cdots pq} + \cdots + \delta_{iq}E^{(2\nu-2)}_{jkl\cdots p}\big), \quad E^{(2)}_{ij} = \delta_{ij}. \tag{76}$$

The terms in (72) and (74) indicated by $\cdots$ are without interest because they would lead to contributions to $\boldsymbol{\mathcal{E}}$ which are of second order in ∇. For the $m^{(1)}_{ij}$, $m^{(1)}_{ijk}$ and $m^{(1)}_{ijkl}$ analogous relations apply with analogously defined $I^{(m1)}$ and $J^{(m1)}$.

Using now (66)–(70) we obtain

$$\alpha^{(v)}_{ij} = \tfrac{1}{3}\delta_{ij}J^{(v1)}, \quad \alpha^{(m)}_{ij} = -\tfrac{1}{3}\delta_{ij}J^{(m1)}, \tag{77}$$

$$\beta^{(v)}_{ij} = \tfrac{1}{3}\delta_{ij}I^{(v1)}, \quad \beta^{(m)}_{ij} = 0, \tag{78}$$

$$\gamma^{(v)} = \tfrac{1}{6}\nabla I^{(v1)}, \quad \gamma^{(m)} = -\tfrac{1}{6}\nabla I^{(m1)}, \tag{79}$$

$$\boldsymbol{\delta}^{(v)} = \boldsymbol{\delta}^{(m)} = \mathbf{0}, \quad \boldsymbol{\kappa}^{(v)} = \boldsymbol{\kappa}^{(m)} = \mathbf{0}. \tag{80}$$

8.2 Rotating Turbulence

In order to include the effect of a rotation of the fluid we have first to study how the original turbulence, that is $v^{(0)}_{ij}$ changes with the rotation, that is with $\boldsymbol{\Omega}$. We assume here that in the absence of rotation the turbulence possesses no helicity. More precisely, we assume that (71) applies with $\mu^{(v)} = 0$ in the limit of vanishing $\boldsymbol{\Omega}$, and we will now calculate the effect of rotation by a perturbation procedure up to the second order in $\boldsymbol{\Omega}$.

Proceeding as in the calculation of the cross-helicity χ_{ij} we start from

$$\frac{\partial v_{ij}(\mathbf{k}, \mathbf{R})}{\partial t} = \Phi\left(\frac{\partial \hat{u}_i}{\partial t}, \hat{u}_j; \mathbf{k}, \mathbf{R}\right) + \Phi\left(\hat{u}_i, \frac{\partial \hat{u}_j}{\partial t}; \mathbf{k}, \mathbf{R}\right), \tag{81}$$

but use the Eq. (15) for $\hat{u}_i$ with $\overline{\mathbf{B}} = \mathbf{0}$ and $\mathbf{T} = \mathbf{0}$ so that v_{ij} turns into $v^{(0)}_{ij}$, and put again $\overline{\mathbf{U}} = \mathbf{0}$. So we arrive at

$$\frac{\partial v^{(0)}_{ij}}{\partial t} + \tilde{M}_{ijklm}\nabla_m v^{(0)}_{kl} + \tilde{N}_{ijkl}v^{(0)}_{kl} + 2\nu k^2 v^{(0)}_{ij} + \tilde{C}_{ij} = 0, \tag{82}$$

where

$$\tilde{M}_{ijklm} = \frac{i}{k^2}\Omega_p[P_{mp}(\mathbf{k})k_q + P_{mq}(\mathbf{k})k_p](\varepsilon_{ikq}\delta_{jl} - \varepsilon_{jlq}\delta_{ik}), \tag{83}$$

$$\tilde{N}_{ijkl} = -2\frac{(\mathbf{k}\cdot\mathbf{\Omega})}{k^2}(\varepsilon_{ikp}\delta_{jl} + \varepsilon_{jlp}\delta_{ik})k_p, \tag{84}$$

$$\tilde{C}_{ij} = -\Phi(\hat{u}_i, P_{ik}\hat{f}_j) - \Phi(P_{ik}\hat{f}_k, \hat{u}_j). \tag{85}$$

We assume now that we know the solution $v_{ij}^{(0)}$ of (82) for $\mathbf{\Omega} = \mathbf{0}$, denote it by $v_{ij}^{(00)}$ and the corresponding $\tilde{C}_{ij}$ by $\tilde{C}_{ij}^{(0)}$. For the case with nonvanishing $\mathbf{\Omega}$ we put then $v_{ij}^{(0)} = v_{ij}^{(00)} + v_{ij}^{(0\Omega)}$ and $\tilde{C}_{ij} = \tilde{C}_{ij}^{(0)} + \tilde{C}_{ij}^{(\Omega)}$. So we find

$$\frac{\partial v_{ij}^{(0\Omega)}}{\partial t} + \tilde{M}_{ijklm}\nabla_m v_{kl}^{(0\Omega)} + \tilde{N}_{ijkl}v_{kl}^{(0\Omega)} + 2\nu k^2 v_{ij}^{(0\omega)} + \tilde{C}_{ij}^{(\Omega)} = I_{ij}^{(\Omega)}, \tag{86}$$

where

$$I_{ij}^{(\Omega)} = -(\tilde{M}_{ijklm}\nabla_m + \tilde{N}_{ijkl})v_{kl}^{(00)}. \tag{87}$$

At this level we introduce again a τ-approximation,

$$\tilde{C}_{ij}^{(\Omega)} = v_{ij}^{(0\Omega)}/\check{\tau}(\mathbf{k}, \mathbf{\Omega}), \tag{88}$$

where $\check{\tau}(\mathbf{k}, \mathbf{\Omega})$ is a relaxation time analogous to $\tau(k)$ introduced with (37). Now we assume that the characteristic time of variations of $v_{ij}^{(0\Omega)}$ is much larger than $\check{\tau}$ so that we can drop the time derivative $\partial v_{ij}^{(0\Omega)}/\partial t$ in (86). Thinking of slow rotation, that is $\Omega\check{\tau} \ll 1$, we expand $v_{ij}^{(0\Omega)}$ in the form

$$v_{ij}^{(0\Omega)} = v_{ij}^{(01)} + v_{ij}^{(02)}, \tag{86}$$

where $v_{ij}^{(01)}$ and $v_{ij}^{(02)}$ are of the first and second order in $\mathbf{\Omega}$, respectively. We further assume that $\check{\tau}$ does not deviate markedly from τ, that it does not depend on the sign of Ω and therefore its expansion with respect to $\mathbf{\Omega}$ possesses no linear term. In that sense we put $\check{\tau} = \tau + \mathrm{O}(\Omega^2)$, where the last term is without interest for the following. In this way we obtain

$$v_{ij}^{(0,\,\mu)} = -\tilde{\tau}_*(\tilde{M}_{ijklm}\nabla_m + \tilde{N}_{ijkl})v_{kl}^{(0,\,\mu-1)}, \quad \tilde{\tau}_*^{\,-1} = \tau^{-1} + 2\nu k^2, \quad \mu = 1, 2. \tag{90}$$

