ACTA ARITHMETICA
LXIV.4 (1993)

The mean square of the Riemann zeta-function in the critical strip III

by

Kohji Matsumoto (Morioka) and Tom Meurman (Turku)

1. Introduction. Let $\zeta(s)$ be the Riemann zeta-function, and define $E(T)$ by

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt = T\log T + (2\gamma - 1 - \log 2\pi)T + E(T)$$

for $T \ge 2$, where γ is Euler's constant. In 1949, Atkinson [1] proved the following now famous formula for $E(T)$. For any positive number ξ, let

$$e(T,\xi) = \left(1+\frac{\pi\xi}{2T}\right)^{-1/4}\left(\frac{2T}{\pi\xi}\right)^{-1/2}\left(\operatorname{arsinh}\sqrt{\frac{\pi\xi}{2T}}\right)^{-1},$$

$$f(T,\xi) = 2T\operatorname{arsinh}\sqrt{\frac{\pi\xi}{2T}} + (\pi^2\xi^2 + 2\pi\xi T)^{1/2} - \frac{\pi}{4},$$

and

$$g(T,\xi) = T\log\frac{T}{2\pi\xi} - T + \frac{\pi}{4}.$$

Then Atkinson's formula asserts that for any positive number X with $X \asymp T$ (i.e. $T \ll X \ll T$), the relation

$$E(T) = \Sigma_1(T,X) - \Sigma_2(T,X) + O(\log^2 T) \tag{1.1}$$

holds, where

$$\Sigma_1(T,X) = \sqrt{2}\left(\frac{T}{2\pi}\right)^{1/4}\sum_{n\le X}(-1)^n d(n) n^{-3/4} e(T,n)\cos(f(T,n)),$$

$$\Sigma_2(T,X) = 2\sum_{n\le B(T,\sqrt{X})} d(n) n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos(g(T,n)),$$

$d(n)$ is the number of positive divisors of the integer n, and

$$B(T,\xi)=\frac{T}{2\pi}+\frac{1}{2}\xi^2-\xi\biggl(\frac{T}{2\pi}+\frac{1}{4}\xi^2\biggr)^{1/2}.$$

The analogue of Atkinson's formula in the strip $1/2<\sigma=\mathrm{Re}(s)<1$ was first investigated by Matsumoto [9]. Define $E_\sigma(T)$ by

$$\int_0^T|\zeta(\sigma+it)|^2dt=\zeta(2\sigma)T+(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}+E_\sigma(T)\,.$$

Matsumoto proved that if $1/2<\sigma<3/4$ and $X\asymp T$, then

$$E_\sigma(T)=\Sigma_{1,\sigma}(T,X)-\Sigma_{2,\sigma}(T,X)+O(\log T)\,, \tag{1.2}$$

where

$$\Sigma_{1,\sigma}(T,X)=\sqrt{2}\biggl(\frac{T}{2\pi}\biggr)^{3/4-\sigma}\sum_{n\le X}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-5/4}e(T,n)\cos(f(T,n))\,,$$

$$\Sigma_{2,\sigma}(T,X)=2\biggl(\frac{T}{2\pi}\biggr)^{1/2-\sigma}\sum_{n\le B(T,\sqrt{X})}\sigma_{1-2\sigma}(n)n^{\sigma-1}\biggl(\log\frac{T}{2\pi n}\biggr)^{-1}\cos(g(T,n))\,,$$

with the notation $\sigma_a(n)=\sum_{d|n}d^a$, and the implied constant depends only on σ.

The reason of the restriction $1/2<\sigma<3/4$ in [9] is as follows. Define

$$D_{1-2\sigma}(\xi)={\sum_{n\le\xi}}'\sigma_{1-2\sigma}(n)\,,$$

where the symbol $\sum'$ means that the last term is to be halved if ξ is an integer. In case $\sigma=1/2$, the classical formula of Voronoï asserts

$$D_0(\xi)=\xi\log\xi+(2\gamma-1)\xi+1/4+\Delta_0(\xi)$$

with

$$\Delta_0(\xi)=\frac{1}{\pi\sqrt{2}}\xi^{1/4}\sum_{n=1}^{\infty}d(n)n^{-3/4}\biggl\{\cos(4\pi\sqrt{n\xi}-\pi/4)-\frac{3}{32\pi}(n\xi)^{-1/2}\sin(4\pi\sqrt{n\xi}-\pi/4)\biggr\}+O(\xi^{-3/4}). \tag{1.3}$$

This formula is one of the essential tools in the proof of Atkinson's formula. Analogously, Matsumoto's proof of (1.2) depends on the following Voronoï-type formula of Oppenheim [16]:

$$D_{1-2\sigma}(\xi) = \zeta(2\sigma)\xi + \frac{\zeta(2-2\sigma)}{2-2\sigma}\xi^{2-2\sigma} - \frac{1}{2}\zeta(2\sigma-1) + \Delta_{1-2\sigma}(\xi) \tag{1.4}$$

with

$$\Delta_{1-2\sigma}(\xi) = \frac{1}{\pi\sqrt{2}}\xi^{3/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-5/4}\Big\{\cos(4\pi\sqrt{n\xi}-\pi/4) - \frac{16(1-\sigma)^2-1}{32\pi}(n\xi)^{-1/2}\sin(4\pi\sqrt{n\xi}-\pi/4)\Big\} + O(\xi^{-1/4-\sigma})\,. \tag{1.5}$$

However, the series in (1.5) converges only for $\sigma < 3/4$, which gives rise to the restriction $1/2 < \sigma < 3/4$ in [9]. Therefore a new method is required to obtain an analogue of Atkinson's formula beyond the line $\sigma = 3/4$.

In this paper we shall prove

THEOREM 1. *For any* σ *and* X *satisfying* $1/2 < \sigma < 1$ *and* $X \asymp T$, *the formula* (1.2) *holds.*

Our starting point is the Voronoï-type formula for

$$\widetilde{D}_{1-2\sigma}(\xi) = \int_0^{\xi}\sum_{n\le t}\sigma_{1-2\sigma}(n)\,dt\,,$$

given in the next section. The crucial point is that the Voronoï series for $\widetilde{D}_{1-2\sigma}(\xi)$ converges for any σ satisfying $1/2 < \sigma < 1$. The basic principle of the proof of Theorem 1 is similar to the proofs of (1.1) and (1.2), but the details are more complicated.

In [9], as an application of (1.2), the upper bound estimate

$$E_\sigma(T) = O(T^{1/(1+4\sigma)}\log^2 T) \tag{1.6}$$

has been proved for $1/2 < \sigma < 3/4$. Now it follows easily from Theorem 1 that (1.6) holds for $1/2 < \sigma < 1$. We should mention that already in 1990, in a different way, Motohashi [15] proved (1.6) for $1/2 < \sigma < 1$, and Ivić [6, Ch. 2] gave an improvement by using the theory of exponent pairs. (See also Ivić [7].) (1)

Another application of Theorem 1 is the mean square result for $E_\sigma(T)$. In [9] it has been shown that

(1) Added in proof (June 1993). In [6] there is an error on top of p. 89 invalidating Theorem 2.11 and its Corollary 1 but not its Corollary 2. However, Professor Ivi/c has informed us that he can now recover his corollaries.

$$\int_2^T E_\sigma(t)^2\,dt = \frac{2}{5-4\sigma}(2\pi)^{2\sigma-3/2}\frac{\zeta^2(3/2)}{\zeta(3)}\zeta\left(\frac{5}{2}-2\sigma\right)\zeta\left(\frac{1}{2}+2\sigma\right)T^{5/2-2\sigma}+F_\sigma(T) \tag{1.7}$$

with $F_\sigma(T) = O(T^{7/4-\sigma}\log T)$ for $1/2 < \sigma < 3/4$, and in [10], the improvement $F_\sigma(T) = O(T)$ has been proved. In case $3/4 \le \sigma < 1$, by using Heath-Brown's [4] method and Theorem 1, it can be shown easily that

$$\int_2^T E_\sigma(t)^2\,dt \ll T\log^2 T\,. \tag{1.8}$$

This can be slightly improved. In particular, for $\sigma = 3/4$ we get an asymptotic formula.

THEOREM 2. *We have*

$$\int_2^T E_{3/4}(t)^2\,dt = \frac{\zeta^2(3/2)\zeta(2)}{\zeta(3)}T\log T + O(T(\log T)^{1/2})\,,$$

and for $3/4 < \sigma < 1$, *we have*

$$\int_2^T E_\sigma(t)^2\,dt \ll T\,.$$

COROLLARY. $E_{3/4}(T) = \Omega((\log T)^{1/2})$.

Comparing Theorem 2 with (1.7), we can observe, as has already been pointed out in [9], that the line $\sigma = 3/4$ is a kind of "critical line" in the theory of the Riemann zeta-function, or at least for the function $E_\sigma(T)$.

It might be possible to reduce the error term $O(T(\log T)^{1/2})$ to $O(T)$ in Theorem 2 without any new idea but only with a lot of extra work.

We also prove in this paper the following result, which has been announced in [10].

