The Mechanical Foundations of Elasticity and Fluid Dynamics

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Dedicated to S. LEFSCHETZ

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The MS was completed in midsummer 1949; while it has been thoroughly revised at intervals subsequently, the viewpoint, structure, and the principal contents are unchanged.

Note added for the reprinting, 1954. Extensive additions and some corrections for this paper appear in volume **2**, pp. 593–616 (1953), and a few further misprints will be noted at the end of volume **3** (1954).

Chapter I. Preliminary Discourse

1. The classical linear theories of elasticity and fluid dynamics. The classical linear theory of isotropic elastic solids is based upon the Hooke-Cauchy law[1]

$$t^i_j = \lambda_E \tilde{e}^k_k \delta^i_j + 2\mu_E \tilde{e}^i_j, \tag{1.1}$$

where t^i_j is the *stress tensor*[2], $\tilde{e}^i_j$ is the *infinitesimal strain tensor*, and λ_E and μ_E are *moduli of elasticity*. The classical theory of isotropic viscous fluids is based upon the Newton-Cauchy-Poisson law[3]

$$t^i_j = -p\delta^i_j + \lambda_V d^k_k \delta^i_j + 2\mu_V d^i_j, \tag{1.2}$$

where p is a *pressure*, d^i_j is the *rate of deformation* tensor, and λ_V and μ_V are *coefficients of viscosity*. Classical linear elasticity describes the slight deformations of media which are perfectly springy, so that when released from deforming forces they revert to their initial shapes; it is a linear theory, in which a uniformly doubled load necessarily produces a doubled displacement. Classical fluid dynamics describes the flow of media altogether without springiness of form, so that when released from all deforming forces except a hydrostatic pressure, they retain their present shapes; it is a partially linear theory, in which a uniformly doubled rate of deformation if dynamically possible would lead to doubled viscous forces.

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[1] Hooke's hypothesis of 1660 was published in [1676, 1, Postscript, No. 3] as an anagram which when deciphered reads "ut tensio sic vis" [1678, 1]. The general form (1.1) is due to Cauchy [1823, 1] [1828, 1, II, eq. (67) (70)] [1830, 1]. Cf. [1831, 1, ¶23, eq. (10)]. Statical equations equivalent to those resulting from (1.1) when $\lambda_E = \mu_E$ are the discovery of Navier [1823, 2] [1827, 5] (cf. also [1829, 3, §7]; Navier's result is dated 1821, in which year Fresnel apparently discovered a special case of (1.1) (see [1866, 1, pp. LXXVIII-LXXXI]).

[2] We employ standard tensorial notations (e.g. [1927, 1]). By "a^i_j is the . . . tensor" we mean "the a^i_j are the mixed components of the . . . tensor".

[3] According to Newton [1687, 1, lib. II, sect IX], "Resistentiam, quae oritur ex defectu lubricitatis partium fluidi, caeteris paribus, proportionalem esse velocitati, qua partes fluidi separantur ab invicem." Cauchy [1823, 1] spoke of "corps solides entièrement dépourvus d'élasticité," but his equations [1828, 1, §III, eq. (95) (96)] differ from (1.2) only in lacking the term $-p\delta^i_j$. The fully general expression is Poisson's [1831, 1, ¶¶60–63]. Dynamical equations equivalent to those resulting from (1.2) when $d^k_k = 0$, $\mu_V =$ const. are the discovery of Navier [1821, 1] [1822, 2] [1825, 1] [1827, 4]. St. Venant [1843, 1] proposed (1.2) in the special case when $\mu_V =$ const., $3\lambda_V + 2\mu_V = 0$; the latter relation (cf. §61) was proposed also by Stokes [1845, 1, §§3–5] and is called the *Stokes relation*; the resulting dynamical equations are the *Navier-Stokes equations*.

2. Possible generalizations. The classical theories describe certain types of response, and thus represent materials of a certain ideal *quality*; this quality is then specified by the simplest possible *quantity*, namely, a linear expression. Accordingly, there are two different ways of defining an ideal material to describe a wider range of physical phenomena; to introduce *different or additional qualities*, or to reformulate the classical theories so as to represent the same qualities in *a more accurate and general quantity*.[1] The former leads to theories of plasticity,[2] which we make no attempt to review here,[3] being content instead to consider non-linear theories of elasticity and fluid dynamics.

The classical theories idealize two limits of physical behavior: the elastic solid responds only to its present strain from a "natural state," generally regarded as unstressed, in a fashion independent of all time rates of change and of its entire intervening history, while the viscous fluid responds only to its present rate of deformation, in a fashion independent of its entire previous history. While most physical materials behave in a manner intermediate between these extremes, there are many phenomena which are of a purely elastic or purely fluid quality, yet for whose description the classical theories are inadequate, as for example in the flow of a moderately rarefied gas, or in the deformation of rubber.

