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The Mean Field Ising Model trough Interpolating Techniques

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Abstract Aim of this paper is to illustrate how some recent techniques developed within the framework of spin glasses do work on simpler model, focusing on the method and not on the analyzed system. To fulfill our will the candidate model turns out to be the paradigmatic mean field Ising model. The model is introduced and investigated with the interpolation techniques. We show the existence of the thermodynamic limit, bounds for the free energy density, the explicit expression for the free energy with its suitable expansion via the order parameter, the self-consistency relation, the phase transition, the critical behavior and the self-averaging properties. At the end a formulation of a Parisi-like theory is tried and discussed.

Keywords Cavity field · Spin glasses · Interpolating techniques

1 Introduction

In the past twenty years the statistical mechanics of disordered systems earned an always increasing weight as a powerful framework by which analyze the world of complex networks [1, 5, 15, 37, 40].

The "harmonic oscillator" of this field of research is the Sherrington-Kirkpatrick model [38] (SK), on which several schemes have been tested along these years [21]; the first method developed has been the *replica trick* [14] which, in a nutshell consists in expanding the logarithm of the partition function $Z(\beta)$ in a power series of such a function via $\ln Z(\beta) = \lim_{n\to 0}(Z(\beta)^n - 1)/n$, allowing, in some way, its analytic continuation to the $n \to 0$ limit [38]. Such analytic continuation is not at all simple and many efforts have been necessary to examine this mathematical problem in the light of theoretical physics tools

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such as symmetries and their breaking [42]. In this scenario a solution has been proposed by Parisi (and recently proved by Guerra [26] and Talagrand [44]) with the well known Replica Symmetry Breaking scheme, both solving the SK-model by showing a peculiar "picture" of the organization of the underlaying microstructure of this complex system [39], as well as conferring a key role to the replica-trick method.

However the replica trick still pays the price of requiring an "a priori" ansatz at some stage of its work and several mathematical problems concerning its foundations and validity are still open [43].

As a consequence, in the past ten years, another method, called the *cavity method* [28], has been largely improved, mainly thanks to its ability to work without ansatz and to its natural predisposition to be implemented into the interpolating technique scheme [6, 8, 26, 32, 33]. Although this method may be not fully empowered to solve the whole SK-problem without working in synergy with the replica framework, it proved a valid alternative to address, at least, a number of questions [10, 20, 34].

The aim of this paper is to show some of the results attainable within the cavity method by applying it to a simple model, the mean field Ising model [1, 42], which can be solved with standard techniques without requiring nor the replica trick neither the cavity method itself. Consequently attention should be payed on the method, which, once applied on a paradigmatic simplest model, should be clearer to the non-expert reader than when applied on complex systems as the SK.

The paper is structured as follows: Hereafter, still in the first section, the model is introduced. In Sect. 2 the interpolating technique for obtaining the thermodynamic limit and the bounds in the size of the system are discussed. In Sect. 3 the interpolating technique to obtain an explicit expression for the free energy and consequently the phase diagram are studied. Section 4 is dedicated to the phase transition: the lacking of commutativity of the infinite volume limit against a vanishing perturbing field, the scaling of the order parameter at criticality and the self-averaging relations are discussed. Section 5 explains a trial technique which aims to reproduce the Parisi scheme within this simpler framework.

1.1 Definition of the Model and Thermodynamics

The Hamiltonian of the Ising model is defined on N spin configurations $\sigma : i \to \sigma_i = \pm 1$, labeled by $i = 1, \dots, N$, as [1, 42]

$$H_N(\sigma) = -\frac{1}{N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j. \tag{1}$$

We assume throughout the paper that, without explicit indications, there is no external field. The thermodynamic of the model is carried by the free energy density $f_N(\beta) = F_N(\beta)/N$, which is related to the Hamiltonian via

$$e^{-\beta F_N(\beta)} = Z_N(\beta) = \sum_{\sigma} e^{\frac{\beta}{N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j}, \tag{2}$$

$Z_N(\beta)$ being the partition function. For the sake of convenience we will not deal with $f_N(\beta)$ but with the thermodynamic *pressure* $\alpha(\beta)$ defined via

$$\alpha(\beta) = \lim_{N\to\infty} \alpha_N(\beta) = \lim_{N\to\infty} -\beta f_N(\beta) = \lim_{N\to\infty} \frac{1}{N} \ln Z_N(\beta). \tag{3}$$

A key role will be played by the magnetization m, its fluctuations and its moments, and so let us introduce it as

$$m_N = \frac{1}{N}\sum_{1\le i\le N}\sigma_i, \qquad \langle m_N\rangle = \frac{\sum_\sigma m_N e^{-\beta H_N(\sigma)}}{\sum_\sigma e^{-\beta H_N(\sigma)}}. \tag{4}$$

Let us consider also its rescaled fluctuation by introducing the following random variable

$$\xi_N(\sigma) = \frac{1}{\sqrt{N}}\sum_i \sigma_i \tag{5}$$

by which the magnetization can be expressed as $\langle m_N\rangle = \langle \xi_N\rangle N^{-1/2}$; further, let us define $\gamma(\beta) = 1/(1-\beta)$ and state, without proof [19], that in the interval $0<\beta<\beta_c=1$, in the thermodynamic limit the distribution of $\xi(\sigma) = \lim_{N\to\infty}\xi_N(\sigma)$ is a centered Gaussian with variance equal to $\gamma(\beta)$. The boundary at which the variance of the distribution diverges (i.e. $\beta=\beta_c=1$) defines the onset of the broken ergodicity phase.

2 Thermodynamic Limit

2.1 Bounding the Free Energy in the System Size

The first step when dealing with the statistical mechanics package is, once defined the relevant observable, checking that the model is well defined (i.e. it admits a good but nontrivial thermodynamic limit). As this task may be not simple (as for the SK model or worst for the Hopfield model of neural network[1]) working out its $\sup_N$ may help as a first pre-step. This is usually a simpler task [22].

2.1.1 First Estimate of the Free Energy

Dealing with the simple Ising model it is possible to obtain a bound for the free energy simply by aligning all the spins among themselves, however this procedure is not allowed in models with competitive interactions (much more interesting and mathematically challenging); let us show how it works:

$$Z_N(\beta) \le \sum_\sigma e^{\beta/N} e^{\frac{N(N-1)}{2}} \le 2^N e^{\frac{\beta(N-1)}{2}}, \tag{6}$$

$$\frac{1}{N}\ln Z_N(\beta) \le \ln 2 + \frac{\beta}{2}\left(1-\frac{1}{N}\right) \quad \Rightarrow \quad \alpha(\beta) \le \ln 2 + \frac{\beta}{2}. \tag{7}$$

Following this approach the next step is trying and bound, in the volume size, the free energy from above and from below. For the Ising model this can be obtained as follows.

[1] a key ingredient for the existence of the thermodynamic limit is the subadditivity or to the superadditivity of the free energy with respect the system size. The Hopfield model shows, varying the storaged memory, both the features. As a consequence there is a region of unknown width in which the model free energy is nor subadditive neither superadditive.

2.1.2 Upper Bound of the Free Energy

While for disordered systems bounding the free energy in the volume limit is not an easy task, for a model with no disorder such bounds can be easily obtained [17, 31]. Consider the trivial estimate of the magnetization m, valid for all trial fixed magnetization M

$$m^2 \geq 2mM - M^2 \tag{8}$$

and plug it into the partition function to get (neglecting terms vanishing in the thermodynamic limit)

$$Z_N(\beta) = \sum_\sigma e^{\frac{\beta}{N}\sum_{1<i<j<N}\sigma_i\sigma_j} = \sum_\sigma e^{\frac{\beta N m^2}{2}} \geq \sum_\sigma e^{\beta m M N} e^{-\frac{1}{2}\beta M^2 N}.$$

Now this sum is easy to compute, since the magnetization appears linearly and therefore the sum factorizes in each spin. Physically speaking, we replaced the two-body interaction, which is difficult to deal with, with a one-body interaction. Then we try to compensate this by modulating the field acting on each spin by means of a trial fixed magnetization and a correction term quadratic in this trial magnetization M.

Remark 1 A recent method [4, 7, 24] introduced by Aizenman and coworkers for the spin-glass theory is a powerful extension of this idea in which the key ingredient is letting interact the system one is dealing with, with an external structure such that, sending the size of the this structure to infinity, thanks to the mean field nature of the interaction, the system no longer interacts with itself, making the mathematical control simpler because the two-body term becomes suppressed.