Identifying now $v_{kl}^{(00)}$ with $v_{kl}^{(0)}$ defined by (71) with $\mu^{(\nu)} = 0$ and neglecting again terms of higher than first order in $\mathbf{\nabla}$ we find

$$v_{ij}^{(01)} = \frac{i\tilde{\tau}_*}{4\pi k^4}\varepsilon_{ijm}k_m\left(\frac{(\mathbf{k}\cdot\mathbf{\Omega})}{k^2}(\mathbf{k}\cdot\mathbf{\nabla}) - (\mathbf{\Omega}\cdot\mathbf{\nabla})\right)W^{(\nu)}(k, \mathbf{R}), \quad v_{ij}^{(02)} = 0. \tag{91}$$

With (91) we arrive at

$$v_{ij}^{(0)}(\mathbf{k},\mathbf{R}) = \frac{1}{8\pi k^2}\left(P_{ij}(\mathbf{k}) + \frac{i}{2k^2}(k_i\nabla_j - k_j\nabla_i) + \frac{2i\tilde{\tau}_*}{k^2}\varepsilon_{ijm}k_m\left[\frac{(\mathbf{k}\cdot\mathbf{\Omega})}{k^2}(\mathbf{k}\cdot\mathbf{\nabla}) - (\mathbf{\Omega}\cdot\mathbf{\nabla})\right]\right)W^{(v)}(k,\mathbf{R}). \tag{92}$$

Interestingly enough this coincides with (71) if we replace there $\mu^{(v)}$ by $\tilde{\mu}^{(v)}$, defined by

$$\tilde{\mu}^{(v)} = -2\tilde{\tau}_*\left(\frac{(\mathbf{k}\cdot\mathbf{\Omega})}{k^2}(\mathbf{k}\cdot\mathbf{\nabla}) - (\mathbf{\Omega}\cdot\mathbf{\nabla})\right)W^{(v)}(k,\mathbf{R}), \tag{93}$$

and cancel the terms with $\mathbf{\nabla}\tilde{\mu}^{(v)}$, which are of second order in $\mathbf{\nabla}$. Note that the absence of terms of higher than second order in $\mathbf{\Omega}$ in the correlation tensor (92) is a consequence of the fact that we restricted ourselves to small $\mathbf{\Omega}\check{\tau}$ and used in that sense (89). The absence of terms of the second order results from the assumption on the dependence of $\check{\tau}$ on $\mathbf{\Omega}$.

As for the magnetic fluctuations of the original turbulence we assume that they are independent of the rotation of the fluid and show no helicity so that

$$m_{ij}^{(0)}(\mathbf{k},\mathbf{R}) = \frac{1}{8\pi k^2}\left(P_{ij}(\mathbf{k}) + \frac{i}{2k^2}(k_j\nabla_i - k_i\nabla_j)\right)W^{(m)}(k,\mathbf{R}). \tag{94}$$

We calculate now the $v_{ijk\cdots p}^{(\mu)}$, $m_{ijk\cdots p}^{(\mu)}, \ldots$ defined by (62)–(63) with v_{ij} replaced by $v_{ij}^{(0)}$ according to (92) and find

$$v_{ij}^{(\mu)} = \frac{1}{3}\delta_{ij}I^{(v\mu)} + \cdots, \tag{95}$$

$$v_{ijk}^{(\mu)} = \frac{1}{12}(\delta_{ik}\nabla_j - \delta_{jk}\nabla_i)I^{(v\mu)} - \frac{1}{3}\varepsilon_{ijp}\left[\delta_{pk}(\mathbf{\Omega}\cdot\mathbf{\nabla}\tilde{I}^{(v\mu)}) - \frac{3}{5}E_{pklm}^{(4)}\Omega_l\nabla_m\tilde{I}^{(v\mu)}\right], \tag{96}$$

$$v_{ijkl}^{(\mu)} = \frac{1}{6}\left(\delta_{ij}\delta_{kl} - \frac{3}{5}E_{ijkl}^{(4)}\right)I^{(v\mu)} + \cdots, \tag{97}$$

$$v_{ijklm}^{(\mu)} = \frac{1}{20}(E_{iklm}^{(4)}\nabla_j - E_{jklm}^{(4)}\nabla_i)I^{(v\mu)} - \frac{1}{5}\varepsilon_{ijp}\left[E_{pklm}^{(4)}(\mathbf{\Omega}\cdot\mathbf{\nabla}\tilde{I}^{(v\mu)}) - \frac{5}{7}E_{pklmqr}^{(6)}\Omega_q\nabla_r\tilde{I}^{(v\mu)}\right], \tag{98}$$

$$v_{ijklmp}^{(\mu)} = \frac{1}{10}\left(\delta_{ij}E_{klmp}^{(4)} - \frac{5}{7}E_{ijklmp}^{(6)}\right)I^{(v\mu)} + \cdots. \tag{99}$$

In accordance with (75) we have used here the definitions

$$I^{(v\mu)} = \int_0^\infty W^{(v)}(k,\mathbf{R})\tau_*^\mu(k)\,dk, \quad \tilde{I}^{(v\mu)} = \int_0^\infty W^{(v)}(k,\mathbf{R})\tau_*^\mu(k)\tilde{\tau}_*(k)\,dk. \tag{100}$$

The expression for $v_{ijklmp}^{(\mu)}$ turns into v'_{ijklmp} if $I^{(v\mu)}$ is replaced by $I^{(v)\prime}$, where

$$I^{(v)\prime} = \int_0^\infty W^{(v)}(k,\mathbf{R})(d\tau_*^2/dk)k\,dk. \tag{101}$$

Again terms indicated by $\cdots$ correspond to contributions of second order in ∇ to $\mathcal{E}$. Relations analogous to (95)–(101) apply for $m_{ij}^{(\mu)}$, $m_{ijk}^{(\mu)}, \ldots$, too, but with $\tilde{I}^{(m\mu)} = 0$.

