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THE MEAN-VARIANCE APPROACH TO PORTFOLIO OPTIMIZATION SUBJECT TO TRANSACTION COSTS

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Abstract Transaction costs are a source of concern for portfolio managers. Due to nonlinearity of the cost function, the ordinary quadratic programming solution technique cannot be applied. This paper addresses the portfolio optimization problem subject to transaction costs. The transaction cost is assumed to be a V-shaped function of difference between an existing and new portfolio. A nonlinear programming solution technique is used to solve the proposed problem. The portfolio optimization system called POSTRAC (*P*ortfolio *O*ptimization *S*ystem with *TRA*nsaction *C*osts) is proposed. The experimental analysis indicates that ignoring the transaction costs results in inefficient portfolios. It is also shown that there does not exist statistically significant difference in portfolio performance with different methods to estimate the expected return of securities, when considering the transaction costs into the portfolio return.

1. Introduction

Transaction costs are a source of concern for portfolio managers. Due to the change in expectation of a future return of securities, most applications of portfolio optimization involve the revision of an existing portfolio. This revision entails both purchases and sales of securities along with transaction costs.

There are observations as to the magnitude of the transaction costs at the U. S. stock market in literature. Schreiner and Smith [11] estimated percentage brokerage commissions over the period from 1968 to 1978. The small investor had been charged more than the large investor. It was also shown that the more per share, the less commissions were. Loeb [5] reported findings concerning the total costs, which included a market maker's spread, price concessions and commissions. The total costs of trading a given size of stock decreased with increase in the size of the market capitalization. For a given size of the market capitalization, the larger the size of trading, the more the costs were. The largest costs were associated with a large size of trading in the small market capitalization. The smallest costs were associated with small trading in the large market capitalization.

In contrast, the transaction costs at the Japanese stock market are rather fixed. It is a piecewise disjunctive linear function of an amount of stock to be traded. When an amount of stock to be traded is less than 1 million Japanese yen, the costs are 1.15% of its amount. Within a range of 1 million to 5 million Japanese yen for trading, the costs are the sum of 2500 Japanese yen and 0.9% of its amount. From 5 million to 10 million Japanese yen, the costs are set to 0.7% of its amount plus, 12500 Japanese yen. As can be seen, the costs consist of a fixed charge and a proportional charge to an amount of stock to be traded. A fixed part of the costs increases up to 78500 Japanese yen as an amount of the traded stock increases. The other part of the costs decreases down to 0.075% with increase in an amount of the traded stock.

The mean-variance approach introduced by Markowitz [7] is often employed into portfolio constructing models. It has gained widespread acceptance as a practical tool for portfolio construction. The problems are usually formulated with a quadratic nonlinear objective function subject to linear constraints and solved by the quadratic programming solution technique. The technique simultaneously allocates securities within the portfolio constructing decision framework. Within the quadratic programming framework, to avoid undesirable movement (or transaction costs) from an existing portfolio to a new portfolio, one can apply turnover constraints (Perold [10]). Since constraints are to be formulated as a linear equation, the problem can be still solved by the quadratic programming solution technique. Introducing the turnover constraints, undesirability of having a large amount of the transaction costs is reduced to the point at which portfolio managers can endure. It should be noted, however, that the derived portfolio from such a problem may not be the one that achieves an optimal tradeoff between costs and benefits. Because of a direct impact of the transaction costs on investment performance, a real net return of securities should be evaluated by considering the costs into an expected return of securities. As mentioned by Arnott and Wagner [1], ignoring the transaction costs would lead to very ineffective portfolio implementation. Due to the complexity of the proposed problem of searching for an optimal portfolio subject to transaction costs, the quadratic programming solution technique cannot be utilized. This is because the transaction cost may be a separable, nonlinear, nonconvex function of a difference in holdings of new and existing portfolio.

The transaction costs are often incorporated into such models that deal with the multiperiod portfolio selection. Recent studies on the costs in portfolio optimization include Mulvey and Vladimirou [8], Dantzig and Infanger [3], and Gennotte and Jung [4]. Mulvey and Vladimirou [8] modeled multiperiod financial planning problems by the stochastic network programming. Within the framework of multiscenario generalized networks, they account for the linear transaction costs by means of arc multipliers. Using multi-stage stochastic linear programs, Dantzig and Infanger [3] incorporated the transaction costs into the multi-period asset allocation problems. They solved the problem by approximating the objective function by a piecewise linear function. Due to this linear transformation, nonlinearity of cost functions dissolved. Gennotte and Jung [4] examined the effect of proportional transaction costs on dynamic portfolio strategies for the two asset case (risky and riskless). They used a V-shaped function of additional investments as proportional transaction costs, and solved the problem by means of dynamic programming.

Although the transaction costs are taken into consideration by these researchers none of them has used the mean-variance approach directly using a V-shaped cost function. The objective of the paper is to address the optimal portfolio problem subject to transaction costs, and to examine the effect of the transaction costs upon portfolio performance. The proposed problem directly takes the transaction costs as a part of a portfolio return. The costs are assumed to be a V-shape function of a new and existing portfolio. Considering the transaction costs, the portfolio selection problem can be multiperiod. In this paper, however only the one-period portfolio revision is considered.

The remainder of the paper presents the portfolio problem subject to transaction costs in the next section, develops the portfolio optimization system to solve the proposed problem in the third section, and provides results of comparison of portfolios with and without transaction costs in the fourth section, followed by conclusions in the last section.

2. Portfolio Optimization Problem

A portfolio optimization problem is often formulated with the following quadratic expected utility function, EU_t,

$$EU_t = \underline{x}_t' \cdot E_t[\underline{r}_t] - \lambda \cdot \underline{x}_t' \cdot \underline{V}_t \cdot \underline{x}_t \tag{2.1}$$

where $\underline{x}_t$ is a portfolio and $\underline{r}_t$ is a security return vector and $\underline{V}_t$ is a variance-covariance matrix of security returns at time t. λ is a given parameter. $E_t[\bullet]$ is a conditional expectation operand at time t. 'denotes transpose. Maximizing the expected utility is the objective of the problem,

$$\begin{aligned} J(\underline{x}_t^*) &= \max EU_t \\ &= \max\{\underline{x}_t' \cdot E_t[\underline{r}_t] - \lambda \cdot \underline{x}_t' \cdot \underline{V}_t \cdot \underline{x}_t\} \end{aligned} \tag{2.2}$$

The following constraints are usually embedded,

$$\underline{x}_t' \cdot \underline{1} = 1 \tag{2.3}$$

$$\underline{x}_t \geq \underline{0} \tag{2.4}$$

where $\underline{1}$ is a unit vector. The first constraint implies that a fund is fully invested among risky and riskless securities. The other constraint is to prohibit short sales and borrowings.