The result is the following bound

$$\frac{1}{N}\ln Z_N(\beta) \geq \sup_M \left\{ \ln 2 + \ln\cosh(\beta M) - \frac{1}{2}\beta M^2 \right\} \tag{9}$$

that holds for any size of the system N. The result is quite typical, the term $\ln 2$ is there because the sum over a spin of a Boltzmann factor linear in the spins is *twice* the hyperbolic cosine, which appears as second term (that essentially gives the entropy). The third term is the internal energy (multiplied by $-\beta$).

2.1.3 Lower Bound of the Free Energy

In order to get the opposite bound to (9), let us notice that the magnetization m can take only $2N+1$ distinct values. We can therefore split the partition function into sums over configurations with constant magnetization in the following way [31]

$$Z_N(\beta) = \sum_\sigma \sum_M \delta_{mM} e^{\frac{1}{2}\beta N m^2} \tag{10}$$

using the trivial identity

$$\sum_M \delta_{mM} = 1. \tag{11}$$

Now inside the sum $m = M$, which means also

$$m^2 = 2mM - M^2. \tag{12}$$

Plugging the last equality into $Z_N(\beta)$ and using the trivial inequality

$$\delta_{mM} \le 1$$

yields

$$Z_N(\beta) \le \sum_M \sum_\sigma e^{\beta N m M} e^{-\frac{1}{2}\beta N M^2}. \tag{13}$$

Now one can carry out the sum over σ bounding the remaining sum over M by $2N + 1$ times its largest term gives then

$$Z_N(\beta) \le \sum_M \sup_M \left\{ \ln 2 + \ln\cosh(\beta M) - \frac{1}{2}\beta M^2 \right\} \tag{14}$$

from which

$$\frac{1}{N} \ln Z_N(\beta) \le \ln \frac{2N+1}{N} + \sup_M \left\{ \ln 2 + \ln\cosh(\beta M) - \frac{1}{2}\beta M^2 \right\}. \tag{15}$$

This gives, together with (9), the exact value of free energy per site at least in the thermodynamic limit.

2.2 Bound by Interpolating the Size of the System

A breakthrough in showing the existence of the thermodynamic limit for mean field disordered systems has been obtained recently within the Guerra-Toninelli interpolation scheme [32, 35]. Previously several beautiful model-specific attempts were made [11–13], but this interpolating scheme showed an immediate wide range of applications and its beauty is its simplicity. We are going to introduce it applied to the Ising-model.

Divide the N spin system into two subsystems of N_1 and N_2 spins each, with $N_1 + N_2 = N$. Denoting by $m_1(\sigma)$, $m_2(\sigma)$ the magnetization corresponding to the subsystems, *i.e.*

$$m_1(\sigma) = \frac{1}{N_1} \sum_{i=1}^{N_1} \sigma_i, \qquad m_2(\sigma) = \frac{1}{N_2} \sum_{i=N_1+1}^{N} \sigma_i,$$

one sees that $m(\sigma)$ is a convex linear combination of $m_1(\sigma)$ and $m_2(\sigma)$:

$$m(\sigma) = \frac{N_1}{N} m_1(\sigma) + \frac{N_2}{N} m_2(\sigma). \tag{16}$$

Since the function $x \to x^2$ is convex, one has

$$Z_N(\beta) \le \sum_{\{\sigma\}} \exp(\beta(N_1 m_1^2(\sigma) + N_2 m_2^2(\sigma))) = Z_{N_1}(\beta) Z_{N_2}(\beta)$$

and

$$N f_N(\beta) = -\frac{1}{\beta} \ln Z_N(\beta) \ge N_1 f_{N_1}(\beta) + N_2 f_{N_2}(\beta). \tag{17}$$

Theorem 1 *The infinite volume limit for $\alpha_N(\beta)$ does exist and equals its sup.*

$$\lim_{N\to\infty}\alpha_N(\beta)=\sup_N \alpha_N(\beta)\equiv\alpha(\beta). \tag{18}$$

Proof In a nutshell the two key ingredients are the subadditivity ($Nf_N \geq N_1 f_{N_1} + N_2 f_{N_2}$) and the property of the free energy density of being limited from above uniformly in N which is established elementary by considering (9, 15). □

Unfortunately, the very simple approach we illustrated above as it is, does not apply to the SK model, where the randomness of the couplings prevents us from exploiting subadditivity directly on the Hamiltonian H_N. However, the related strategy, which allows in some sense an extension to mean field spin glass models is to interpolate between the original systems of N spins, and two non-interacting systems, containing N_1 and N_2 spins, respectively, and to compare the corresponding free energies. To this purpose, consider the interpolating parameter $0\leq t\leq 1$, and the auxiliary partition function

$$Z_N(t)=\sum_{\{\sigma\}}\exp(\beta(Ntm^2(\sigma)+N_1(1-t)m_1^2(\sigma)+N_2(1-t)m_2^2(\sigma))). \tag{19}$$

Of course, for the boundary values $t=0,1$ one has

$$-\frac{1}{N\beta}\ln Z_N(1)=f_N(\beta), \tag{20}$$

$$-\frac{1}{N\beta}\ln Z_N(0)=\frac{N_1}{N}f_{N_1}(\beta)+\frac{N_2}{N}f_{N_2}(\beta) \tag{21}$$

and, taking the derivative with respect to t,

$$-\frac{d}{dt}\frac{1}{N\beta}\ln Z_N(t)=-\left\langle m^2(\sigma)-\frac{N_1}{N}m_1^2(\sigma)-\frac{N_2}{N}m_2^2(\sigma)\right\rangle_t\geq 0, \tag{22}$$

where $\langle\,\rangle_t$ denotes the Boltzmann-Gibbs thermal average with the extended weight encoded in the t-dependent partition function (19). Therefore, integrating in t between 0 and 1, and recalling the boundary conditions (20), (21), one finds again the superadditivity property (17).

The interpolation method, which may look unnecessarily complicated for the Curie-Weiss model, is actually the only one working in the case of mean field spin glass systems.

3 The Structure of the Free Energy

In this chapter we adapt the work [6] (in which a novel interpolating cavity field technique was developed for the SK model) to the mean field Ising model.

The main idea of the *cavity field* method is to look for an explicit expression of $\alpha_N(\beta)=-\beta f_N(\beta)$ upon increasing the size of the system from N particles (the cavity) to $N+1$ so that, in the limit of N that goes to infinity [25, 27]

$$\lim_{N\to\infty}(-\beta F_{N+1}(\beta))-(-\beta F_N(\beta))=-\beta f(\beta) \tag{23}$$

because the existence of the thermodynamic limit (Sect. 2.2) implies only vanishing correction of the free energy density.

Note Strictly speaking the limit does exist surely just in the Cesàro sense [23] (Cesàro limits are employed when analyzing sequences which can oscillate and do not converge, i.e. the Liebnitz series converges to zero in the Cesàro sense [36]) but this level of mathematical rigor will not be presented along the paper.

3.1 Interpolating Cavity Field

As we will see, the interpolating technique can be very naturally implemented in the cavity method; let us consider the partition function of a system made by $N+1$ spins:

$$Z_{N+1}(\beta) = \sum_{\sigma} e^{-\beta H_{N+1}(\sigma)} = \sum_{\sigma_{N+1}=\pm 1} \sum_{\sigma} e^{\frac{\beta}{N+1}\sum_{1\le i<j\le N}\sigma_i\sigma_j} e^{\frac{\beta}{N+1}\sum_{1\le i\le N}\sigma_i\sigma_{N+1}}. \tag{24}$$

With the gauge transformation $\sigma_i \to \sigma_i\sigma_{N+1}$, which, of course, is a symmetry of the Hamiltonian, we get

$$Z_{N+1}(\beta) = 2Z_N(\beta^*)\tilde{\omega}(e^{\frac{\beta}{N+1}\sum_{1\le i\le N}\sigma_i}) \tag{25}$$

where $\tilde{\omega}$ is the Boltzmann state at the inverse temperature $\beta^* = \beta\frac{N}{N+1}$ (note that in the thermodynamic limit the shifted temperature converges to the real one $\beta^* \to \beta$). Let us reverse the temperature shift and apply the logarithm to both the sides of (25) to obtain

$$\ln Z_{N+1}\left(\beta\frac{N+1}{N}\right) = \ln 2 + \ln Z_N(\beta) + \ln \omega_N(e^{\frac{\beta}{N}\sum_{1\le i\le N}\sigma_i}). \tag{26}$$

Equation (26) tell us that via the third term of its r.h.s. we can bridge an Ising system with N particles at an inverse temperature β to an Ising system with $N+1$ particles at a shifted inverse temperature $\beta^* = \beta(N+1)/N$. Focusing on such a term let us make the following definitions.