Calculating now the $a_{ij}^{(v)}$, $a_{ij}^{(m)}, \ldots$ according to (54)–(61) and finally the $\alpha_{ij}^{(v)}$, $\alpha_{ij}^{(m)}, \ldots$ according to (41)–(42) we arrive at

$$\alpha_{ij}^{(v)} = \frac{2}{15}\left[4\delta_{ij}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}) - \Omega_i\nabla_j - \Omega_j\nabla_i\right]\tilde{I}^{(v1)} + \frac{4}{15}\left[\delta_{ij}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}) - \frac{7}{8}(\Omega_i\nabla_j + \Omega_j\nabla_i)\right]I^{(v2)}$$
$$+ \frac{2}{15}(\varepsilon_{ilm}\Omega_j + \varepsilon_{jlm}\Omega_i)\Omega_l\nabla_m I^{(v3)}, \tag{102}$$

$$\alpha_{ij}^{(m)} = -\frac{1}{10}\left[\frac{8}{3}\delta_{ij}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}) + \Omega_i\nabla_j + \Omega_j\nabla_i\right]I^{(m2)} - \frac{2}{15}(\varepsilon_{ilm}\Omega_j + \varepsilon_{jlm}\Omega_i)\Omega_l\nabla_m I^{(m3)}, \tag{103}$$

$$\beta_{ij}^{(v)} = \frac{1}{3}\delta_{ij}I^{(v1)} - \frac{2}{5}\left(\delta_{ij}\Omega^2 + \frac{1}{3}\Omega_i\Omega_j\right)I^{(v3)}, \tag{104}$$

$$\beta_{ij}^{(m)} = \frac{2}{15}(\delta_{ij}\Omega^2 - 3\Omega_i\Omega_j)I^{(m3)}, \tag{105}$$

$$\boldsymbol{\gamma}^{(v)} = \frac{1}{6}\boldsymbol{\nabla}(I^{(v1)} - \frac{8}{5}\Omega^2 I^{(v3)}) + \frac{1}{6}\boldsymbol{\Omega}\times\boldsymbol{\nabla}I^{(v2)} + \frac{2}{15}\boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}I^{(v3)}), \tag{106}$$

$$\boldsymbol{\gamma}^{(m)} = -\frac{1}{6}\boldsymbol{\nabla}\left(I^{(m1)} - \frac{8}{5}\Omega^2 I^{(m3)}\right) + \frac{1}{6}\boldsymbol{\Omega}\times\boldsymbol{\nabla}I^{(m2)} - \frac{2}{15}\boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}I^{(m3)}), \tag{107}$$

$$\boldsymbol{\delta}^{(v)} = -\frac{1}{6}\boldsymbol{\Omega}I^{(v2)}, \tag{108}$$

$$\boldsymbol{\delta}^{(m)} = \frac{1}{6}\boldsymbol{\Omega}I^{(m2)}, \tag{109}$$

$$\kappa_{ijk}^{(v)} = -\frac{1}{6}(\delta_{ij}\Omega_k + \delta_{ik}\Omega_j)\left(I^{(v2)} + \frac{2}{5}I^{(v)\prime}\right) - \frac{2}{15}(\varepsilon_{ijl}\Omega_l\Omega_k + \varepsilon_{ikl}\Omega_l\Omega_j)I^{(v3)}, \tag{110}$$

$$\kappa_{ijk}^{(m)} = -\frac{7}{30}(\delta_{ij}\Omega_k + \delta_{ik}\Omega_j)\left(I^{(m2)} + \frac{2}{7}I^{(m)\prime}\right) + \frac{2}{15}(\varepsilon_{ijl}\Omega_l\Omega_k + \varepsilon_{ikl}\Omega_l\Omega_j)I^{(m3)}. \tag{111}$$

8.3 Specification to Kolmogorov Type Turbulence Spectra

Let us now specify the original turbulence to be of Kolmogorov type, i.e., to possess a constant energy flux through the spectrum, and consider the inertial range of wave numbers, $k_0 \leq k \leq k_d$, where $k_0^{-1} = l_0$ defines the largest length scale and k_d^{-1} the dissipative scale of the turbulence. In this range we have $W^{(v)} = (q-1)\ (\langle \mathbf{u}^{(0)2}\rangle/k_0)\times (k/k_0)^{-q}$, and $W^{(m)}$ analogously. Furthermore we put $\mu^{(v)} = (q-1)(\langle \mathbf{u}^{(0)}\cdot\ (\boldsymbol{\nabla}\times\mathbf{u}^{(0)})\rangle/ k_0)(k/k_0)^{-q}$ but, by reasons connected with the conservation of the magnetic helicity in the high-conductivity limit, $\mu^{(m)} = \langle \mathbf{b}^{(0)}\cdot(\boldsymbol{\nabla}\times\mathbf{b}^{(0)})\rangle\delta(k-k_0)$. Finally we assume that $\tau_* = \tilde{\tau}_* = 2\tau_0(k/k_0)^{1-q}$, with $2\tau_0$ being a correlation (or turnover) time for $k = k_0$.

In all cases q is a constant constrained by $1 < q < 3$. Assuming that it is sufficient to take the integrals in (75) and (100)–(101) over the inertial range only and that $k_0/k_d \ll 1$ we obtain

$$I^{(v\mu)} = \frac{2^\mu}{\mu+1}\langle \mathbf{u}^{(0)2}\rangle \tau_0^\mu, \quad J^{(v1)} = \langle \mathbf{u}^{(0)} \cdot (\nabla \times \mathbf{u}^{(0)})\rangle \tau_0, \tag{112}$$

$$\tilde{I}^{(v1)} = I^{(v2)}, \quad I^{(v)\prime} = -2(q-1)I^{(v2)}, \tag{113}$$

$$I^{(m\mu)} = \frac{2^\mu}{\mu+1}\frac{\langle \mathbf{b}^{(0)2}\rangle}{\mu\rho}\tau_0^\mu, \quad J^{(m1)} = \frac{\langle \mathbf{b}^{(0)} \cdot (\nabla \times \mathbf{b}^{(0)})\rangle}{\mu\rho}\tau_0, \tag{114}$$

$$I^{(m)\prime} = -2(q-1)I^{(m2)}. \tag{115}$$

Note that the $I^{(v\mu)}$, $J^{(v1)}$, $\tilde{I}^{(v1)}$, $I^{(m\mu)}$ and $J^{(m1)}$ are independent of q.