Dealing with transaction costs, the predominant strategy is to use the turnover constraints. Letting $x_{i,t}^I$ be the amount by which a proportion in the i-th security is increased, and $x_{i,t}^D$ be the amount by which a proportion in the i-th security is decreased at time t, we have the following,

$$\left\{ \begin{array}{c} \underline{x}_t = \underline{x}_{t-1} - \underline{x}_t^D + \underline{x}_t^I \\ \underline{x}_t^D \geq \underline{0} \\ \underline{x}_t^I \geq \underline{0} \end{array} \right\} \tag{2.5}$$

where $\underline{x}_t^D = (x_{1,t,}^D x_{2,t,}^D x_{3,t,,,,}^D x_{n,t}^D)'$, and $\underline{x}_t^I = (\underline{x}_{1,t,}^I x_{2,t,}^I x_{3,t,,,,}^I x_{n,t}^I)'$. n is the number of securities. Notice that $\underline{x}_{t-1}$ is a given existing portfolio. Restricting the sum of purchases or sales, the turnover constraint is formulated by,

$$\underline{1}' \cdot \underline{x}_t^D \leq U \quad , \quad (or\ \underline{1}' \cdot \underline{x}_t^I \leq U) \tag{2.6}$$

where U is an upper bound. Perold [10] further introduced constraints called minimum trading size constraints to reduce undesirable small trades or holdings. The constraint for the i-th security is disjunctive,

$$\left\{ \begin{array}{c} l_{i,t} \leq x_{i,t} \leq l_{i,t}^0 \\ or \\ x_{i,t} = x_{i,t-1} \\ or \\ u_{i,t}^0 \leq x_{i,t} \leq u_{i,t} \end{array} \right\} \tag{2.7}$$

where $\{l_{i,t}, l_{i,t}^0\}$ and $\{u_{i,t}^0, u_{i,t}\}$ are the admissible lower and upper region of the i-th security, $x_{i,t,}$ at time t. With the turnover constraints and minimum trading size constraints, portfolio managers could avoid undesirable trading intentionally. The upper or lower limit of trades must be specified in advance by experiences. In no consideration of an effect of transaction costs on the portfolio return, however a non-optimal solution could result.

In the proposed problem, the transaction cost at time t, $\{c_{i,t}\}$, is assumed to be a V-shaped function of a difference between a given existing portfolio, $\underline{x}_{t-1}$, and a new portfolio, $\underline{x}_t$ and formulated explicitly into the portfolio return.

$$c_{i,t} = k_i|x_{i,t} - x_{i,t-1}| \tag{2.8}$$

where $c_{i,t}$ is a transaction cost of the i-th security at time t and k_i is a constant cost per change in a proportion of the i-th security. To have the problem solved by the nonlinear programming solution technique, the absolute term on the right hand side of Equation (2.8) is further transformed into the following,

$$c_{i,t} = k_i(d^+_{i,t} + d^-_{i,t}) \quad , \quad \forall i \tag{2.9}$$

where

$$x_{i,t} - x_{i,t-1} = d^+_{i,t} - d^-_{i,t} \quad , \quad \forall i \tag{2.10}$$

$$d^+_{i,t} \cdot d^-_{i,t} = 0 \quad , \quad \forall i \tag{2.11}$$

$$d^+_{i,t}, d^-_{i,t} \geq 0 \quad , \quad \forall i \tag{2.12}$$

If the difference, $x_{i,t} - x_{i,t-1}$, is positive (negative), $d^-_{i,t}$ ($d^+_{i,t}$) becomes zero and $d^+_{i,t}$ ($d^-_{i,t}$) becomes the difference. Due to Equation (2.11), both of d's cannot be positive at the same time. Accounting for the above transaction costs, our problem (Problem NLP) is,

Problem NLP

$$J(\underline{x}^*_t) = \max\{\underline{x}'_t \cdot E_t[\underline{r}_t] - \underline{c}'_t \cdot \underline{1} - \lambda \cdot \underline{x}'_t \cdot \underline{V}_t \cdot \underline{x}_t\} \tag{2.13}$$

subject to

$$c_{i,t} = k_i(d^+_{i,t} + d^-_{i,t}) \quad , \quad \forall i \tag{2.9}$$

$$x_{i,t} - x_{i,t-1} = d^+_{i,t} - d^-_{i,t} \quad , \quad \forall i \tag{2.10}$$

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$$d^+_{i,t}, d^-_{i,t} \geq 0 \quad , \quad \forall i \tag{2.12}$$

$$\underline{x}'_t \cdot \underline{1} = 1 \tag{2.3}$$

$$\underline{x}_t \geq \underline{0} \tag{2.4}$$

where $\underline{c}_t = (c_{1,t}, c_{2,t}, c_{3,t}, , , c_{n,t,})'$. No additional constraints, such as turnover constraints and minimum trading size constraints, are considered. Since the transaction costs are incorporated into the optimization framework, an optimal tradeoff between costs and benefits can be searched. It is noticeable that although the problem involves the dynamic change in portfolio, it is only applied to two sequential periods with a given $\{x_{i,t-1}\}$, not over the whole time horizon.

3. Portfolio Optimization System

In the proposed system, an expected return of each security was calculated in two ways. The first was to regard an arithmetical mean of the past data on a security return as the expected return, while the second was to estimate the expected return by using the simple regression model with a yield spread as an exogenous variable. Since the mean-variance asset allocation models can be highly sensitive to small perturbations in the expected return of securities and covariance, even a small change in the expected return and covariance structure may lead to large variations in allocations. Using a significantly correlated exogenous variable in the simple regression model may result in a far different expected return than an arithmetical mean. It is of interest to see how the transaction costs affect these variations. In what follows, each method to calculate the expected return is presented, then the proposed system is elaborated.

i) Arithmetical Mean Method

Using an arithmetical mean, a return of the i-th security at period T, $r_{i,T}$, is calculated from its past data,

(3.1)
$$r_{i,T} = ER_{i,T-1} + \varepsilon_{i,T}$$

where $ER_{i,T-1}$ is an arithmetical mean and calculated by,

(3.2)
$$ER_{i,T-1} = \frac{1}{T-t_0}\sum_{t=t_0}^{T-1} r_{i,t} \quad , \quad (T-1 > t_0)$$

$\varepsilon_{i,T}$ is a disturbance term, t_0 is the starting period and T is the current period for the problem, at which a new portfolio will be constructed. An arithmetical mean is regarded as the expected return.