Definition 1 We define an extended partition function $Z_N(\beta, t)$ as

$$Z_N(\beta, t) = \sum_{\sigma} e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}. \tag{27}$$

Note that the above partition function, at $t = \beta$, turns out to be, via the global gauge symmetry $\sigma_i \to \sigma_i\sigma_{N+1}$, a partition function for a system of $N+1$ spins at a shifted temperature β^* apart a constant term. On the same line

Definition 2 We define the generalized Boltzmann state $\langle\,\rangle_t$ as

$$\langle F(\sigma)\rangle_t = \frac{\langle F(\sigma)e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}\rangle}{\langle e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}\rangle}, \tag{28}$$

$F(\sigma)$ being a generic function of the spins.

Definition 3 Related to the Boltzmann state $\langle\rangle$ we define the cavity function $\Psi(\beta,t) = \lim_{N\to\infty}\Psi_N(\beta,t)$ as

$$\Psi(\beta,t) = \lim_{N\to\infty}\Psi_N(\beta,t) = \lim_{N\to\infty}\ln\langle e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}\rangle. \tag{29}$$

It will appear clear, while reading the paper, when we deal with the finite-N cavity function and when with its thermodynamic limit.

Definition 4 We define respectively as fillable and filled monomials the odd and even momenta of the magnetization weighted by the extended Boltzmann measure such that

- $\langle m_N^{2n+1}\rangle_t$ with $n\in\mathbb{N}$ is fillable
- $\langle m_N^{2n}\rangle_t$ with $n\in\mathbb{N}$ is filled

Proposition 1 *The cavity function $\Psi(\beta,t)$ is the generating function of the centered momenta of the magnetization, examples of which are*

$$\frac{\partial\Psi_N(\beta,t)}{\partial t} = \langle m_N\rangle_t, \tag{30}$$

$$\frac{\partial^2\Psi_N(\beta,t)}{\partial t^2} = \langle m_N^2\rangle_t - \langle m_N\rangle_t^2. \tag{31}$$

Proof The proof is straightforward and can be obtained by simple derivation:

$$\frac{\partial\Psi_N(\beta,t)}{\partial t} = \partial_t\ln\omega_N(e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}) = \partial_t\ln\sum_\sigma e^{-\beta H_N(\sigma)}e^{\frac{t}{N}\sum_i\sigma_i}$$

$$= \frac{\sum_\sigma\frac{1}{N}\sum_{1\le i\le N}\sigma_i e^{-\beta H_N(\sigma)}e^{\frac{t}{N}\sum_i^N\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)}e^{\frac{t}{N}\sum_i^N\sigma_i}} = \langle m_N\rangle_t.$$

The second derivative is worked out exactly as the first. □

Remark 2 We stress that in the disordered counterpart (i.e. the SK model) a proper interpolating cavity function is defined by introducing $\sqrt{t}$ instead of t. This reflects the property of the Gaussian coupling of adding another extra derivation due to Wick theorem. It is worth nothing that again the Gaussian coupling makes necessary the normalization factor $\sqrt{N}$ instead of N in front of the Hamiltonian such that the adaptation from Ising t/N to SK $\sqrt{t/N}$ is the same for t and N.

3.2 The Free Energy via the Interpolating Cavity Method

The fact that the free energy is expressed as the difference between an entropy term coming from a one-body interaction and the internal energy times β is typical of thermodynamics. We found this feature when looking at the bounds (9), (15); now, stating the next fundamental theorem, we find the same structure via this interpolating version of the cavity field method (and again we will find it in the next section when dealing with the Parisi-like theory).

Theorem 2 *The following relation holds in the thermodynamic limit*:

$$\alpha(\beta) = \ln 2 + \Psi(\beta, t = \beta) - \beta \frac{\partial \alpha(\beta)}{\partial \beta}. \tag{32}$$

Proof Let us consider again the partition function of a system made up by $(N + 1)$ spins and point out with β the true temperature and with $\beta^* = \beta(1 + N^{-1})$ the shifted one:

$$\begin{aligned} Z_{N+1}(\beta) &= \sum_{\sigma_{N+1}} e^{\frac{\beta}{\sqrt{N+1}} \sum_{1\le i<j\le N+1} \sigma_i \sigma_j} \\ &= 2 \sum_{\sigma_N} e^{\frac{\beta^*}{\sqrt{N}} \sum_{1\le i<j\le N} \sigma_i \sigma_j} e^{\frac{\beta}{\sqrt{N+1}} \sum_{1\le i\le N} \sigma_i}. \end{aligned} \tag{33}$$

Now we multiply and divide by $Z_N(\beta^*)$ the right hand side of (33), then we take the logarithm on both sides and subtract from every member the quantity $\ln Z_{N+1}(\beta^*)$; expanding $\ln Z_{N+1}(\beta)$ around $\beta = \beta^*$ as

$$\ln Z_{N+1}(\beta) - \ln Z_{N+1}(\beta^*) = (\beta - \beta^*) \partial_{\beta^*} \ln Z_{N+1}(\beta^*) + O((\beta - \beta^*)^2) \tag{34}$$

with

$$\beta - \beta^* = \beta^* \left(\sqrt{\frac{N+1}{N}} - 1 \right) = \frac{\beta^*}{2N} + O(N^{-1}) \tag{35}$$

we substitute β with β^* inside the state ω and neglecting corrections $O(N^{-1})$ we have:

$$\begin{aligned} &\ln Z_{N+1}(\beta^*) + (\beta - \beta^*) \partial_{\beta^*} \ln Z_{N+1}(\beta^*) \\ &\quad = \ln 2 + \ln Z_N(\beta^*) + \ln \omega_{N,\beta^*}(e^{\frac{\beta}{\sqrt{N+1}} \sum_{1<i<N} \sigma_i}) + O(N^{-1}), \end{aligned} \tag{36}$$

where, with the symbol ω_{N,β^*} we stressed that the temperature inside the Boltzmann average is the shifted one. Using the variable $\alpha(\beta^*)$ and renaming $\beta^* \to \beta$ in the thermodynamic limit we get:

$$\alpha(\beta) + \beta \frac{d\alpha(\beta)}{d\beta} = \ln 2 + \Psi(\beta, t = \beta) \tag{37}$$

and this is the thesis of the theorem. □

3.3 Saturability and Gauge-Invariance

The next step is to motivate why we introduced the whole machinery: The first reason we are going to show are peculiar properties of both the filled and the fillable monomials (see Definition 4). In the thermodynamic limit, the first class do not depend on the perturbation induced by the cavity field and, at $t = \beta$, the latter (via the $\sigma_i \to \sigma_i \sigma_{N+1}$ symmetry) is projected into the first class. The second reason is that the free energy can be expanded via these monomials, so a good control of them means a good knowledge of the thermodynamic of the system.

Theorem 3 *In the $N \to \infty$ limit the averages $\langle m_N^{2n} \rangle$ of the filled monomials are t-independent for almost all values of β, such that*

$$\lim_{N\to\infty} \partial_t \langle m_N^{2n} \rangle_t = 0.$$

Proof Without loss of generality we will prove the theorem in the simplest case (for $\langle m_N^2 \rangle$); it will appear immediately clear how to generalize the proof to higher order monomials. Let us write the cavity function as

$$\Psi_N(\beta, t) = \ln Z_N(\beta, t) - \ln Z_N(\beta) \tag{38}$$

and derive it with respect to β:

$$\frac{\partial \Psi_N(\beta, t)}{\partial \beta} = \frac{N}{2}(\langle m_N^2 \rangle - \langle m_N^2 \rangle_t). \tag{39}$$

We can introduce an auxiliary function $\Upsilon_N(\beta, t) = (\langle m_N^2 \rangle - \langle m_N^2 \rangle_t)$ such that:

$$\Upsilon_N(\beta, t) = \frac{2}{N} \partial_\beta \Psi_N(\beta, t) \tag{40}$$

and integrate it in a generic interval $[\beta_1, \beta_2]$:

$$\int_{\beta_1}^{\beta_2} \Upsilon_N(\beta, t) d\beta^2 = \frac{4}{N}[\Psi_N(\beta_2, t) - \Psi_N(\beta_1, t)]. \tag{41}$$

Now we must control $\Psi_N(\beta, t)$ in the $N \to \infty$ limit; the simplest way is to look at its t-streaming $\partial_t \Psi_N(\beta, t) = \langle m_N \rangle_t$ such the N-dependence is just taken into account by the Boltzmann factor inside the averages and, as $\langle m_N \rangle_t \in [-1, 1]$, in the thermodynamic limit $\Psi(\beta, t)$ remains bounded and the second member of (41) goes to zero such that, $\forall$ $[\beta_1,\beta_2]$, $\Upsilon_N(\beta, t)$ converges to zero implying $\langle m_N^2 \rangle_t \to \langle m_N^2 \rangle$. □

Remark 3 A consequence of this property, in the spin glass theory, turns out to be the stochastic stability of a large class of overlap polynomials [16, 41].