By the way, with the above specification of $\mu^{(m)}$ we have simply $\langle \mathbf{b}^{(0)} \cdot (\nabla \times \mathbf{b}^{(0)})\rangle = \langle \mathbf{a}^{(0)} \cdot \mathbf{b}^{(0)}\rangle / l_0^2$. Here $\langle \mathbf{a}^{(0)} \cdot \mathbf{b}^{(0)}\rangle$ is the magnetic helicity, where $\mathbf{a}^{(0)}$ is the vector potential of $\mathbf{b}^{(0)}$, i. e., $\mathbf{b}^{(0)} = \nabla \times \mathbf{a}^{(0)}$. The factor $\delta(k-k_0)$ in the function $\mu^{(b)}$ is chosen in order to meet the realizability condition for the magnetic helicity (see, e.g., Moffatt, 1978; Zeldovich *et al.*, 1983).

8.4 Specific Results for Nonrotating Turbulence

Let us now summarize and discuss our results. When speaking in this context simply of contributions of velocity and magnetic field fluctuations we refer always to the original turbulence and so to $\mathbf{u}^{(0)}$ and $\mathbf{b}^{(0)}$. Starting with the case of nonrotating turbulence we specify now (77)–(80) by (112)–(115) and obtain

$$\alpha_{ij} = \alpha\delta_{ij}, \quad \alpha = \frac{1}{3}\left(\langle \mathbf{u}^{(0)} \cdot (\nabla \times \mathbf{u}^{(0)})\rangle - \frac{\langle \mathbf{b}^{(0)} \cdot (\nabla \times \mathbf{b}^{(0)})\rangle}{\mu\rho}\right)\tau_0, \tag{116}$$

$$\beta_{ij} = \beta\delta_{ij}, \quad \beta = \frac{1}{3}\langle \mathbf{u}^{(0)2}\rangle \tau_0, \tag{117}$$

$$\gamma = \frac{1}{6}\nabla\left(\langle \mathbf{u}^{(0)2}\rangle - \frac{\langle \mathbf{b}^{(0)2}\rangle}{\mu\rho}\right)\tau_0, \tag{118}$$

$$\boldsymbol{\delta} = \mathbf{0}, \quad \boldsymbol{\kappa} = \mathbf{0}. \tag{119}$$

We have an isotropic α-effect, and α is a sum $\alpha^{(v)} + \alpha^{(m)}$ of two contributions determined by the kinematic helicity and the current helicity of the original turbulence. Whereas the signs of $\alpha^{(v)}$ and the kinematic helicity coincide, those of $\alpha^{(m)}$ and the current helicity are opposite to each other. This is in agreement with results, e.g., by Pouquet *et al.* (1976), by Zeldovich *et al.* (1983) or by Vainshtein and Kichatinov (1983). The contributions of the kinematic and the current helicities to the α-effect, acting in opposite directions, may even compensate each other.

We also have an isotropic mean-field diffusivity β, and this is determined only by the intensity of the velocity fluctuations of the original turbulence. There is no contribution of the magnetic fluctuations of the original turbulence. This again agrees with results by Vainshtein and Kichatinov (1983).

An inhomogeneity of the original turbulence leads to a transport of mean magnetic flux which corresponds to that by a mean velocity $-\gamma$. The expulsion of flux from regions with high intensity of the velocity fluctuations has been sometimes discussed as "turbulent diamagnetism" see, e.g., Zeldovich (1957), Rädler (1968, 1970, 1976) or Krause and Rädler (1971, 1980). Our result shows, again in agreement with Vainshtein and Kichatinov (1983), that magnetic fluctuations act in the opposite sense, that is, in the sense of a "turbulent paramagnetism". The magnetic flux is expelled from regions where $\langle \mathbf{u}^{(0)2} \rangle - (1/\mu\rho)\langle \mathbf{b}^{(0)2} \rangle$ is higher, and pushed into regions where it is lower compared to the surroundings. With equipartition of kinetic and magnetic energy, $\langle \mathbf{u}^{(0)2} \rangle = (1/\mu\rho)\langle \mathbf{b}^{(0)2} \rangle$, this effect vanishes.

8.5 Specific Results for Rotating Turbulence

In the case of the rotating turbulence in which helicity occurs only due to inhomogeneity and rotation we have

$$\alpha_{ij} = \frac{16}{15}\left[\delta_{ij}\mathbf{\Omega}^\star \cdot \nabla\left(\langle \mathbf{u}^{(0)2} \rangle - \frac{1}{3}\frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right) - \frac{11}{24}(\Omega_i^\star \nabla_j + \Omega_j^\star \nabla_i)\left(\langle \mathbf{u}^{(0)2} \rangle + \frac{3}{11}\frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\right.$$
$$\left. + \frac{1}{3}(\varepsilon_{ilm}\Omega_l^\star \Omega_j^\star + \varepsilon_{jlm}\Omega_l^\star \Omega_i^\star)\nabla_m\left(\langle \mathbf{u}^{(0)2} \rangle - \frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\right]\tau_0, \tag{120}$$

$$\beta_{ij} = \frac{1}{3}\left\{\delta_{ij}\left[\langle \mathbf{u}^{(0)2} \rangle\left(1 - \frac{12}{5}\Omega^{\star 2}\right) + \frac{4}{5}\Omega^{\star 2}\frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right] - \frac{4}{5}\Omega_i^\star \Omega_j^\star\left(\langle \mathbf{u}^{(0)2} \rangle + 3\frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\right\}\tau_0, \tag{121}$$

$$\gamma = \frac{1}{6}\left\{\nabla\left(\langle \mathbf{u}^{(0)2} \rangle - \frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\left(1 - \frac{16}{5}\Omega^{\star 2}\right) + \frac{4}{3}\mathbf{\Omega}^\star \times \nabla\left(\langle \mathbf{u}^{(0)2} \rangle + \frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\right.$$
$$\left. + \frac{8}{5}\mathbf{\Omega}^\star\left[\mathbf{\Omega}^\star \cdot \nabla\left(\langle \mathbf{u}^{(0)2} \rangle - \frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\right]\right\}\tau_0, \tag{122}$$

