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THE MEAN-GINI EFFICIENT PORTFOLIO FRONTIER

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Abstract

A main advantage of the mean-variance (MV) portfolio frontier is its simplicity and ease of derivation. A major shortcoming, however, lies in its familiar restrictions, such as the quadraticity of preferences or the normality of distributions. As a workable alternative to MV, we present the mean-Gini (MG) efficient portfolio frontier. Using an optimization algorithm, we compute MG and mean-extended Gini (MEG) efficient frontiers and compare the results with the MV frontier. MEG allows for the explicit introduction of risk aversion in building the efficient frontier. For U.S. classes of assets, MG and MEG efficient portfolios constructed using Ibbotson (2000) monthly returns appear to be more diversified than MV portfolios. When short sales are allowed, distinct investor risk aversions lead to different patterns of portfolio diversification, a result that is less obvious when short sales are foreclosed. Furthermore, we derive analytically the MG efficient portfolio frontier by restricting asset distributions. The MG frontier derivation is identical in structure to that of the MV efficient frontier derivation. The penalty paid for simplifying the search for the MG efficient frontier is the loss of some information about the distribution of assets.

JEL Classification: G11

I. Introduction

Since its development by Markowitz (1952), the mean-variance (MV) model for portfolio selection has become the standard tool by which risky financial assets are allocated. MV has gained a prominent place in finance because of its conceptual simplicity and ease of computation. Many authors, however, challenge the model's premises, primarily, the normality of asset return probability distributions or the quadraticity of preferences. On the other hand, Kroll, Levy, and Markowitz (1984) reassert MV validity by showing that MV faithfully approximates expected utility.

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The challenge to the validity of MV has led researchers to seek alternative solutions to efficient portfolio selection, resulting in approaches such as the three-moments, lower partial moments, semi-variance, value-at-risk, stochastic dominance, and mean-Gini (MG) models, to name only a few. Still, no other model has managed to attain the popularity of MV with practitioners, owing to the lack of intuitive reasoning and the complex computation of alternative models.

Our purpose is twofold. First, we present the MG and mean-extended Gini (MEG) portfolio models as workable alternatives to MV. We numerically compute the MG and the MEG efficient frontiers and compare the results with the MV frontier. Second, we analytically derive the MG efficient frontier in a way that is similar to MV so it can be used as easily as MV. Deriving MG portfolios is complex mainly because of the additional information the Gini statistic infers on properties of the distributions. If one is ready to forgo this additional information, finding and interpreting MG portfolios can be as simple as constructing and analyzing MV portfolios.

The MG approach in finance has proven to be a powerful alternative to MV modeling. By supplying necessary conditions for stochastic dominance, MG has been shown to be compatible with expected utility maximization. Hence, MG analysis provides a consistent alternative to MV modeling whenever investment returns are not normally distributed or when the investor's utility is not quadratic.

Shalit and Yitzhaki (1984) develop the MG model of portfolio analysis and derive the MG equilibrium pricing of risky assets. Since then, however, no attempt has been made to analytically derive the MG efficient frontier because the Gini is generally more complex to calculate in a portfolio framework.

There is some interest in providing alternatives to the MV efficient frontier that are tailored to specific investor needs. One idea is to include risk-aversion considerations in building an individual portfolio. This approach is the basis for models that use Bayesian processes to include investor views in updating the prior distribution of returns. The MEG offers a simple way to include risk aversion in the construction of an efficient portfolio by providing an infinite number of variability measures that depend on one parameter. By varying the MEG parameter, the investigator modifies the risk aversion and offers an efficient frontier that suits the risk preference to the investor. The efficient frontier can then supply the price attached to a unit of risk, given a level of risk aversion.

We use the monthly returns of six U.S. asset classes (small- and large-company stocks, corporate bonds, intermediate and long-term government bonds, and bills) over the last 75 years to compute the MG, MEG, and MV portfolio frontiers. We show that MV efficient portfolios tend to be less diversified than MG and MEG portfolios. Furthermore, the results for MEG portfolios show that for a given required return the proportion of the two stock classes in the efficient portfolios declines with the degree of risk aversion as exhibited by the MEG parameter. It is surprising that relying on variance to construct efficient portfolios entails a lower level of risk aversion than would be implied by using the Gini. The intuitive explanation for this result is that the portfolio distribution tends to have fat upper and lower tails. Because the variance is sensitive to both extremes, it is affected by the upper and lower tails. The effect of the upper tail is to mitigate the effect of the lower tail, leading to the unexpected result that the variance demonstrates low risk aversion (Shalit and Yitzhaki 2002).