(3.3)
$$E_T[r_{i,T}] = ER_{i,T-1}, \forall i$$

A variance, $VAR_T[\bullet]$, is estimated by,

(3.4)
$$VAR_T[r_{i,T}] = VAR_T[\varepsilon_{i,T}]$$

A covariance for the i-th and j-th securities is given by,

(3.5)
$$COV_T[r_{i,T}, r_{j,T}] = COV_T[\varepsilon_{i,T}, \varepsilon_{j,T}]$$

and an element of a variance-covariance matrix, $\underline{V}_T$, becomes:

(3.6)
$$v_{i,j}^T = \begin{cases} VAR_T[\varepsilon_{i,T}] & \text{if } i = j \\ COV_T[\varepsilon_{i,T}, \varepsilon_{j,T}] & \text{if } i \neq j \end{cases}$$

ii) Simple Regression Model

The second method is to use a simple regression model to calculate the expected return of a security. It was assumed that a return of each security responds to an exogenous variable with a lapse of time so that the expected return of a security at time t is calculated at time $t-1$. Letting $R_{i,T-1}$ be an exogenous variable for the i-th security at time T, the simple regression model used here is built as follows,

(3.7)
$$r_{i,T} = \beta_0 + \beta_1 \cdot R_{i,T-1} + \varepsilon_{i,T}$$

where β_0 and β_1 are unknown coefficients, and $\varepsilon_{i,T}$ is a disturbance term. The length of lags was set to be one period. Unknown coefficients are estimated by means of the method of least squares (see Maddala [6]).

Provided that coefficients were estimated, the expected return of a security is calculated by,

$$E_T[r_{i,T}] = \beta_0 + \beta_1 \cdot R_{i,T-1} \tag{3.8}$$

while a variance is estimated by Equation (3.4). An element of the resultant variance-covariance matrix is calculated by Equation (3.6).

iii) System Structure

Using the above two methods to estimate the expected return of a security, the portfolio optimization system called POSTRAC (*P*ortfolio *O*ptimization *S*ystem with *TRA*nsaction *C*osts) was proposed. The structure of POSTRAC is depicted in Figure 1. POSTRAC consists of four main parts. The first is to calculate the expected return of each security, while the second part is the input file generator for the optimizer. The third part is for optimization process. Due to nonlinearity of the constraints of the problem, the quadratic programming technique was unable to be utilized. Thus, the nonlinear optimizer called GAMS/MINOS was incorporated in POSTRAC. It uses MINOS (Murtagh and Saunders [9]) as an optimizer and GAMS (Brooke et al. [2]) as a user interface. This interface reads the input file generated in the second part. An optimal solution and other outputs from the optimizer are transformed into the output file. This is performed by the output file generator. Given the four parts, POSTRAC is built in the batch mode. It connects all four parts together in the order as in Figure 1.

4. Model Experimentation

i) Data Description

In this section, an effect of the transaction costs on an optimal portfolio was examined using the two methods. The analysis was achieved for the global asset allocation problem. Counties for investment were Japan, the U. K., the U. S., Germany, Canada, and France. In each country, two security indices, i. e., the stock market index and the Salomon Brothers bond performance index, were used. The data for both indices were from the database serviced by Datastream International. The Japanese one-month CD (certificate deposit) was used as a riskless security. Its data were obtained from the CAPITAL database. Foreign securities were hedged by one-month forward exchange rate, for which the Barclays Bank US dollar exchange rate quotes were used from the database by Datastream International. No foreign exchange exposure was considered.

A return of each security was defined as a relative growth rate of its index value. Since the data were discrete, the following approximation was used to estimate a return,

$$r_{i,t} = \frac{I_{i,t+1} - I_{i,t}}{I_{i,t}} = \frac{I_{i,t+1}}{I_{i,t}} - 1 \tag{4.1}$$

where $r_{i,t}$ is a return and $I_{i,t}$ is an index value for the i-th security at time t. For hedged securities, a return was calculated by,

$$r_{i,t} = \frac{I_{i,t+1}}{I_{i,t}} \cdot \frac{F_{i,t}}{S_{i,t}} - 1 \tag{4.2}$$

where $F_{i,t}$ and $S_{i,t}$ are the forward rate and spot rate of foreign exchange for the i-th security at time t on the basis of the Japanese currency.

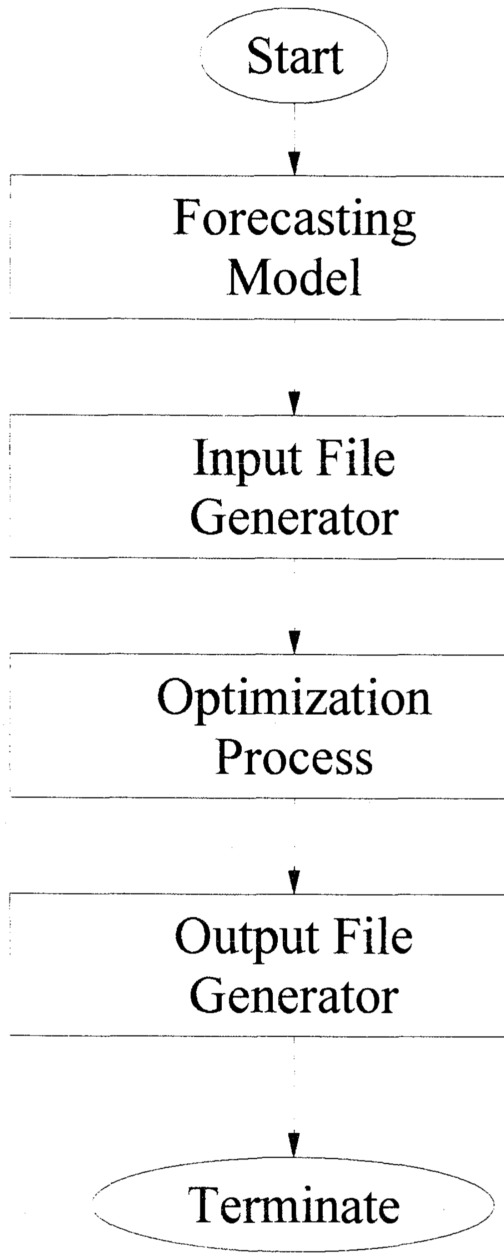


Figure 1. Structure of the proposed system POSTRAC
(Portfolio Optimization System with TRAnsaction Costs)

As for exogenous variables for security returns, a yield spread was used for the stock indices. A yield spread is defined by difference between long-term (10 years) yield-to-maturity for the bond and a reciprocal of a price-earnings ratio for the stock, so that it can be interpreted as profitability of the stock relative to the bond. Since a yield spread has been said to be highly correlated to a security return among portfolio managers, it is of interest to incorporate a yield spread into the model to estimate a security return, and to see its effect on portfolio performance discussed later. A yield spread was calculated by,

$$YS_{i,t} = LNG_{i,t} - \frac{1}{PER_{i,t}} \tag{4.3}$$

A spread between long-term (10 years) and short-term (3 months) yield-to- maturity was used for the bond indices, since this spread can give a convenient proxy for long-term profitability of the bond. It was calculated by,

$$SP_{i,t} = LNG_{i,t} - SHT_{i,t} \tag{4.4}$$

Each variable is defined as follows;

$YS_{i,t}$: a yield spread for the i-th stock index at time t,
$SP_{i,t}$: a spread for the i-th bond index at time t,
$PER_{i,t}$: a price-earnings ratio for the i-th stock index at time t,
$LNG_{i,t}$: long-term yield-to-maturity for the i-th bond index at time t,
$SHT_{i,t}$: short-term yield-to-maturity for the i-th bond index at time t.