THEOREM 3. *For any fixed* σ *satisfying* $1/2 < \sigma < 3/4$, *we have*

$$E_\sigma(T) = \Omega_+(T^{3/4-\sigma}(\log T)^{\sigma-1/4}).$$

COROLLARY. $F_\sigma(T) = \Omega(T^{9/4-3\sigma}(\log T)^{3\sigma-3/4})$.

We can deduce Theorem 3 from (1.2). The problem of deducing a certain Ω_+-result in case $3/4 \le \sigma < 1$ seems to be much more difficult. This situation also suggests the critical property of the line $\sigma = 3/4$.

2. A Voronoï-type formula. Hereafter, except for the last section, we assume $3/4 \le \sigma < 1$. Let $\xi \ge 1$, and define $\widetilde{\Delta}_{1-2\sigma}(\xi)$ by

$$\widetilde{D}_{1-2\sigma}(\xi) = \frac{1}{2}\zeta(2\sigma)\xi^2 + \frac{\zeta(2-2\sigma)}{(2-2\sigma)(3-2\sigma)}\xi^{3-2\sigma} - \frac{1}{2}\zeta(2\sigma-1)\xi + \frac{1}{12}\zeta(2\sigma-2) + \widetilde{\Delta}_{1-2\sigma}(\xi)\,. \tag{2.1}$$

Then the following Voronoï-type formula holds.

Lemma 1. *We have*

$$\widetilde{\Delta}_{1-2\sigma}(\xi) = c_1\xi^{5/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\cos(c_2\sqrt{n\xi}+c_3) + c_4\xi^{3/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-9/4}\cos(c_2\sqrt{n\xi}+c_5) + O(\xi^{1/4-\sigma})\,, \tag{2.2}$$

where the two infinite series on the right-hand side are uniformly convergent on any finite closed subinterval in $(0,\infty)$, *and the values of the constants are* $c_1 = -1/(2\sqrt{2}\pi^2)$, $c_2 = 4\pi$, $c_3 = \pi/4$, $c_4 = (5-4\sigma)(7-4\sigma)/(64\sqrt{2}\pi^3)$ *and* $c_5 = -\pi/4$.

Voronoï-type formulas are studied in Hafner [3] in a fairly general situation. We can prove the formula (2.2) as a special case of Hafner's theorem. In fact, let $F(s) = \pi^{-s}\zeta(s)\zeta(s-1+2\sigma)$ and $G(s) = \Gamma(s/2)\Gamma((s-1+2\sigma)/2)$. Then the functional equation

$$G(s)F(s) = G(2-2\sigma-s)F(2-2\sigma-s)$$

holds, which agrees with Hafner's Definition 1.1 with $a(n)=b(n)=\sigma_{1-2\sigma}(n)$, $\lambda_n = \mu_n = \pi n$, $\phi(s) = \psi(s) = F(s)$, $\sigma_a^* = \sigma_a = 1$, $\Delta(s) = G(s)$, $N = 2$, $\alpha_1 = \alpha_2 = 1/2$, $\beta_1 = 0$, $\beta_2 = \sigma - 1/2$, $S = \{1-2\sigma, 0, 2-2\sigma, 1\}$, $D = \mathbb{C} - S$, $\chi(s) = G(s)F(s)$ and $r = 2-2\sigma$. Also we choose $\varrho = 1$, $b = 3$, $c = 3/2$ and $R = 2$ in Hafner's notation. In this case Hafner's $A_\varrho(x)$ is equal to

$$\sum_{\pi n \le x}\sigma_{1-2\sigma}(n)(x-\pi n) = \pi\int_0^{x/\pi}\sum_{n\le t}\sigma_{1-2\sigma}(n)\,dt\,,$$

which is obviously continuous in $(0,\infty)$. Therefore, (2.2) and the claim of uniform convergence in Lemma 1 follow from Theorem B and Lemma 2.1 (with $m = 1$) of Hafner [3]. Hafner does not give the values of the constants $c_1,\ldots,c_5$ explicitly, but the values of c_1, c_2, c_3 and c_5 can be determined by combining Lemma 2.1 of Hafner [3] with the explicit values of μ and h given in Lemma 1 of Chandrasekharan–Narasimhan [2]. (There is a minor misprint in Hafner's paper. The right-hand side of (2.3) in [3] should be multiplied by $\sqrt{2}$.) The value of c_4 may also be determined by tracing the proof of Lemma 1 in Chandrasekharan–Narasimhan [2] carefully, but the value of c_4 is not necessary for the purpose of the present paper.

Meurman [13] gives a considerably simpler proof of (1.3). All the steps of Meurman's proof are explicit, and the same method can be applied to our present case. Therefore we can obtain a different proof of Lemma 1, with explicit values of all the constants $c_1, \ldots, c_5$. The details, omitted here, are given in a manuscript form [14].

By using Lemma 1, we can prove the following useful estimate.

LEMMA 2. *We have* $\widetilde{\Delta}_{1-2\sigma}(\xi) = O(\xi^r \log \xi)$, *where*

$$r = \frac{-4\sigma^2 + 7\sigma - 2}{4\sigma - 1} \le \frac{1}{2}\,.$$

Proof. We first note the elementary estimate

$$\Delta_{1-2\sigma}(v) \ll v^{1-\sigma}\,. \tag{2.3}$$

In fact, by the Euler–Maclaurin summation formula we have

$$\sum_{m \le \sqrt{v}} m^{-2\sigma} = \frac{1}{1-2\sigma} v^{1/2-\sigma} + \zeta(2\sigma) + O(v^{-\sigma})\,,$$

$$\sum_{m \le \sqrt{v}} m^{2\sigma-2} = \frac{1}{2\sigma-1} v^{\sigma-1/2} + \zeta(2-2\sigma) + O(v^{\sigma-1})\,,$$

and, for $1 \le n \le \sqrt{v}$,

$$\sum_{m \le v/n} m^{1-2\sigma} = \frac{1}{2-2\sigma}\left(\frac{v}{n}\right)^{2-2\sigma} + c(\sigma) + O(v^{1/2-\sigma})\,,$$

where $c(\sigma)$ is a constant depending on σ. By the well-known splitting up argument of Dirichlet (see Titchmarsh [18, §12.1]), we get

$$\sum_{n \le v} \sigma_{1-2\sigma}(n)$$
$$= v \sum_{m \le \sqrt{v}} m^{-2\sigma} + \sum_{n \le \sqrt{v}} \Big(\sum_{m \le v/n} m^{1-2\sigma} - \sum_{m \le \sqrt{v}} m^{1-2\sigma} \Big) + O(v^{1-\sigma})\,.$$

Applying the above summation formulas we get

$$\sum_{n \le v} \sigma_{1-2\sigma}(n) = \zeta(2\sigma)v + \frac{\zeta(2-2\sigma)}{2-2\sigma} v^{2-2\sigma} + O(v^{1-\sigma})\,,$$

which implies (2.3). Hence, by (1.4) and (2.1),

$$\widetilde{\Delta}_{1-2\sigma}(\xi) - \widetilde{\Delta}_{1-2\sigma}(u) = \int_u^\xi \Delta_{1-2\sigma}(v)\,dv \ll |\xi - u|\xi^{1-\sigma}$$

for $u \asymp \xi$. Hence

$$\widetilde{\Delta}_{1-2\sigma}(\xi) = Q^{-1} \int\limits_{\xi}^{\xi+Q} \widetilde{\Delta}_{1-2\sigma}(u)\, du + O(Q\xi^{1-\sigma})$$

for $0 < Q \ll \xi$. Formula (2.2) gives trivially

$$\widetilde{\Delta}_{1-2\sigma}(u) = c_1 u^{5/4-\sigma} \sum_{n>N} \sigma_{1-2\sigma}(n) n^{\sigma-7/4} \cos(4\pi\sqrt{nu} + c_3) + O(u^{3/4-\sigma}) + O(u^{5/4-\sigma} N^{\sigma-3/4} \log N)\,,$$

where $N \geq 1$. It follows that

$$\widetilde{\Delta}_{1-2\sigma}(\xi) = c_1 Q^{-1} \sum_{n>N} \sigma_{1-2\sigma}(n) n^{\sigma-7/4} \int\limits_{\xi}^{\xi+Q} u^{5/4-\sigma} \cos(4\pi\sqrt{nu} + c_3)\, du + O(Q\xi^{1-\sigma}) + O(\xi^{5/4-\sigma} N^{\sigma-3/4} \log N)\,.$$

The integral here is $\ll \xi^{7/4-\sigma} n^{-1/2}$ by the first derivative test. Therefore the series contributes

$$O(Q^{-1}\xi^{7/4-\sigma} N^{\sigma-5/4})\,.$$

Choosing $N = \xi Q^{-2}$ and $Q = \xi^{(2\sigma-1)/(4\sigma-1)}$ completes the proof of Lemma 2.

The following lemma gives the average order of $\widetilde{\Delta}_{1-2\sigma}(\xi)$. We shall need it because the factor $\log \xi$ in Lemma 2 causes trouble when $\sigma = 3/4$.