$$\boldsymbol{\delta} = -\frac{2}{9}\left(\langle \mathbf{u}^{(0)2} \rangle - \frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\mathbf{\Omega}^\star \tau_0, \tag{123}$$

$$\kappa_{ijk} = -\frac{2}{9}\left\{(\delta_{ij}\Omega_k^\star + \delta_{ik}\Omega_j^\star)\left[\langle \mathbf{u}^{(0)2} \rangle + \frac{7}{5}\frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho} - \frac{4(q-1)}{5}\left(\langle \mathbf{u}^{(0)2} \rangle + \frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\right]\right.$$
$$\left. + \frac{6}{5}(\varepsilon_{ijl}\Omega_l^\star \Omega_k^\star + \varepsilon_{ikl}\Omega_l^\star \Omega_j^\star)\left(\langle \mathbf{u}^{(0)2} \rangle - \frac{\langle \mathbf{b}^{(0)2} \rangle}{\mu\rho}\right)\right\}\tau_0. \tag{124}$$

Here $\mathbf{\Omega}^\star$ stands for $\mathbf{\Omega}\tau_0$. Our approximations are justified for $|\Omega^\star| \ll 1$ only. We recall that we have ignored all terms containing factors $(\eta - \nu)k^2\tau_*$ in (50) and (51). The only influence of these terms on the above result would consist in the occurrence of contributions to α_{ij} which, in comparison to others, are smaller by a factor of the order of the small quantity $(\eta - \nu)k_0^2\tau_0$. As far as contributions of the velocity fluctuations are concerned the result (120)–(124) is in qualitative agreement with earlier results (e.g., Rädler, 1980).

The α-effect occurs now as a consequence of the simultaneous presence of a rotation of the fluid and gradients in the intensities of the velocity or magnetic fluctuations, and it is clearly anisotropic. A rough measure of the α-effect is the trace of α_{ij}. For this quantity we have

$$\alpha_{ii} = \frac{20}{9}(\boldsymbol{\Omega}^\star \cdot \boldsymbol{\nabla})\left(\langle \mathbf{u}^{(0)2}\rangle - \frac{3}{5}\frac{\langle \mathbf{b}^{(0)2}\rangle}{\mu\rho}\right)\tau_0. \tag{125}$$

At least with this quantity the effect of the velocity fluctuations is again diminished by the magnetic fluctuations.

Like the α-effect the mean-field diffusivity, too, is in general anisotropic. Interestingly enough, in contrast to the case of nonrotating turbulence β_{ij} is now no longer independent of the magnetic fluctuations. For equipartition of kinetic and magnetic energy, $\langle \mathbf{u}^{(0)2}\rangle = (1/\mu\rho)\langle \mathbf{b}^{(0)2}\rangle$, the mean-field diffusivity is again isotropic. Even in the general case the tensor β_{ij} has no other nonzero elements than diagonal ones. If $\boldsymbol{\Omega}^\star$ is parallel to the x_3-axis we have

$$\begin{aligned}\beta_{11} = \beta_{22} &= \frac{1}{3}\langle \mathbf{u}^{(0)2}\rangle\left(1 - \frac{12}{5}\boldsymbol{\Omega}^{\star 2}\right) + \frac{4}{15}\frac{\langle \mathbf{b}^{(0)2}\rangle}{\mu\rho}\boldsymbol{\Omega}^{\star 2}, \\ \beta_{33} &= \frac{1}{3}\langle \mathbf{u}^{(0)2}\rangle\left(1 - \frac{16}{5}\boldsymbol{\Omega}^{\star 2}\right) - \frac{8}{15}\frac{\langle \mathbf{b}^{(0)2}\rangle}{\mu\rho}\boldsymbol{\Omega}^{\star 2}.\end{aligned} \tag{126}$$

In the absence of magnetic fluctuations not caused by the mean magnetic field, i.e. $\mathbf{b}^{(0)} = \mathbf{0}$, all elements of β_{ij} decrease with growing $|\boldsymbol{\Omega}^\star|$.

Compared to the case of nonrotating turbulence, the vector $\boldsymbol{\gamma}$ describing the transport of mean magnetic flux has an additional term perpendicular to the rotation axis. This term contains no longer the gradient of $\langle \mathbf{u}^{(0)2}\rangle - (1/\mu\rho)\langle \mathbf{b}^{(0)2}\rangle$ but that of $\langle \mathbf{u}^{(0)2}\rangle + (1/\mu\rho)\langle \mathbf{b}^{(0)2}\rangle$, that is, it does not vanish with $\langle \mathbf{u}^{(0)2}\rangle = (1/\mu\rho)\langle \mathbf{b}^{(0)2}\rangle$.

In contrast to the case of nonrotating turbulence even in the absence of gradients of the turbulence intensities, the vector $\boldsymbol{\delta}$ is no longer equal to zero. So we rediscover the contribution to the mean electromotive force proportional to $\boldsymbol{\Omega} \times (\boldsymbol{\nabla} \times \overline{\mathbf{B}})$, which has been sometimes discussed as "$\boldsymbol{\Omega} \times \mathbf{J}$-effect". We note that $\boldsymbol{\Omega} \times (\boldsymbol{\nabla} \times \overline{\mathbf{B}}) = -(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\overline{\mathbf{B}} + \boldsymbol{\nabla}(\boldsymbol{\Omega} \cdot \overline{\mathbf{B}})$ and that the term $\boldsymbol{\nabla}(\boldsymbol{\Omega} \cdot \overline{\mathbf{B}})$ plays no part in the induction equation for $\overline{\mathbf{B}}$ as long as the coefficient connecting $\boldsymbol{\delta}$ and $\boldsymbol{\Omega}$ does not depend on space-coordinates. We recall that an electromotive force proportional to $\boldsymbol{\Omega} \times (\boldsymbol{\nabla} \times \overline{\mathbf{B}})$ is, even in the absence of an α-effect, in combination with a differential rotation capable of dynamo action; see, e.g., Rädler, 1969, 1970, 1980, 1986, Roberts, 1972 and Moffatt and Proctor, 1982. With respect to $\boldsymbol{\delta}$ velocity and magnetic fluctuations act again in the opposite sense, and $\boldsymbol{\delta}$ vanishes with $\langle \mathbf{u}^{(0)2}\rangle = (1/\mu\rho)\langle \mathbf{b}^{(0)2}\rangle$. For a rotating turbulence, again even in the absence of gradients of the turbulence intensities, also κ_{ijk} is unequal to zero.