A return of each security at time t was regressed on the corresponding exogenous variable with the one-period lapse of time. Monthly data from January 1985 to July 1991 were used, so that the expected return was calculated on the monthly basis. The regression period was set to 24 months, resulting in the analysis period form January 1987 to June 1991.

ii) Comparison of Efficient Frontiers

In using thirteen securities, an optimal portfolio was searched. A constant transaction cost coefficient, k_i, was assumed to be 1% per change in proportion of a security. It was applied to all securities. In order to investigate if there is an improvement in the portfolio construction, three efficient frontiers were compared. The first one was derived by solving the following Problem NLP' in a sequential fashion.

Problem NLP'

$$J(\underline{x}_t^*) = \min[\underline{x}_t' \cdot \underline{V}_t \cdot \underline{x}_t] \tag{4.5}$$

subject to

$$\underline{x}_t' \cdot E_t[\underline{r}_t] - \underline{c}_t' \cdot \underline{1} = R_0 \tag{4.6}$$

$$c_{i,t} = k_i(d_{i,t}^+ + d_{i,t}^-) \quad , \quad \forall i \tag{4.7}$$

$$x_{i,t} - x_{i,t-1} = d_{i,t}^+ - d_{i,t}^- \quad , \quad \forall i \tag{4.8}$$

$$d_{i,t}^+ \cdot d_{i,t}^- = 0 \quad , \quad \forall i \tag{4.9}$$

$$d_{i,t}^+, d_{i,t}^- \geq 0 \quad , \quad \forall i \tag{4.10}$$

(4.11) $$\underline{x}_t' \cdot \underline{1} = 1$$

(4.12) $$\underline{x}_t \geq \underline{0}$$

where R_0 is a given target return. The second was constructed by solving the same problem with the zero transaction costs, i. e., $k_i = 0$ for all i. The last frontier was built from the second frontier after subtracting the transaction costs from the derived portfolio return.

Each frontier except the third one, was built in the following way. Firstly the upper and lower bounds of a portfolio return along the efficient frontier were searched by using the nonlinear programming technique for the first frontier and the linear programming technique for the second frontier. A target return, R_0, was increased by one-twentieth of the difference between the upper and lower bounds to the lower bound sequentially up to the upper bound. The number of target returns used, thus was twenty-one. A set of points specified by a return-risk (standard deviation) combination of an optimal solution constitutes an efficient frontier. The simple regression model was used for the expected return calculation in this analysis. The analysis period was February 1987, for which the historical data from February 1985 to January 1987 were utilized for the regression. The proportion of each security in an existing portfolio was assigned to 7.7% (one-thirteenth). At this period, the riskless security had the least return among others.

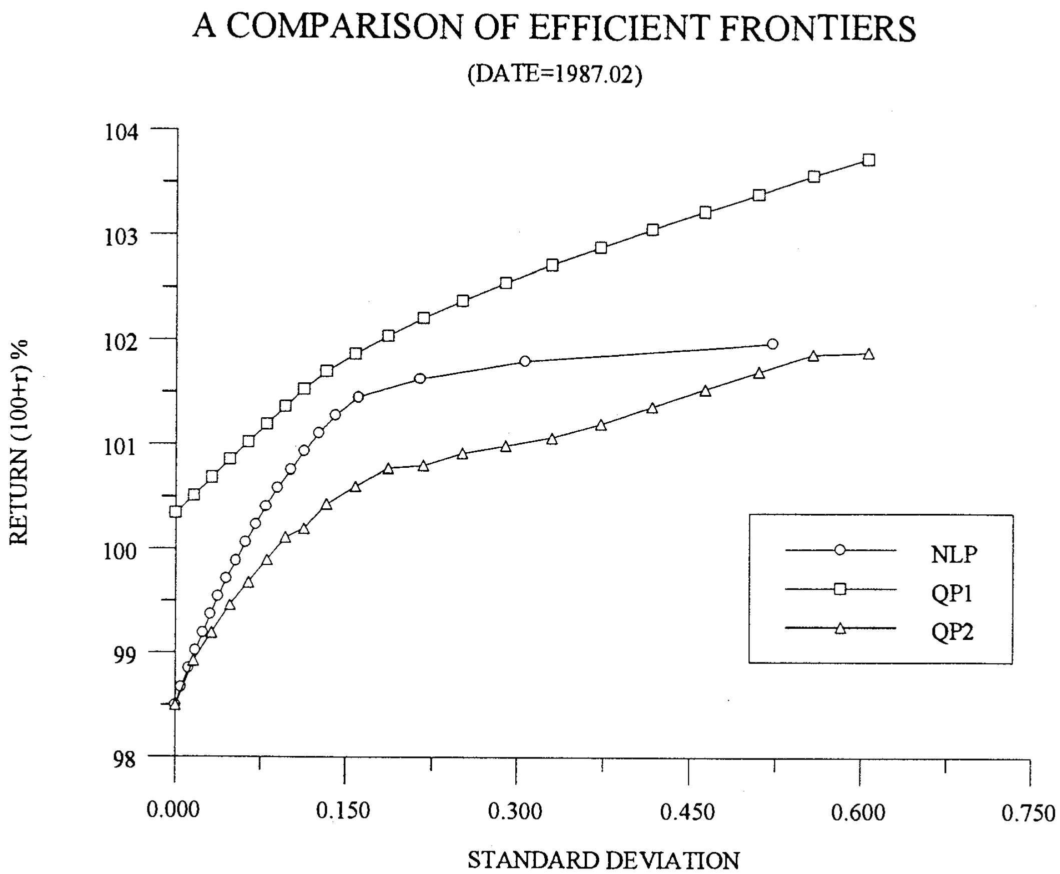


Figure 2. A comparison of efficient frontiers

Figure 2 depicts three frontiers searched. The first frontier was labeled by NLP, the second by QP1, and the last frontier was labeled by QP2. As can be observed in Figure 2, the third frontier (QP2) was inferior to the first frontier (NLP) and to the second (QP1). Since the QP2 frontier was derived after charging the transaction costs to the QP1 frontier, QP2 had less return than QP1. If an investor selects any portfolio on the QP1 frontier, its actual position on the return-risk (standard deviation) diagram results in the QP2 frontier. Because of an effect of the transaction costs, the NLP frontier was inferior to the QP1 frontier, while it was superior to the QP2 frontier. This implies that ignoring the transaction costs results in inefficient portfolio, QP2.