II. The MG Efficient Frontier

In the MG model, investors use the portfolio's Gini as the measure of risk to be minimized, subject to a given mean return. The Gini is Gini's mean difference (GMD), which is defined as half of the expected absolute difference between the returns of two randomly drawn amounts invested in a portfolio. The Gini can also be defined as the covariance between the return and its probability distribution:

$$\Gamma = 2\,\mathrm{cov}[r, F(r)], \tag{1}$$

where r is the return, Γ is the Gini, and $F(r)$ is the cumulative distribution function (CDF). The advantage of the Gini over the variance as a measure of risk is rooted in the necessary and sufficient conditions for second-degree stochastic dominance (SSD) in the following way (Yitzhaki 1982a): Consider two portfolios (1 and 2) yielding returns r_1 and r_2, means μ_1 and μ_2, and Ginis Γ_1 and Γ_2. Then, $\mu_1 \geq \mu_2$ and $\mu_1 - \Gamma_1 \geq \mu_2 - \Gamma_2$ are necessary conditions whereby no risk-averse expected utility maximizer will prefer portfolio 2 to portfolio 1. If one restricts the distributions of the portfolios to the family of cumulative distributions that intersect at most once, these conditions are also sufficient. Therefore, MG ranks consistently risky alternatives whenever MV might fail (Shalit and Yitzhaki 1984).

The implication of this result is that the efficient set of MG is included in the efficient set of risk-averse investors, so that every efficient MG portfolio maximizes the expected value of a utility function. This result does not hold for the MV efficient set.

To see this, consider the choice between two portfolios. The first offers a return between 0 and \$1, and the second offers returns between \$1 million and \$2 million. Both portfolios are included in the efficient MV set because the first offers a lower variance and the second offers a higher expected return. Thus, if one relies only on the mean and variance, one may end up choosing the portfolio that every risk-averse investor would reject. The necessary conditions for stochastic dominance prevent MG users from making this mistake.

Consider a portfolio p whose returns r_p are obtained by $r_p = \mathbf{r}'\mathbf{w}$ where $\mathbf{r}$ is a vector of asset returns, and $\mathbf{w}$ is a vector of portfolio weights. Then, the Gini is written as:

$$\Gamma_p = 2\,\mathrm{cov}(r_p, F_p) = 2\sum w_i \mathrm{cov}(r_i, F_p) = 2\mathbf{w}'\mathbf{K}(\mathbf{r}, F_p), \tag{2}$$

where $\mathbf{K}(\mathbf{r}, F_p)$ is a vector of covariances of assets returns with the CDF of the portfolio. We can now obtain the MG-efficient frontier by solving the following optimization problem:

$$\begin{aligned} &Min\ 2\mathbf{w}'\mathbf{K}(\mathbf{r}, F_p) \\ &s.t.\ \mu_p = \mathbf{w}'\boldsymbol{\mu} \\ &\quad 1 = \mathbf{w}'\mathbf{l} \\ &\quad \mathbf{w} \geq 0, \end{aligned} \tag{3}$$

where $\mathbf{l}$ is a vector of ones and $\boldsymbol{\mu}$ is a vector of assets mean returns. Problem (3), although similar in structure to the MV optimization problem, is much more complicated than the MV problem because the cumulative distribution is not a simple function of assets distribution functions.

MG analysis can be extended to include the investor's preference toward risk. This is done by introducing the extended Gini as a measure of risk (Yitzhaki 1983; Shalit and Yitzhaki 1984) that attaches higher weights to the lower portions of the return probability distribution. This implies that higher risk aversion attributes more weight to the lower payoff realizations than does low risk aversion.

The extended Gini is defined much like the definition in equation (1):

$$\Gamma(\nu) = -\nu\,\mathrm{cov}\{r, [1 - F(r)]^{\nu-1}\}, \tag{4}$$

where ν is a parameter determining the relative weight attributed to various portions of the probability distribution. The parameter ν ranges from 1 to infinity, with $\nu \to 1$ implying variability as viewed by a risk-neutral investor. For $\nu = 2$ we obtain the standard Gini risk aversion, and for $\nu \to \infty$ we allow for the max-min investor who wants to avoid the worst possible outcome.

The utility function that is implied by using the extended Gini can be viewed as a special case of the utility functions suggested by Yaari's (1987) dual theory of risk aversion that distinguishes the notion of declining marginal utility of income from behavior under risk. Where ν is a positive integer, the link of ν to risk aversion can be shown as follows. The extended Gini equals the mean return minus the expected least outcome from ν independent random draws from the return distribution:

$$\Gamma(\nu) = \mu - E[Min(r_1, \ldots, r_\nu)]. \tag{5}$$

Equation (5) can then be used to develop additional necessary conditions for stochastic dominance. In particular, comparing $\mu - \Gamma(\nu)$ of a risky portfolio with the return on a safe portfolio enables one to view $\mu - \Gamma(\nu)$ as the certainty equivalent of the portfolio. With a higher ν, one assigns higher odds of obtaining bad outcomes. Hence, the certainty equivalent of the portfolio is lowered, which raises the risk premium required by the investor.

This interpretation of certainty equivalence relates risk aversion to the discounting of probabilities of good outcomes, which does not originate from assuming declining marginal utility of income, as in expected utility theory. Rather, the higher the risk aversion, the more the investor tends to amplify the probability of bad events and to discount the probability of good events.