3. The scope of this review. The idea of generalizing the classical theories by representing more accurately the same physical qualities which their linear formulae (1.1) and (1.2) approximate is familiar, and is often dismissed with the vague phrase "retaining the non-linear terms." Those who have come to grips with the details, however, have found the classical notions to be not so precise as might casually appear, for there are several different possible concepts of springiness and fluidity, which lead to quite different formulae in their general mathematical embodiments but nevertheless all yield the classical linearizations (1.1) or (1.2) as respective first approximations. I shall organize, compare, and extend recent researches in this field, taking care to found them in the bed rock of the basic but too often neglected discoveries of previous generations.

To achieve a rational generalization of a classical theory it is not sufficient simply to add a second approximation, for the mathematical simplicity which is the main justification for the classical procedure is lost as soon as the first non-linear term is written down.[1] Rather, a general non-linear theory must

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[1] Cf. [1932, **5** pp. 250–251].

[2] See the survey by v. Mises [1930, 1]. Classification of continua is attempted in [1931, **2**], [1932, **5**], [1945, **8**].

[3] An absolute distinction is impossible, and some of the theories discussed in this memoir represent set or hysteresis; our criterion for inclusion is whether or not some modification of the classical theories is proposed within the range of purely elastic or purely fluid behavior, and, in particular, theories employing a yield condition or the concept of over-stress or over-strain are excluded.

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[1] Cf. the first and second approximations in the motion of a pendulum.

attain *clarity*. It must be the unequivocal mathematical expression of a concept, and in this respect it will be simpler than the linear theory, which can express only a first approximation. If it be successful in this first requirement it will simultaneously achieve completeness, for this general expression will automatically yield approximations of any desired order. Since integration of the equations of these non-linear theories is difficult, emphasis must lie upon the phenomena they represent and the types of physical behavior they lead us to expect. In particular, we shall see that in typical specific problems not only is there a quantitative difference in the result, but also wholly new ranges of phenomena not included even approximately in the linear theories appear in any non-linear generalization. Finally not the least value of a general theory is that it clarifies the original theory which it generalizes; the nature of an approximation is never really understood until we specify just what it is that is being approximated.

Since elastic bodies and viscous fluids represent opposite limits of physical behavior, the formal similarity between the classical linear expressions (1.1) and (1.2) is astonishing. Hasty persons have concluded that the non-linear theories are similarly analogous, but this notion is utterly false, for as soon as a single non-linear term is written down the basic physical difference between the two types of materials appears in the mathematical formulae, and the analogy disappears. Indeed, the fact that strain is dimensionless while rate of deformation is of the dimension $\mathbf{T}^{-1}$, $\mathbf{T}$ being a unit of time, must necessarily lead to profound differences in the two general theories: the formal similarity of (1.1) and (1.2) is a fortuity incident upon linearization.

There are two methods of constructing a theory of elasticity or fluid dynamics. The first, used originally by Boscovich, Navier, Cauchy, and Poisson[2] and after long discredit now again in favor among physicists, deduces macroscopic equations from special assumptions relative to the behavior of the supposed ultimate discrete entities comprising the medium. In the present article I employ only the continuum approach of Clairaut, D'Alembert, Euler, Lagrange, Fresnel, Cauchy, Green, St. Venant, and Stokes,[3] in which molecular speculations are avoided, and gross phenomena are described by gross variables and gross hypotheses alone.[4]

A theory aiming to represent physical phenomena must be invariant not only

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[2] [1763, **1**] [1821, **1**] [1823, **2**] [1825, **1**] [1827, **1a**, **4–5**] [1828, **2–3**] [1829, **2–3**] [1831, **1**] [1842, **2**].

[3] [1743, **1**] [1752, **1**] [1757, **1–2**] [1762, **1–2**] [1769, **1**] [1770, **1**] [1783, **1**] [1788, **1**, Part II, sects. 10–12] [1823, **1**] [1827, **1**, **3**] [1828, **1**] [1829, **1**] [1839, **1**] [1841, **2**] [1843, **1**] [1845, **1**] [1866, **1**]. FOURIER's theory of heat flow [1822, **1**] [1833, **1**] and MAXWELL's electromagnetic theory [1873, **3**] are the principal achievements of continuum physics apart from mechanics.

[4] Excluded from the present survey are both MACCULLAGH's theory of the quasi-elastic ether [1848, **1**] [1889, **5**, §§217–220] [1890, **1**, §§14–28] [1892, **6**] [1893, **4**] [1894, **6**] [1895, **3**] [1909, **5**, §§37, 45–46] [1947, **28**, §15] and relativistic continuum theories [1911, **7**, **9**, **11**] [1912, **5**] [1924, **2**] [1933, **8**] [1934, **3**] [1937, **3**] [1939, **17–20**] [1940, **3**, **6**, **10**] [1941, **2–4**] [1944, **1–2**] [1945, **3**] [1946, **4–9**] [1948, **19**] [1949, **17**] [1950, **17**].

under changes of geometrical co-ordinates but also under changes of physical units. So as automatically to achieve the former invariance, as well as to shorten the deductions, I employ tensor analysis.[5] At the same time I fully exploit the method of dimensions, whose value as a mathematical tool is sometimes not realized.[6] In face of these two invincible adversaries the general theories fall into a definite and explicit form.

