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The Measure Representation: A Correction

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Abstract

Wakker [9] and Puppe [2] point out a mistake in Theorem 1 in Segal [6]. This theorem deals with representing preference relations over lotteries by the measure of their epigraphs. An error in the theorem is that it gives wrong conditions concerning the continuity of the measure. This paper corrects the error. Another problem is that the axioms do not imply that the measure is bounded, therefore the measure representation applies only to subsets of the space of lotteries, although these subsets can become arbitrarily close to the whole space of lotteries. Some additional axioms (Segal [6, 7]), implying that the measure is a product measure (and hence anticipated utility), also guarantee that the measure is bounded.

Key Words: Anticipated utility, measure representation.

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Quiggin's [3] anticipated utility (or rank dependent) model for decision-making under uncertainty proved itself to be one of the most successful alternatives to expected utility theory. According to this model, the value of a lottery X with a cumulative distribution function F is given by

$$AU(X) = \int u(x)df(F(x)) \tag{1}$$

where $f : [0,1] \longrightarrow [0,1]$ is strictly increasing, continuous, and onto.[1] One possible interpretation of this model is that the preference relation $\succeq$ over lotteries can be represented by a measure of the epigraphs of the lotteries cumulative distribution functions, and moreover, that this measure is a product measure. That is, there are two increasing functions u (defined on the outcomes axis) and f (defined on the probabilities axis) such that the measure of the rectangle $[x,y] \times [p,q]$ is $[u(y)-u(x)][f(q)-f(p)]$. Indeed, let $X = (x_1,p_1;\ldots;x_n,p_n)$ such that $x_1 \leq \cdots \leq x_n$. Then (1) is reduced to

$$AU(X) = \sum_{i=1}^{n} u(x_i)[f(\sum_{j=0}^{i} p_j) - f(\sum_{j=0}^{i-1} p_j)] \tag{2}$$

where $p_0 = 0$. If we assume $u(0) = 0$, then the above expression can be viewed as the sum of the measures of the rectangles $[0,x_i] \times [\sum_{j=0}^{i-1} p_j, \sum_{j=0}^{i} p_j]$, each with the measure $[u(x_i)-u(0)][f(\sum_{j=0}^{i} p_j) - f(\sum_{j=0}^{i-1} p_j)]$.[2]

A natural extension of this model is to represent the preference relation $\succeq$ on lotteries by a general (not necessarily product) measure of the lotteries' epigraphs. This functional is suggested and axiomatized in Segal [5, 6]. It turns out, however, that there are some mistakes in these papers (see Wakker [9] and Puppe [2]), concerning the questions what sets have zero measure, and what sets which have zero Lebesgue measure must also

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[1]Quiggin's axioms imply $f(\frac{1}{2}) = \frac{1}{2}$. Yaari [10] assumes linear utility function u. The above general form of the rank dependent model first appeared in Segal [5].

[2]Recently, Tversky and Kahneman [8] suggested a more general form of this functional, where decision-makers use two different distribution transformation functions (for positive and negative outcomes). This too is a special case of the general measure representation.

have zero measure according to the representation functional. The aim of this paper is to answer these concerns. It turns out that lines that can serve as the lower boundary of the epigraph of a lottery (i.e., lines that can be created by connecting up pieces of the graph of a cumulative distribution function) are the only sets that must have zero measure. This result is quite natural — if such a set has positive measure, then the order does not satisfy continuity. However, other lines may have positive measure. In particular, the down-slopping line connecting the points (0,1) and (1,0) may have a positive measure (see Wakker [9] for examples).

Another issue is whether the representation measure may go to ∞. This leads to the conclusion that the measure representation applies only to subsets of the space of lotteries, although these subsets can become arbitrarily close to the whole space of lotteries. Some additional axioms (Segal [6, 7]), implying that the measure is a product measure (and hence anticipated utility), also guarantee that the measure is bounded.

1 Axioms and Theorem

Let L be the family of all the real random variables with outcomes in $[0, M]$ and let $\dot{L} = L \backslash \{\delta_0, \delta_M\}$. ($\delta_x$ is the degenerate lottery yielding x with probability 1). For every $X \in L$ define the cumulative distribution function F_X by $F_X(x) = \Pr(X \leq x)$. For $s > 0$, let $L_s = \{X \in \dot{L} : \text{ for } x \in [0,s),\ F_X(x) \leq 1-s\}$. Note that if $s < s'$ then $L_{s'} \subseteq L_s$. For $s > 0$, let Q_s be the square $[0,s) \times (1-s, 1]$. Let $D = [0, M] \times [0,1]$, $\dot{D} = D \backslash \{(0,1),(M,1)\}$, and $D_s = \dot{D} \backslash Q_s$. For $X \in L$, let $X^\circ = \mathrm{Cl}(\{(x,p) \in D : p > F_X(x)\})$.