LEMMA 3. *We have*

$$\int\limits_1^x \widetilde{\Delta}_{1-2\sigma}(\xi)^2\, d\xi \ll x^{7/2-2\sigma}\,.$$

Proof. By Lemma 1, for $1 \leq \xi \leq x$ we have

$$\widetilde{\Delta}_{1-2\sigma}(\xi) = c_1 \xi^{5/4-\sigma} \sum_{n \leq N(x)} \sigma_{1-2\sigma}(n) n^{\sigma-7/4} \cos(4\pi\sqrt{n\xi} + \pi/4) + O(\xi^{5/4-\sigma})$$

with a sufficiently large $N(x)$ depending only on x (and σ). The rest of the proof is standard and proceeds similarly to the proof of Theorem 13.5 in Ivić [5].

It is not hard to refine Lemma 3 by showing that

$$\int\limits_1^x \widetilde{\Delta}_{1-2\sigma}(\xi)^2\, d\xi = \frac{\zeta^2(5/2)\zeta(7/2-2\sigma)\zeta(3/2+2\sigma)}{8\pi^4(7-4\sigma)\zeta(5)} x^{7/2-2\sigma} + O(x^{3-2\sigma}).$$

However, Lemma 3 is sufficient for our purpose.

It should be noted that except for the inequality $r \le 1/2$ in Lemma 2, the results in this section are also valid for $1/2 < \sigma < 3/4$. However, estimate (4.2) depends on the inequality $r \le 1/2$, and in §6 there are several estimates which require the condition $3/4 \le \sigma < 1$. Therefore the proof of Theorem 1 is valid only on this condition.

3. The basic decomposition. Now we start the proof of Theorem 1. At first we assume $X \asymp T$ and X is not an integer. Let u be a complex variable, $\xi \ge 1$,

$$h(u,\xi) = 2\int_0^\infty y^{-u}(1+y)^{u-2\sigma}\cos(2\pi\xi y)\,dy\,,$$

and define

$$\begin{aligned} g_1(u) &= \sum_{n\le X}\sigma_{1-2\sigma}(n)h(u,n)\,,\\ g_2(u) &= \Delta_{1-2\sigma}(X)h(u,X)\,,\\ g_3(u) &= \int_X^\infty (\zeta(2\sigma)+\zeta(2-2\sigma)\xi^{1-2\sigma})h(u,\xi)\,d\xi\,,\\ g_4(u) &= \int_X^\infty \Delta_{1-2\sigma}(\xi)\frac{\partial h(u,\xi)}{\partial\xi}\,d\xi\,. \end{aligned}$$

Since the integral $h(u,\xi)$ is absolutely convergent for $\mathrm{Re}(u) < 1$, $g_1(u)$ and $g_2(u)$ can be defined in the same region. Also, Matsumoto [9, (4.2)] gives the analytic continuation of $g_3(u)$ to the region $\mathrm{Re}(u) < 1$. Hence, if $g_4(u)$ can be analytically continued to $\mathrm{Re}(u) < 1$, then we can define

$$G_j = \int_{\sigma-iT}^{\sigma+iT} g_j(u)\,du \quad (1\le j\le 4)$$

for $1/2 < \sigma < 1$, and obtain (see [9, (4.3)])

$$E_\sigma(T) = -i(G_1 - G_2 + G_3 - G_4) + O(1). \tag{3.1}$$

Now we show the analytic continuation of $g_4(u)$. From (1.4) and (2.1) it follows that

$$\frac{1}{12}\zeta(2\sigma-2) + \widetilde{\Delta}_{1-2\sigma}(\xi) = \int_0^\xi \Delta_{1-2\sigma}(t)\,dt\,.$$

Hence, by integration by parts we have

$$g_4(u) = -\widetilde{\Delta}_{1-2\sigma}(X)h'(u,X) - \int_X^\infty \widetilde{\Delta}_{1-2\sigma}(\xi)h''(u,\xi)\,d\xi, \tag{3.2}$$

where h' and h'' mean $\partial h/\partial\xi$ and $\partial^2 h/\partial\xi^2$, respectively. Here we have used Lemma 2 and the estimate

$$h'(u,\xi) = O(\xi^{\mathrm{Re}(u)-2}) \tag{3.3}$$

for $\mathrm{Re}(u) < 1$ and bounded u, proved in Atkinson [1]. Differentiating the expression

$$h(u,\xi) = \int_0^{i\infty} y^{-u}(1+y)^{u-2\sigma}e^{2\pi i\xi y}\,dy + \int_0^{-i\infty} y^{-u}(1+y)^{u-2\sigma}e^{-2\pi i\xi y}\,dy$$

with respect to ξ, and estimating the resulting integrals, we obtain (3.3). One more differentiation gives

$$\begin{aligned} h''(u,\xi) = &-4\pi^2 \int_0^{i\infty} y^{2-u}(1+y)^{u-2\sigma}e^{2\pi i\xi y}\,dy \\ &-4\pi^2 \int_0^{-i\infty} y^{2-u}(1+y)^{u-2\sigma}e^{-2\pi i\xi y}\,dy\,, \end{aligned} \tag{3.4}$$

and from this formula we can deduce that

$$h''(u,\xi) = O(\xi^{\mathrm{Re}(u)-3}). \tag{3.5}$$

It follows from (3.5) and Lemma 2 that the integral on the right-hand side of (3.2) is absolutely convergent for $\mathrm{Re}(u) < 1$. Hence (3.2) gives the desired analytic continuation of $g_4(u)$. And we divide G_4 as

$$\begin{aligned} G_4 = &-\widetilde{\Delta}_{1-2\sigma}(X)\int_{\sigma-iT}^{\sigma+iT} h'(u,X)\,du \\ &-\int_X^{\infty}\widetilde{\Delta}_{1-2\sigma}(\xi)\int_{\sigma-iT}^{\sigma+iT} h''(u,\xi)\,du\,d\xi \\ = &-G_4^* - G_4^{**}\,, \end{aligned} \tag{3.6}$$

say.

The integrals G_1, G_2 and G_3 can be treated by the method described in [9, §4–§5], and the results are

$$G_1 = i\Sigma_{1,\sigma}(T,X) + O(T^{1/4-\sigma})\,, \tag{3.7}$$

$$G_2 = O(T^{1/2-\sigma})\,, \tag{3.8}$$

$$G_3 = -2\pi i\zeta(2\sigma-1) + O(T^{\sigma-1})\,. \tag{3.9}$$

We note that the proof of (3.8) uses (2.3) instead of [9, Lemma 2].

4. Evaluation of G_4^*. We have

$$\begin{aligned}\int\limits_{\sigma-iT}^{\sigma+iT} h'(u,X)\,du &= \frac{\partial}{\partial X}\int\limits_{\sigma-iT}^{\sigma+iT} h(u,X)\,du\\
&= \frac{\partial}{\partial X}\biggl(2i\int\limits_0^\infty y^{-\sigma}(1+y)^{-\sigma}\cos(2\pi Xy)\int\limits_{-T}^{T}\biggl(\frac{1+y}{y}\biggr)^{it}dt\,dy\biggr)\\
&= \frac{\partial}{\partial X}\biggl(4i\int\limits_0^\infty \frac{\cos(2\pi Xy)\sin(T\log((1+y)/y))}{y^\sigma(1+y)^\sigma\log((1+y)/y)}\,dy\biggr)\\
&= \frac{\partial}{\partial X}\biggl(4i\int\limits_0^\infty \frac{X^{2\sigma-1}\cos(2\pi y)\sin(T\log((X+y)/y))}{y^\sigma(X+y)^\sigma\log((X+y)/y)}\,dy\biggr)\\
&= 4i(2\sigma-1)X^{2\sigma-2}\int\limits_0^\infty \frac{\cos(2\pi y)\sin(T\log((X+y)/y))}{y^\sigma(X+y)^\sigma\log((X+y)/y)}\,dy\\
&\quad + 4iX^{2\sigma-1}T\int\limits_0^\infty \frac{\cos(2\pi y)\cos(T\log((X+y)/y))}{y^\sigma(X+y)^{\sigma+1}\log((X+y)/y)}\,dy\\
&\quad - 4i\sigma X^{2\sigma-1}\int\limits_0^\infty \frac{\cos(2\pi y)\sin(T\log((X+y)/y))}{y^\sigma(X+y)^{\sigma+1}\log((X+y)/y)}\,dy\\
&\quad - 4iX^{2\sigma-1}\int\limits_0^\infty \frac{\cos(2\pi y)\sin(T\log((X+y)/y))}{y^\sigma(X+y)^{\sigma+1}\log^2((X+y)/y)}\,dy\,.\end{aligned}$$

We split up these four integrals at $y=T$. Then we estimate in each case $\int_T^\infty$ by the first derivative test and $\int_0^T$, after the further splitting up into integrals over the intervals $(2^{-k}T, 2^{-k+1}T]$ $(k=1,2,\ldots)$, by the second derivative test (see Ivić [5, (2.3), (2.5)]). This gives

$$\int\limits_{\sigma-iT}^{\sigma+iT} h'(u,X)\,du \ll T^{-1/2}\,. \tag{4.1}$$

Together with Lemma 2 and the definition of G_4^* this gives

$$G_4^* \ll \log T\,. \tag{4.2}$$

The integral in (4.1) has already been calculated in Matsumoto [9, §4], but there are some misprints in the formula stated between (4.6) and (4.7) in [9]. The above calculation contains the correction.