We recall that we determined the correlation tensor $v_{ij}^{(0)}$ used for the calculation of (120)–(124) under the assumption that the expansion of the relaxation time $\check{\tau}$ with respect to $\mathbf{\Omega}$ has no linear term. As a consequence $v_{ij}^{(0)}$ contains no terms of second order in $\mathbf{\Omega}$. It can be followed up easily, however, that a deviation from this assumption would change in (120)–(124) nothing else than numerical factors of the terms of second order in $\mathbf{\Omega}^{\star}$, and it would leave (125) unchanged.

Our results (120)–(124) differ in some details from those given in the articles by Kichatinov (1991), Rüdiger and Kichatinov (1993) and Kichatinov *et al.* (1994). There the second-order correlation approximation, or some modification of it, was used. This implies that the nonlinear terms in the equations for the velocity and magnetic field fluctuations are dropped from the very beginning. Of course, the ranges of validity of this approximation and our τ-approximation are different. The second-order correlation approximation can only be justified for a certain range of parameters like the hydrodynamic and magnetic Reynolds numbers and the Strouhal number. Simple special cases in which it applies are small Reynolds numbers, or large Reynolds numbers but a small Strouhal number. In the articles mentioned, however, it was at least partially assumed that $\nu = \eta = l_c^2/\tau_c$, where ν and η are again kinematic viscosity and magnetic diffusivity, and l_c and τ_c correlation length and time of the turbulent velocity field. This corresponds to hydrodynamic and magnetic Reynolds numbers of the order of unity, for which the conditions of the applicability of the second-order correlation approximation cannot easily be given. Our τ-approximation requires, as already mentioned, large hydrodynamic Reynolds numbers.

8.6 Implications for Mean-field Dynamo Models

Let us add some remarks on the possibilities of dynamo action of the induction effects described by (120)–(124). For the sake of simplicity we consider an axisymmetric dynamo model. We use corresponding cylindrical co-ordinates r, φ and z. The mean motion is assumed to consist in a differential rotation

$$\overline{\mathbf{U}} = \omega \mathbf{e}_z \times \mathbf{r} \tag{127}$$

where the angular velocity ω may depend on r and z, $\mathbf{e}_z$ is the unit vector in z-direction and $\mathbf{r}$ the radius vector. Further, again for simplicity, only the contributions to $\mathcal{E}$ which are linear in $\mathbf{\Omega}^{\star}$ and independent of $\mathbf{b}^{(0)}$ are taken into account. We write

$$\begin{aligned}\mathcal{E} = &-\alpha_0(\boldsymbol{g}\cdot\mathbf{\Omega}^{\star})\overline{\mathbf{B}} + \alpha_1\big[(\mathbf{\Omega}^{\star}\cdot\overline{\mathbf{B}})\boldsymbol{g} + (\boldsymbol{g}\cdot\overline{\mathbf{B}})\,\mathbf{\Omega}^{\star}\big] - \beta_0\nabla\times\overline{\mathbf{B}}\\ &-\gamma_0\,\boldsymbol{g}\times\overline{\mathbf{B}} - \gamma_1\big[(\mathbf{\Omega}^{\star}\cdot\overline{\mathbf{B}})\boldsymbol{g} - (\boldsymbol{g}\cdot\overline{\mathbf{B}})\,\mathbf{\Omega}^{\star}\big]\\ &-\delta_0\big[(\mathbf{\Omega}^{\star}\cdot\nabla)\,\overline{\mathbf{B}} - \nabla(\mathbf{\Omega}^{\star}\cdot\overline{\mathbf{B}})\big] + \kappa_0\big[(\mathbf{\Omega}^{\star}\cdot\nabla)\,\overline{\mathbf{B}} + \nabla(\mathbf{\Omega}^{\star}\cdot\overline{\mathbf{B}})\big],\end{aligned} \tag{128}$$

where $\alpha_0 = (24/11)\alpha_1 = (16/15)\langle \mathbf{u}^{(0)2}\rangle\tau_0$, $\beta_0 = (1/3)\langle \mathbf{u}^{(0)2}\rangle\tau_0$, $\gamma_0 = (3/4)\gamma_1 = (1/6)\,\langle \mathbf{u}^{(0)2}\rangle\tau_0$, $\delta_0 = (2/9)\langle \mathbf{u}^{(0)2}\rangle\tau_0$, $\kappa_0 = (2/9)[1 - 4(q-1)/5]\langle \mathbf{u}^{(0)2}\rangle\tau_0$ and $\boldsymbol{g} = \nabla\langle \mathbf{u}^{(0)2}\rangle/\langle \mathbf{u}^{(0)2}\rangle$. $\mathbf{\Omega}^{\star}$ is assumed to be parallel to the z-axis, $\mathbf{\Omega}^{\star} = \Omega^{\star}\mathbf{e}_z$ with $\Omega^{\star} > 0$.

We split $\overline{\mathbf{B}}$ into its poloidal and toroidal part, $\overline{\mathbf{B}}^P$ and $\overline{\mathbf{B}}^T$, and represent them in the form

$$\overline{\mathbf{B}}^P = \nabla \times (A\mathbf{e}_\varphi), \quad \overline{\mathbf{B}}^T = B\mathbf{e}_\varphi, \tag{129}$$

where A and B are two functions of r, z and t, and $\mathbf{e}_\varphi$ is the unit vector in φ-direction. We assume that the differential rotation is so strong that concerning the generation of $\overline{\mathbf{B}}^T$ from $\overline{\mathbf{B}}^P$ all contributions to $\boldsymbol{\mathcal{E}}$ except the β_0-term can be neglected. Considering η again as constant and starting from the induction Eq. (6) for $\overline{\mathbf{B}}$ we arrive at

$$\begin{aligned} \eta_m \Delta' A - \alpha_0 \Omega^\star g_z B + \frac{\gamma_0}{r} \boldsymbol{g} \cdot \nabla(rA) - (\delta_0 - \kappa_0)\Omega^\star \frac{\partial B}{\partial z} - \frac{\partial A}{\partial t} = 0, \\ \eta_m \Delta' B - \left(\frac{\partial \omega}{\partial r} \frac{\partial}{\partial z} - \frac{\partial \omega}{\partial z} \frac{\partial}{\partial r} \right)(rA) - \frac{\partial B}{\partial t} = 0, \end{aligned} \tag{130}$$

where $\eta_m = \eta + \beta_0$ and

$$\Delta' f = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rf)\right] + \frac{\partial^2 f}{\partial z^2}. \tag{131}$$