Let L° be the family of all the non-empty closed sets S in D satisfying $[(x,p) \in S,\ 0 \leq y \leq x,\ p \leq q \leq 1] \Rightarrow (y,q) \in S$. Obviously, for every $S \in L^\circ$ there is a unique lottery $X \in L$ such that $X^\circ = S$. The cumulative distribution function of this lottery is given by $F_X(x) = \min\{p : (x,p) \in S\}$. Denote this lottery X by S^+.

Let L^* be the set of all the finite lotteries X in $\dot{L}$ of the form $(x_1, p_1; \ldots; x_n, p_n)$ and let $\Lambda = \{[x,y] \times [p,q] \subset \dot{D} : x < y,\ p < q\}$. Obviously, if $X \in L^*$, then X° can be represented as a finite union of elements of Λ.[3] Let $\succeq$ be a complete and transitive preference relation over L. Define the relations $\succ$ and $\sim$ by $X \succ Y$ if and only if $X \succeq Y$ but not $Y \succeq X$, and $X \sim Y$ if and only if $X \succeq Y$ and $Y \succeq X$. Let $\tilde{L} \subseteq L$. We say that the function $V : \tilde{L} \to \Re$ represents the preference relation $\succeq$ on $\tilde{L}$ if for all $X, Y \in \tilde{L}$, $V(X) \geq V(Y) \Leftrightarrow X \succeq Y$.

Consider the following three axioms:

(a) Continuity The preference relation $\succeq$ on L is continuous in the topology of weak convergence. That is, let $X, Y, Y_1, Y_2, \ldots \in L$ such that at each continuity point

[3] This representation is of course not unique. For $X = (x_1, p_1; \ldots; x_n, p_n) \in L^*$ let $p_0 = 0$, $x_0 = 0$, and obtain $X = \cup_{i=1}^n ([0, x_i] \times [\sum_{j=0}^{i-1} p_j, \sum_{j=0}^{i} p_j]) = \cup_{i=1}^n ([x_{i-1}, x_i] \times [\sum_{j=0}^{i-1} p_j, 1])$.

x of F_Y, $F_{Y_i}(x) \to F_Y(x)$. If, for every i, $X \succeq Y_i$, then $X \succeq Y$. If, for every i, $Y_i \succeq X$, then $Y \succeq X$.

(b) First-Order Stochastic Dominance If, for every x, $F_X(x) \leq F_Y(x)$ and there exists x such that $F_X(x) < F_Y(x)$, then $X \succ Y$.

(c) Irrelevance Let $X, Y, X', Y' \in L$ and let S be a finite union of segments in $[0, M]$. If on S, $F_X(x) = F_Y(x)$ and $F_{X'}(x) = F_{Y'}(x)$, and on $[0, M]\backslash S$, $F_X(x) = F_{X'}(x)$ and $F_Y(x) = F_{Y'}(x)$, then $X \succeq Y$ if and only if $X' \succeq Y'$.

Definition A curve $C \subset D$ is the continuous image of a function $f : [0,1] \to D$. The curve C is increasing if $(x, p) \in C \Rightarrow C \cap \{(y, q) : y < x, q > p\} = \emptyset\}$.

Note that a point in D is an increasing curve as is the set $\{(x,p) \in X^\circ : y > x, q < p \Rightarrow (y, q) \notin X^\circ\}$ for all $X \in L$.

Let ϑ be a countably additive measure on $\dot{D}$ such that for every $s > 0$, $Q_s \cap D$ is a measurable set. For $s > 0$, define the measure ϑ_s on D as follows: For every ϑ-measurable set $S \subseteq D$, $\vartheta_s(S) = \vartheta(S \backslash Q_s)$.

Theorem 1 *The following three conditions are equivalent:*

1. *The preference relation $\succeq$ on $\dot{L}$ satisfies the continuity, first-order stochastic dominance, and irrelevance axioms.*

2. *There is a (countably) additive measure ϑ on $\dot{D}$ satisfying*

   (a) *For $S = [a, b] \times [p, q] \subset \dot{D}$ such that $a < b$ and $p < q$, $0 < \vartheta(S) < \infty$;*

   (b) *If $C \subset \dot{D}$ is an increasing curve, then $\vartheta(C) = 0$; and*

   (c) *The preference relation $\succeq$ on L_s can be represented by $V_s(X) = \vartheta_s(X^\circ)$.*

3. *There is a measure ϑ as in condition 2 satisfying (a), (b), and*

   (c') *For every $X, Y \in \dot{L}$, $X \succeq Y$ if and only if $\vartheta(X^\circ \backslash Y^\circ) \geq \vartheta(Y^\circ \backslash X^\circ)$.*