The extended Gini of a portfolio is defined similarly to the Gini shown in equation (2):

$$\Gamma_p(\nu) = -\nu \sum_{i=1}^{n} w_i \text{cov}\{r_i, [1 - F_p(r_p)]^{\nu-1}\}, \tag{6}$$

so that the optimization problem becomes:

$$\begin{aligned} &Min\ \Gamma_p(\nu) \\ s.t.\quad &\mu_p = \mathbf{w}'\boldsymbol{\mu} \\ &1 = \mathbf{w}'\mathbf{l} \\ &\mathbf{w} \geq 0. \end{aligned} \tag{7}$$

In applications, the empirical cumulative distribution function is used as an estimator for the cumulative distribution. It is obtained by ranking the returns of the portfolio in increasing order and dividing the rank of each observation by the number of observations. Because a ranking procedure is invoked each time the portfolio Gini (or extended Gini) is calculated, nonlinear programming techniques should be used with caution.[1] In the following application, we use an algorithm that does not require derivatives to find the MG frontier and we ignore the operational and computational efficiency of solving the problem.[2]

[1]When applied to empirical data, the problem is one of a piecewise linear optimization. See Okunev (1991) for a linear programming solution.

[2]The algorithm is based on the variable metric method and is described in Yitzhaki (1982b).

TABLE 1. Nominal Monthly Returns, January 1926 to August 2001.

Assets	Mean (%)	Std. Dev. (%)	Gini (%)
Large-company stocks	1.0107	5.6361	2.8475
Small-company stocks	1.3422	8.6259	4.2114
Long-term corporate bonds	0.4893	1.9563	0.9797
Long-term government bonds	0.4586	2.2149	1.1398
Intermediate government bonds	0.4427	1.2540	0.6244
U.S. Treasury bills	0.3132	0.2581	0.1417

III. Comparing the MG and the MV Efficient Frontiers for U.S. Assets

We construct the efficient frontier for U.S. assets using Ibbotson's aggregate data on stocks, bonds, and bills. The data consist of 908 monthly nominal returns from January 1926 through August 2001 for six classes of assets: large-company stocks (LCS), small-company stocks (SCS), long-term corporate bonds (LCB), long-term government bonds (LGB), intermediate-term government bonds (IGB), and U.S. Treasury bills (TB). The basic statistics are given in Table 1.

As can be seen, ranking the assets according to standard deviations yields the same result as ranking by the Gini. This means that differences between assets consist mainly of magnitudes attached to risk and of correlations among the assets.[3]

The optimizations are conducted using an algorithm with a variable metric method that does not require specifying derivatives and hence is easy to implement (Yitzhaki 1982b). Its main advantage is that it constructs the estimate of the Hessian by previous changes in the gradient and therefore is not sensitive to piecewise linearity of the target function.[4]

Table 2 displays the efficient portfolios as a function of the required return when short sales are allowed. These portfolios are simpler to interpret because the non-negativity constraints are not applied.

Panel A of Table 2 shows the share of MV efficient portfolios. Note that the higher the required return, the higher is the share of LCS, SCS, LCB, and IGB, and the lower is the share of LGB and TB. That is, the higher the required portfolio return, the greater are the short sales of LGB and TB used to increase the shares of other assets. This expected result indicates that to increase the return of the portfolio, the investor needs to borrow by short selling.

Panel B of Table 2 presents the MG portfolios. The same pattern of increasing the shares of LCS, SCS, LCB, and IGB is evident, but the rate of increase

[3]Note that LCB dominates LGB because of its higher mean return and lower variability. In a portfolio context, however, the statistics that matter are the covariances between assets.

[4]We do not claim this is the best algorithm available for solving the problem, as we did not try to save computer time.

TABLE 2. Mean-Variance, Mean-Gini, and Mean-Extended Gini Efficient Frontiers with Short Sales Allowed (in Percentages).

Panel A. Mean-Variance

Mean Return	Std. Dev.	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.26	0.41	0.29	1.89	−1.83	1.87	97.37
0.44	0.29	5.50	3.00	11.36	−27.32	61.24	46.23
0.52	1.24	8.77	4.74	17.44	−43.69	99.35	13.40
0.69	2.25	16.05	8.61	30.99	−80.15	184.27	−59.75
0.84	3.15	22.49	12.03	42.97	−112.43	259.42	−124.49
0.94	3.74	26.66	14.25	50.74	−133.32	308.08	−166.41
1.02	4.21	30.05	16.05	57.03	−150.27	347.55	−200.40
1.10	4.68	33.42	17.84	63.31	−167.18	386.92	−234.32
1.25	5.59	39.87	21.27	75.31	−199.49	462.13	−299.11

Panel B. Mean-Gini ($\nu = 2$)