The next theorem is crucial for this section, so, for the sake of simplicity, we split it in two part: at first we prove the following lemma than it will make us able to proof the core of the theorem itself which will be showed immediately after. For a clearer statement of the lemma we take the freedom of pasting the volume dependence of the averages as a subscript close to the perturbing tuning parameter t.

Lemma 1 *Let $\langle\rangle_N$ and $\langle\rangle_{N,t}$ be the states defined, on a system of N spins, respectively by the canonical partition function $Z_N(\beta)$ and by the extended one $Z_N(\beta, t)$; if we consider the ensemble of indexes $\{i_1, \dots, i_r\}$ with $r \in [1, N]$, then for $t = \beta$, where the two measures become comparable, thanks to the global gauge symmetry (i.e. the substitution $\sigma_i \to \sigma_i \sigma_{N+1}$) the following relation holds*

$$\omega_{N,t=\beta}(\sigma_{i_1} \cdots \sigma_{i_r}) = \omega_{N+1}(\sigma_{i_1} \cdots \sigma_{i_r} \sigma_{N+1}^r) + O\left(\frac{1}{N}\right) \tag{42}$$

where r is an exponent, so if r is even $\sigma_{N+1}^r = 1$, while if it is odd $\sigma_{N+1}^r = \sigma_{N+1}$.

Proof Let us write $\omega_{N,t}$ for $t = \beta$, defining for the sake of simplicity $\pi = \sigma_{i_1} \cdots \sigma_{i_r}$:

$$\omega_{N,t=\beta}(\pi) = \left[\sum_\sigma \frac{1}{Z_N(\beta)} e^{\frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le N} \sigma_i \sigma_j + \frac{\beta}{\sqrt{N}} \sum_i \sigma_i} \pi\right]. \tag{43}$$

Introducing first a sum over σ_{N+1} at the numerator and at the denominator, (which is the same as multiply and divide for 2^N because there is still no dependence to σ_{N+1}) and making the transformation $\sigma_i \to \sigma_i \sigma_{N+1}$, the variable σ_{N+1} appears at the numerator and it is possible to build the status at $N+1$ particles with the little temperature shift which vanishes in the thermodynamic limit:

$$\omega_{N,t=\beta}(\pi) = \omega_{N+1}(\pi \sigma_{N+1}^r) + O\left(\frac{1}{N}\right). \tag{44}$$

□

Using this lemma we are able to proof the following

Theorem 4 *Let $\langle M \rangle_t$ be a fillable monomial of the magnetization*, (*this means that $\langle mM \rangle$ is filled*). *We have*:

$$\lim_{N\to\infty} \lim_{t\to\beta} \langle M \rangle_t = \langle mM \rangle. \tag{45}$$

Proof The proof is a straightforward application of Lemma 1. □

3.4 Self-Consistency of the Order Parameter via its Streaming

Usually it is much simpler to evaluate the internal energy than the free energy because there is no contribution by the entropy, which, especially in complex system, can make things much harder; consequently if we learn how to extrapolate information from the cavity function, which is deeply related to the entropy, we can obtain information for the free energy. To fulfill this task we state the following theorem.

Theorem 5 *When a generic well defined function of the spins $F(\sigma)$ is considered*, *the following streaming equation holds*:

$$\frac{\partial \langle F_N(\sigma) \rangle_t}{\partial t} = \langle F_N(\sigma) m_N \rangle_t - \langle F_N(\sigma) \rangle_t \langle m_N \rangle_t. \tag{46}$$

Proof The proof is straightforward and can be obtained by simple derivation:

$$\begin{aligned}
\frac{\partial \langle F_N(\sigma) \rangle_t}{\partial t} &= \partial_t \frac{\sum_\sigma F_N(\sigma) e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}} \\
&= \left(\frac{\sum_\sigma F_N(\sigma) \frac{1}{N} \sum_{1<i<N} \sigma_i e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)}} \right) \\
&\quad - \left(\frac{\sum_\sigma F_N(\sigma) e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)}} \right) \\
&\quad \times \left(\frac{\sum_\sigma \frac{1}{N} \sum_{1<i<N} \sigma_i e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)}} \right) \\
&= \langle F_N(\sigma) m_N \rangle_t - \langle F_N(\sigma) \rangle_t \langle m_N \rangle_t.
\end{aligned}$$

□

We now want to expand the cavity function via filled monomials of the magnetization by applying the streaming equation (46) directly to its derivative, thanks to (30). It is immediate to find that the streaming of $\langle m_N \rangle_t$ obeys the following differential equation

$$\partial_t \langle m_N \rangle_t = \langle m_N^2 \rangle_t - \langle m_N \rangle_t^2 \tag{47}$$

which, thanks to Theorem 4, becomes trivial in the thermodynamic limit. In fact, calling $m = \lim_{N\to\infty} m_N$ and skipping the subscript t on $\lim_{N\to\infty} \langle m_N^2 \rangle_t = \langle m^2 \rangle$ we obtain

$$\frac{1}{\langle m^2 \rangle} \partial_t \langle m \rangle_t = 1 - \left(\frac{\langle m \rangle_t^2}{\langle m^2 \rangle} \right)$$

which is easily solved by splitting the variables and the solution is

$$\langle m \rangle_t = \sqrt{\langle m^2 \rangle} \tanh(\sqrt{\langle m^2 \rangle} t). \tag{48}$$

Once evaluated (48) by using the gauge at $t = \beta$ (i.e. $\langle m \rangle_{t=\beta} = \langle m^2 \rangle$) we get

$$\sqrt{\langle m^2 \rangle} = \tanh(\beta \sqrt{\langle m^2 \rangle}) \tag{49}$$

which is the well known self-consistency equation for the Ising-model.

3.5 The Free Energy Expansion

From (48) it is possible to obtain an explicit expression for the cavity function to plug into (32) solving for the free energy. In fact we have

$$\lim_{N\to\infty} \Psi_N(\beta, t) = \lim_{N\to\infty} \int_0^t dt' \langle m_N \rangle_{t'} = \int_0^t dt' \sqrt{\langle m^2 \rangle} \tanh(\sqrt{\langle m^2 \rangle} t') \tag{50}$$

from which is immediate to solve for $\Psi(\beta, t)$:

$$\Psi(\beta, t) = \ln\cosh(\sqrt{\langle m^2 \rangle} t). \tag{51}$$

The last term still missing to fulfill the expression of the free energy via (32), which is immediate to obtain, is the internal energy.

Proposition 2 *The internal energy of the Ising model is*

$$\beta \frac{d\alpha_N(\beta)}{d\beta} = \frac{\beta}{2} \langle m_N^2 \rangle. \tag{52}$$

Proof The proof is straightforward and can be obtained by direct calculation on the same line of the previous proofs. □

Pasting all together we have

Proposition 3 *The free energy of the Ising model is*

$$\alpha(\beta) = \ln 2 + \ln\cosh(\beta \sqrt{\langle m^2 \rangle}) - \frac{\beta}{2} (\sqrt{\langle m^2 \rangle})^2. \tag{53}$$

Proof The proof proceeds by making explicit (32). □

4 The Phase Transition

4.1 Breaking Commutativity of Infinite Volume Against Vanishing Perturbation Limits

The motivation of this section can be found, always in the context of spin glasses in [9].