Proof: (2) $\Leftrightarrow$ (3): Let $X, Y \in \dot{L}$. By definition, there exists $\varepsilon > 0$ such that $F_X(0), F_Y(0) < 1 - \varepsilon$. Since cumulative distribution functions are continuous from the right, there is $\varepsilon' > 0$ such that for $z \leq \varepsilon'$, $F_X(z), F_Y(z) < 1 - \varepsilon$. Define $s = \min\{\varepsilon, \varepsilon'\}$ and obtain that $X, Y \in L_s$, hence $Q_s \subseteq X^\circ \cap Y^\circ$. It follows that $\vartheta(X^\circ \backslash Y^\circ) \geq \vartheta(Y^\circ \backslash X^\circ)$ if and only if $\vartheta(X^\circ \backslash Q_s) \geq \vartheta(Y^\circ \backslash Q_s)$ if and only if $\vartheta_s(X^\circ) \geq \vartheta(Y^\circ)$. (Note that $X^\circ \backslash Q_s = (X^\circ \backslash Y^\circ) \cup ([X^\circ \cap Y^\circ] \backslash Q_s)$).

(2) $\Rightarrow$ (1): Let $X_n \to X$. It follows by the first-order stochastic dominance axiom that the condition in the continuity assumption is trivially satisfied if $X \in \{\delta_0, \delta_M\}$ (although

$\delta_0, \delta_M \notin \dot{L}$). Assume therefore that there exists $s > 0$ such that $X \in L_s$. Without loss of generality, we may assume that for every n, $X_n \in L_s$. To show that the order $\succeq$ is continuous, one has to prove that $V(X_n) - V(X) \to 0$. Let S_n be the symmetric difference between X_n° and X°, $S_n = (X_n^\circ \cup X^\circ)\backslash(X_n^\circ \cap X^\circ)$ and let $T_n = \cup_{i=n}^\infty S_i$. Note that $\vartheta(S_n) = \vartheta_s(S_n) \leq \vartheta_s(X^\circ \cup X_n^\circ) < \infty$. Since $V(X) = \vartheta_s(X^\circ)$, it follows that $| V(X_n) - V(X) | \leq \vartheta_s(S_n) = \vartheta(S_n) \leq \vartheta(T_n)$. Let $\hat{X}$ be the south-east boundary of X°, that is, $\hat{X} = \{(x,p) \in X^\circ : y > x, q < p \Rightarrow (y,q) \notin X^\circ\}$. As mentioned above, $\hat{X}$ is an increasing curve, hence $\vartheta(\hat{X}) = 0$. Moreover, $\cap_{n=1}^\infty T_n \subset \hat{X}$. Otherwise, let $(x,p) \in (\cap_{n=1}^\infty T_n)\backslash\hat{X}$. Since $(x,p) \notin \hat{X}$, either $p > F_X(x)$ or $p < \lim_{y\to x^-} F_X(y)$. We assumed that $(x,p) \in \cap_{n=1}^\infty T_n = \cap_{n=1}^\infty \cup_{i=n}^\infty S_i$ hence there is a subsequence $\{X_{n_j}\}$ such that for every j, $(x,p) \in S_{n_j}$. If $p > F_X(x)$, then $(x,p) \in X^\circ$. Therefore by definition, for every j, $(x,p) \notin X_{n_j}^\circ$. Hence, for every j, $\lim_{y\to x^-} F_{X_{n_j}}(y) > p$. Since cumulative distribution functions are continuous from the right and increasing, it follows that there exists $\varepsilon > 0$ such that for all $y \in [x, x+\varepsilon)$,

$$F_{X_{n_j}}(y) > p > \frac{p + F_X(x)}{2} > F_X(y).$$

Since there must be a continuity point of F_X in $[x, x+\varepsilon)$, it follows that $X_n \not\to X$. If $p < \lim_{y\to x^-} F_X(y)$, then $(x,p) \notin X^\circ$. Therefore for every j, $(x,p) \in X_{n_j}^\circ$ and $\lim_{y\to x^-} F_{X_{n_j}}(y) \leq p$. As before it follows that there exists $\varepsilon > 0$ such that for $y \in (x-\varepsilon, x]$,

$$F_{X_{n_j}}(y) \leq p < \frac{p + \lim_{y\to x^-} F_X(y)}{2} < F_X(y).$$

Here too, since there must be a continuity point of F_X in $(x-\varepsilon, x]$, it follows that $X_n \not\to X$. Since $\cap_{n=1}^\infty T_n \subset \hat{X}$ and $\vartheta(\hat{X}) = 0$, it follows that $\lim \vartheta(T_n) = 0$ (see Royden [4, p. 192]). First-order stochastic dominance follows by condition 2-(a) and the irrelevance condition follows by the fact that $\succeq$ on L_s can be represented by a measure.