5. Evaluation of G_4^{**} (the first step). In this section we evaluate the inner integral of G_4^{**}. Integrating (3.4) twice by parts, we have

$$\begin{aligned} h''(u,\xi) = 2\xi^{-2} \int_0^\infty \{&(2-u)(1-u)y^{-u}(1+y)^{u-2\sigma} \\ &+ 2(2-u)(u-2\sigma)y^{1-u}(1+y)^{u-2\sigma-1} \\ &+ (u-2\sigma)(u-2\sigma-1)y^{2-u}(1+y)^{u-2\sigma-2}\}\cos(2\pi\xi y)\,dy\,. \end{aligned}$$

Hence,

$$\int_{\sigma-iT}^{\sigma+iT} h''(u,\xi)\,du = 2\xi^{-2}\int_0^\infty (1+y)^{-2\sigma-2} I(y)\cos(2\pi\xi y)\,dy\,, \tag{5.1}$$

where

$$I(y) = \int_{\sigma-iT}^{\sigma+iT} (u^2 + P_1(y)u + P_2(y))\left(\frac{1+y}{y}\right)^u du$$

and $P_j(y)$ is a polynomial in y of degree j whose coefficients may depend on σ. We have

$$\int_{\sigma-iT}^{\sigma+iT} x^u\,du = 2ix^\sigma \frac{\sin(T\log x)}{\log x}\,,$$

$$\int_{\sigma-iT}^{\sigma+iT} ux^u\,du = 2ix^\sigma \frac{\sigma\sin(T\log x) + T\cos(T\log x)}{\log x} - 2ix^\sigma\frac{\sin(T\log x)}{\log^2 x}\,,$$

$$\begin{aligned} \int_{\sigma-iT}^{\sigma+iT} u^2x^u\,du = 2ix^\sigma &\frac{\sigma^2\sin(T\log x) + 2\sigma T\cos(T\log x) - T^2\sin(T\log x)}{\log x} \\ &-4ix^\sigma\frac{\sigma\sin(T\log x) + T\cos(T\log x)}{\log^2 x} + 4ix^\sigma\frac{\sin(T\log x)}{\log^3 x}\,. \end{aligned}$$

Hence

$$\begin{aligned} I(y) = 2i\left(\frac{1+y}{y}\right)^\sigma \left(\log\frac{1+y}{y}\right)^{-1}\bigg\{ &-T^2\sin\left(T\log\frac{1+y}{y}\right) \\ &+ H_1(y)T\cos\left(T\log\frac{1+y}{y}\right) + H_0(y)\sin\left(T\log\frac{1+y}{y}\right)\bigg\}, \end{aligned}$$

where $H_0(y)$ and $H_1(y)$ are linear combinations of terms of the form

$$y^\mu\left(\log\frac{1+y}{y}\right)^{-\nu}$$

with non-negative integers μ and ν satisfying $\mu+\nu \le 2$. We substitute this expression for $I(y)$ into (5.1). The method used in §4 gives

$$\int_0^\infty \frac{\exp(iT\log((1+y)/y))\cos(2\pi\xi y)}{y^{\sigma-\mu}(1+y)^{\sigma+2}(\log((1+y)/y))^{\nu+1}}\,dy \ll T^{-1/2}$$

for $\xi \ge X$. Hence

$$\int_{\sigma-iT}^{\sigma+iT} h''(u,\xi)\,du$$
$$= -4iT^2\xi^{-2}\int_0^\infty \frac{\cos(2\pi\xi y)\sin(T\log((1+y)/y))}{y^\sigma(1+y)^{\sigma+2}\log((1+y)/y)}\,dy + O(\xi^{-2}T^{1/2}).$$

Then we apply [9, Lemma 3] to estimate the integral on the right hand side. Substituting the result into the definition of G_4^{**} we arrive at

$$G_4^{**} = (i\sqrt{\pi})^{-1}T^{5/2}J + O\Big(T^{1/2}\int_X^\infty \xi^{-2}|\widetilde{\Delta}_{1-2\sigma}(\xi)|\,d\xi\Big),$$

where

$$J = \int_X^\infty \frac{\widetilde{\Delta}_{1-2\sigma}(\xi)\sin(TV+2\pi\xi U-\pi\xi+\pi/4)}{\xi^3 VU^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi$$

with

$$U = \Big(\frac{T}{2\pi\xi}+\frac14\Big)^{1/2}, \qquad V = 2\,\mathrm{arsinh}\sqrt{\frac{\pi\xi}{2T}}.$$

Using Lemma 3 we get

$$G_4^{**} = (i\sqrt{\pi})^{-1}T^{5/2}J + O(T^{3/4-\sigma}). \tag{5.2}$$

6. Evaluation of G_4^{**} (the second step). Now our problem is reduced to the evaluation of J. Consider the truncated integral

$$J(b) = \int_X^b \frac{\widetilde{\Delta}_{1-2\sigma}(\xi)\sin(TV+2\pi\xi U-\pi\xi+\pi/4)}{\xi^3 VU^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi \qquad (b > X),$$

and substitute (2.2) into the right-hand side. By Lemma 1 the series in the expression for $\widetilde{\Delta}_{1-2\sigma}(\xi)$ are uniformly convergent when b is finite, so that in $J(b)$ we can perform termwise integration to obtain

$$J(b) = c_1\sum_{n=1}^\infty \sigma_{1-2\sigma}(n)n^{\sigma-7/4}J_1(n,b) + c_4\sum_{n=1}^\infty \sigma_{1-2\sigma}(n)n^{\sigma-9/4}J_2(n,b) + O(T^{-\sigma-7/4}), \tag{6.1}$$

where

$$J_1(n,b) = \int\limits_X^b \xi^{-7/4-\sigma} \frac{\cos(4\pi\sqrt{n\xi}+\pi/4)\sin(TV+2\pi\xi U-\pi\xi+\pi/4)}{VU^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi$$

and

$$J_2(n,b) = \int\limits_X^b \xi^{-9/4-\sigma} \frac{\cos(4\pi\sqrt{n\xi}-\pi/4)\sin(TV+2\pi\xi U-\pi\xi+\pi/4)}{VU^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi\,.$$

Hence our task is to evaluate the integral

$$\int\limits_X^b \frac{\exp(i(\pm 4\pi\sqrt{n\xi}-TV-2\pi\xi U+\pi\xi))}{\xi^{\sigma+\mu}VU^{1/2}(U^2-1/4)^\sigma(U+1/2)^2}\,d\xi = \left(\frac{2\pi}{T}\right)^\sigma I_\mu(n,b;\pm)\,,$$

where

$$\begin{aligned} I_\mu(n,b;\pm) = \int\limits_{\sqrt{X}}^{\sqrt{b}} & x^{1-2\mu}\left(\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{-1/4} \\ & \times\left(\left(\frac{T}{2\pi x^2}+\frac14\right)^{1/2}+\frac12\right)^{-2}\exp\bigg\{i\bigg(\pm 4\pi x\sqrt{n} \\ & -2T\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)-(2\pi Tx^2+\pi^2x^4)^{1/2}+\pi x^2\bigg)\bigg\}\,dx\,. \end{aligned}$$

LEMMA 4. *For $b\ge T^2$ and for $\mu=7/4$ or $\mu=9/4$ we have*

$$\begin{aligned} I_\mu(n,b;\pm) = 2\delta_n&\left(\frac{2\pi}{T}\right)^2 n^{\mu-\sigma-1}\left(\frac{T}{2\pi}-n\right)^{7/2+2\sigma-2\mu}\left(\log\frac{T}{2\pi n}\right)^{-1} \\ &\times\exp\left(i\left(T-T\log\frac{T}{2\pi n}+\frac{\pi}{4}\right)\right) \\ &+O\left(\delta_n n^{\mu-\sigma-1}\left(\frac{T}{2\pi}-n\right)^{2+2\sigma-2\mu}T^{-3/2}\right)+O(e^{-cT-c\sqrt{nT}}) \\ &+O\left(X^{1/2+\sigma-\mu}\min\left\{1,\left|\pm 2\sqrt{n}+\sqrt{X}-\left(X+\frac{2T}{\pi}\right)^{1/2}\right|^{-1}\right\}\right) \\ &+O(b^{1/2+\sigma-\mu}n^{-1/2})\,, \end{aligned}$$

where c is a positive constant and

$$\delta_n = \begin{cases} 1 & \text{if } 1\le n< T/(2\pi),\ nX\le (T/(2\pi)-n)^2 \\ & \text{and the double sign takes } +\,, \\ 0 & \text{otherwise.} \end{cases}$$

This is a slight modification of Lemma 3 of Atkinson [1], and we omit the proof.