Mean Return	Gini ($\nu = 2$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.14	0.11	0.23	1.90	−1.43	1.68	97.51
0.44	0.41	4.40	3.80	12.99	−26.59	57.42	47.95
0.52	0.64	6.88	6.22	19.91	−43.21	94.06	16.14
0.69	1.15	12.10	11.47	35.16	−79.85	175.32	−54.20
0.84	1.61	16.74	16.14	49.00	−112.65	247.49	−116.74
0.94	1.92	19.95	19.33	58.51	−135.22	297.27	−159.85
1.02	2.17	22.42	21.83	65.71	−152.73	336.37	−193.60
1.10	2.41	24.86	24.33	72.84	−170.09	374.97	−226.91
1.25	2.87	29.44	28.92	86.37	−202.40	446.80	−289.13

Panel C. Mean-Extended Gini ($\nu = 4$)

Mean Return	Gini ($\nu = 4$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.24	0.04	0.12	2.71	−1.48	3.30	95.31
0.44	0.74	2.89	4.04	14.42	−25.46	59.88	44.23
0.52	1.17	4.48	7.00	21.29	−41.74	97.74	11.29
0.69	2.14	8.06	13.39	36.63	−77.51	181.02	−61.59
0.84	2.97	11.07	18.80	49.18	−107.62	251.88	−123.30
0.94	3.53	13.10	22.45	57.65	−127.96	299.81	−165.05
1.02	3.98	14.69	25.46	64.46	−144.40	338.77	−198.98
1.10	4.44	16.36	28.41	71.38	−160.88	377.80	−233.06
1.25	5.27	19.38	33.81	84.25	−191.49	449.75	−295.71

Panel D. Mean-Extended Gini ($\nu = 6$)

Mean Return	Gini ($\nu = 6$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.27	0.00	0.06	2.63	−1.07	2.83	95.55
0.44	0.93	2.44	4.17	14.35	−24.76	60.91	42.87
0.52	1.47	3.73	7.31	20.47	−40.39	98.65	10.23
0.69	2.67	6.65	13.94	34.10	−74.06	180.26	−60.88
0.84	3.74	9.18	19.84	46.16	−103.65	252.24	−123.77
0.94	4.47	10.90	23.87	54.27	−123.60	301.04	−166.48
1.02	5.05	12.30	27.01	60.80	−139.60	339.86	−200.39
1.10	5.61	13.71	30.07	66.95	−154.92	377.44	−233.25
1.25	6.69	16.37	35.91	79.04	−184.46	449.51	−296.38

(Continued)

TABLE 2. Continued.

Panel E. Mean-Extended Gini ($\nu = 8$)							
Mean Return	Gini ($\nu = 8$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.29	−0.02	0.05	2.75	−1.1	3.03	95.28
0.44	1.07	2.36	4.25	14.21	−24.49	61.38	42.29
0.52	1.69	3.61	7.44	19.67	−39.46	98.78	9.95
0.69	3.08	6.40	14.26	32.29	−72.06	179.79	−60.68
0.84	4.30	8.80	20.28	43.20	−100.31	250.50	−122.47
0.94	5.16	10.56	24.43	50.82	−119.91	299.46	−165.37
1.02	5.82	11.99	27.60	56.80	−135.29	337.40	−198.51
1.10	6.45	13.35	30.64	62.62	−150.06	373.88	−230.42
1.25	7.70	15.99	36.61	73.64	−178.94	445.94	−293.23

Note: LCS = large-company stocks, SCS = small-company stocks, LCB = long-term corporate bonds, LGB = long-term government bonds, IGB = intermediate government bonds, TB = U.S. Treasury bills.

is mitigated. This means that, for a given required return, the share of LCS tends to be lower in MG than in MV portfolios. On the other hand, the share of SCS is greater in MG than in MV portfolios. The share of LCB is lower in MV than in MG portfolios, whereas IGB tends to have a higher share in MV. When looking at the assets used for short sales, LGB are used less in MG, whereas TB are used more aggressively in MV.

The last three panels of Table 2 present the efficient portfolio results for MEG with $\nu = 4$, MEG with $\nu = 6$, and MEG with $\nu = 8$. The same results observed earlier continue to hold. This means that for a given required mean return, as risk-aversion increases, the holdings of LCS decline and the holdings of SCS increase. This unexpected result indicates that increasing risk aversion, as exhibited by a larger ν, implies higher shares of riskier SCS.

Figure I presents the share of stocks (LCS and SCS) in the portfolios as a function of the required rate of return. As can be seen, the higher the risk aversion, the lower is the share of stocks. Note that the pattern of the MV portfolios implies a lower risk-aversion parameter than the Gini portfolios with $\nu = 2$. This, too, is an unexpected outcome because as variance is a quadratic function, we would expect it to have a higher weight on the extreme observations. As Shalit and Yitzhaki (2002) point out, however, the variance imposes higher weights on both extremes of the distribution, which contradicts the idea that risk aversion is more concerned with the lower returns. We speculate that sensitivity of the variance to higher returns of the distribution causes the portfolios to appear as if they were constructed assuming lower values of risk aversion. This result is unexpected, as we did not anticipate differential risk aversion to cause a clear pattern in efficient portfolios.