Given the same level of portfolio risk (standard deviation), a difference in portfolio returns from NLP and QP2 had the largest, 0.85% per month (10.69% per year) when the risk was 0.16. In other words, incorporating the transaction costs into the optimization framework could make 0.85% more profit than would be if they were ignored. This value might vary dependent upon an existing portfolio, however inferiority of the QP2 frontier may remain.

iii) Comparison of Total Transaction Cost

Comparison of the efficient frontiers showed that the portfolio was improved by taking the transaction costs into account within the optimization framework. In the following, a change in the total transaction costs imposed over the time horizon, was presented so that a cumulative effect of the transaction costs on portfolio performance can be revealed. For the comparison purpose, two portfolios were constructed. One was an optimal solution of the problem subject to transaction costs. This was searched by solving Problem NLP with a given lambda, λ, and labeled by NLP_COST in figures. The other was an optimal solution of the problem with the zero transaction costs. This was searched by solving the following Problem QP. The problem was set up in such a way that the objective was to maximize an expected return of a portfolio, given the same level of the risk as obtained from an optimal solution of the first problem (Problem NLP). This was labeled by QP_COST in figures. Problem QP has a target risk constraint as well as two other constraints, Equation (4.11) and (4.12),

Problem QP

$$J(\underline{x}_t^*) = \max[\underline{x}_t' \cdot E_t(\underline{r}_t)] \tag{4.13}$$

subject to

$$\underline{x}_t' \cdot \underline{V}_t \cdot \underline{x}_t = V_0 \tag{4.14}$$

$$\underline{x}_t' \cdot \underline{1} = 1 \tag{4.11}$$

$$\underline{x}_t \geq \underline{0} \tag{4.12}$$

where V_0 is a given level of the risk equal to that of an optimal solution from Problem NLP. Since the constraints were nonlinear, Problem QP was also solved by the same solution technique, GAMS/MINOS. Comparison conducted here was to see if there exists any portfolio that is more efficient than the derived portfolio from Problem NLP. That is, given the same level of the risk, if any portfolio has a higher return than the derived portfolio from Problem NLP, that portfolio is more efficient than the other. The transaction costs of the portfolio revision for the above Problem QP were calculated at each period after a new portfolio was derived. Three different values for lambda, λ, were used, 20, 40 and 60. Note that an initial portfolio at January 1987 was derived by setting all transaction costs equal to zero for both problems.

In using the simple regression model, a change in the total transaction costs over the time horizon is depicted in Figure 3. It was observed that the less lambda, λ, the more the total transaction costs were for solutions from Problem QP. It is true that the less the lambda, λ, the more an investor prefers risky assets. Thus, choosing a small value for λ, such securities that have a higher return and higher risk tend to be included into an optimal portfolio. Results in Figure 3 imply that such securities cannot keep a high level of return, resulting in the frequent revision of the portfolio. Since the transaction costs were not considered in the optimization framework for Problem QP, the total transaction costs varied within a high range from 1% to 2% across all lambda's at most periods. In contrast, solutions from Problem NLP did not have such a tendency. The total transaction costs were constantly low across three values of lambda, λ, ranging from 0% to 0.5%. Relatively high value above 0.5%, however, was observed during the 1987's the last 6 months of the 1990's and the first 3 months of the 1991's. The total transaction costs of 2% mean that a new portfolio is revised completely different of the existing one.

The cumulative effect of the transaction costs was as follows. The cumulative cost of a solution from Problem NLP became 17% $(= 1 - 0.87)$ for $\lambda = 20$, 10% $(= 1 - 0.90)$ for $\lambda = 40$, 7% $(= 1 - 0.93)$ for $\lambda = 60$ on June 1991, respectively. By contrast, for Problem QP, it was 48% $(= 1 - 0.52)$ for $\lambda = 20$, 44% $(= 1 - 0.56)$ for $\lambda = 40$, and 41% $(= 1 - 59)$ for $\lambda = 60$, respectively. Comparing the two cumulative costs, a difference on June 1991 was 31% for $\lambda = 20$, it was 34% for $\lambda = 40$ and 60, which could be interpreted as a cost of ignoring the transaction costs over the time horizon.

When applying the arithmetical mean method, one may realize that the expected return of each security did not change largely over the time horizon as opposed to that estimated by the simple regression model. Figure 4 depicts the total transaction costs in the use of the arithmetical mean method. A solution for Problem QP resulted in a low cost ranging from 0.5% to 1.5% in most cases. This implies the less revision of the portfolio over the time horizon in comparison to the one derived by using the simple regression model. The more the value for λ, the less the total transaction costs for solutions from Problem QP became. A high cost was observed during the period from 1987 to 1989. For Problem NLP, the total transaction costs were almost zero over the time horizon. A little increase in the costs was observed on November 1987, April 1990 and September 1991.

The total transaction costs are directly associated with the degree of fluctuation in proportion of each security period by period. Table 1 shows the degree of fluctuation of an optimal portfolio over the time horizon. The degree of fluctuation is defined by,

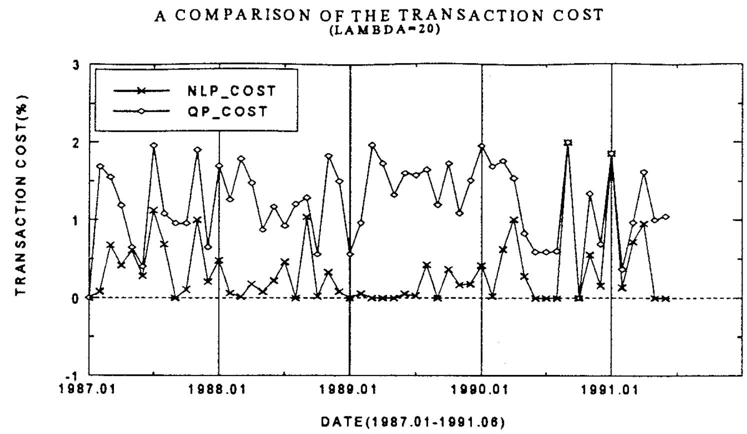
$$\sqrt{\frac{\sum_{t=t_0+1}^{T} \sum_{i=1}^{n} (x_{i,t} - x_{i,t-1})^2}{T - t_0}} \tag{4.16}$$

where

$x_{i,t}$: proportion of the i-th security in the portfolio at time t

$x_{i,t-1}$: given proportion of the i-th security in the portfolio at time t-1.

a)



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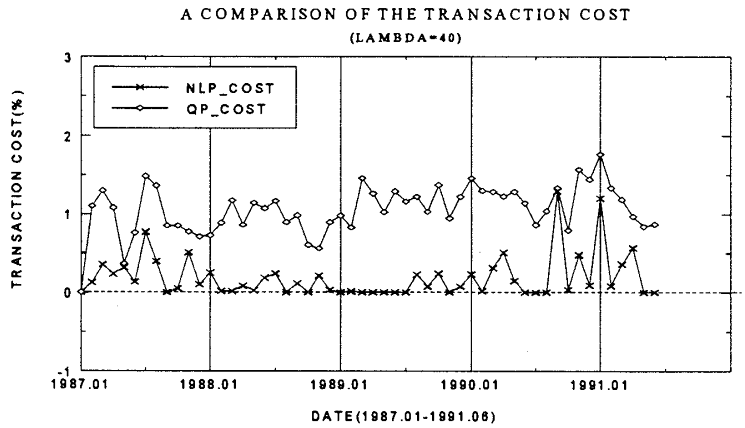