We have $\delta_n = 1$ if and only if $1 \le n \le B(T, \sqrt{X})$ and the double sign takes $+$. Apply the above lemma to $J_2(n, b)$, and substitute the result into (6.1). The contribution of the error term including b vanishes as b tends to infinity. Noting that $B(T, \sqrt{X}) \ll T$ and $T/2\pi - B(T, \sqrt{X}) \gg T$, we conclude that the total contribution of $J_2(n, b)$ to J is $O(T^{-\sigma-7/4})$.

Next, applying Lemma 4 to $J_1(n, b)$, we have

$$\text{(6.2)}\quad J_1(n,b) = -\left(\frac{2\pi}{T}\right)^{\sigma} \left\{ \delta_n \left(\frac{2\pi}{T}\right)^2 n^{3/4} \left(\log \frac{T}{2\pi n}\right)^{-1} \times \sin\left(T - T\log\frac{T}{2\pi n} + \frac{\pi}{4}\right) + O(R_1 + R_2 + R_3^+ + R_3^- + R_4) \right\},$$

where

$$R_1 = \delta_n n^{3/4} \left(\frac{T}{2\pi} - n\right)^{-3/2} T^{-3/2},$$
$$R_2 = e^{-cT - c\sqrt{nT}},$$
$$R_3^{\pm} = X^{-5/4} \min\left\{1, \left|\pm 2\sqrt{n} + \sqrt{X} - \left(X + \frac{2T}{\pi}\right)^{1/2}\right|^{-1}\right\},$$
$$R_4 = b^{-5/4} n^{-1/2}.$$

The contribution of the error term R_4 to $J(b)$ vanishes as b tends to infinity. The contribution of R_1 and R_2 can be easily estimated by $O(T^{-3})$. The contribution of R_3^+ is

$$\begin{aligned} &\ll T^{-\sigma-5/4} \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n) n^{\sigma-7/4} \min\{1, |2(\sqrt{n} - \sqrt{B(T, \sqrt{X})})|^{-1}\} \\ &= T^{-\sigma-5/4} \Big(\sum_{n \le B/2} + \sum_{B/2 < n \le B - \sqrt{B}} + \sum_{B-\sqrt{B} < n \le B + \sqrt{B}} + \sum_{B+\sqrt{B} < n < 2B} + \sum_{2B \le n} \Big) \\ &= T^{-\sigma-5/4}(R_{31} + R_{32} + R_{33} + R_{34} + R_{35}), \end{aligned}$$

say, where $B = B(T, \sqrt{X})$. Since $B \asymp T$ it is easy to see that $R_{31} = O(T^{\sigma-5/4} \log T)$ and $R_{35} = O(T^{\sigma-5/4})$. Next,

$$\text{(6.3)}\quad \begin{aligned} R_{32} &\ll B^{1/2} \sum_{B/2 < n \le B - \sqrt{B}} \sigma_{1-2\sigma}(n) n^{\sigma-7/4} (B-n)^{-1} \\ &\ll B^{\sigma-5/4} \sum_{\sqrt{B} \le n \le B/2} n^{-1} \sigma_{1-2\sigma}([B] - n), \end{aligned}$$

where $[B]$ means the greatest integer $\le B$. For any positive numbers x and

y, the elementary estimate

$$\sum_{x<n\le x+y}\sigma_{1-2\sigma}(n)\ll y+\sqrt{x} \tag{6.4}$$

holds (see Matsumoto–Meurman [10, (2.1)]). By using this inequality and partial summation, the last sum in (6.3) can be estimated by $O(\log T)$, whence $R_{32}=O(T^{\sigma-5/4}\log T)$. Quite similarly, we have $R_{34}=O(T^{\sigma-5/4}\log T)$. Also, since

$$R_{33}\ll\sum_{B-\sqrt{B}<n\le B+\sqrt{B}}\sigma_{1-2\sigma}(n)n^{\sigma-7/4},$$

the estimate $R_{33}=O(T^{\sigma-5/4})$ follows by using (6.4) again. Hence, the total contribution of R_3^+ is $O(T^{-5/2}\log T)$, and likewise for R_3^-, because $R_3^-\le R_3^+$ for any n. Therefore we now arrive at

$$J=-c_1\left(\frac{2\pi}{T}\right)^{\sigma+2}\sum_{n\le B}\sigma_{1-2\sigma}(n)n^{\sigma-1}\times\left(\log\frac{T}{2\pi n}\right)^{-1}\sin\left(T-T\log\frac{T}{2\pi n}+\frac{\pi}{4}\right)+O(T^{-5/2}\log T),$$

which by (5.2) implies

$$G_4^{**}=-i\Sigma_{2,\sigma}(T,X)+O(\log T),$$

since $c_1=-1/(2\pi^2\sqrt{2})$. Combining this with (4.2) and (3.6) gives

$$G_4=i\Sigma_{2,\sigma}(T,X)+O(\log T).$$

Combining this with (3.7)–(3.9) and (3.1), we obtain (1.2) when X is not an integer. This last condition can be removed, because we can easily show that $\Sigma_{j,\sigma}(T,X)-\Sigma_{j,\sigma}(T,X')\ll 1$ $(j=1,2)$ if $X-X'\ll\sqrt{T}$, by using (6.4) and the fact that $B(T,\sqrt{X})-B(T,\sqrt{X'})\ll\sqrt{T}$. The proof of Theorem 1 is, therefore, now complete.

7. An averaged formula. Now we consider the mean square of $E_\sigma(T)$. To prove the weak estimate (1.8), Theorem 1 is enough. But the proof of Theorem 2 requires the following ideas: the averaging technique introduced in Meurman [12]; the application of Montgomery–Vaughan's inequality as Preissmann [17] did; the application of the mean value theorem for Dirichlet polynomials similarly to Matsumoto–Meurman [10]. In this section we prove an averaged formula for $E_\sigma(T)$.

From (3.1) and (3.6)–(3.9) we get

$$E_\sigma(T)=\Sigma_{1,\sigma}(T,X)-iG_4^*-iG_4^{**}+O(1)$$

for $X \asymp T$. We average with respect to X. Let $X = (L+\mu)^2$, where $L \asymp \sqrt{T}$, $0 \le \mu \le M$ and $M \asymp \sqrt{T}$.

We note that in Matsumoto–Meurman [10] we chose $M \asymp T^{1/4}$. This was necessary to get $O(T^{-1/4})$ in [10, (3.29)]. In the present situation $O(1)$ is enough (and in fact the best we can get for $\sigma = 3/4$), and hence we may choose $M \asymp \sqrt{T}$.

We have

$$\frac{1}{M}\int_0^M \Sigma_{1,\sigma}(T,(L+\mu)^2)\,d\mu = \Sigma^*_{1,\sigma}(T,L,M)\,,$$

where $\Sigma^*_{1,\sigma}(T,L,M)$ is the same as $\Sigma_{1,\sigma}(T,(L+M)^2)$ except that its terms are multiplied by the function

$$w_1(n) = \begin{cases} 1 & \text{if } n \le L^2\,, \\ 1 + \dfrac{L}{M} - \dfrac{\sqrt{n}}{M} & \text{if } L^2 < n \le (L+M)^2\,. \end{cases}$$

From (3.6) and (4.1) we have

$$G_4^* \ll T^{-1/2}|\widetilde{\Delta}_{1-2\sigma}(X)|\,.$$

Hence, using Lemma 3, we obtain

$$\frac{1}{M}\int_0^M G_4^*\,d\mu \ll 1\,.$$

From (5.2) we have

$$\frac{1}{M}\int_0^M G_4^{**}\,d\mu = (i\sqrt{\pi})^{-1}T^{5/2}\frac{1}{M}\int_0^M J\,d\mu + O(1)$$

and

$$\frac{1}{M}\int_0^M J\,d\mu = c_1\left(\frac{2\pi}{T}\right)^{\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}K_n + O(T^{-7/4-\sigma})\,,$$

where

$$\begin{aligned} K_n = \frac{1}{M}\int_0^M\int_{L+\mu}^{\infty} & x^{-5/2}\left(\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{-1/4} \\ & \times\left(\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{1/2}+\frac{1}{2}\right)^{-2}\sin(f(T,x^2)-\pi x^2+\pi/2) \\ & \times\left\{\cos(4\pi x\sqrt{n}+\pi/4) + c_1^{-1}c_4\frac{1}{x\sqrt{n}}\cos(4\pi x\sqrt{n}-\pi/4)\right\}dx\,d\mu\,. \end{aligned}$$