Finally, we can see that all of the MEG-efficient portfolios meet the necessary conditions for SSD because as we move to higher required returns, the mean

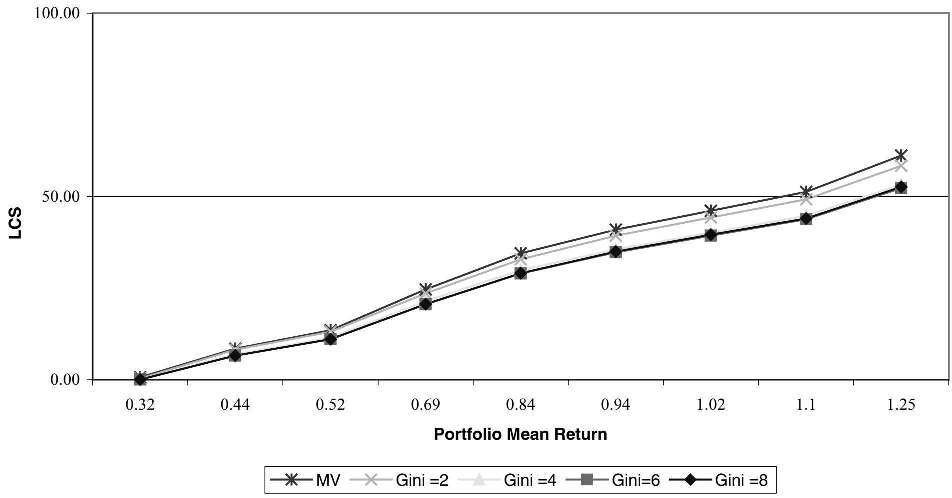

Figure I. Share of Large-Company Stock (LCS) and Small-Company Stock (SCS).

minus the extended Gini declines, implying that no MEG optimal portfolio is found to be SSD inefficient.

Table 3 presents no-short-sales-allowed efficient portfolios obtained under MV, MG, and MEG with νs = 4, 6, and 8. Under MV (first panel), except for the highest required return, the share of LCS is greater than the share of SCS. Also, LGB are not included in any of the efficient portfolios. LCB tend to get the lion's share of all bond holdings whenever the required rate of return is high, whereas IGB play the same role for the lower required return.

The second panel of Table 3 exhibits the MG frontier and portfolio composition for the same selected number of mean returns. In general, it is easy to verify that the MG portfolios tend to be more diversified than the MV efficient portfolios with the same required return. For example, in three cases, there are five assets in the MG efficient portfolio, whereas there are at most four assets in the MV efficient portfolio. Moreover, LGB are included in three portfolios but not in MV portfolios. It is interesting that under MG, LCS have a lower share than under MV in almost all portfolios. The reverse can be said with respect to SCS; they have a greater share under MG than under MV. LCB tend to have a lower share under MG than under MV. The final panels of Table 3 present the efficient frontier and the composition of MEG portfolios with $\nu = 4$, $\nu = 6$, and $\nu = 8$. The tendency for more diversified Gini portfolios continues to hold. Under $\nu = 4$, three portfolios have five participating assets, and under $\nu = 6$, four portfolios have five participating assets. Also, for a given return, the higher the risk aversion, the lower is the share of LCS, which in most cases is compensated for with an increase in the share of

TABLE 3. Mean-Variance, Mean-Gini, and Mean-Extended Gini Efficient Frontiers, No Short Sales Allowed (in Percentages).

Panel A. Mean-Variance

Mean Return	Std. Dev.	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.26	0.31	0.29				98.49
0.44	0.84	5.26	3.96			38.14	52.64
0.52	1.33	8.55	6.39			62.89	22.18
0.69	2.52	18.96	15.30	4.29		61.45	
0.84	3.76	29.29	24.55	21.69		24.47	
0.94	4.67	36.17	30.73	33.10			
1.02	5.38	41.23	37.01	21.75			
1.10	6.12	46.30	43.29	10.41			
1.25	7.57	27.83	72.17				

Panel B. Mean-Gini ($\nu = 2$)

Mean Return	Gini ($\nu = 2$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.14	0.08	0.29	0.26		2.04	97.33
0.44	0.44	4.85	5.76	0.15		34.41	58.82
0.52	0.69	7.84	7.96	0.08		54.22	29.91
0.69	1.26	16.30	17.01	10.27	3.46	52.97	
0.84	1.92	23.66	28.20	16.96	12.52	18.66	
0.94	2.32	31.02	34.00	28.96	0.65	53.72	
1.02	2.67	31.88	43.35	12.23		12.46	
1.09	2.99	35.15	49.32	9.66			
1.25	3.72	28.08	71.95				

Panel C. Mean-Extended Gini ($\nu = 4$)