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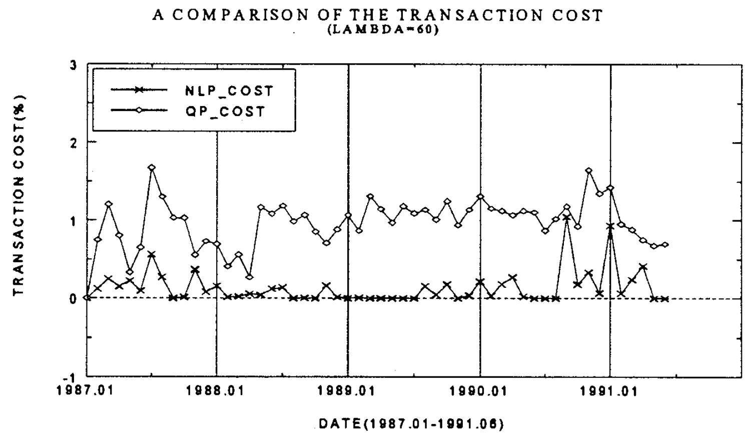
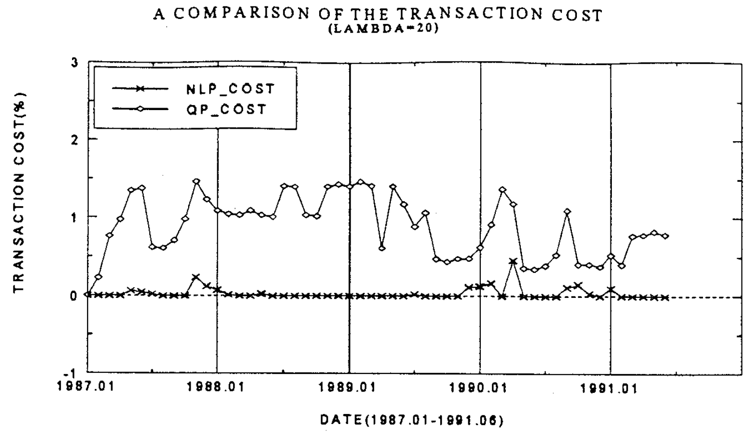


Figure 3. Transaction costs over the time horizon

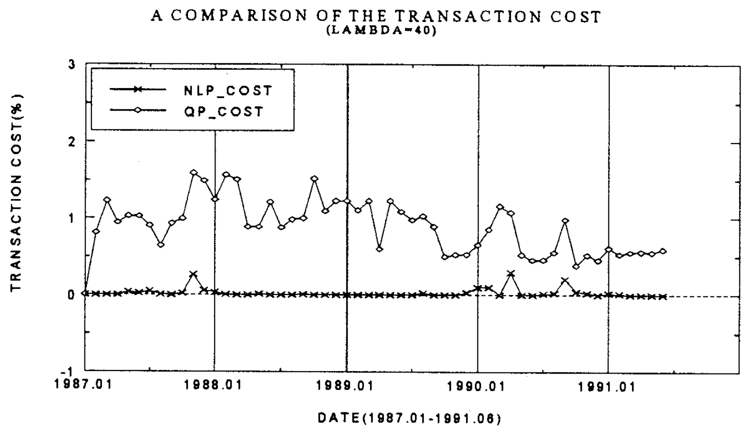
- Simple regression model -

a) λ=20, b) λ=40, c) λ=60

a)



b)



c)

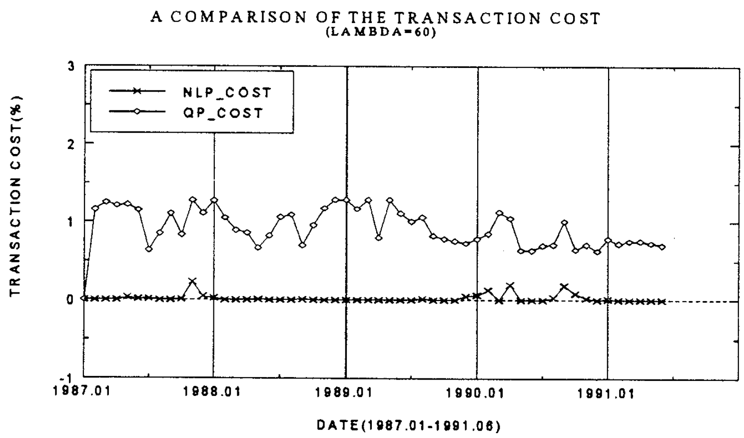
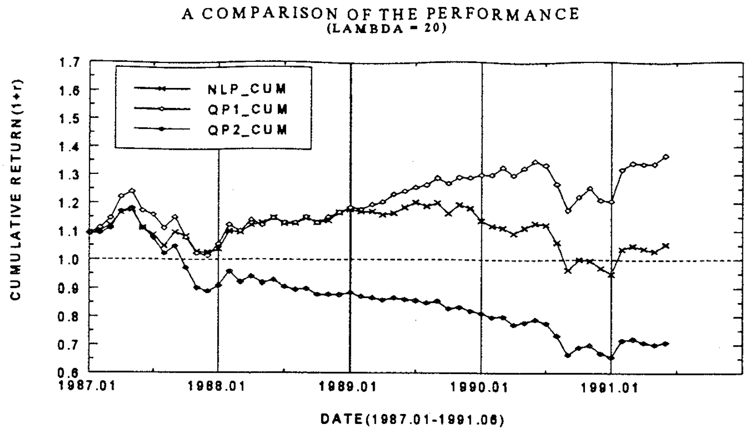


Figure 4. Transaction costs over the time horizon

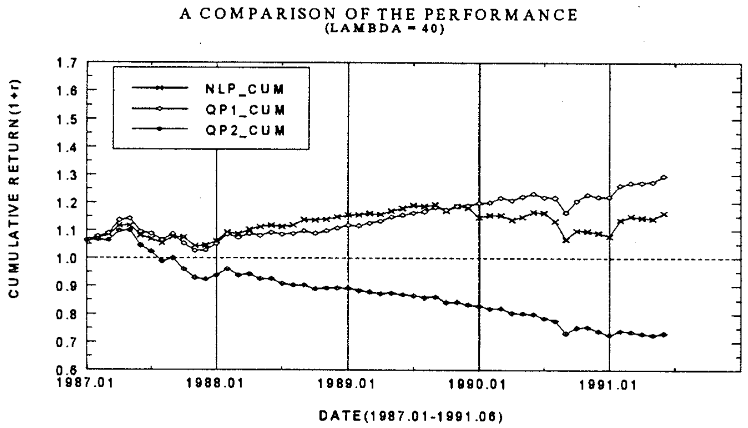
- Arithmetical mean method -

a) λ=20, b) λ=40, c) λ=60

a)



b)



c)