Let us move one step backward and consider (53) at finite N. The receipt to obtain the expression of the free energy via the filled monomial is to perform at first the $N \to \infty$ limit to saturate the fillable term and then the $t \to \beta$ limit to free the measure from the perturbation (making it works as a cavity field). So in other words $\alpha(\beta) = \lim_{t\to\beta} \lim_{N\to\infty} \alpha_N(\beta, t)$. But what if we exchange the limits such that $\alpha^*(\beta) = \lim_{N\to\infty} \lim_{t\to\beta} \alpha_N(\beta, t)$?

Simply, thanks to the gauge invariance $\lim_{N\to\infty} \lim_{t\to\beta} \langle m_N \rangle = 0$ implying $\Psi(\beta, t) = 0$, defining the high temperature expression for $\alpha^*(\beta)$, so

$$\alpha(\beta) = \lim_{t\to\beta} \lim_{N\to\infty} \alpha_N(\beta, t) \neq \lim_{N\to\infty} \lim_{t\to\beta} \alpha_N(\beta, t) = \alpha^*(\beta). \tag{54}$$

Alternatively one can solve (47) for the variable $\langle \xi_N(\sigma) \rangle_t$ by sending first $N \to \infty$ and check that these fluctuations scale accordingly to the paragraph after (5).

Furthermore there is a range in temperature (the paramagnetic phase) in which $\alpha(\beta) = \alpha^*(\beta)$ such that the two limits $\lim_{t\to\beta} \lim_{N\to\infty}$ and $\lim_{N\to\infty} \lim_{t\to\beta}$ do commute. This can be understood as follows: If we consider just the "high temperature region" (i.e. the paramagnetic phase) saturability implies $\langle m^2 \rangle = 0$ (because $\lim_{N\to\infty} \langle m \rangle_t \to \langle m^2 \rangle \in [0, 1]$ such that $\langle m_N^2 \rangle = 0$ or 1 but $\langle m^2(\beta = 0) \rangle = 0$) and the high temperature expression holds. In the range $\beta \in [0, 1]$ the global symmetry of the Hamiltonian $\sigma_i \to \sigma_i \sigma_{N+1}$ is a symmetry of the Boltzmann state too, while in the range $\beta \in]1, \infty]$ the Boltzmann state shares no longer this invariance and ergodicity is lost.

Once understood the existence of a phase transition thanks to the lacking of the commutativity encoded in (54) we dedicate the next section to the finding of the critical point, which defines the onset of ergodicity breaking, together with the control of the system at criticality.

4.2 Critical Behavior: Scaling Laws

Critical exponents are needed to characterize singularities of the theory at the critical point and, for us, this information is encoded in the behavior of the order parameter $\sqrt{\langle m^2 \rangle}$.

Assuming for the moment that $\beta_c = 1$ (where β_c stands for the critical point in temperature), close to criticality, we take the freedom of writing $G(\beta) \sim G_0 \cdot (\beta - 1)^\gamma$, where the symbol $\sim$ has the meaning that the term at the second member is the dominant but there are corrections of order higher than τ^γ.

The standard way to look at the scaling of the order parameter is by expanding the hyperbolic tangent around $\sqrt{\langle m^2 \rangle} \sim 0$ obtaining

$$\sqrt{\langle m^2 \rangle} = \tanh(\beta \sqrt{\langle m^2 \rangle}) \sim \beta \sqrt{\langle m^2 \rangle} - \frac{(\beta \sqrt{\langle m^2 \rangle})^3}{3} \tag{55}$$

by which one gets

$$\sqrt{\langle m^2 \rangle}(1 - \beta) + \frac{1}{3}(\beta(\sqrt{\langle m^2 \rangle})^3) \sim 0. \tag{56}$$

The first solution of (56) is $\sqrt{\langle m^2\rangle}=0$ (which is also the only solution in the ergodic phase) while the other two solutions can be obtained by solving

$$(\sqrt{\langle m^2\rangle})^2 \sim \frac{(\beta-1)^3}{\beta^3} \sim 3\left(1-\frac{1}{\beta}\right) \tag{57}$$

close to the critical point, obtaining

$$\sqrt{\langle m^2\rangle} \sim (\beta-1)^{\frac{1}{2}} \tag{58}$$

which gives as the critical exponent $\gamma = 1/2$.

Within our framework the procedure is by using directly the streaming equation (46), choosing $F(\sigma)=\langle m\rangle_t$, expanding iteratively in filled monomials, obtaining

$$\begin{aligned}\langle m\rangle_t &= \langle m^2\rangle t - \int_0^t \langle m\rangle_{t'}^2 dt' \\ &= \langle m^2\rangle t - \int_0^t dt' \left(\langle m^2\rangle^2 t'^2 - 2\langle m^2\rangle t' \int_0^{t'} dt'' \langle m\rangle_{t''}^2 + \left(\int_0^{t'} dt'' \langle m\rangle_{t''}^2\right)\right) \\ &= \langle m^2\rangle t - \langle m^2\rangle^2 \frac{t^3}{3} + O(\langle m^2\rangle^4),\end{aligned} \tag{59}$$

where higher order terms, close to criticality, can be neglected. Now by applying saturability (Theorem 4) at $t=\beta$ we get

$$\langle m^2\rangle(\beta-1) = \langle m^2\rangle^2 \frac{\beta^3}{3} + O(\langle m^2\rangle^4) \tag{60}$$

from which we can derive both the critical point and the scaling exponent: To find the critical point it is enough to rewrite (60) switching to the rescaled order parameter $\xi(\sigma)$, such that, by applying a central limit argument, its fluctuations become

$$\sqrt{\langle \xi(\sigma)^2\rangle} = \frac{\langle \xi(\sigma)^2\rangle}{\sqrt{(\beta-1)}} \frac{\beta^3}{3}$$

which diverge as soon as the denominator approaches zero (i.e. for $\beta \to 1^-$).

Finding the critical exponent happens on the same line by rewriting (60) as

$$\sqrt{\langle m^2\rangle}\sqrt{(\beta-1)} \sim \langle m^2\rangle \frac{\beta^3}{3}$$

and considering, close to criticality, $\beta^3 \sim 1$, which immediately yields

$$\sqrt{\langle m^2\rangle} \sim (\beta-1)^{\frac{1}{2}} \tag{61}$$

according to (58).

Remark 4 Using (59) to work out an expansion of the cavity function we obtain

$$\Psi(t) = \int_0^t dt \langle m\rangle_t = \int_0^t dt \left(\langle m^2\rangle t - \langle m^2\rangle^2 \frac{t^3}{3} + O(\langle m^2\rangle^4)\right) \tag{62}$$

which gives

$$\Psi(t) = \langle m^2 \rangle \frac{t^2}{2} - \langle m^2 \rangle^2 \frac{t^4}{12} + O(\langle m^2 \rangle^4) \tag{63}$$

in perfect agreement with the expansion of the logarithm of the hyperbolic cosine.

Note The same method, respectively applied on the SK and on the Viana-Bray model [45] of diluted spin glass, has been discussed in [2] and [10].

Remark 5 Using the expansion (63) for the free energy expression in (32) we obtain

$$\alpha(\beta) = \ln 2 + \frac{\beta}{2}(\beta - 1)\langle m^2 \rangle - \frac{\beta^4}{12}\langle m^2 \rangle^2 + \cdots \tag{64}$$

by which we argue the critical point must be $\beta_c = 1$. This can be seen as follows: Let us note that $A(\beta) = (\beta/2)(\beta - 1)$ is the coefficient of the second order of the expansion in power of the order parameter (i.e. $\sqrt{\langle m^2 \rangle}$). In the ergodic phase (with preserved symmetry) the minimum of the free energy corresponds to a zero order parameter (i.e. $\sqrt{\langle m^2 \rangle} = 0$). This implies that $A(\beta) \geq 0$. Anyway, immediately below the critical point values of the order parameter different from zero are possible if and only if $A(\beta) \leq 0$ and consequently at the critical point $A(\beta)$ must be zero.

This identifies the critical point $\beta_c = 1$.

Coherently, for the same reason the first order term in the expansion must be identically zero.

Note An identical approach holds also for the SK spin glass model [6].

4.3 Self-Averaging Properties

As a sideline, to try and make the work as close as possible to a guide for more complex models, it is possible to derive the "locking" of the order parameter, which, in other context (i.e. spin glasses) is found as a set of equations called Ghirlanda-Guerra [20] and Aizenman-Contucci [3], while in simpler systems as the one we are analyzing, not surprisingly [16], do coincide with just one kind of self-averaging.