Mean Return	Gini ($\nu = 4$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.24	0.01	0.12	2.13		1.45	96.23
0.44	0.81	3.03	4.78			47.26	44.94
0.54	1.35	4.67	8.82			77.35	9.16
0.69	2.37	11.60	19.72	9.32		59.36	
0.84	3.54	15.85	33.07	24.13	0.56	26.39	
0.94	4.40	12.12	46.68	17.96		23.26	
1.02	4.99	19.34	50.39	29.75	0.10	0.42	
1.10	5.68	24.77	56.77	18.37	0.10		
1.26	7.00	24.71	74.95	0.30			

Panel D. Mean-Extended Gini ($\nu = 6$)

Mean Return	Gini ($\nu = 6$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.27		0.07	2.20		2.56	95.17
0.44	0.97	3.09	4.56	1.71		43.03	47.62
0.52	1.57	4.48	8.35	0.00		69.85	17.33
0.69	3.02	6.80	21.93	29.12		42.16	
0.84	4.51	14.30	34.09	24.58	0.05	26.99	
0.94	5.55	15.55	43.96	35.38		5.11	
1.02	6.41	26.00	47.27	12.82	2.27	11.64	
1.10	7.30	22.55	58.86	13.82		4.78	
1.25	8.87	24.39	74.35	1.27			

(Continued)

TABLE 3. Continued.

Panel E. Mean-Extended Gini ($\nu = 8$)

Mean Return	Gini ($\nu = 8$)	LCS	SCS	LCB	LGB	IGB	TB
0.32	0.29		0.07	3.14		2.54	94.24
0.44	1.13	2.56	5.01	3.02		43.02	46.39
0.52	1.77	3.85	8.49	1.03		69.14	17.49
0.69	3.49	5.39	22.95	29.90	0.12	41.64	
0.84	5.20	8.77	37.07	34.79	0.03	19.34	
0.94	6.41	17.54	43.13	23.58		15.75	
1.02	7.41	14.15	54.07	30.67	1.04	0.07	
1.10	8.34	16.79	61.21	21.75		0.26	
1.25	10.27	21.91	75.73	2.37			

Note: LCS = large-company stocks, SCS = small-company stocks, LCB = long-term corporate bonds, LGB = long-term government bonds, IGB = intermediate government bonds, TB = U.S. Treasury bills.

SCS. LGB continue to participate in some portfolios, which is not the case for MV.

IV. Analytical Derivation of the MG Frontier

We derive the MG efficient portfolios analytically in a manner similar to derivation of MV portfolios. We show that if one is ready to impose restrictions on the underlying asset distributions, one can derive the MG portfolios using the same technique that solves the MV constrained minimization problem. To see the parallel between the MG and MV derivations, only the similarities and differences of the two dispersion measures need to be explored because all other components of the optimization problems are identical.

The Gini and the variance derive their properties from the covariance. The variance is calculated as the covariance of the return with itself, whereas the Gini is calculated as the covariance of the return with its CDF. Although Gini's reliance on the return and the cumulative distribution complicates its use, this relation enables the Gini to extract more information from the underlying distribution.

The main concern in portfolio analysis is that the Gini is associated with two correlation coefficients, whereas the variance is related to one correlation coefficient (namely, Pearson's correlation coefficient). To see this, note that for two random variables r_i and r_j, one can define two correlation coefficients:

$$\rho_{ij} = \frac{\mathrm{cov}[r_i,\ F_j(r_j)]}{\mathrm{cov}[r_i,\ F_i(r_i)]} \quad \rho_{ji} = \frac{\mathrm{cov}[r_j,\ F_i(r_i)]}{\mathrm{cov}[r_j,\ F_j(r_j)]}. \tag{8}$$

Both correlation coefficients are needed to decompose the portfolio Gini into the contributions of individual assets to the Gini.

The properties of the Gini correlations are examined by Schechtman and Yitzhaki (1999), who show that the correlation coefficients are equal if the distributions of r_i and r_j are exchangeable up to a linear transformation.[5] Intuitively, exchangeability up to a linear transformation requires that the shapes of marginal distributions of assets are equal up to a linear transformation. A disparity in the two correlations means a different shape in the two marginal distributions (of asset returns) causes the difference in the correlations.[6]

The correlation coefficients allow us to decompose the portfolio Gini as follows:

PROPOSITION. *Let* $r_p = \sum_{i=1}^{n} w_i r_i$. *Then,*

$$\Gamma_P^2 - \Gamma_P \sum_{i=1}^{n} w_i D_{iP} \Gamma_i = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \Gamma_i \Gamma_j (\rho_{ij} + \rho_{ji}), \tag{9}$$

where $D_{ip} = \rho_{ip} - \rho_{pi}$ $(i = 1, \ldots, n)$ *is the difference between the two Gini correlations defined by the return of the portfolio and the return of asset i.*

Proof. The proof of the proposition follows that found in Wodon and Yitzhaki (2003) and is provided in Appendix A.