This is obtained by applying Lemma 1, and the constants c_1 and c_4 are as in Lemma 1. The change of the summation and the integrations can be justified as in Matsumoto–Meurman [10]. We can evaluate K_n by Jutila [8, Theorem 2.2]. The saddle point is $x_0 = n^{-1/2}(T/(2\pi) - n)$. Note that $c_0 = 1$ in Jutila's theorem. We get

$$\begin{aligned} K_n = &- w_2(n,T)\left(\frac{2\pi}{T}\right)^2 n^{3/4}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos(g(T,n)) \\ &+ O\Big(M^{-1}T^{-5/4}\sum_{j=0}^{1}\min\{1,(\sqrt{n}-\sqrt{B(T,L+jM)})^{-2}\}\Big) \\ &+ O(R(n)T^{-5/2}n^{3/4}), \end{aligned}$$

where

$$w_2(n,T) = \begin{cases} 1 & \text{if } n < B(T,L+M), \\ \dfrac{1}{M}\left(\dfrac{T}{2\pi\sqrt{n}} - \sqrt{n} - L\right) & \text{if } B(T,L+M) \le n < B(T,L), \\ 0 & \text{if } n \ge B(T,L) \end{cases}$$

and

$$R(n) = \begin{cases} T^{-1/2} & \text{if } n < B(T,L+M), \\ 1 & \text{if } B(T,L+M) \le n < B(T,L), \\ 0 & \text{if } n \ge B(T,L). \end{cases}$$

Hence

$$\begin{aligned} \frac{1}{M}\int_0^M J\,d\mu = -c_1\left(\frac{2\pi}{T}\right)^{\sigma+2} &\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-1}w_2(n,T) \\ &\times\left(\log\frac{T}{2\pi n}\right)^{-1}\cos(g(T,n)) + O(T^{-5/2}). \end{aligned}$$

Collecting the above results, we now obtain

$$E_\sigma(T) = \Sigma^*_{1,\sigma}(T,L,M) - \Sigma^*_{2,\sigma}(T,L,M) + O(1), \tag{7.1}$$

where $\Sigma^*_{2,\sigma}(T,L,M)$ is the same as $\Sigma_{2,\sigma}(T,B(T,L))$ except that its terms are multiplied by $w_2(n,T)$.

8. Proof of Theorem 2. Let $T \le t \le 2T$. From (7.1) with $L = M = \frac{1}{2}\sqrt{T}$ we have

$$E_\sigma(t) = \Sigma^*_{1,\sigma}(t,\tfrac12\sqrt{T},\tfrac12\sqrt{T}) - \Sigma^*_{2,\sigma}(t,\tfrac12\sqrt{T},\tfrac12\sqrt{T}) + O(1).$$

We shall prove that

$$\int_T^{2T}(\Sigma^*_{1,\sigma}(t,\tfrac12\sqrt{T},\tfrac12\sqrt{T}))^2\,dt = \begin{cases} \zeta^2\left(\dfrac32\right)\dfrac{\zeta(2)}{\zeta(3)}T\log T + O(T) & \text{if } \sigma = \dfrac34, \\ O(T) & \text{if } \sigma > \dfrac34, \end{cases} \tag{8.1}$$

and that

$$\int_T^{2T} (\Sigma^*_{2,\sigma}(t, \tfrac{1}{2}\sqrt{T}, \tfrac{1}{2}\sqrt{T}))^2\, dt = O(T)\,. \tag{8.2}$$

Theorem 2 then follows easily.

Consider the left-hand side of (8.1). We square out and integrate term by term. The non-diagonal terms give $O(T)$, as in Matsumoto–Meurman [10]. The diagonal terms contribute

$$\frac{1}{2}\sum_{n\le T} w_1(n)^2 n^{2\sigma-2}\sigma_{1-2\sigma}(n)^2 \times \int_T^{2T} \left(\frac{t}{2\pi}\right)^{1-2\sigma} \left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2} \left(\frac{2t}{\pi n}+1\right)^{-1/2} dt$$
$$+\frac{1}{2}\sum_{n\le T} w_1(n)^2 n^{2\sigma-2}\sigma_{1-2\sigma}(n)^2 \times \int_T^{2T} \left(\frac{t}{2\pi}\right)^{1-2\sigma} \left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2} \left(\frac{2t}{\pi n}+1\right)^{-1/2} \cos(2f(t,n))\, dt\,.$$

Here we have used the formula $\cos^2 z = \frac{1}{2} + \frac{1}{2}\cos(2z)$. The second sum is $O(T^{2-2\sigma})$, which we can see by estimating the integral by Ivić [5, Lemma 15.3]. For the first sum we use

$$(\operatorname{arsinh} z)^{-2} = z^{-2} + O(1) \qquad (z \to 0)$$

and

$$(z+1)^{-1/2} = z^{-1/2} + O(z^{-3/2}) \qquad (z \to \infty)$$

to deduce that it is equal to

$$\frac{1}{2}\sum_{n\le T} w_1(n)^2 n^{2\sigma-2}\sigma_{1-2\sigma}(n)^2 \int_T^{2T} \left(\frac{t}{2\pi}\right)^{1-2\sigma} \left(\frac{2t}{\pi n}\right)^{1/2} dt + O(T)\,.$$

For $\sigma > 3/4$ this is $O(T)$, proving the second part of (8.1). For $\sigma = 3/4$ the above is equal to

$$T\sum_{n\le T} w_1(n)^2 n^{-1}\sigma_{-1/2}(n)^2 + O(T) = T\sum_{n\le T/4} n^{-1}\sigma_{-1/2}(n)^2 + O(T)\,,$$

because the terms with $T/4 < n \le T$ contribute $O(T)$ and for $n \le T/4$ we have $w_1(n) = 1$. By Titchmarsh [18, (1.3.3)] and Perron's formula we get

$$\sum_{n\le T/4} n^{-1}\sigma_{-1/2}(n)^2 = \frac{\zeta^2(3/2)\zeta(2)}{\zeta(3)}\log T + O(1)\,,$$

which proves the first part of (8.1).

Next we prove (8.2). The left-hand side of (8.2) is

$$\ll T^{1-2\sigma}\int_T^{2T}\Big|\sum_n w_2(n,t)\sigma_{1-2\sigma}(n)n^{\sigma-1+it}\Big(\log\frac{t}{2\pi n}\Big)^{-1}\Big|^2\,dt\,. \tag{8.3}$$

We proceed to remove the factor $w_2(n,t)/\log(t/(2\pi n))$ in the above sum by partial summation. We have $B(t,\sqrt{T})\ge \alpha T$ for some sufficiently small positive α. Consequently, $w_2(n,t)=1$ for $n\le \alpha T$. For $n>\alpha T$ we have $w_2(n+1,t)-w_2(n,t)\ll T^{-1}$. It follows that

$$w_2(n+1,t)\Big(\log\frac{t}{2\pi(n+1)}\Big)^{-1}-w_2(n,t)\Big(\log\frac{t}{2\pi n}\Big)^{-1} \ll \Big(n\log^2\frac{t}{2\pi n}\Big)^{-1}\ll\Big(n\log^2\frac{T}{n}\Big)^{-1}. \tag{8.4}$$

In particular, since $w_2(n,t)=0$ for $n\ge B(t,\frac12\sqrt{T})$, we have

$$w_2(\beta,t)\Big(\log\frac{t}{2\pi\beta}\Big)^{-1}\ll T^{-1}, \tag{8.5}$$

where β means the greatest integer $\le B(t,\frac12\sqrt{T})$. Now using (8.4), (8.5) and partial summation we see that the sum $\sum_n$ in (8.3) is

$$\ll T^{-1}\Big|\sum_{n=1}^{\beta}\sigma_{1-2\sigma}(n)n^{\sigma-1+it}\Big|+\sum_{n=1}^{\beta-1}\Big(n\log^2\frac{T}{n}\Big)^{-1}\Big|\sum_{m=1}^{n}\sigma_{1-2\sigma}(m)m^{\sigma-1+it}\Big|.$$

The first sum here is trivially $O(T^{\sigma})$, so its contribution to the left-hand side of (8.2) is $O(1)$. Hence it remains to show that

$$\int_T^{2T}\Big(\sum_{n\le T/2}\Big(n\log^2\frac{T}{n}\Big)^{-1}\Big|\sum_{m=1}^{n}\sigma_{1-2\sigma}(m)m^{\sigma-1+it}\Big|\Big)^2dt\ll T^{2\sigma}\,,$$

since $\beta-1\le T/2$. Here we use Schwarz's inequality, take the integration under the summation and use the mean value theorem for Dirichlet polynomials (see Ivić [5, Theorem 5.2]). We also need the elementary estimate

$$\sum_{n\le x}\sigma_{1-2\sigma}(n)^2\ll x$$

(see [10, §2]). Then (8.2) follows and the proof of Theorem 2 is complete.