(1) $\Rightarrow$ (2): Let $\Psi = \{(X,\lambda) \in \dot{L} \times \Lambda : \text{Int}(X^\circ) \cap \text{Int}(\lambda) = \emptyset \text{ and } X^\circ \cup \lambda \in L^\circ\}$. The irrelevance axiom implies that if $(X,\lambda), (Y,\lambda) \in \Psi$, then $X \succeq Y$ if and only if $(X^\circ \cup \lambda)^+ \succeq (Y^\circ \cup \lambda)^+$. Indeed, for $\lambda = [x,y] \times [p,q] \in \Lambda$, let $S = (x,y]$. Since $(X,\lambda), (Y,\lambda) \in \Psi$, it follows that for every $z \in S$, $F_X(z) = F_Y(z) = q$. Also, for every $z \in S$, $F_{(X^\circ\cup\lambda)^+}(z) = F_{(Y^\circ\cup\lambda)^+}(z) = p$. Of course, for $z \notin (x,y]$, $F_X(z) = F_{(X^\circ\cup\lambda)^+}(z)$ and $F_Y(z) = F_{(Y^\circ\cup\lambda)^+}(z)$.

Define on Λ a partial order R_X by $\lambda_1 \, R_X \, \lambda_2$ if and only if $(X,\lambda_1), (X,\lambda_2) \in \Psi$ and $(X^\circ \cup \lambda_1)^+ \succeq (X^\circ \cup \lambda_2)^+$. By the irrelevance axiom we obtain:

Fact 1 For every X_1 and X_2, R_{X_1} and R_{X_2} do not contradict each other. That is, if λ_1 and λ_2 can be compared by both R_{X_1} and R_{X_2}, then $\lambda_1 \, R_{X_1} \, \lambda_2$ if and only if $\lambda_1 \, R_{X_2} \, \lambda_2$.

(To see why Fact 1 follows from the irrelevance axiom, let $\lambda_i = [x_i, y_i] \times [p_i, q_i]$, $i = 1, 2$ and define $S = (x_1, y_1] \cup (x_2, y_2]$).

Let $R = \cup_X R_X$. That is, $\lambda_1\, R\, \lambda_2$ if and only if there exists X such that $\lambda_1\, R_X\, \lambda_2$. Define $\lambda_1\, I\, \lambda_2$ if and only if $\lambda_1\, R\, \lambda_2$ and $\lambda_2\, R\, \lambda_1$.

Let $\lambda_i = [x_i, y_i] \times [p_i, q_i]$, $i = 1, 2$. Obviously, λ_1 and λ_2 can be compared by R if and only if either $y_1 \leq x_2$ and $q_1 \leq p_2$, or $y_2 \leq x_1$ and $q_2 \leq p_1$. It thus follows that for every $\lambda_1, \lambda_2, \lambda_3$ such that any two of them can be compared by R there is a lottery X such that $(X, \lambda_i) \in \Psi$, $i = 1, 2, 3$. Therefore we obtain:

Fact 2 if $\lambda_1\, I\, \lambda_2$, $\lambda_2\, I\, \lambda_3$, and λ_1 and λ_3 can be compared by R, then $\lambda_1\, I\, \lambda_3$.

Let $x_1 = 0$, $x_4 = M$, and $p_k = \frac{k-1}{3}$, $k = 1, \ldots, 4$. By the continuity and first-order stochastic dominance axioms there are $0 < x_2 < x_3 < M$ such that $([x_k, x_{k+1}] \times [p_k, p_{k+1}])\, I\, ([x_\ell, x_{\ell+1}] \times [p_\ell, p_{\ell+1}])$, $k, \ell \in \{1, 2, 3\}$. Define the strictly increasing sequences $y_j^{k,i}$, $j = 0, \ldots, 2^i$; $i = 0, \ldots, \infty$; $k = 1, 2, 3$, such that

1. $y_0^{k,i} = x_k$, $y_{2^i}^{k,i} = x_{k+1}$, $i = 0, \ldots, \infty$; $k = 1, 2, 3$
2. $y_{2j}^{k,i} = y_j^{k,i-1}$, $j = 0, \ldots, 2^{i-1}$; $i = 1, \ldots, \infty$; $k = 1, 2, 3$
3. $([y_j^{k,i}, y_{j+1}^{k,i}] \times [p_k, p_{k+1}])\, I\, ([y_{j'}^{\ell,i}, y_{j'+1}^{\ell,i}] \times [p_\ell, p_{\ell+1}])$, $j, j' = 0, \ldots, 2^i - 1$; $i = 0, \ldots, \infty$; $k \neq \ell$; $k, \ell \in \{1, 2, 3\}$.