Following exchangeability up to a linear transformation between the distribution of each asset and the portfolio, $D_{ip} = 0$. Hence, exchangeability among the assets leads to:

$$\Gamma_p^2 = \sum_{i=1}^{n} w_i^2 \Gamma_i^2 + 2 \sum_{i=1}^{n} \sum_{j \neq i}^{n} w_i w_j \Gamma_i \Gamma_j \rho_{ij}, \tag{10}$$

which is identical in structure to decomposition of the variance, where each variance is substituted with a Gini and each Pearson correlation coefficient is substituted with a symmetric Gini correlation coefficient. Because the rest of the optimization problem (3) is identical to the MV optimization problem, one can adapt the textbook derivation of MV and apply it to MG (e.g., see Merton 1972; Huang and Litzenberger 1988, p. 63). In other words, one can adopt the MV solution, allowing for short sales, and substitute for every variance the Gini and for every Pearson correlation coefficient the appropriate Gini correlation coefficient to obtain the

[5]A set of random variables is exchangeable if for every permutation of the n subscripts the joint distributions of $(x_{j1}, \ldots, x_{jn})$ are identical (Stuart and Ord 1994). The multivariate normal is an example of an exchangeable distribution up to a linear transformation.

[6]Further research is needed to see what can be learned from the discrepancy in the correlations about the shape of the distributions.

solution to the MG portfolio optimization.[7] Appendix B presents the full MG frontier derivation in matrix notation when short sales are allowed.

Ignoring sampling variability, the MG and MV solutions will be identical if the underlying distributions are multivariate-normal. If the distribution of even one asset diverges from normality, however, the solutions of the MG and MV will differ.

The derivations presented in this section can be obtained using the extended Gini, and results that pertain to the construction of efficient portfolios continue to hold.[8] However, instead of trying to verify the nature of the underlying distributions of different assets, it is easier to compare the optimal portfolios and see whether the change in the intensity of risk aversion makes a difference in the composition of optimal portfolios.

V. Conclusions

We show that the MG and MEG portfolio frontiers can be a workable alternative to the MV efficient frontier. Besides providing necessary conditions for stochastic dominance, MEG analysis allows incorporation of a risk-aversion differential in the construction of efficient frontier portfolios. Hence, the analyst can offer clients portfolios tailored to their risk-aversion preferences. When short sales are allowed and when return distributions are restricted to be exchangeable, MV and MG portfolios are identical in the sense that they have the same structure except that the appropriate Gini and Gini correlations are substituted for the variance and Pearson correlations.

For building U.S. asset portfolios, we use Ibbotson's (2000) aggregate data on stocks, bonds, and bills. We find that when short sales are not allowed, MG and MEG optimal portfolios tend to be more diversified than MV portfolios for all levels of required mean returns. Furthermore, more assets are included in MG and MEG efficient frontiers, whereas long-term government bonds are entirely absent from MV portfolios.

Appendix A

Proof of Proposition. Define w_i as the share of asset i in the portfolio. For simplicity, we restrict ourselves to two assets. The extension to n assets is immediate.

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[7] To be precise, for each covariance in the MV framework we substitute a Gini correlation multiplied by the appropriate Gini.

[8] See Schechtman and Yitzhaki (2003) for a comparison of the properties of the extended Gini correlations with the properties of the Gini correlation.

We start by substituting $r_p = w_1 r_1 + w_2 r_2$ into the covariance formula of the Gini:

$$\Gamma_p = 2\text{cov}[w_1 r_1 + w_2 r_2, F(r_p)].$$

Using the properties of the covariance we can write:

$$\begin{aligned}\Gamma_p &= 2\text{cov}[w_1 r_1 + w_2 r_2, F(r_p)] = w_1 2\text{cov}[r_1, F(r_p)] + w_2 2\text{cov}[r_2, F(r_p)]\\ &= w_1 \rho_{1p} \Gamma_1 + w_2 \rho_{2p} \Gamma_2. \qquad \text{(A1)}\end{aligned}$$

We define the identity:

$$\rho_{ip} = \rho_{pi} + D_{ip} \quad \text{for } I = 1, 2, \qquad \text{(A2)}$$

where D_{ip} is the difference between the two Gini correlations defined between r_p and r_i. Using (A1) and (A2), we obtain:

$$\Gamma_p = w_1(\rho_{p1} + D_{1p})\Gamma_1 + \alpha_2(\rho_{p2} + D_{2p})\Gamma_2.$$

Rearranging terms:

$$\Gamma_p - w_1 D_{1p}\Gamma_1 - w_2 D_{2p}\Gamma_2 = w_1 \rho_{p1}\Gamma_1 + a_2 \rho_{p2}\Gamma_2.$$

Using the properties of the covariance:

$$\begin{aligned}\rho_{p1} &= \frac{\text{cov}[r_p, F(r_1)]}{\text{cov}[r_p, F(r_p)]} = \frac{1}{\text{cov}[r_p, F(r_p)]}\{w_1\text{cov}[r_1, F(r_1)] + w_2\text{cov}[r_2, F(r_1)]\}\\ &= \frac{w_1\Gamma_1 + w_2\Gamma_2 r_{21}}{\Gamma_p}.\end{aligned}$$