Let us restrict ourselves to a local analysis of these equations, that is, to an investigation in some finite region of the rz-plane only. For this purpose we assume that there the coefficients α_0, γ_0, δ_0 and κ_0 as well as $\boldsymbol{g}$ are constant. In the sense of a simple example we further assume that there ω has a logarithmic dependence on r, that is, varies like $\omega_0 + \omega_1 \log(r/r_0)$ with r, where ω_0, ω_1 and r_0 are constants, and is independent of z. Then the quantity $G = r\partial\omega/\partial r$ is also constant, and $\partial\omega/\partial z = 0$. We use the ansatz

$$(A, B) = \mathrm{Re}[(A_0, B_0) J_1(k_r r) \exp(\mathrm{i}k_z z + \lambda t)], \tag{132}$$

where A_0 and B_0 are complex constants, J_1 is the first-order Bessel function of first kind, k_r and k_z are real constants and λ is a complex constant. If $g_r = 0$ this ansatz reduces (130) to a system of two linear homogeneous algebraic equations for A_0 and B_0. We are interested in nontrivial solutions only and have therefore to require that the determinant of this system vanishes. This leads to

$$\lambda = -\eta_m k^2 - \mathrm{i}\frac{\gamma_0 g_z k_z}{2} \pm \sqrt{G\Omega^\star\left[\mathrm{i}\alpha_0 g_z k_z - (\delta_0 - \kappa_0)k_z^2\right] - \left(\frac{\gamma_0 g_z k_z}{2}\right)^2} \tag{133}$$

with $k^2 = k_r^2 + k_z^2$. If $g_r \neq 0$ this reduction works only in the limit $k_r \to 0$, and then (133) applies with $k^2 = k_z^2$.

Solutions of the Eqs. (130) with a λ possessing a nonnegative real part correspond to nondecaying mean magnetic fields. We first consider the case $\boldsymbol{g} \neq \mathbf{0}$, in which we have an α-effect. As can be easily seen from (133) values of λ with nonnegative real part and nonvanishing imaginary part occur if k is sufficiently small. They correspond to undamped dynamo waves traveling parallel to the z-axis. Let us further proceed to the case $\boldsymbol{g} = \mathbf{0}$, in which there is no α-effect. If then $-G\Omega^\star(\delta_0 - \kappa_0) > 0$, the quantity λ is real, and it takes nonnegative values for sufficiently small k. That is, even in the absence of an α-effect dynamo action proves be possible due to combination of differential rotation with the δ or κ-effect. This includes the possibility of such dynamos with δ-effect mentioned above. We see now that δ and κ-effect are in competition and that they compensate each other if $\delta_0 = \kappa_0$. We note that under our assumptions $\delta_0 - \kappa_0 = (8/45)(q-1)\langle \mathbf{u}^{(0)2} \rangle$. Since $q > 1$ we may conclude that the dynamo requires $G < 0$, that is $\partial\omega/\partial r < 0$.

Of course, the results of the local analysis of the dynamo equations should be confirmed by solving them in all conducting space using proper boundary conditions. This has been done so far in the investigations referred to above for dynamo models involving differential rotation and δ-effect.

9. CONCLUDING REMARKS

In this article we have shown a procedure to calculate the mean electromotive force $\mathcal{E}$ for a magnetohydrodynamic turbulence. The bounds of its applicability result mainly from the use of a closure assumption for the deviation of the turbulence from that for zero magnetic field and zero rotation. As explained above it can only be justified for sufficiently small mean magnetic fields and small rotation rates of the fluid. For simplicity we have restricted ourselves to the case in which there is, apart from the rotation, no mean motion. Specific results have been derived for the limit in which the mean electromotive force is linear in the mean magnetic field and a Kolmogorov-type turbulence.

Let us compare results obtained in the kinematic approach on the basis of the second-order correlation approximation, or first-order smoothing, for the high-conductivity limit with results of our procedure. Take as a simple example the case of homogeneous and isotropic turbulence. Then we have $\boldsymbol{\mathcal{E}} = -\alpha\overline{\mathbf{B}} - \beta\nabla\times\overline{\mathbf{B}}$ for sufficiently weak variations of $\overline{\mathbf{B}}$ in space and in time. In the kinematic approach under the conditions mentioned we have

$$\alpha = \frac{1}{3}\int_0^\infty \langle \mathbf{u}(\mathbf{x}, t)\cdot[\nabla\times\mathbf{u}(\mathbf{x}, t-\tau)]\rangle \, d\tau, \tag{134}$$

$$\beta = \frac{1}{3}\int_0^\infty \langle \mathbf{u}(\mathbf{x}, t)\cdot\mathbf{u}[\mathbf{x}, t-\tau])\rangle \, d\tau. \tag{135}$$

This is often expressed in the form

$$\alpha = \tfrac{1}{3}\langle \mathbf{u}\cdot(\nabla\times\mathbf{u})\rangle \tau_{\rm corr}^{(\alpha)}, \quad \beta = \tfrac{1}{3}\langle \mathbf{u}^2\rangle \tau_{\rm corr}^{(\beta)}, \tag{136}$$

with properly defined correlation times $\tau_{\rm corr}^{(\alpha)}$ and $\tau_{\rm corr}^{(\beta)}$. The validity of these results can only be readily justified under the condition $u\tau/l \ll 1$, where u, l, and τ are typical values of the velocity and of the length and time scales of the velocity field. This condition, however, is problematic in view of applications to realistic situations, and the validity of these results beyond this condition is questionable. Basically it is possible to improve the approximation by including higher-order terms in $\mathbf{u}$ but this is very tedious.

The results (136) have the structure of our result given by (116), specified by $\mathbf{b}^{(0)} = \mathbf{0}$, and our result (117). The validity of our results, however, is not restricted by a condition like $u\tau/l \ll 1$, but only by the applicability of the τ-approximation (37). That is, we have in any case a much wider range of validity.

We point out that despite the formal similarity of the mentioned results gained in the kinematic approach and those derived here, there is a basic difference between them. In the first case we have originally, that is in (134) and (135), correlations between values of $\mathbf{u}$ taken at different times, which are often as in (136) expressed by $\mathbf{u}$ at a given time and a correlation time, but in the second case we consider from the very beginning only correlations between values of $\mathbf{u}$ at the same time. In that sense there is no simple connection between the two kind of results.