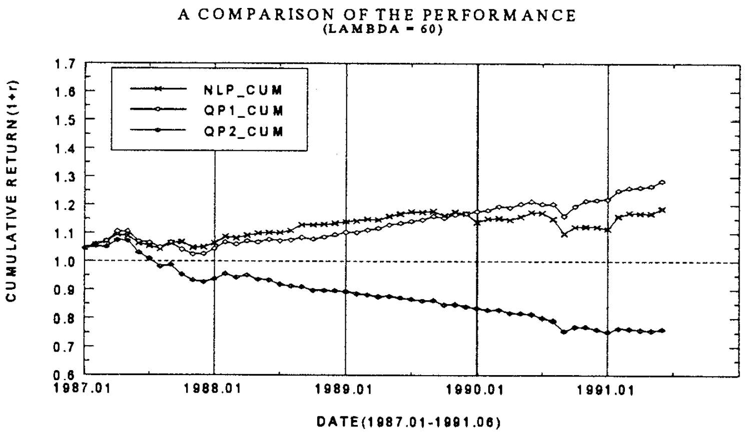


Figure 5. A comparison of portfolio performance
- Simple regression model -
a) λ=20, b) λ=40, c) λ=60

Table 1. The fluctuation of the optimal rebalanced portfolio over the time horizon

		With Transaction Costs	Without Transaction Costs
Simple	λ=20	35.42773	59.86429
Regression	λ=40	20.35782	39.54483
Model	λ=60	15.22604	31.07568
Arithmetical	λ=20	5.985303	35.45433
Mean	λ=40	4.374423	33.55673
Method	λ=60	3.565163	29.34202

This is an average Euclidian distance between the new and existing portfolio over the time horizon.

Fluctuation in solutions for Problem QP was almost twice as much as that in solutions for Problem NLP when using the simple regression model. In contrast, using the arithmetical mean method, fluctuation in solutions for Problem QP was about 6 to 8 times larger than that for the other. Solutions for Problem NLP yielded the smallest fluctuation. Small fluctuation was also associated with the large value of λ. These results imply that the transaction costs play an important role in stabilizing the portfolio over the time horizon for both methods to calculate the expected return of securities.

iv) Comparison of Portfolio Performance

Thus far, it was shown that the total transaction costs imposed were largely reduced once the transaction costs were taken into consideration within the optimization framework. In what follows, comparison of portfolio performance was conducted in terms of an actual return of the derived portfolio. This is because solutions from Problem NLP and Problem QP were derived under the same level of risk, so that a difference in portfolio performance is reflected in an actual return. Under this situation, using the Sharpe measure would lead to the same results. An actual return plays a main role in judging portfolio performance. While an average return of the portfolio over the time horizon is utilized in the Sharpe measure, we used the cumulative return of the portfolio so as to investigate the effect of the transaction costs over the time horizon.

Using the simple regression model, the cumulative return of the portfolio is depicted in Figure 5. NLP_CUM represents solutions for Problem NLP, QP1_CUM is for Problem QP with the zero transaction costs, and QP2_CUM is for Problem QP after subtracting the transaction costs. The cumulative return, CR_t, was calculated by,

$$\left\{ \begin{array}{c} CR_t = \prod_{i=t_0}^{t} [1 + \underline{x}_i' \cdot \underline{\hat{r}}_i - \underline{c}_i' \cdot \underline{1}] \\ \underline{c}_{t_0} = \underline{0} \end{array} \right\} \tag{4.17}$$

where $\underline{\hat{r}}_i$ is an actual return vector of securities at period i, so that $\underline{x}_i' \cdot \underline{\hat{r}}_i$ represents an actual return of the portfolio.

At the end of the analysis period, June 1991, NLP_CUM yielded 5% (= 1.05 − 1), while QP2_CUM yielded −29% (= 0.71 − 1) when $\lambda = 20$. Once λ was set to 40, NLP_CUM had 16% (= 1.16 − 1) and QP2_CUM had −27% (= 0.73 − 1). For $\lambda = 60$, they were 19% (= 1.19 − 1) for NLP_CUM and −24% (= 0.76 − 1) for QP2_CUM, respectively. Most likely due to the effect of the Black Monday in 1987, a downward peak was observed at the

Table 2. Results of the paired-t test: Comparison of portfolios with and without transaction costs

		λ=20	λ=40	λ=60
Simple	Average	0.725185	0.854778	0.817796
Regression	Standard Deviation	1.341033	1.008601	0.882364
Model	Degree of Freedom	53	53	53
	t-value	3.973802	6.227745	6.810739
Arithmetical	Average	0.774130	0.862556	0.874833
Mean	Standard Deviation	1.687069	1.346718	1.081426
Method	Degree of Freedom	53	53	53
	t-value	3.371925	4.706601	5.944638

end of 1987. Large decrease in the cumulative return on August 1990 might be caused by the eruption of the Persian Gulf War. A difference in the cumulative return of NLP_CUM and QP2_CUM was 34% for $\lambda = 20$, 43% for $\lambda = 40$, and 45% for $\lambda = 60$. That is, ignoring the transaction costs would result in about 34% to 45% reduction in the cumulative return over 65 months, which is approximately equivalent to 7.4% to 10.4% annual reduction.

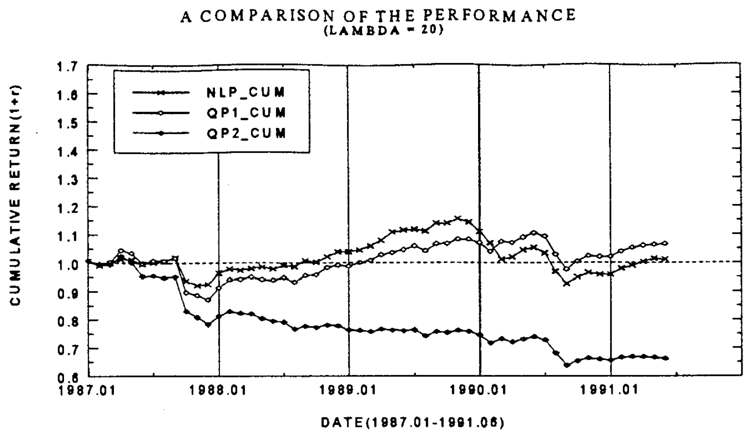
Ignoring the transaction costs, QP1_CUM yielded the highest cumulative return when $\lambda = 20$, followed by $\lambda = 40$ and $\lambda = 60$. Once the transaction costs were subtracted from the same portfolio, this order became opposite. QP2_CUM resulted in the highest value at $\lambda = 60$, the second highest at $\lambda = 40$, and the lowest at $\lambda = 20$. NLP_CUM had the same order for the cumulative return on June 1991 as QP2_CUM. These results imply that to seek the higher expected return of the portfolio, the frequent revision of the portfolio would be necessary, resulting in the higher transaction costs. Comparison of QP1_CUM and QP2_CUM indicated that there exists a large difference in the actual cumulative return between QP1_CUM and QP2_CUM. If one does not recognize the transaction costs, "poor" portfolio performance may result, and its effect is accumulated over the time horizon constantly.