9. Proof of Theorem 3. In this final section we assume $1/2<\sigma<3/4$. Let G be a parameter satisfying $G=o(T)$. Our first goal is to deduce from (1.2) a suitable expression for $E_\sigma(T+u)$, where $|u|\le G$. In (1.2) we take $X=T$. For $n\le T$ and $|u|\le G$ we find by straightforward calculation that

$$e(T+u,n)=e(T,n)(1+O(|u|T^{-1}))=O(1)\,,$$

$$(T+u)^{3/4-\sigma} = T^{3/4-\sigma}(1+O(|u|T^{-1}))\,,$$

and

$$f(T+u,n) = f(T,n) + 2u\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}} + u^2 d(T,n) + O(|u|^3T^{-2})\,,$$

where $d(T,n)$ is real and

$$d(T,n) \ll T^{-1} \tag{9.1}$$

(see Meurman [11, p. 363]). We have

$$B(T+u,\sqrt{T}) = c_6T + O(|u|)\,,$$

where

$$c_6 = \frac{1}{4\pi^2}\left(\frac{1}{2\pi} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{2\pi}\right)^{1/2}\right)^{-1} < \frac{1}{4\pi^2}\,. \tag{9.2}$$

For $n \le B(T+u,\sqrt{T})$ and $|u| \le G$ we have

$$(T+u)^{1/2-\sigma} = T^{1/2-\sigma}(1+O(|u|T^{-1}))\,,$$

$$\left(\log\frac{T+u}{2\pi n}\right)^{-1} = \left(\log\frac{T}{2\pi n}\right)^{-1} + O(|u|T^{-1})\,,$$

and

$$g(T+u,n) = g(T,n) + u\log\frac{T}{2\pi n} + \frac{u^2}{2T} + O(|u|^3T^{-2})\,.$$

Using these facts and (6.4), it may be easily deduced from (1.2) that for $|u| \le G$ we have

$$\begin{aligned} E_\sigma(T+u) &= \sqrt{2}\left(\frac{T}{2\pi}\right)^{3/4-\sigma}\sum_{n\le T} a(n)e(T,n) \\ &\quad\times\cos\left(f(T,n) + 2u\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}} + u^2d(T,n)\right) \\ &\quad - 2\left(\frac{T}{2\pi}\right)^{1/2-\sigma}\sum_{n\le c_6T}\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\log\frac{T}{2\pi n}\right)^{-1} \\ &\quad\times\cos\left(g(T,n) + u\log\frac{T}{2\pi n} + \frac{u^2}{2T}\right) \\ &\quad + O(\log T) + O(G^3T^{-3/2}) + O(GT^{-1/2})\,, \end{aligned} \tag{9.3}$$

where

$$a(n) = (-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-5/4}\,.$$

Set $Z = \sqrt{2T/\pi Y}$, and suppose that Y satisfies

$$1 \le Y \le T^{1/4}. \tag{9.4}$$

Our next goal is to deduce from (9.3) an expression for

$$E_\sigma(T,Y) = \int_{-G/Z}^{G/Z} E_\sigma(T+Zt)e^{-t^2}\,dt. \tag{9.5}$$

For this purpose we have to consider the integrals

$$I_1(n) = \int_{-G/Z}^{G/Z} \exp\left(2iZ\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}}\right)t - (1 - id(T,n)Z^2)t^2\right)dt \qquad (n \le T) \tag{9.6}$$

and

$$I_2(n) = \int_{-G/Z}^{G/Z} \exp\left(iZ\left(\log\frac{T}{2\pi n}\right)t - \left(1 - \frac{iZ^2}{2T}\right)t^2\right)dt \qquad (n \le c_6T). \tag{9.7}$$

By the general formula

$$\int_{-\infty}^{\infty} \exp(At - Bt^2)\,dt = (\pi/B)^{1/2}\exp(A^2/4B) \qquad (\mathrm{Re}(B) > 0)$$

(see Ivić [5, (A.38)]) we get

$$I_1(n) = \left(\frac{\pi}{1 - id(T,n)Z^2}\right)^{1/2}\exp\left(-\frac{(Z\operatorname{arsinh}\sqrt{\pi n/2T})^2}{1 - id(T,n)Z^2}\right) + O(e^{-(G/Z)^2})$$

and

$$I_2(n) = \left(\frac{\pi}{1 - iZ^2/2T}\right)^{1/2}\exp\left(-\frac{(Z\log(T/2\pi n))^2}{4 - 2iZ^2/T}\right) + O(e^{-(G/Z)^2}).$$

Suppose now that $G \ge T^{1/2+\varepsilon}$ for some fixed positive ε. Then $\exp(-(G/Z)^2) \ll \exp(-T^\varepsilon)$. In case $n \le Y^2$ we have

$$\left(\frac{\pi}{1 - id(T,n)Z^2}\right)^{1/2} = \pi^{1/2} + O(Z^2T^{-1}) = \pi^{1/2} + O(Y^{-1})$$

by (9.1), since $d(T,n)$ is real. Also, using (9.4) and the formula $\operatorname{arsinh} x = x + O(x^3)$, we have

$$\begin{aligned} -\frac{(Z\operatorname{arsinh}\sqrt{\pi n/2T})^2}{1 - id(T,n)Z^2} &= -Z^2\frac{\pi n}{2T} + O\left(\left(\frac{Zn}{T}\right)^2\right) + O(Z^4T^{-2}n) \\ &= -nY^{-1} + O(nY^{-2}). \end{aligned}$$

Hence it follows that, for $n \le Y^2$,

$$I_1(n) = \pi^{1/2} e^{-n/Y}(1 + O(Y^{-1}) + O(nY^{-2})) + O(\exp(-T^{\varepsilon})) = \pi^{1/2} e^{-n/Y} + O(n^{-1})\,. \tag{9.8}$$

In case $Y^2 < n \le T$ we have

$$I_1(n) \ll \exp(-c_7 Z^2 n/T) + \exp(-T^{\varepsilon}) = \exp(-2c_7 n/\pi Y) + \exp(-T^{\varepsilon}) \tag{9.9}$$

with some positive c_7. For any $n \le c_6 T$ we have

$$I_2(n) \ll \exp\biggl(-c_8\biggl(Z \log \frac{T}{2\pi c_6 T}\biggr)^2\biggr) + \exp(-T^{\varepsilon}) \ll \exp(-T^{\varepsilon}) \tag{9.10}$$

with some positive c_8. By (9.3) and (9.5)–(9.7) we get

$$\begin{aligned} E_\sigma(T,Y) = \sqrt{2}\biggl(\frac{T}{2\pi}\biggr)^{3/4-\sigma} &\sum_{n \le T} a(n) e(T,n) \operatorname{Re}(e^{if(T,n)} I_1(n)) \\ - 2\biggl(\frac{T}{2\pi}\biggr)^{1/2-\sigma} &\sum_{n \le c_6 T} \sigma_{1-2\sigma}(n) n^{\sigma-1} \biggl(\log \frac{T}{2\pi n}\biggr)^{-1} \\ &\times \operatorname{Re}(e^{ig(T,n)} I_2(n)) + O(G^3 T^{-3/2})\,. \end{aligned}$$

Here we have combined the error terms using $G \ge T^{1/2+\varepsilon}$. Then we use (9.8)–(9.10) to obtain

$$\begin{aligned} E_\sigma(T,Y) = \sqrt{2\pi}\biggl(\frac{T}{2\pi}\biggr)^{3/4-\sigma} &\sum_{n \le Y^2} a(n) e(T,n) e^{-n/Y} \cos(f(T,n)) \\ &+ O(G^3 T^{-3/2}) + O(T^{3/4-\sigma})\,. \end{aligned}$$

Now we choose $G = T^{3/4-\sigma/3}$ whence $T^{1/2+\varepsilon} \le G = o(T)$ with $\varepsilon = 1/4 - \sigma/3$, as required. Then, since $e(T,n) = 1 + O(n/T)$ and

$$f(T,n) = \sqrt{8\pi n T} - \pi/4 + O(n^{3/2} T^{-1/2})$$

(see [5, (15.74), (15.75)]), and noting (9.4), we get easily

$$E_\sigma(T,Y) = \sqrt{2\pi}\biggl(\frac{T}{2\pi}\biggr)^{3/4-\sigma} (S(T,Y) + O(1))\,, \tag{9.11}$$

where

$$S(T,Y) = \sum_{n \le Y^2} a(n) e^{-n/Y} \cos(\sqrt{8\pi n T} - \pi/4)\,. \tag{9.12}$$

From (9.5) and (9.11) it is clear that Theorem 3 follows from

LEMMA 5. *For any positive* T_1 *we can choose* $T \ge T_1$ *and* Y *satisfying* (9.4) *for which*

$$S(T,Y) \ge 10^{-11}\zeta(2\sigma)^{-2}(\log T)^{\sigma-1/4}. \tag{9.13}$$

To prove Lemma 5, we shall first obtain a lower bound for the sum

$$\varrho(x) = \sum_{n\le x}(-1)^n\sigma_{1-2\sigma}(n).$$

LEMMA 6. *There exists a constant* $c_9 = c_9(\sigma) \ge 1$ *such that* $\varrho(x) \ge x/12$ *for any* $x \ge c_9$.