4. The plan of this review. Since isotropic media are of principal interest, if only because of the simplicity of the theories describing them, Chapter II presents the theory of isotropic functions in a space of three dimensions. Chapter III is a hasty exposition[1] of such classical results and recent developments concerning continuous media in general as are needed for the sequel. Chapters IV and V are devoted to the general theories of elasticity and fluid dynamics, respectively, while Chapter VI concerns media exhibiting both elastic and fluid properties. Chapter VII is an evaluation of the present state of the non-linear theories. In accord with the principles already stated, I emphasize ideas and their realization, mentioning only such special applications as serve to illustrate and point the general theories.

5. Notation. I have attempted to form a consistent and clear scheme from notations already in use, summarized in the following partial table:

Latin light face: co-ordinates and tensor point functions x^i, X^L, X^α, f^i, $x^i{}_{,\alpha}$, $t^i{}_j$, $T^L{}_M$ (for further distinctions see §§12–13).

Latin bold face: matrices of the above tensors **f**, **t**, **T** (§6). The context will indicate whether **t** stands for $\| t^{ij} \|$, $\| t^i{}_j \|$, $\| t_i{}^j \|$, or $\| t_{ij} \|$, or whether it is intended as a general symbol for all these matrices.

Superscript -1: matrix inverse, $(a^{-1})^i{}_j$ being the mixed components of $\mathbf{a}^{-1}$ (§6).

Superposed dot: total (material) time derivative $\dot{x}^i$, $\dot{f}^i$ (§20). A superposed bar (e.g. $\dot{\overline{x^i{}_{,jk}}}$ or $\dot{\overline{x^i{}_{,j}}}{}_{,k}$ as distinguished from $\dot{x}^i{}_{,jk}$) serves as a bracket indicating the extent of the symbol on which the dot operates.

Roman numerals: $I_{\mathbf{t}}$, $II_{\mathbf{t}}$, $III_{\mathbf{t}}$; principal invariants of **t**; barred Roman numerals $\overline{II}_{\mathbf{t}}$, $\overline{III}_{\mathbf{t}}$: moments of **t** (§6).

Greek majuscule: other invariants of tensors, $\Delta_{\mathbf{a}}$, $\Phi(\mathbf{a}, \mathbf{b})$ (§6).

Greek minuscule: thermodynamic variables π, ρ, η . . . moduli of elasticity λ_E, μ_E, . . . moduli of viscosity λ_V, μ_V . . . .

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[5] The very real advantages of tensor methods in continuum mechanics I first learned in the courses of Professor Michal.

[6] One-dimensional and engineering usage of dimensional principles in non-linear rheology is familiar (e.g. [1932, 7] [1936, **3**, Ch. IV] [1943, **6**, pp. 128–132] [1949, **31**, §4] [1949, **43**, Lect. VII]) but I believe the first systematic use of dimensional invariance in a general continuum theory is in my preliminary study [1947, **6**, §XI].

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[1] Other surveys, including some topics not treated in the present one, are [1896, **1**] [1913, **1**] [1914, **1**] [1938, **2**, Ch. X] [1943, **1**, **14**] [1950, **12**].

Script majuscule $\mathcal{G}_{\mathrm{T}}^{\mathrm{b(a)}}$: scalar coefficients in expansion of **b** as an isotropic function of **a** (§6).

Script minuscule $\mathscr{g}_{\mathrm{T}}^{\mathrm{b(a)}}$: dimensionless scalar coefficients in expansion of **b** as an isotropic function of **a** (§§47, 65, 69).

Other script letters: regions of space $\mathcal{U}$, $\mathcal{C}$, ...

German letters: line, surface, and volume integrals $\mathfrak{K}$, $\mathfrak{P}^i$, ...

Bold-face sans-serif letters, physical units $\mathbf{M}$, $\mathbf{L}$, $\mathbf{T}$, $\boldsymbol{\Theta}$, . . .

Old English: similarity parameters $\mathfrak{R}$, $\mathfrak{T}$, . . .

Small sans-serif, Greek, or German majuscules: subscripts and superscripts which do *not* indicate tensor character.