A change in the cumulative return in using the arithmetical mean method is delineated in Figure 6. The same effect of the Black Monday on 1987 and the eruption of the Persian Gulf War as in Figure 5 was observed. For NLP_CUM, the cumulative return on June 1991 yielded the highest 11% $(= 1.11 - 1)$ at $\lambda = 60$. The second highest 9% $(= 1.09 - 1)$ was observed at $\lambda = 40$, then the lowest 1% $(= 1.01 - 1)$ at $\lambda = 20$. On the other hand, QP2_CUM had the highest -31% $(= 0.69 - 1)$ at $\lambda = 60$, the second highest -32% $(= 0.68 - 1)$ at $\lambda = 40$, and the lowest -44% $(= 0.66 - 1)$ at $\lambda = 20$. A difference among the values was almost the same for both cases as in Figure 5, meaning that the effect of λ was almost the same for the arithmetical mean method and the single regression model.

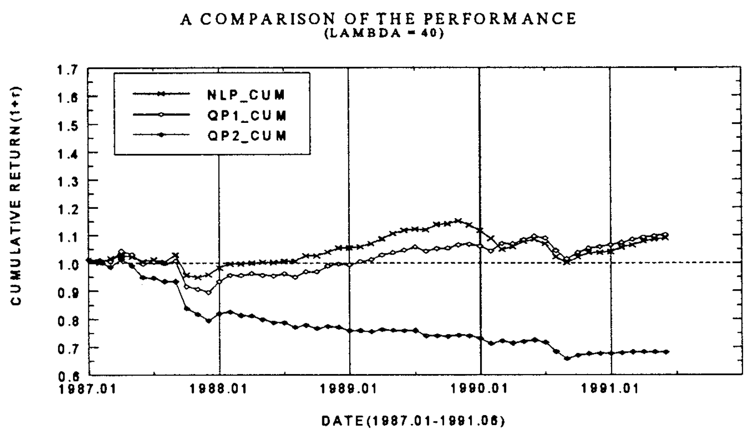
Table 2 shows results of the paired t-test conducted so as to compare performance of NLP_CUM and QP2_CUM in terms of an actual return. An actual return reflects a cost of revising the portfolio as well as a return from each security in itself. Using the simple regression model, the t-value was 3.99, 6.22 and 6.81 for $\lambda = 20$, 40 and 60, respectively. This indicates statistically significant superiority of NLP_CUM over QP2_CUM. This can be also said as to the use of the arithmetical mean method. At $\lambda = 20$, the t-value was 3.37,at $\lambda = 40$, it was 4.71 and the t-value was 5.94 at $\lambda = 60$. Ignoring the transaction costs, one would choose an inefficient portfolio with a high probability.

Table 3 presents results of the paired t-test to compare the actual return in both ways to calculate the expected return of securities. Results showed that there did not exist a

a)



b)



c)

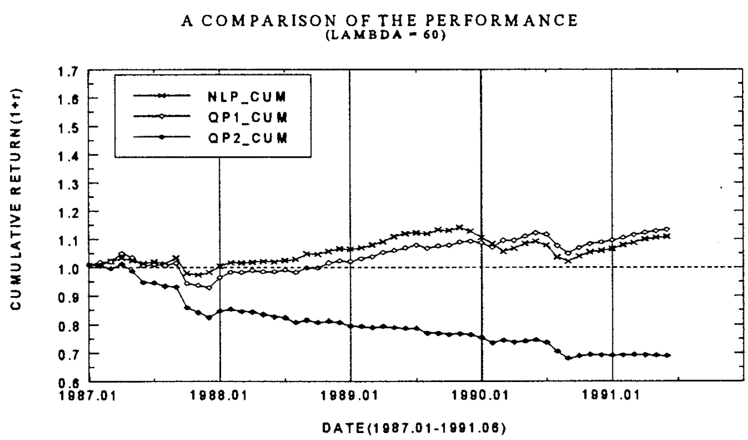


Figure 6. A comparison of portfolio performance

- Arithmetical mean method -

a) λ=20, b) λ=40, c) λ=60

Table 3. Results of the paired-t test: Comparison of portfolios derived with different methods to calculate the expected return

		λ=20	λ=40	λ=60
With Transaction Costs	Average	0.100685	0.122611	0.124778
	Standard Deviation	2.828060	1.889620	1.465155
	Degree of Freedom	53	53	53
	t-value	0.261622	0.476818	0.625822
Without Transaction Costs	Average	0.487130	0.299130	0.228389
	Standard Deviation	2.474196	1.603501	1.233084
	Degree of Freedom	53	53	53
	t-value	1.446796	1.370841	1.361066

statistically significant difference among the two methods when the transaction costs were considered. In contrast, with the 10% significance level, a statistically significant difference was observed when ignoring the transaction costs. Notice, however, it turns out to be insignificant with the 5% significance level.

With the zero transaction costs, the derived portfolio was sensitive to a small change in the expected return of securities. However, if the transaction costs were taken into consideration, the derived portfolio became insensitive. This is because changing the weight of the selected securities in the optimal portfolio would produce not only an extra return, but also an extra cost, so that the portfolio revision would not be implemented unless its expected return were more than the costs. It is important to notice that a return of a security has uncertain characteristics, while the transaction costs are certain to be charged.

5. Conclusions

The objective of this paper was to address the portfolio optimization problem subject to transaction costs, and to analyze an effect of the transaction costs on the derived portfolio. By formulating the cost function directly into the portfolio return, more realistic problems are to be set up. The proposed problem had a nonlinear V-shaped cost function, so that the nonlinear programming solution technique was applied to solve the problem.

The predominant strategy to deal with the transaction costs has been to use additional constraints, such as turnover constraints and minimum trading size constraints. However, ignoring the transaction costs often results in an inefficient portfolio, although in some degree inefficiency of such a portfolio could be avoided by using those constraints. Searching for such a solution that achieves an optimal tradeoff between a cost of the portfolio revision and a benefit of security returns, remained unresolved.

In this paper, the portfolio optimization problem was formulated so as to search for an optimal portfolio subject to transaction costs. The problem was solved by using the nonlinear programming solution technique, GAMS/MINOS. An optimal solution from the proposed problem was statistically superior to the one derived without the transaction costs in terms of the actual return of the portfolio. This implies that one has to consider the transaction costs in the portfolio construction. It should be recognized that a cost of the portfolio revision is charged with certainty, while a return of each security is not. This is an important part of the tradeoff between risk and return.

As for comparison of the two methods, the arithmetical mean method and the simple regression model, to calculate the expected return of securities, it was concluded that there is not any statistically significant difference in portfolio performance if the transaction costs are taken into consideration in the portfolio return. The transaction costs can play as a penalty factor for the portfolio revision, so that portfolio components may not vary much over the time horizon. When portfolio managers are involved in forecasting a return of a security, it is also of importance to consider the transaction costs in their performance in order to search for the efficient portfolio revision.

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