Proof. Since

$$\sigma_{1-2\sigma}(2n) \ge 1 + 2^{1-2\sigma}\sigma_{1-2\sigma}(n)$$

and

$$\sum_{n\le x}\sigma_{1-2\sigma}(n) \sim \zeta(2\sigma)x$$

(see (1.4)), it follows that

$$\begin{aligned}\varrho(x) &= 2\sum_{n\le x/2}\sigma_{1-2\sigma}(2n) - \sum_{n\le x}\sigma_{1-2\sigma}(n)\\ &\ge 2[x/2] + 2^{2-2\sigma}\sum_{n\le x/2}\sigma_{1-2\sigma}(n) - \sum_{n\le x}\sigma_{1-2\sigma}(n)\\ &\sim (1 + (2^{1-2\sigma}-1)\zeta(2\sigma))x.\end{aligned}$$

By Titchmarsh [18, (2.2.1)], the coefficient of x equals

$$1 - \sum_{n=1}^{\infty}(-1)^{n-1}n^{-2\sigma} \ge 2^{-2\sigma} - 3^{-2\sigma} \ge \frac{1}{2}\cdot 3^{-3/2} > \frac{1}{12},$$

which completes the proof of Lemma 6.

We denote by q the greatest integer $\le 10^8\zeta(2\sigma)^2$. Clearly we may suppose that $T_1 \ge \exp((c_9q)^4)$. Let $Y = \log T_1$. Then $Y \ge 1$, as required in (9.4). We apply Dirichlet's theorem (see Ivić [5, Lemma 9.1]) to find a T satisfying

$$T_1 \le T \le T_1 q^{2qY}, \qquad \|\sqrt{2nT/\pi}\| \le q^{-1} \qquad (1 \le n \le qY),$$

where $\|x\|$ denotes the distance of x from the nearest integer. Then $Y \le \log T \le T^{1/4}$ as required in (9.4). Moreover, it follows that

$$\log T \le \log T_1 + 2qY\log q \le q^2 Y$$

whence

$$Y^{\sigma-1/4} \ge q^{1/2-2\sigma}(\log T)^{\sigma-1/4} \ge q^{-1}(\log T)^{\sigma-1/4}. \tag{9.14}$$

Another consequence is that

$$\left|\frac{1}{\sqrt{2}} - \cos(\sqrt{8\pi nT} - \pi/4)\right| \tag{9.15}$$
$$= |\cos(-\pi/4) - \cos(\pm 2\pi\|\sqrt{2nT/\pi}\| - \pi/4)|$$
$$\le 2\pi\|\sqrt{2nT/\pi}\| \le 2\pi q^{-1} \qquad (1 \le n \le qY).$$

By a simple elementary argument we have

$$\sum_{n\le x} \sigma_{1-2\sigma}(n) \le \zeta(2\sigma)x. \tag{9.16}$$

Hence

$$\sum_{n\le x} |a(n)| \le 4\zeta(2\sigma)x^{\sigma-1/4}$$

and

$$\sum_{n>x} |a(n)|n^{-1} \le 3\zeta(2\sigma)x^{\sigma-5/4}.$$

Using the last two inequalities and (9.15) we get

$$S(T,Y) = S_1(Y) - S_2(Y) - S_3(Y), \tag{9.17}$$

where

$$S_1(Y) = \frac{1}{\sqrt{2}} \sum_{n\le qY} a(n)e^{-n/Y},$$

$$S_2(Y) = \sum_{n\le qY} a(n)e^{-n/Y}\left(\frac{1}{\sqrt{2}} - \cos(\sqrt{8\pi nT} - \pi/4)\right) \tag{9.18}$$
$$\le 2\pi q^{-1} \sum_{n\le qY} |a(n)| \le 8\pi\zeta(2\sigma)q^{-1/2}Y^{\sigma-1/4},$$

$$S_3(Y) = -\sum_{qY<n\le Y^2} a(n)e^{-n/Y}\cos(\sqrt{8\pi nT} - \pi/4) \tag{9.19}$$
$$\le Y\sum_{n>qY} |a(n)|n^{-1} \le 3\zeta(2\sigma)q^{-1/2}Y^{\sigma-1/4}.$$

Consider $S_1(Y)$. We define $\phi(x) = x^{\sigma-5/4}e^{-x/Y}$. Then

$$\phi'(x) = -((\tfrac{5}{4} - \sigma)x^{-1} + Y^{-1})\phi(x),$$

which is negative, and for $1 \le x \le c_9$ we have $|\phi'(x)| \le 2/x$. Using these facts, (9.16) and partial summation we get

$$S_1(Y) = \frac{1}{\sqrt{2}} \sum_{n\le qY} (-1)^n \sigma_{1-2\sigma}(n)\phi(n) = S_{11}(Y) - S_{12}(Y) - S_{13}(Y),$$

where

$$S_{11}(Y) = -\frac{1}{\sqrt{2}} \int_{c_9}^{qY} \phi'(x)\varrho(x)\,dx\,,$$

$$S_{12}(Y) = \frac{1}{\sqrt{2}} \int_{1}^{c_9} \phi'(x)\varrho(x)\,dx \le \sqrt{2}\zeta(2\sigma)c_9 \le \zeta(2\sigma)q^{-1/2}Y^{\sigma-1/4}\,,$$

$$S_{13}(Y) = -\frac{1}{\sqrt{2}}\phi(qY)\varrho(qY) \le \zeta(2\sigma)q^{-1/2}Y^{\sigma-1/4}\,.$$

Consider $S_{11}(Y)$. Since $Y \ge c_9$, we get, by Lemma 6,

$$S_{11}(Y) \ge -\frac{1}{\sqrt{2}} \int_{Y}^{qY} \phi'(x)\varrho(x)\,dx \ge -\frac{Y}{12\sqrt{2}} \int_{Y}^{qY} \phi'(x)\,dx$$

$$= \frac{Y}{12\sqrt{2}}(\phi(Y) - \phi(qY)) \ge \frac{1}{100}Y^{\sigma-1/4}\,.$$

Hence

$$S_1(Y) \ge \left(\frac{1}{100} - 2\zeta(2\sigma)q^{-1/2}\right)Y^{\sigma-1/4}\,.$$

Combined with (9.17)–(9.19) and (9.14) this gives

$$S(T,Y) \ge c_{10}Y^{\sigma-1/4} \ge c_{10}q^{-1}(\log T)^{\sigma-1/4}\,,$$

where $c_{10} = \frac{1}{100} - (8\pi+5)\zeta(2\sigma)q^{-1/2}$. By the choice of q we have

$$c_{10}q^{-1} \ge \frac{1}{200}q^{-1} > 10^{-11}\zeta(2\sigma)^{-2}\,,$$

which completes the proof of Lemma 5, and hence of Theorem 3.

References

[1] F. V. Atkinson, *The mean-value of the Riemann zeta function*, Acta Math. 81 (1949), 353–376.

[2] K. Chandrasekharan and R. Narasimhan, *Approximate functional equations for a class of zeta-functions*, Math. Ann. 152 (1963), 30–64.

[3] J. L. Hafner, *On the representation of the summatory functions of a class of arithmetical functions*, in: Analytic Number Theory, M. I. Knopp (ed.), Lecture Notes in Math. 899, Springer, 1981, 148–165.

[4] D. R. Heath-Brown, *The mean value theorem for the Riemann zeta-function*, Mathematika 25 (1978), 177–184.

[5] A. Ivić, *The Riemann Zeta-Function. The Theory of the Riemann Zeta-Function with Applications*, Wiley, 1985.

[6] —, *Mean Values of the Riemann Zeta Function*, Lectures on Math. 82, Tata Inst. Fund. Res., Springer, 1991.

[7] —, *La valeur moyenne de la fonction zêta de Riemann*, Sém. Théorie des Nombres 1990/91, Université Orsay, Paris, to appear.

[8] M. Jutila, *A Method in the Theory of Exponential Sums*, Lectures on Math. 80, Tata Inst. Fund. Res., Springer, 1987.

[9] K. Matsumoto, *The mean square of the Riemann zeta-function in the critical strip*, Japan. J. Math. 15 (1989), 1–13.

[10] K. Matsumoto and T. Meurman, *The mean square of the Riemann zeta-function in the critical strip II*, Journées Arithmétiques de Genève 1991, Astérisque 209 (1992), 265–274.

[11] T. Meurman, *A generalization of Atkinson's formula to L-functions*, Acta Arith. 47 (1986), 351–370.

[12] —, *On the mean square of the Riemann zeta-function*, Quart. J. Math. Oxford (2) 38 (1987), 337–343.

[13] —, *A simple proof of Voronoï's identity*, to appear.

[14] —, *Voronoï's identity for the Riesz mean of* $\sigma_\alpha(n)$, unpublished manuscript, 24 pp.

[15] Y. Motohashi, *The mean square of* $\zeta(s)$ *off the critical line*, unpublished manuscript, 11 pp.

[16] A. Oppenheim, *Some identities in the theory of numbers*, Proc. London Math. Soc. (2) 26 (1927), 295–350.

[17] E. Preissmann, *Sur la moyenne quadratique de la fonction zêta de Riemann*, preprint.

[18] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford Univ. Press, Oxford 1951.

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
IWATE UNIVERSITY
MORIOKA 020
JAPAN

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TURKU
SF-20500 TURKU
FINLAND

Received on 22.12.1992 (2355)