Chapter II. Isotropic Functions

6. Isotropy in three dimensions. The laws (1.1) and (1.2) are isotropic relations connecting the components of two tensors: we now examine the most general form of such relations, following the analysis of Reiner as suggested by Racah.[1] Two tensors $a^i{}_j$ and $b^i{}_j$ are isotropically related if their matrices **a** and **b** satisfy

$$f(\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}') = 0, \tag{6.1}$$

where primes denote transposition, and where the functions f may depend also upon scalar functions of **a** and **b**. In particular, **b** is an analytic isotropic function of **a** if

$$\mathbf{b} = f(\mathbf{a}, \mathbf{a}') = k_0\mathbf{I} + k_1\mathbf{a} + k_1'\mathbf{a}' + k_2\mathbf{a}^2 + k_2'\mathbf{a}\mathbf{a}' + k_2''\mathbf{a}'\mathbf{a} + k_2'''\mathbf{a}'^2 + \cdots, \tag{6.2}$$

where the k_i, k_i' $\cdots$ are scalar functions of **a**. Hence *if* **a** *be symmetric, so also is* **b**.

Henceforth we assume **a** is symmetric. Then (6.2) reduces to[2]

$$\mathbf{b} = k_\mathrm{T}\mathbf{a}^\mathrm{T}. \tag{6.3}$$

Since $\mathbf{a}^\mathrm{T}$ has the same principal axes as **a**, it follows that *the principal axes of* **a** *and* **b** *coincide.*

By the *proper values* a_i of **a** we shall mean the common[3] proper values of the matrices $\| a^i{}_j \|$ and $\| a_i{}^j \|$.

Now any analytic scalar function of an $n \times n$ matrix **a** is a function of its n principal invariants.[4] For the case $n = 3$ these invariants I_a, II_a, III_a are the elementary symmetric functions of the proper values[5]:

$$\begin{aligned} I_\mathrm{a} &\equiv \frac{1}{1!}\,\delta^i{}_j a^j{}_i = a_1 + a_2 + a_3, \\ II_\mathrm{a} &\equiv \frac{1}{2!}\,\delta^{i\,k}_{j\,l}\, a^l{}_k a^j{}_i = a_2 a_3 + a_3 a_1 + a_1 a_2, \\ III_\mathrm{a} &\equiv \frac{1}{3!}\,\delta^{i\,k\,m}_{j\,l\,n}\, a^n{}_m a^l{}_k a^j{}_i = \det a^i{}_j = a_1 a_2 a_3. \end{aligned} \tag{6.4}$$

[1] [1945, **1**, §4]. Cf. [1945, **5**, §2] [1948, **13**, §2]. These results follow also from a more elaborate analysis in the theory of isotropic tensors, whose rectangular Cartesian components are invariant under rotation of axes, a theory which may be traced back to Cauchy and which was formalized by Cisotti [1939, **12**]; developments are given in [1930, **13–15, 17–18**] [1931, **15**, §2] [1932, **14**] [1933, **10**]. A typical result: the most general fourth order isotropic tensor is (in the usual notation of Cartesian tensors) $\lambda\delta_{ik}\delta_{mp} + \mu(\delta_{im}\delta_{kp} + \delta_{ip}\delta_{km}) + \nu(\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km})$, where λ, μ, ν are scalars ([1889, **5**, §§18–23] [1892, **4**, §16] [1930, **13**] [1931, **9**, Ch. VII]). Cf. also [1940, **9**].

Hence the scalars k_T in (6.3) are functions of I_a, II_a, III_a only. Furthermore, the Cayley-Hamilton equation takes the form[6]

$$\mathbf{a}^3 = III_a\mathbf{I} - II_a\mathbf{a} + I_a\mathbf{a}^2. \tag{6.5}$$

Hence $\mathbf{a}^3$ and all higher powers of $\mathbf{a}$ in (6.3) may be expressed in terms of $\mathbf{I}$, $\mathbf{a}$, $\mathbf{a}^2$, and the invariants, and thus $\mathbf{b}$ *is an analytic isotropic function of* $\mathbf{a}$ *if and only if it be of the form*

$$\mathbf{b} = \mathcal{G}_0^{\mathbf{b}(\mathbf{a})}\mathbf{I} + \mathcal{G}_1^{\mathbf{b}(\mathbf{a})}\mathbf{a} + \mathcal{G}_2^{\mathbf{b}(\mathbf{a})}\mathbf{a}^2, \tag{6.6}$$

where the coefficients $\mathcal{G}_T^{\mathbf{b}(\mathbf{a})}$ are power series in the principal invariants I_a, II_a, III_a. Equivalently,

$$b^i{}_j = \mathcal{G}_0^{\mathbf{b}(\mathbf{a})}\delta^i{}_j + \mathcal{G}_1^{\mathbf{b}(\mathbf{a})}a^i{}_j + \mathcal{G}_2^{\mathbf{b}(\mathbf{a})}a^i{}_k a^k{}_j\,. \tag{6.7}$$

The *joint invariant* $\Phi(\mathbf{a}, \mathbf{b})$ is defined by

$$\Phi(\mathbf{a}, \mathbf{b}) \equiv a^i{}_j b^j{}_i = \Phi(\mathbf{b}, \mathbf{a}). \tag{6.8}$$