We may use the framework explained above also beyond the limit of very small mean magnetic fields and study, for example, $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$-quenching at least for not too strong fields. Then, of course, v_{ij} and m_{ij} can no longer be replaced by $v_{ij}^{(0)}$ and $m_{ij}^{(0)}$. Instead we have to derive equations for v_{ij} and m_{ij} corresponding to Eq. (30) for χ_{ij}, insert their solutions depending on $\overline{\mathbf{B}}$ in I_{ij} in (35) and follow the above pattern of the determination of a_{ij} and b_{ijk}, or $\boldsymbol{\alpha}$, $\boldsymbol{\beta}, \ldots$. By the way, then even in the case $\mathbf{b}^{(0)} = \mathbf{0}$ there are contributions of m_{ij} to these coefficients which, of course, vanish like $m_{ij}^{(0)}$ with $\overline{\mathbf{B}}$.

In this context it suggests itself to study in addition to the mean electromotive force $\boldsymbol{\mathcal{E}}$ also the mean pondermotive force $\boldsymbol{\mathcal{F}}$. It has a part independent of $\overline{\mathbf{B}}$, which can be calculated with the help of $v_{ij}^{(0)}$ and $m_{ij}^{(0)}$ only. Its general form can be derived on the basis of solutions of the equations for v_{ij} and m_{ij} mentioned.

Acknowledgment

We thank Dr. M. Rheinhardt for inspiring discussions. We are grateful to the two anonymous referees for their helpful comments. We acknowledge support from INTAS Program Foundation (Grant No. 99-348). N.K. and I.R. also thank the Astrophysical Institute of Potsdam for its hospitality.

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APPENDIX A: RELATIONS WITH ε_{ijk}

For the derivation of (19) it is useful to know the relation

$$\varepsilon_{ijk}\Omega_k + (\varepsilon_{ikl}k_j - \varepsilon_{jkl}k_i)\frac{k_k\Omega_l}{k^2} = \varepsilon_{ijk}k_k\frac{(\mathbf{k}\cdot\boldsymbol{\Omega})}{k^2}, \tag{A1}$$

which applies to arbitrary vectors $\mathbf{k}$ and $\boldsymbol{\Omega}$.

In view of the derivation of (67) we recall the identity

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn}. \tag{A2}$$

APPENDIX B: DERIVATION OF EQUATION (30)

In the calculations of $\partial\chi_{ij}/\partial t$ on the basis of the Eqs. (15) and (16) contributions to this quantity occur which have, e.g., the form of

$$\begin{aligned} X_{ij}(\mathbf{k},\mathbf{R}) &= \int \langle \hat{S}_i(\mathbf{u},\overline{\mathbf{B}};\mathbf{k}+\mathbf{K}/2)\hat{u}_j(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{K}\cdot\mathbf{R})\,d^3K \\ &= i\int (k_k + K_k/2)\langle \hat{u}_i(\mathbf{k}+\mathbf{K}/2-\mathbf{Q})\hat{u}_j(-\mathbf{k}+\mathbf{K}/2)\rangle \hat{\overline{B}}_k(\mathbf{Q})\exp(i\mathbf{K}\cdot\mathbf{R})\,d^3K\,d^3Q. \end{aligned} \tag{B1}$$

In the last expression we may change the sequence of integration so that the inner integral is over $\mathbf{K}$ and the outer integral is over $\mathbf{Q}$. In the inner integral we may further change the integration variable $\mathbf{K}$ into $\mathbf{K}-\mathbf{Q}$, denoted by $\mathbf{K}'$ in the following. In this way, and using $Q_k\hat{\overline{B}}_k = 0$, we obtain

$$\begin{aligned} X_{ij}(\mathbf{k},\mathbf{R}) = i\int (k_k + K'_k/2)\langle \hat{u}_i(\mathbf{k}-\mathbf{Q}/2+\mathbf{K}'/2)\hat{u}_j(-\mathbf{k}+\mathbf{Q}/2 \\ + \mathbf{K}'/2)\rangle \hat{\overline{B}}_k(\mathbf{Q})\exp[i(\mathbf{K}'\cdot\mathbf{R}+\mathbf{Q}\cdot\mathbf{R})]\,d^3K'\,d^3Q. \end{aligned} \tag{B2}$$

Remembering the definition of $v_{ij}(\mathbf{k},\mathbf{R})$ we can rewrite this into

$$X_{ij}(\mathbf{k},\mathbf{R}) = \int \left[ik_k v_{ij}(\mathbf{k}-\mathbf{Q}/2,\mathbf{R}) + \frac{1}{2}\left(\frac{\partial v_{ij}(\mathbf{k}-\mathbf{Q}/2,\mathbf{R})}{\partial R_k}\right)\right]\hat{\overline{B}}_k(\mathbf{Q})\exp(i\mathbf{Q}\cdot\mathbf{R})\,d^3Q. \tag{B3}$$

The fact that $\overline{\mathbf{B}}$ varies only on large scales, that is, $\hat{\overline{\mathbf{B}}}$ is only nonzero for certain small $|\mathbf{Q}|$, suggests to use the Taylor expansion

$$v_{ij}(\mathbf{k}-\mathbf{Q}/2,\mathbf{R}) \simeq v_{ij}(\mathbf{k},\mathbf{R}) - \frac{1}{2}\left(\frac{\partial v_{ij}(\mathbf{k},\mathbf{R})}{\partial k_k}\right)Q_k + \mathrm{O}(\mathbf{Q}^2). \tag{B4}$$

This yields

$$X_{ij}(\mathbf{k},\mathbf{R}) \simeq [i(\mathbf{k}\cdot\overline{\mathbf{B}}) + \frac{1}{2}(\overline{\mathbf{B}}\cdot\nabla)]v_{ij}(\mathbf{k},\mathbf{R}) - k_k v_{ijl}(\mathbf{k},\mathbf{R})\overline{B}_{k,l}, \tag{B5}$$

where $v_{ijl} = (1/2)\partial v_{ij}/\partial k_l$. According to our assumption a term of the second-order in ∇ was neglected. The contributions to the Taylor expansion indicated by $\mathrm{O}(\mathbf{Q}^2)$ only lead to terms of higher order in ∇ and need not to be considered.