The only non-trivial requirement is condition 3. By the choice of x_2 and x_3, this condition is satisfied for the case $i = 0$. Suppose $y_j^{k,i}$, $j = 0, \ldots, 2^i$; $i = 0, \ldots, i_0$; $k = 1, 2, 3$, satisfy the above three conditions, and construct y_j^{k,i_0+1}, $j = 0, \ldots, 2^{i_0+1}$; $k = 1, 2, 3$, as follows: For $j = 2m$, let $y_j^{k,i_0+1} = y_m^{k,i_0}$, $m = 0, \cdots, 2^{i_0}$; $k = 1, 2, 3$. By the continuity and first-order stochastic dominance assumptions, there are $y_1^{k,i_0+1} \in (x_k, y_1^{k,i_0})$, $k = 1, 2, 3$, such that $\lambda_m\, I\, \cdots\, I\, \lambda_{m+1}$, $m = 1, \ldots, 4$, where

$$\lambda_k = \begin{cases} [x_k, y_1^{k,i_0+1}] \times [p_k, p_{k+1}] & k = 1, 2, 3 \\ \\ [y_1^{k-3,i_0+1}, y_1^{k-3,i_0}] \times [p_{k-3}, p_{k-2}] & k = 4, 5 \end{cases}$$

By Fact 2 it follows that $\lambda_1\, I\, \lambda_3$, $\lambda_3\, I\, \lambda_5$, $\lambda_1\, I\, \lambda_5$, and $\lambda_2\, I\, \lambda_4$. Also, $\lambda_5\, I\, \lambda_6 = [y_1^{3,i_0+1}, y_1^{3,i_0}] \times [p_3, p_4]$. This follows by

$$\begin{aligned}
&(y_1^{1,i_0+1}, \frac{1}{3}; y_1^{2,i_0+1}, \frac{1}{3}; y_1^{3,i_0}, \frac{1}{3}) \sim \\
&(y_1^{1,i_0}, \frac{1}{3}; x_2, \frac{1}{3}; y_1^{3,i_0}, \frac{1}{3}) \sim \\
&(y_1^{1,i_0}, \frac{1}{3}; y_1^{2,i_0}, \frac{1}{3}; x_3, \frac{1}{3}) \sim \\
&(y_1^{1,i_0}, \frac{1}{3}; y_1^{2,i_0+1}, \frac{1}{3}; y_1^{3,i_0+1}, \frac{1}{3}) \sim \\
&(y_1^{1,i_0+1}, \frac{1}{3}; y_1^{2,i_0}, \frac{1}{3}; y_1^{3,i_0+1}, \frac{1}{3}).
\end{aligned}$$

Of course, $\lambda_m \, I \, \lambda_6$, $m = 1, 2, 4$. One can now use λ_1 to define $y_j^{k,i_0+1} \in (y^{k,i_0}_{\frac{j-1}{2}}, y^{k,i_0}_{\frac{j+1}{2}})$, $j = 3, 5, \ldots, 2^{i_0+1} - 1$; $k = 2, 3$, and λ_3 to define $y_j^{1,i_0+1} \in (y^{1,i_0}_{\frac{j-1}{2}}, y^{1,i_0}_{\frac{j+1}{2}})$, $j = 3, 5, \ldots, 2^{i_0+1} - 1$. Condition 3 is clearly satisfied. Moreover, by the definition of I and by the first-order stochastic dominance axiom, for a given i,

$$\begin{aligned} &(y^{1,i}_{j_1}, \frac{1}{3}; y^{2,i}_{j_2}, \frac{1}{3}; y^{3,i}_{j_3}, \frac{1}{3}) \succeq (y^{1,i}_{j'_1}, \frac{1}{3}; y^{2,i}_{j'_2}, \frac{1}{3}; y^{3,i}_{j'_3}, \frac{1}{3}) \Leftrightarrow \\ &j_1 + j_2 + j_3 \geq j'_1 + j'_2 + j'_3 \end{aligned} \tag{3}$$