Hence $\mathbf{b}$ *is an analytic isotropic function of* $\mathbf{a}$ *if and only if*

$$\Phi = \Phi(I_a, II_a, III_a). \tag{6.9}$$

The *moments* $\overline{II}_a$, $\overline{III}_a$ and the *octahedral invariant*[7] Δ_a are defined by

$$\begin{aligned} \overline{II}_a &\equiv a^i{}_j a^j{}_i = \Phi(\mathbf{a}, \mathbf{a}) = I_a{}^2 - 2II_a = \sum_i (a_i)^2, \\ \overline{III}_a &\equiv a^i{}_j a^j{}_k a^k{}_i = I_a{}^3 - 3I_a II_a + 3III_a = \sum_i (a_i)^3, \\ 3\Delta_a &\equiv [2I_a{}^2 - 6II_a]^{\frac{1}{2}} = [\sum_{i>j} (a_i - a_j)^2]^{\frac{1}{2}}. \end{aligned} \tag{6.10}$$

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[2] Here and henceforth the summation convention is applied to *every diagonally repeated index*, tensorial or otherwise, the sum being understood to run over the full range of variability of the index. Thus (6.3) reads $\mathbf{b} = \Sigma_{T=0}^{\infty} k_T \mathbf{a}^T$.

[3] The proper values of both matrices of mixed components and both matrices of physical components of any tensor of the second order coincide. Cf. [1952, **3**].

[4] RANKINE [1856, **2**, §3] refers to this result as a discovery of CAYLEY, but I have been unable to trace the reference or to find the theorem itself in modern works on algebra, although it is frequently used without proof in studies on continuum mechanics. Professor WHAPLES has kindly communicated to me a modern proof, too long to include here.

[5] These invariants were introduced by CAUCHY [1827, **2**, eq. (20)].

[6] *E.g.* [1934, **1**, §2.05].

[7] $\Delta_a{}^2$ was introduced by v. MISES [1913, **2**, §1] as the sum of the squares of the principal off-diagonal components of $\mathbf{a}$; a geometrical interpretation is given by NADAI & LODE [1933, **5**, §II]; cf. [1937, **4**, pp. 206–207]. From the indication given in an abstract [1872, **2**, p. 431] of an unpublished paper, I judge it possible that similar interpretation was given by KLEITZ.

$(\overline{II}_a)^{\frac{1}{2}}$ is the *intensity* of **a**. Thus $3^{\frac{1}{2}}\Delta_a$ equals the intensity of the *deviator* of **a**, whose components are $a^i{}_j - I_a\delta^i{}_j/3$; plainly $\Delta_{a+cI} = \Delta_a$, for any c. We have[8]

$$\frac{\partial I_a}{\partial a^j{}_i} = \delta^i{}_j, \qquad \frac{\partial II_a}{\partial a^j{}_i} = I_a\delta^i{}_j - a^i{}_j, \qquad \frac{\partial \overline{II}_a}{\partial a^j{}_i} = 2a^i{}_j,$$
$$\frac{\partial III_a}{\partial a^j{}_i} = a^i{}_k a^k{}_j - I_a a^i{}_j + II_a\delta^i{}_j = III_a(a^{-1})^i{}_j, \qquad \frac{\partial \overline{III}_a}{\partial a^j{}_i} = 3a^i{}_k a^k{}_j, \tag{6.11}$$

where in $(6.11)_5$, as henceforth, $(a^{-1})^i{}_j$ stands for the components of $\mathbf{a}^{-1}$, which is assumed to exist.

If a relation of the type

$$b^i{}_j = \frac{\partial \Gamma}{\partial a^j{}_i} \tag{6.12}$$

hold, then **b** as a function of **a** *admits the potential* Γ. To determine whether or not a given analytic isotropic function **b**(**a**) admits a potential, form $\partial\Gamma/\partial a^i{}_j$ from $\Gamma = \Gamma(I_a, II_a, III_a)$ and (6.11), compare the result with (6.7), and thus obtain expressions for $\partial\Gamma/\partial I_a$, $\partial\Gamma/\partial II_a$, and $\partial\Gamma/III_a$; by cross differentiation follow the necessary and sufficient conditions[9]

$$\frac{\partial \mathcal{G}_0}{\partial II_a} + I_a\frac{\partial \mathcal{G}_1}{\partial II_a} + (I_a^2 - II_a)\frac{\partial \mathcal{G}_2}{\partial II_a} = -\frac{\partial \mathcal{G}_1}{\partial I_a} - I_a\frac{\partial \mathcal{G}_2}{\partial I_a},$$
$$\frac{\partial \mathcal{G}_0}{\partial III_a} + I_a\frac{\partial \mathcal{G}_1}{\partial III_a} + (I_a^2 - II_a)\frac{\partial \mathcal{G}_2}{\partial III_a} = \frac{\partial \mathcal{G}_2}{\partial I_a}, \tag{6.13}$$
$$-\frac{\partial \mathcal{G}_1}{\partial III_a} - I_a\frac{\partial \mathcal{G}_2}{\partial III_a} = \frac{\partial \mathcal{G}_2}{\partial II_a},$$

where for ease of writing the superscript b(a) has been omitted.