The idea we follow [6–8] is deriving filled monomial with respect to the interpolating parameter, remembering that, in the thermodynamic limit, they do not depend on such a parameter end evaluating the "fillable" result (which do depends on t) at $t = \beta$ to free the measure from the perturbing cavity field.

Proposition 4 *The self-averaging properties, consequence of the invariance of filled monomials with respect the perturbing field, hold in the thermodynamic limit; an example being*

$$0 = \lim_{N \to \infty} \partial_t \langle m_N^2 \rangle = \langle m^3 \rangle_t - \langle m^2 \rangle \langle m \rangle_t = \langle m^4 \rangle - \langle m^2 \rangle^2. \tag{65}$$

Even though we followed the derivation presented in [6] (and deepen in [8] for its dilute variant) to obtain such constraints, for the Ising model it is straightforward to check that the original idea presented in [20] concerning the self-averaging of the internal energy shares the same relation. In fact, defining $\langle E\rangle = \lim_{N\to\infty} E_N$ and $E_N = H_N(\sigma)/N$, by direct evaluation we have

Remark 6 The self-averaging property of the order parameter is a consequence of self-averaging of the internal energy

$$\lim_{N\to\infty}(\langle E_N\rangle^2 - \langle E_N^2\rangle) = 0 \quad \Rightarrow \quad (\langle m^2\rangle^2 - \langle m^4\rangle) = 0.$$

Note In this system without disorder the AC relations and the GG identities do coincide because of the absence of the external average over the noise, which introduce different kinds of self-averaging as discussed for instance in [18].

A less known alternative, richer of surprises, emerges again when investigating the cavity function. Of course in simple system such investigation will not tell us much more than what showed so far, but, remembering we want to show a working method more than the results themselves it offers for this particular system, we want to explore this last variant.

Remembering Theorem 4 and Proposition 3 let us rewrite the free energy according to

$$\alpha(\beta) = \ln 2 + \ln\cosh(\sqrt{\langle m\rangle_t}t)|_{t=\beta} - \frac{\beta}{2}\sqrt{\langle m^2\rangle} \tag{66}$$

and emphasize that the total derivative with respect to β is

$$\frac{d\alpha(\beta)}{d\beta} = \frac{\partial\alpha(\beta)}{\partial\beta} + \frac{\partial\alpha(\beta)}{\partial\sqrt{\langle m^2\rangle}}\frac{\partial\sqrt{\langle m^2\rangle}}{\partial\beta}, \tag{67}$$

while, from the general law of thermodynamics [42], we know the total derivative of the free energy with respect to β is the internal energy

$$\frac{d\alpha(\beta)}{d\beta} = \frac{1}{2}(\sqrt{\langle m^2\rangle})^2. \tag{68}$$

With this preamble let us move evaluating the partial derivative of the free energy still with respect β:

$$\begin{aligned}\frac{\partial\alpha(\beta)}{\partial\beta} &= -\frac{1}{2}(\sqrt{\langle m^2\rangle})^2 + (\sqrt{\langle m\rangle_t}\tanh(\sqrt{\langle m\rangle_t}t))|_{t=\beta} \\ &= -\frac{1}{2}(\sqrt{\langle m^2\rangle})^2 + (\sqrt{\langle m^2\rangle}\tanh(\sqrt{\langle m^2\rangle}\beta)\end{aligned}$$

which thanks to self-consistency for the order parameter (49) becomes

$$-\frac{1}{2}(\sqrt{\langle m^2\rangle})^2 + (\sqrt{\langle m^2\rangle})^2 = \frac{1}{2}(\sqrt{\langle m^2\rangle})^2 \tag{69}$$

hence

$$\frac{\partial\alpha(\beta)}{\partial\sqrt{\langle m^2\rangle}}\frac{\partial\sqrt{\langle m^2\rangle}}{\partial\beta} = 0. \tag{70}$$

Let us split the evaluation of (70) in two terms A,B (such that the equation reduces to $AB=0$) by defining and evaluating

$$A=\frac{\partial\alpha(\beta)}{\partial\sqrt{\langle m^2\rangle}}=\beta(\sqrt{\langle m^2\rangle}-\tanh(\beta\sqrt{\langle m^2\rangle})), \tag{71}$$

$$B=\frac{\partial\sqrt{\langle m^2\rangle}}{\partial\beta}=\frac{N}{4\sqrt{\langle m^2\rangle}}(\sqrt{\langle m^4\rangle}-(\sqrt{\langle m^2\rangle})^2). \tag{72}$$

Putting together the results $AB=0$ we obtain

$$\beta(\sqrt{\langle m^2\rangle}-\tanh(\beta\sqrt{\langle m^2\rangle}))\frac{N}{4\sqrt{\langle m^2\rangle}}(\sqrt{\langle m^4\rangle}-(\sqrt{\langle m^2\rangle})^2)=0. \tag{73}$$

This equation acts as a bound and, thought in terms of the expression (70), has a vague variational taste. As in simple system it does not tell us much more than that the product of self-consistency and self-averaging goes to zero faster than N^{-1}, in complex system has a key role both in defining the locking of the order parameters [6] as in controlling the system at criticality [10]. Furthermore in such equation the two key ingredient for the behavior of the system, i.e. self-consistency and self-averaging, appear together as a whole.

4.4 Hamilton-Jacobi Formalism: Order Parameter Self-Averaging and Response to Field

This section has been adapted from the work [29] where the method, in the framework of spin glasses, were originally developed.

We want to investigate the self-averaging of the magnetization itself. This can be achieved in several ways also within the interpolating techniques. For the sake of completeness we want to show a very elegant technique based on two interpolating parameters.

4.4.1 The Structure of the Hamilton-Jacobi Equation

Let us consider a generalized partition function depending on two parameter t,x (that we are going to think about in terms of *generalized time* and *space*) such that the corresponding free energy can be written as follows

$$\alpha_N(t,x)=\frac{1}{N}\ln Z_N(t,x)=\frac{1}{N}\ln\sum_\sigma e^{\frac{t}{2N}\sum_{1\le i<j\le N}\sigma_i\sigma_j+x\sum_{1\le i\le N}\sigma_i} \tag{74}$$

and let us consider its t and x streaming (with obvious meaning, in the averages, of the subscript $\langle\,\rangle_{t,x}$):

$$\frac{\partial\alpha_N(t,x)}{\partial t}=-\frac{1}{2}\langle m_N^2\rangle_{t,x}, \tag{75}$$

$$\frac{\partial\alpha_N(t,x)}{\partial x}=\langle m_N\rangle_{t,x}. \tag{76}$$

Let us also define a *potential* $V_N(t,x)$ as the variance of the magnetization in these extended averages:

$$V_N(t,x) = \frac{1}{2}(\langle m_N^2\rangle_{t,x} - \langle m_N\rangle_{t,x}^2) \tag{77}$$

and introduce an Hamilton function $S_N(t,x)$ as $S_N(t,x) = -\alpha_N(t,x)$. It is now possible to formulate the next

Proposition 5 *In the generalized space of the interpolants the following Hamilton-Jacobi equation holds*

$$\frac{\partial S_N(t,x)}{\partial t} + \frac{1}{2}\left(\frac{\partial S_N(t,x)}{\partial x}\right)^2 + V_N(t,x) = 0. \tag{78}$$

The plan now is as follows: Let us try and solve at first the free-field solution ($V(t,x)=0$), from which the proper solution of the mean field Ising model (53) will follow and we will argue that $\lim_{N\to\infty}(\langle m_N^2\rangle - \langle m_N\rangle^2) = 0$ (because $V(t,x)=0$).

4.4.2 The Free Field Solution: Self-Averaging

If the t-dependent potential is zero then the energy is a constant of motion such that the *Lagrangian* $\mathcal{L}$, which is trivially $\frac{1}{2}(\frac{\partial S_N(t,x)}{\partial x})^2$, does not depend on t (remember that in this formal bridge with classical mechanics the interpolating parameter t takes the same meaning of time) and the trajectories of motion are the straight lines $x(t) = x_0 + \langle m\rangle t$.