Writing ρ_{p2} in a similar manner, we get equation (9):

$$\begin{aligned}&\Gamma_p^2 - (w_1 D_{1p}\Gamma_1 + w_2 D_{2p}\Gamma_2)\Gamma_p\\ &\quad = w_1\Gamma_1(w_1\Gamma_1 + w_2\Gamma_2\rho_{21}) + w_2\Gamma_2(w_1\rho_{12}\Gamma_1 + w_2\Gamma_2)\\ &\quad = w_1^2\Gamma_1^2 + w_2^2\Gamma_2^2 + w_1 w_2\Gamma_1\Gamma_2(\rho_{12} + \rho_{21}).\end{aligned}$$

Assuming equality of the Gini correlation coefficients between r_p and r_1 sets $D_{1p} = 0$. A similar assumption for r_2 and r_p sets $D_{2p} = 0$. The assumption $\rho = \rho_{12} = \rho_{21}$ completes the proof of (10).

Appendix B

Under the assumption of exchangeability up to a linear transformation, between each asset and the portfolio and among every pair of assets, the Gini of the portfolio can be written as:

$$\Gamma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \Gamma_i \Gamma_j,$$

where $\rho_{ii} = 1$. In matrix notation, let **R** be the matrix of Gini correlations, $\mathbf{\Gamma}$ be a matrix with the Gini of the assets on the diagonal and zero in all off-diagonal terms, and **w** be the vectors of weights. Then, the equivalent of the variance-covariance matrix is **V**:

$$\mathbf{V} = \mathbf{\Gamma}\,\mathbf{R}\,\mathbf{\Gamma}$$

and the Gini of the portfolio:

$$\Gamma_p = \mathbf{w}'\mathbf{V}\mathbf{w}.$$

The optimization problem becomes:

$$\begin{aligned} &Min\ \mathbf{w}'\mathbf{V}\mathbf{w} \\ &s.t.\ \ \mu_p = \mathbf{w}'\boldsymbol{\mu} \\ &\qquad\quad 1 = \mathbf{w}'\mathbf{l}. \end{aligned} \tag{B1}$$

The problem without short sales is identical to (B1) with the additional requirement that $\mathbf{w} \geq 0$.

The matrix **V** is a matrix of constants; hence, our problem is identical to the familiar problem of minimizing the variance portfolio subject to the same restrictions. The derivation of the solution can be found in Merton (1972) and Huang and Litzenberger (1988, pp. 63–66). For completeness, we give the solution to the more complicated problem of maximization subject to the non-negativity constraint.

Then,

$$L(w,\lambda,\gamma) = \mathbf{w}'\mathbf{V}\mathbf{w} + \lambda(\mu_p - \mathbf{w}'\boldsymbol{\mu}) + \gamma(1 - \mathbf{w}'\mathbf{l})\ .$$

The first-order conditions are:

$$\frac{\partial L}{\partial w} = \mathbf{V}\mathbf{w} - \lambda\boldsymbol{\mu} - \gamma\mathbf{l} = 0$$

$$\frac{\partial L}{\partial \lambda} = \mu_p - \mathbf{w}'\boldsymbol{\mu} = 0$$
$$\frac{\partial L}{\partial \gamma} = 1 - \mathbf{w}'\mathbf{l} = 0,$$

yielding the solution:

$$\mathbf{w}_p = \mathbf{x} + \mu_p \mathbf{y}, \tag{B2}$$

where

$$\mathbf{x} = [B(\mathbf{V}^{-1}\mathbf{l}) - A(\mathbf{V}^{-1}\boldsymbol{\mu})]/D$$
$$\mathbf{y} = [C(\mathbf{V}^{-1}\boldsymbol{\mu}) - A(\mathbf{V}^{-1}\mathbf{l})]/D$$
$$A = \mathbf{l}'\mathbf{V}^{-1}\boldsymbol{\mu};\ B = \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu};\ C = \mathbf{l}'\mathbf{V}^{-1}\mathbf{l};\ D = BC - A^2\,.$$

Equation (B2) presents the weights of an MG frontier portfolio for a given μ_p. Hence, similarly to MV, one can generate the efficient frontier by constructing convex combinations of two distinct frontier portfolios.

Consider a portfolio q built from portfolios p_1 and p_2 with weights α and $(1-\alpha)$, respectively. The mean return is $\mu_q = \alpha\mu_1 + (1-\alpha)\mu_2$. By construction:

$$\mathbf{w}_q = \alpha\mathbf{w}_1 + (1-\alpha)\mathbf{w}_2 = \mathbf{x} + \mu_q\mathbf{y},$$

which can be shown to yield

$$\Gamma_q^2 = \frac{1}{D}\left(C\mu_q^2 - 2A\mu_q + B\right).$$

Using this equation, one can derive the minimum Gini frontier and find the hyperbola in the mean Gini space.

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