7. Inversion of isotropic functions. If the series (6.3) may be reverted, a similar analysis yields for each inverse $\mathbf{a}^{-1}$ an expression of type (6.6) with coefficients $\mathcal{G}_\Gamma^{a(b)}$ which are power series in I_b, II_b, III_b. Reiner[1] has obtained formal expressions for the $\mathcal{G}_\Gamma^{a(b)}$ in terms of the $\mathcal{G}_\Gamma^{b(a)}$, as follows. Put (6.6) into the analogous expression for **a**, and eliminate $\mathbf{a}^3$ and $\mathbf{a}^4$ by (6.5). There results a quadratic polynomial in **a** which must be identically zero, and whose three scalar coefficients must therefore vanish. Solving the resulting three scalar equations for the $\mathcal{G}_\Gamma^{a(b)}$, we obtain

$$\mathcal{G}_0^{a(b)} = -\mathcal{J}[\mathcal{G}_0^{b(a)}\mathcal{K}_2 - \mathcal{G}_2^{b(a)}\mathcal{K}_0], \qquad \mathcal{G}_1^{a(b)} = \mathcal{J}\mathcal{K}_2, \qquad \mathcal{G}_2^{a(b)} = -\mathcal{J}\mathcal{G}_2^{b(a)}, \tag{7.1}$$

[8] [1937, **1**, §3] [1945, **1**, §4].

[9] This analysis is given by REINER [1945, **1**, §4] in terms of the deviator.

[1] [1945, **1**, §5]. The results are expressed in terms of the deviators.

where

$$
\begin{aligned}
\mathcal{H}_0 &\equiv [\mathcal{G}_0^{b(a)}]^2 + 2III_a\mathcal{G}_1^{b(a)}\mathcal{G}_2^{b(a)} + I_a III_a[\mathcal{G}_2^{b(a)}]^2,\\
\mathcal{H}_1 &\equiv 2\mathcal{G}_0^{b(a)}\mathcal{G}_1^{b(a)} - 2II_a\mathcal{G}_1^{b(a)}\mathcal{G}_2^{b(a)} + (III_a - I_a II_a)[\mathcal{G}_2^{b(a)}]^2,\\
\mathcal{H}_2 &\equiv [\mathcal{G}_1^{b(a)}]^2 + 2\mathcal{G}_0^{b(a)}\mathcal{G}_2^{b(a)} + 2I_a\mathcal{G}_1^{b(a)}\mathcal{G}_2^{b(a)} + (I_a^2 - II_a)[\mathcal{G}_2^{b(a)}]^2,\\
\mathcal{J} &\equiv [\mathcal{G}_1^{b(a)}\mathcal{H}_2 - \mathcal{G}_2^{b(a)}\mathcal{H}_1]^{-1}.
\end{aligned}
\tag{7.2}
$$

The formulae (7.1) give $\mathcal{G}_T^{a(b)}$ as a function of I_a, II_a, III_a. To obtain these latter in terms of I_b, II_b, III_b one must solve the transcendental system

$$
\begin{aligned}
I_b &= 3\mathcal{G}_0^{b(a)} + \mathcal{G}_1^{b(a)}I_a + \mathcal{G}_2^{b(a)}(I_a^2 - 2II_a),\\
II_b &= 3[\mathcal{G}_0^{b(a)}]^2 + 2\mathcal{G}_0^{b(a)}\mathcal{G}_1^{b(a)}I_a + 2\mathcal{G}_0^{b(a)}\mathcal{G}_2^{b(a)}(I_a^2 - 2II_a)\\
&\quad + [\mathcal{G}_1^{b(a)}]^2 II_a + \mathcal{G}_1^{b(a)}\mathcal{G}_2^{b(a)}(I_a II_a - 3III_a) + [\mathcal{G}_2^{b(a)}]^2(II_a^2 - 2I_a III_a),\\
III_b &= [\mathcal{G}_0^{b(a)}]^3 + [\mathcal{G}_0^{b(a)}]^2\mathcal{G}_1^{b(a)}I_a + [\mathcal{G}_0^{b(a)}]^2\mathcal{G}_2^{b(a)}(I_a^2 - 2II_a)\\
&\quad + \mathcal{G}_0^{b(a)}[\mathcal{G}_1^{b(a)}]^2 II_a + \mathcal{G}_0^{b(a)}\mathcal{G}_1^{b(a)}\mathcal{G}_2^{b(a)}(I_a II_a - 3III_a)\\
&\quad + \mathcal{G}_0^{b(a)}[\mathcal{G}_2^{b(a)}]^2(II_a^2 - 2I_a II_a) + [\mathcal{G}_1^{b(a)}]^3 III_a\\
&\quad + [\mathcal{G}_1^{b(a)}]^2\mathcal{G}_2^{b(a)}I_a III_a + \mathcal{G}_1^{b(a)}[\mathcal{G}_2^{b(a)}]^2 II_a III_a + [\mathcal{G}_2^{b(a)}]^3 III_a^2.
\end{aligned}
\tag{7.3}
$$

In general, if real solutions exist, they are not unique, and the purely formal character of the whole analysis must be borne in mind.