If we denote by a bar the Hamilton function which satisfies the free-field problem, such solution $\bar{S}(t,x)$ can be worked out finding a point in the space of solution plus the integral of the Lagrangian over the time

$$\bar{S}(t,x) = \bar{S}(t_0,x_0) + \int dt' \mathcal{L}(t',x). \tag{79}$$

Anyway, as we already stressed, the Lagrangian, in the free-field problem does not depend on time and the integral inside (79) turns out to be a simple product, furthermore, as initial point (t_0,x_0) in the plane (t,x) we choose a generic x_0 but $t_0=0$ as this choice enable us to neglect the two body interaction in the partition function and the problem becomes straightforward.

So we have

$$\frac{\partial \bar{S}_N(t,x)}{\partial t} + \frac{1}{2}\left(\frac{\partial \bar{S}_N(t,x)}{\partial x}\right)^2 = 0 \tag{80}$$

on the trajectories $x = x_0 + \langle m\rangle t$. To enforce now the generalized partition function defined in (74) to be the true one of statistical mechanics, remembering that $S(t,x) = -\alpha(t,x)$ and so $\bar{S}(t,x) = -\bar{\alpha}(t,x)$, we must evaluate the solution at $t=\beta, x=0$. The solution is immediate and is

$$\bar{S}(t,x) = \bar{S}(0,x_0) + \int dt\mathcal{L}(t,x) = -\ln 2 - \ln\cosh(\langle m\rangle t) + \frac{t}{2}\langle m_N^2\rangle, \tag{81}$$

$$\bar{\alpha}(\beta) = \ln 2 + \ln\cosh(\beta\langle m\rangle) - \frac{\beta}{2}\langle m^2\rangle \tag{82}$$

which coincides with the solution of the model (53) assuming that

$$\lim_{N\to\infty}\sqrt{\langle m_N^2\rangle}=\langle m\rangle \tag{83}$$

which is perfect agreement to our request $V(t,x)=0$.

4.4.3 Response to a Field

We understood that, due to the global gauge symmetry, we can think at the cavity field both as an added spin of the system as well as an external perturbation. Once considered the cavity field $x\sum_i^N\sigma_i$ as a perturbation it may be interesting asking what the associated observable is for such a field. It is immediately to check that the observable is the magnetization.

$$\partial_x\frac{1}{N}\ln\sum_\sigma e^{-tH_N(\sigma)+x\sum_i^N\sigma_i}|_{t=\beta,x=0}=\langle m_N\rangle_{t=\beta,x=0}=\langle m_N\rangle. \tag{84}$$

While it may still look unnecessary for the Ising model we stress that the cavity field naturally puts in evidence the symmetry of the perturbing field needed to have a projector (a proper "active" selector in the free energy landscape). In fact, it is immediate to think at the perturbing field as a magnetic field of strength x/β in some proper units. In complex systems, as spin glasses, understanding the right coupling field it is not immediate and this property can be of precious help as discussed in [9].

5 Parisi-Like Representation

As a final section, following the early ideas of Guerra [30], we try and introduce a formalism close to the Parisi scheme for spin glasses. This trial is of course not necessary for the mean field Ising model, but the existence of this possibility acts as a bridge to a better understanding of the Parisi theory itself.

Using the replica trick Parisi expressed the free energy of the Sherrington-Kirkpatrick model by relating it to a trial function $f(q,y(q))$ which obeys a nonlinear PDE and has a nontrivial dependence on a peculiar order parameter $[0,1]\ni q\to x\in[-1,1]$. We propose hereafter the same scheme with an order parameter $[-1,+1]\ni\tilde{m}\to x\in[-1,1]$; for the sake of clearness our function $f(q,y(q))$ will be written as $f(\tilde{m},y(\tilde{m}))$, $\tilde{m}$ being the trial magnetization (now depending on x) and $y=(\tilde{m}\sum_i^N\sigma_i/N)$.

We are now ready to state the main proposition of this section

Proposition 6 *The free energy of the Ising model can be expressed via an order parameter* $[-1,1]\ni\tilde{m}\to x\in[-1,1]$ *and a function* $f(\tilde{m},y(\tilde{m}))$ *as follows*

$$\alpha(\beta)=\min_{x(\tilde{m})}\left(\ln 2-\frac{\beta}{2}\int_0^{\tilde{m}}x(\tilde{m}')d\tilde{m}'+f(0,0)\right) \tag{85}$$

where the function $f(\tilde{m},y(\tilde{m}))$ *obeys the following Cauchy problem*

$$\begin{cases}\partial_{\tilde{m}}f(\tilde{m},y)+x(\tilde{m})|\partial_y f(\tilde{m},y)|=0,\\ f(1,y)=\ln\cosh(\beta y).\end{cases} \tag{86}$$

Remark 7 The above equation immediately reveals a big difference between the Ising model and the SK: linearity. In fact the Parisi equation for the spin glasses [38] is a non linear PDE and shows several bifurcation points, while, in the problem (86), once chosen a branch for $\tilde{m}$, the evolution is unique.

Proof The simplest way to obtain a control of the streaming of $f(\tilde{m}, y(\tilde{m}))$ with respect to both its variables is by introducing an auxiliary function as

$$\Phi(\tilde{m}) = \ln\langle e^{f(\tilde{m}, \frac{\tilde{m}}{N}\sum_i^N \sigma_i)}\rangle \tag{87}$$

such that

$$\Phi(1) = \ln\langle e^{f(1, \frac{1}{N}\sum_i \sigma_i)}\rangle = \Psi(t = \beta), \tag{88}$$

as can be explicitly checked for every even Boltzmann state, and that

$$\Phi(0) = f(0, y(0) = 0). \tag{89}$$

If we can state that $f(0,0) = \Psi(t = \beta)$ the result is obtained: To impose the bridge $f(0,0) = \Phi(0) = \Phi(1) = \Psi(t = \beta)$ it is sufficient (but not necessary) to derive Φ with respect to $\tilde{m}$ and constrain the derivative to be zero. For the sake of convenience, let us introduce

$$\langle a\rangle_f = \frac{\langle ae^f\rangle}{\langle e^f\rangle}$$

so to write

$$\frac{d\Phi}{d\tilde{m}} = \langle\partial_{\tilde{m}} f\rangle_f + \frac{1}{N}\sum_i^N \langle\sigma_i \partial_y f\rangle_f \tag{90}$$

and let us consider the following bounds

$$\frac{1}{N}\sum_i^N \langle\sigma_i \partial_y f\rangle_f \le \left|\frac{1}{N}\sum_i^N \langle\sigma_i \partial_y f\rangle_f\right| \le \frac{1}{N}\sum_i^N |\langle\sigma_i \partial_y f\rangle_f|$$
$$\le \frac{1}{N}\sum_i^N \langle|\partial_y f|\rangle_f = |\partial_y f|.$$

The introduction of the function modulus allows one to use a function $[-1,1] \ni \tilde{m} \to x \in [-1,1]$ such that

$$\frac{1}{N}\sum_i^N \langle\sigma_i \partial_y f\rangle_f = x(\tilde{m})|\partial_y f|. \tag{91}$$

□

Remark 8 The existence of the function *modulus* inside the r.h.s. of (91) allows one to take into account just one branch at time with complete symmetry between the two branches. This reflects the properties of the magnetization in the broken ergodicity phase.

By substituting the expression of (91) into (90) we obtain the Parisi-like equation for ferromagnetism (86).

To start solving (86) let us make the following change of variable

$$p = -\int_{\tilde{m}}^{1} d\tilde{m}' x(\tilde{m}') \tag{92}$$

by which, thanks to the Jacobian of the transformation (92), the Parisi-like equation for the Ising model turns out to be solvable with the D'Alambert technique. In fact we get

$$\partial_p f(p, y) + \partial_y f(p, y) = 0$$

solved by $f(p, y) = \ln\cosh(t(p + y)) \to f(q, y) = \ln\cosh(t(y \pm \int_q^1 dq' x(q')))$, where the $\pm$ signs are chosen accordingly to the branch of the chosen derivative of f with respect to y.

Solving for the $\Psi(t = \beta)$ we get

$$\Psi(t = \beta) = \ln\cosh\left(\beta \int_0^1 d\tilde{m}' x(\tilde{m}')\right) \tag{93}$$

which shows a deep connection among Parisi theory and our formulation (see Theorem 2).

Let us now equate (51) with (93): We immediately obtain

$$\sqrt{\langle m^2 \rangle} = \langle m \rangle = \int_0^1 d\tilde{m} x(\tilde{m}) \tag{94}$$

by which we argue that the function $x(\tilde{m})$ has the meaning of a probability density for the order parameter (i.e. the magnetization) and the solution for the free energy follows straightforwardly.