The next step in the proof is to show that the sequence $\{y_j^{k,i}\}$, $j = 0, \ldots, 2^i$; $i = 0, \ldots, \infty$ is dense in $[x_k, x_{k+1}]$, $k = 1, 2, 3$. Suppose, for example, that there are no values of $\{y_j^{1,i}\}$ in (α, β) and assume that (α, β) is maximal in that sense. There is a sequence $\{j_i\}_{i=0}^{\infty}$ such that $j_0 = 0$, $j_i \in \{2j_{i-1}, 2j_{i-1} + 1\}$ and $y^{2,i}_{j_i+1} - y^{2,i}_{j_i} \leq (x_3 - x_2) \cdot 2^{-i}$. By Cantor's Lemma $\{y^{2,i}_{j_i}\}_{i=0}^{\infty}$ and $\{y^{2,i}_{j_i+1}\}_{i=0}^{\infty}$ have a common limit, denote it y^2. Let m_i satisfy $y^{1,i}_{m_i} \leq \alpha$ and $\beta \leq y^{1,i}_{m_i+1}$, $i = 1, \ldots, \infty$. By construction, $(y^{1,i}_{m_i}, \frac{1}{3}; y^{2,i}_{j_i+1}, \frac{1}{3}; x_3, \frac{1}{3}) \sim (y^{1,i}_{m_i+1}, \frac{1}{3}; y^{2,i}_{j_i}, \frac{1}{3}; x_3, \frac{1}{3})$. By letting i approach ∞ one obtains $(\alpha, \frac{1}{3}; y^2, \frac{1}{3}; x_3, \frac{1}{3}) \sim (\beta, \frac{1}{3}; y^2, \frac{1}{3}; x_3, \frac{1}{3})$, a violation of the first-order stochastic dominance axiom.

Define $\vartheta([y_j^{k,i}, y_{j+1}^{k,i}] \times [p_k, p_{k+1}]) = 2^{-i}$, $j = 0, \ldots, 2^i - 1$; $i = 0, \ldots, \infty$; $k = 1, 2, 3$. For $x \in [x_k, x_{k+1})$, let $j_i(x)$ be such that $y^{k,i}_{j_i(x)} \leq x < y^{k,i}_{j_i(x)+1}$. Define $\varphi^k : [x_k, x_{k+1}] \to \Re$ by $\varphi^k(x) = \lim_{i \to \infty} j_i(x) \cdot 2^{-i}$, $k = 1, 2, 3$. By the above argument φ^k is strictly increasing. It is also continuous. Let z_n go down to $z \in [x_k, x_{k+1})$. For every i there exists n such that $z_n < y^{k,i}_{j_i(z)+1}$ hence

$$\begin{aligned} \varphi^k(z) \leq \; & \lim_{n \to \infty} \varphi^k(z_n) \leq \\ & \lim_{i \to \infty} \varphi^k(y^{k,i}_{j_i(z)+1}) = \\ & \lim_{i \to \infty} [\varphi^k(y^{k,i}_{j_i(z)}) + 2^{-i}] = \varphi^k(z). \end{aligned}$$

A similar proof holds for the case where z_n goes up to $z \in (x_k, x_{k+1}]$. It follows by continuity from (3) that for $y^k, z^k \in [x_k, x_{k+1}]$, $k = 1, 2, 3$,

$$\begin{aligned} &(y^1, \frac{1}{3}; y^2, \frac{1}{3}; y^3, \frac{1}{3}) \succeq (z^1, \frac{1}{3}; z^2, \frac{1}{3}; z^3, \frac{1}{3}) \Leftrightarrow \\ &\varphi^1(y^1) + \varphi^2(y^2) + \varphi^3(y^3) \geq \varphi^1(z^1) + \varphi^2(z^2) + \varphi^3(z^3) \end{aligned} \tag{4}$$

Let $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. By continuity and first-order stochastic dominance it follows that for every $\lambda \subset \bar{\lambda}^k = [x_k, x_{k+1}] \times [p_k, p_{k+1}]$ there is $y \in (x_{\sigma(k)}, x_{\sigma(k)+1}]$ such that $\lambda \, I \, ([x_{\sigma(k)}, y] \times [p_{\sigma(k)}, p_{\sigma(k)+1}])$. Define

$$\vartheta(\lambda) = \varphi^{\sigma(k)}(y) > 0.$$

Note that by Fact 2, $\lambda \, I \, ([x_{\sigma^{-1}(k)}, z] \times [p_{\sigma^{-1}(k)}, p_{\sigma^{-1}(k)+1}])$, where $\varphi^{\sigma(k)}(y) = \varphi^{\sigma^{-1}(k)}(z)$. The set-function ϑ satisfies the following condition:

Claim 1 Let $\lambda_1, \lambda_2 \subset \bar{\lambda}^k$. If $\lambda_1 \cup \lambda_2 \in \Lambda$ and $\mathrm{Int}(\lambda_1) \cap \mathrm{Int}(\lambda_2) = \emptyset$, then $\vartheta(\lambda_1 \cup \lambda_2) = \vartheta(\lambda_1) + \vartheta(\lambda_2)$.