8. Power series for the coefficients. We write the power series for $\mathcal{G}_T^{b(a)}$ in the form

$$
\mathcal{G}_T^{b(a)} = \mathcal{G}_{TIJK}^{b(a)} I_a^{I} II_a^{J} III_a^{K},
\tag{8.1}
$$

where the $\mathcal{G}_{TIJK}^{b(a)}$ are independent of **a**.

From (6.8) and (6.6) we have then

$$
\begin{aligned}
\Phi(\mathbf{a}, \mathbf{b}) &= \Phi_{IJK}^{(a,b)} I_a^{I} II_a^{J} III_a^{K},\\
\Phi_{IJK}^{(a,b)} &\equiv \mathcal{G}_{0,I-1,J,K} + \mathcal{G}_{1,I-2,J,K} - 2\mathcal{G}_{1,I,J-1,K}\\
&\qquad + \mathcal{G}_{2,I-3,J,K} - 3\mathcal{G}_{2,I-1,J-1,K} + 3\mathcal{G}_{2,I,J,K-1}.
\end{aligned}
\tag{8.2}
$$

Thus

$$
\begin{aligned}
\Phi = \mathcal{G}_{0000}I_a &+ (\mathcal{G}_{0100} + \mathcal{G}_{1000})I_a^2 - 2\mathcal{G}_{1000}II_a\\
&+ (\mathcal{G}_{0200} + \mathcal{G}_{1100} + \mathcal{G}_{2000})I_a^3 + (\mathcal{G}_{0010} - 2\mathcal{G}_{1100} - 3\mathcal{G}_{2000})I_a II_a\\
&+ 3\mathcal{G}_{2000}III_a + \cdots,
\end{aligned}
\tag{8.3}
$$

where for ease of writing the superscript b(a) has been omitted. There are certain changes of the coefficients $\mathcal{G}_{\mathrm{TIJK}}$ which leave Φ invariant, e.g. one in which the four coefficients $\mathcal{G}_{0010}$, $\mathcal{G}_{0200}$, $\mathcal{G}_{1100}$, and $\mathcal{G}_{2000}$ are varied in any way that leaves the three sums $\mathcal{G}_{0200} + \mathcal{G}_{1100} + \mathcal{G}_{2000}$, $\mathcal{G}_{0010} - 2\mathcal{G}_{1100} - 3\mathcal{G}_{2000}$, $3\mathcal{G}_{2000}$ invariant.

9. Linear and quasi-linear functions. The classical theories of elasticity and fluid dynamics now appear as defined by linear approximations to isotropic relationships. To obtain the Hooke-Cauchy law (1.1) we write $b^i{}_j = t^i{}_j$, $a^i{}_j = \tilde{e}^i{}_j$, $\mathcal{G}_{0100}^{\mathrm{t}(\tilde{\mathrm{e}})} = \lambda_{\mathrm{E}}$, $\mathcal{G}_{1000}^{\mathrm{t}(\tilde{\mathrm{e}})} = 2\mu_{\mathrm{E}}$, and suppose all the remaining $\mathcal{G}_{\mathrm{TIJK}}^{\mathrm{t}(\tilde{\mathrm{e}})}$ vanish. To obtain the Newton-Cauchy-Poisson law (1.2), we write $b^i{}_j = t^i{}_j$, $a^i{}_j = d^i{}_j$, $\mathcal{G}_{0000}^{\mathrm{t(d)}} = -p$, $\mathcal{G}_{0100}^{\mathrm{t(d)}} = \lambda_{\mathrm{V}}$, $\mathcal{G}_{1000}^{\mathrm{t(d)}} = 2\mu_{\mathrm{V}}$, and suppose all the remaining $\mathcal{G}_{\mathrm{TIJK}}^{\mathrm{t(d)}}$ vanish. The conditions (6.13) are satisfied, and in fact $\Gamma = \frac{1}{2}\Phi$; for (1.1), Γ is the *strain energy* or *elastic energy*, while for (1.2) $\Gamma + pI_{\mathrm{d}}$ is *Rayleigh's dissipation function.*[1]