Further one could go beyond this scheme, but this will not be discussed here, working out the equivalent of the broken replica bound to make sharper statements concerning the $x(\tilde{m})$ following [26].

Remark 9 Another possibility is by exploring the *replica trick* method [38] assigning a delta-like probability distribution for the interaction matrix J_{ij} (i.e. $P(J_{ij}) \sim \delta(J_{ij} - 1)$) which factorizes replicas and no ansatz is required in this simple case.

6 Conclusion

In this paper we have studied the mean field Ising model with the interpolating techniques. These methods, which have been at the basis of a recent breakthrough in spin glass theory turn out to be of great generality, as this test on simpler model demonstrated. Several techniques, linked one to another by the interpolation method, have been shown throughout the paper: key ingredients for the free energy thermodynamic limit are the sub-additivity and the bounds in the volume size. Another central role is played by the gauge invariance when analyzing the expression of the free energy itself: via this symmetry the cavity field becomes a perturbing external field (what is called *stochastic stability* in spin glass literature) and vice versa and the synergy between the two approaches enables one to work out several properties of the model as the critical behavior and the self-averaging relations. The technique with two interpolating parameters has also been discussed: a suitable streaming of a generalized free energy with respect to properly defined parameters can bring to the formulation of an Hamilton-Jacobi equation in the interpolation space by which again the solution of the model and the self-averaging can be deduced. At the end a formulation of the theory in terms of Parisi representation is tried, with particular emphasis on the meaning of the order parameter.

As a last remark we stress that this work has been written with the aim of developing a simpler but dense exercise of statistical mechanics to make these techniques ready to be used by the reader not familiar with the field of spin-glasses.

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References

1. Amit, D.J.: Modeling Brain Function: The World of Attractor Neural Network. Cambridge University Press, Cambridge (1992)
2. Agostini, A., Barra, A., De Sanctis, L.: Positive-overlap transition and critical exponents in mean field spin glasses. J. Stat. Mech. P11015 (2006)
3. Aizenman, M., Contucci, P.: On the stability of the quenched state in mean field spin glass models. J. Stat. Phys. **92**, 765–783 (1998)
4. Aizenman, M., Sims, R., Starr, S.L.: An extended variational principle for the SK spin-glass model. Phys. Rev. B **68**, 214403 (2003)
5. Albert, R., Barabasi, A.-L.: Statistical mechanics of complex networks. Rev. Mod. Phys. **74**, 47 (2002)
6. Barra, A.: Irreducible free energy expansion and overlap locking in mean field spin glasses. J. Stat. Phys. **123**, 601–614 (2006)
7. Barra, A., De Sanctis, L.: Overlap fluctuation from Boltzmann random overlap structure. J. Math. Phys. **47**, 103305 (2006)
8. Barra, A., De Sanctis, L.: Stability properties and probability distributions of multi-overlaps in diluted spin glasses. J. Stat. Mech. P08025 (2007)
9. Barra, A., De Sanctis, L.: Spin-glass transition as the lacking of the volume limit commutativity (2007, to appear)
10. Barra, A., De Sanctis, L., Folli, V.: Critical behavior of random spin systems. J. Phys. A **41**(21), 215005 (2008)
11. Bovier, A., Kurkova, I.: Rigorous results on some simple spin glass models. Markov Proc. Relat. Fields **9**, (2003)
12. Bovier, A., Kurkova, I., Loewe, M.: Fluctuations of the free energy in the REM and the p-spin SK model. Ann. Probab. **30** (2002)
13. Comets, F., Neveu, J.: The Sherrington-Kirkpatrick model of spin glasses and stochastic calculus: the high temperature case. Commun. Math. Phys. **166**, 549 (1995)
14. Coolen, A.C.C.: The trick which became a theory: a brief history of the replica method. Available at http://www.mth.kcl.ac.uk/~tcoolen/
15. Contucci, P., Ghirlanda, S.: Modeling society with statistical mechanics: an application to cultural contact and immigration. Qual. Quantit. **41**, 569–578 (2007)
16. Contucci, P., Giardinà, C.: Spin-glass stochastic stability: a rigorous proof. math-ph/0408002
17. De Sanctis, L.: General structures for spherical and other mean-field spin models. J. Stat. Phys. **126**
18. De Sanctis, L., Franz, S.: Self averaging identities for random spin systems. math-ph/0705:2978
19. Ellis, R.S.: Large Deviations and Statistical Mechanics. Springer, New York (1985)
20. Ghirlanda, S., Guerra, F.: General properties of overlap distributions in disordered spin systems. Towards Parisi ultrametricity. J. Phys. A **31**, 9149–9155 (1998)
21. Fischer, K.H., Hertz, J.A.: Spin Glasses. Cambridge University Press, Cambridge (1991)
22. Gallo, I., Contucci, P.: Bipartite mean field spin system: existence and solution. cond-mat/0710.0800
23. Guerra, F.: Mathematical aspects of mean field spin glass theory. cond-mat/0410435
24. Guerra, F.: About the cavity fields in mean field spin glass models. cond-mat/0307673
25. Guerra, F.: Fluctuations and thermodynamic variables in mean field spin glass models. In: Albeverio, S., et al. (eds.) Stochastic Provesses, Physics and Geometry, II. Singapore (1995)
26. Guerra, F.: Broken replica symmetry bounds in the mean field spin glass model. Commun. Math. Phys. **233:1**, 1–12 (2003)

27. Guerra, F.: About the overlap distribution in mean field spin glass models. Int. J. Mod. Phys. B **10**, 1675–1684 (1996)
28. Guerra, F., Albeverio, S. et al.: The cavity method in the mean field spin glass model. Functional representations of thermodynamic variables. In: Albeverio, S., et al. (eds.) Advances in Dynamical Systems and Quantum Physics. Singapore (1995)
29. Guerra, F.: Sum rules for the free energy in the mean field spin glass model. In: Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects. Fields Institute Communications, vol. 30. American Mathematical Society, Providence (2001)
30. Guerra, F.: Private communications
31. Guerra, F.: An introduction to mean field spin glass theory: methods and results. In: Lecture at Les Houches Winter School (2005)
32. Guerra, F., Toninelli, F.L.: The thermodynamic limit in mean field spin glass models. Commun. Math. Phys. **230**(1), 71–79 (2002)
33. Guerra, F., Toninelli, F.L.: The high temperature region of the Viana-Bray diluted spin glass model. J. Stat. Phys. 115 (2004)
34. Guerra, F., Toninelli, F.L.: Central limit theorem for fluctuations in the high temperature region of the Sherrington-Kirkpatrick spin glass model. J. Math. Phys. **43**, 6224–6237 (2002)
35. Guerra, F., Toninelli, F.L.: The infinite volume limit in generalized mean field disordered models. Markov Process. Relat. Fields **9**(2), 195–207 (2003)
36. Kuttner, J.: Some theorems on the Cesaro limit of a function. Lond. Math. Soc. **s1-33**, 107–118 (1958)
37. Mertens, S., Mezard, M., Zecchina, R.: Threshold values of random K-SAT from the cavity method. Random Struct. Algorithms **28**, 340–373 (2006)
38. Mézard, M., Parisi, G., Virasoro, M.A.: Spin Glass Theory and Beyond. World Scientific, Singapore (1987)
39. Mézard, M., Parisi, G., Sourlas, N., Toulouse, G., Virasoro, M.A.: Replica symmetry breaking and ultrametricity. J. Phys. **45**, 843 (1984)
40. Pagnani, A., Parisi, G., Ricci-Tersenghi, F.: Glassy transition in a disordered model for the RNA secondary structure. Phys. Rev. Lett. **84**, 2026 (2000)
41. Parisi, G.: Stochastic stability. In: Proceedings of the Conference Disordered and Complex Systems, London (2000)
42. Parisi, G.: Statistical Field Theory. Addison-Wesley, New York (1988)
43. Talagrand, M.: Spin Glasses: A Challenge for Mathematicians. Cavity and Mean Field Models. Springer, Berlin (2003)
44. Talagrand, M.: The Parisi formula. Ann. Math. **163**(1), 221–263 (2006)
45. Viana, L., Bray, A.J.: Phase diagrams for dilute spin-glasses. J. Phys. C **18**, 3037 (1985)