Proof: Let $X = (x_{\sigma(k)}, \frac{1}{3}; x_{\sigma^{-1}(k)}, \frac{1}{3}; z_1, q_1; \ldots; z_n, q_n) \in L$ such that $z_1, \ldots, z_n \in [x_k, x_{k+1}]$ and (X, λ_1), $(X, \lambda_1 \cup \lambda_2) \in \Psi$ (assume, without loss of generality, that λ_1 is either above or to the left of λ_2). It follows by (4) and Fact 1 that

$$
\begin{aligned}
&(X^\circ \cup \lambda_1 \cup \lambda_2)^+ \sim \\
&(X^\circ \cup \lambda_1 \cup ([x_{\sigma^{-1}(k)}, [\varphi^{\sigma^{-1}(k)}]^{-1}(\vartheta(\lambda_2))] \times [p_{\sigma^{-1}(k)}, p_{\sigma^{-1}(k)+1}]))^+ \sim \\
&(X^\circ \cup ([x_{\sigma(k)}, [\varphi^{\sigma(k)}]^{-1}(\vartheta(\lambda_1))] \times [p_{\sigma(k)}, p_{\sigma(k)+1}]) \cup \\
&\qquad ([x_{\sigma^{-1}(k)}, [\varphi^{\sigma^{-1}(k)}]^{-1}(\vartheta(\lambda_2))] \times [p_{\sigma^{-1}(k)}, p_{\sigma^{-1}(k)+1}]))^+ \sim \\
&(X^\circ \cup ([x_{\sigma(k)}, [\varphi^{\sigma(k)}]^{-1}(\vartheta(\lambda_1) + \vartheta(\lambda_2))] \times [p_{\sigma(k)}, p_{\sigma(k)+1}]))^+,
\end{aligned}
$$

hence the claim. ∎

Define $\psi^k : [x_k, x_{k+1}] \times [p_k, p_{k+1}] \to \Re$ by $\psi^k(x,p) = \vartheta([x_k, x] \times [p, p_{k+1}])$. The function ψ^k satisfies the following condition:

Claim 2 For every $x_k \le x < y \le x_{k+1}$ and $p_k \le q < p \le p_{k+1}$,

$$\psi^k(x,p) + \psi^k(y,q) - \psi^k(x,q) - \psi^k(y,p) \ge 0$$

Proof: It follows by Claim 1 that $\psi^k(x,p) + \psi^k(y,q) - \psi^k(x,q) - \psi^k(y,p) = \vartheta([x,y] \times [p,q]) > 0$. ∎

By the continuity of $\succeq$ it follows that ψ^k is continuous. Therefore, ϑ can be uniquely extended to a countable additive measure on $\bar{\lambda}^k$ (see Billingsley [1, Section 12]). Moreover, it follows by (4) that the order $\succeq$ on $D' = \{X = (x_1, p_1; \ldots; x_n, p_n) \in L :$ for $x \in [x_k, x_{k+1})$, $p_k \le F_X(x) \le p_{k+1}$, $k = 1,2,3\}$ can be represented by $\vartheta(X^\circ \backslash (x_1, \frac{1}{3}; x_2, \frac{1}{3}; x_3, \frac{1}{3})^\circ)$. Also, by the continuity of $\succeq$, if $C \subset D'$ is an increasing curve, then $\vartheta(C) = 0$.

The next step in the proof is to extend ϑ to $\dot{D}$. It follows by continuity and first-order stochastic dominance that for every $x_3 \le y_1 < y_2 < M$ and $\frac{2}{3} \le q_1 < q_2 < 1$ such that $\lambda = ([y_1, y_2] \times [q_1, q_2])\, I\, ([y_2, M] \times [q_2, 1])$ there is a finite sequence of probabilities $\frac{1}{3} = r_1 < \cdots < r_n \le q_1$ such that $\lambda\, I\, ([0, x_3] \times [r_i, r_{i+1}])\, R ([0, x_3] \times [r_n, q_1])$, $i = 1, \ldots, n-1$.[4] Suppose instead that the sequence $\{r_i\}$ is not finite and let $\lim r_i = r \le q_1$. For every i,

$$
\begin{aligned}
&(0, 1 - r_i; x_3, q_1 - r_i; y_1, q_2 - q_1; y_2, 1 - q_2) \sim \\
&(0, 1 - r_{i+1}; x_3, q_1 - r_i; y_2, 1 - q_1) \sim \\
&(0, 1 - r_{i+2}; x_3, q_1 - r_i; y_2, q_2 - q_1; M, 1 - q_2)
\end{aligned}
$$

[4] Note that $([y_1, y_2] \times [q_1, q_2])$, $([y_2, M] \times [q_2, 1]) \subset \bar{\lambda}^3$.

As i approaches infinity we obtain that

$$(0,1-r;x_3,q_1-r_i;y_2,1-q_1) \sim$$
$$(0,1-r;x_3,q_1-r_i;y_2,q_2-q_1;M,1-q_2),$$

a contradiction. The measure ϑ can thus be extended to $([0,x_2]\times[\frac{1}{3},\frac{2}{3}])\cup([0,x_3]\times[\frac{2}{3},q_1])$. Since q_1 can be taken arbitrarily close to 1, we obtain that the measure ϑ can be extended to $([0,x_2]\times[\frac{1}{3},\frac{2}{3}])\cup([0,x_3]\times[\frac{2}{3},1))$.

In a similar way the measure ϑ can be extended also to $((0,x_2]\times[\frac{1}{3},1])\cup([x_2,x_3]\times[\frac{2}{3},1])$. For this, we use as benchmark the sets $([0,y_3]\times[0,q_3])\, I\, ([y_3,y_4]\times[q_3,q_4])$ where $0<y_3<y_4<x_2$ and $0<q_3<q_4<\frac{1}{3}$. (Both these sets are therefore in $\bar{\lambda}^1$). In a similar way we can extend the measure to $([x_2,x_3]\times[0,\frac{1}{3}])\cup([x_3,1)\times[0,\frac{2}{3}])$ and to $([x_2,1]\times(0,\frac{1}{3}])\cup([x_3,1]\times[\frac{1}{3},\frac{2}{3}])$. The measure ϑ is thus extended to $\dot{D}$. It may happen that the measure ϑ does not represent the order $\succeq$ in L^* because it may be unbounded (see Wakker [9] for an example). Nevertheless, by its construction, ϑ, and hence ϑ_s, is bounded on D_s. The proof that ϑ_s represents the order on L_s^* (the set of finite lotteries in L_s) is similar to the proof that ϑ represents the order for lotteries X such that $\hat{X}\subset D'$ and so is the proof that if $C\subset\dot{D}$ is an increasing curve then $\vartheta(C)=0$. The extension for L_s follows by continuity. ∎

The rank dependent (or anticipated utility) functional is a product measure. Theorem 2 in [6] and Theorem 9 in [7] prove that under some further conditions, the measure ϑ is a product measure. These proofs implicitly assume that ϑ on D is bounded. Although this is no longer true, the theorems still hold. To see this, observe that ϑ is bounded on $[\varepsilon,M-\varepsilon]\times[0,1]$ and on $[0,M]\times[\varepsilon,1-\varepsilon]$ for all ε. Let $L_\varepsilon=\{X\in L:\lim_{x\to\varepsilon}F_X(x)=0,\ F_X(M-\varepsilon)=1\}$ and let $L_\varepsilon^*=\{X\in L:\ F_X(0)=\varepsilon,\lim_{x\to M}F_X(x)=1-\varepsilon\}$. The above mentioned theorems thus imply the existence of functions $f_\varepsilon:[0,1]\to\Re$, $u_\varepsilon:[\varepsilon,M-\varepsilon]\to\Re$, $f_\varepsilon^*:[\varepsilon,1-\varepsilon]\to\Re$, and $u_\varepsilon^*:[0,M]\to\Re$, each of them unique up to positive linear transformations, such that the order $\succeq$ on L_ε can be represented by $\int_\varepsilon^{M-\varepsilon}u_\varepsilon df_\varepsilon(F_X(x))$ and the order $\succeq$ on L_ε^* can be represented by $\int_0^M u_\varepsilon^* df_\varepsilon^*(F_X(x))$. For every $\bar{\varepsilon}>0$ and for every $\varepsilon\in(0,\bar{\varepsilon}]$, all these representations cardinally coincide on $L_{\bar{\varepsilon}}\cap L_{\bar{\varepsilon}}^*$. Hence, without loss of generality, for every $p\in[0,1]$ and $\varepsilon,\varepsilon'>0$, $f_\varepsilon(p)=f_{\varepsilon'}(p):=f(p)$. Similarly, for every $x\in[0,M]$ and $\varepsilon,\varepsilon'>0$, $u_\varepsilon^*(x)=u_{\varepsilon'}^*(x):=u(x)$. Also, $\forall p\in(\bar{\varepsilon},1-\bar{\varepsilon})$ and $\forall\varepsilon\in(0,\bar{\varepsilon}]$, $f_\varepsilon^*(p)=f(p)$ and $\forall x\in(\bar{\varepsilon},M-\bar{\varepsilon})$ and $\forall\varepsilon\in(0,\bar{\varepsilon}]$, $u_\varepsilon(x)=u(x)$. It follows by the continuity axiom that the order $\succeq$ on L can be represented by $\int_0^M u(x)df(F_X(x))$.

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