A part of the simplicity of the classical theories arises not specifically from their linearity in the actual quantities concerned, but from their tensorial linearity. Consider the *quasi-linear* case, in which $\mathcal{G}_2^{\mathrm{b(a)}} = 0$ although $\mathcal{G}_0^{\mathrm{b(a)}}$ and $\mathcal{G}_1^{\mathrm{b(a)}}$ remain arbitrary power series in I_{a}, II_{a}, III_{a}. From (7.1) we now obtain $\mathcal{G}_0^{\mathrm{a(b)}} = -\mathcal{G}_0^{\mathrm{b(a)}}/\mathcal{G}_1^{\mathrm{b(a)}}$, $\mathcal{G}_1^{\mathrm{a(b)}} = [\mathcal{G}_1^{\mathrm{b(a)}}]^{-1}$, $\mathcal{G}_2^{\mathrm{a(b)}} = 0$, so that any inverse relation must also be quasi-linear. Let l, m, n be any three quantities such that $l + m + n = 0$. Then *for a quasi-linear analytic isotropic relationship between* **b** *and* **a** *to exist it is necessary and sufficient that*

$$\frac{lb^1{}_1 + mb^2{}_2 + nb^3{}_3}{la^1{}_1 + ma^2{}_2 + na^3{}_3} = \mathcal{G}_1^{\mathrm{b(a)}} = \frac{1}{\mathcal{G}_1^{\mathrm{a(b)}}}; \tag{9.1}$$

from $l = \frac{2}{3}$, $m = -\frac{1}{3}$, $n = -\frac{1}{3}$ follows the proportionality of the deviators of **a** and **b**:

$$\frac{b^i{}_j - I_{\mathrm{b}}\delta^i{}_j/3}{a^i{}_j - I_{\mathrm{a}}\delta^i{}_j/3} = \mathcal{G}_1^{\mathrm{b(a)}}, \tag{9.2}$$

while from $l = 1$, $m = -1$, $n = 0$ follows[2]

$$\frac{b^1{}_1 - b^2{}_2}{a^1{}_1 - a^2{}_2} = \frac{b^2{}_2 - b^3{}_3}{a^2{}_2 - a^3{}_3} = \frac{b^3{}_3 - b^1{}_1}{a^3{}_3 - a^1{}_1} = \mathcal{G}_1^{\mathrm{b(a)}}. \tag{9.3}$$

An alternative formulation of (9.1) is: *a necessary and sufficient condition for* **b** *to be a quasi-linear analytic isotropic function of* **a** *is that identically*

---

[1] RAYLEIGH [1873, **4**, §II] treated an analogous situation in mass-point mechanics.

[2] A special case is given by ST. VENANT [1843, **1**, eq. (1)] and MAXWELL [1853, **1**, p. 38]. Cf. [1931, **11**, Ch. 14], [1948 **1**, §6].

$$f(lb^1_1 + mb^2_2 + nb^3_3, mb^1_1 + nb^2_2 + lb^3_3, nb^1_1 + lb^2_2 + mb^3_3)$$
$$= f(la^1_1 + ma^2_2 + na^3_3, ma^1_1 + na^2_2 + la^3_3, na^1_1 + la^2_2 + ma^3_3), \quad (9.4)$$

where f is any homogeneous function of degree zero, and $l + m + n = 0$.
From (9.2) follows $\mathcal{G}_1^{b(a)} = \Delta_b/\Delta_a$, so that[3]

$$b^i_j = \tfrac{1}{3}I_b\delta^i_j + \frac{\Delta_b}{\Delta_a}(a^i_j - \tfrac{1}{3}I_a\delta^i_j) = \frac{1}{3}\left(\frac{I_b}{I_a} - \frac{\Delta_b}{\Delta_a}\right)I_a\delta^i_j + \frac{\Delta_b}{\Delta_a}a^i_j, \quad (9.5)$$

i.e., $3\mathcal{G}_0^{b(a)} = (I_b/I_a - \Delta_b/\Delta_a)I_a$. For the quasi-linear case the conditions (6.13) for the existence of a potential require that $\mathcal{G}_0^{b(a)}$ and $\mathcal{G}_1^{b(a)}$ be functions of I_a and II_a only, and that $\partial\mathcal{G}_0^{b(a)}/\partial II_a + I_a\,\partial\mathcal{G}_1^{b(a)}/\partial II_a + \partial\mathcal{G}_1^{b(a)}/\partial I_a = 0$.

If terms of second order in **a** be retained in the power series expansion of (6.7) by means of (8.1), it must follow that $\mathcal{G}_2^{b(a)} \neq 0$ except in the special case when $\mathcal{G}_{2000}^{b(a)} = 0$, so that in general an approximation which goes beyond the linear terms will fail to be quasi-linear.

10. Special cases. We note here the special cases of (6.6) which occur in typical illustrative situations:

$$\mathbf{a} = \begin{Vmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{Vmatrix},$$

$$\mathbf{b} - \mathcal{G}_0^{b(a)}\mathbf{I} = \mathcal{G}_1^{b(a)}\begin{Vmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{Vmatrix} + \mathcal{G}_2^{b(a)}\begin{Vmatrix} r^2 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & s^2 \end{Vmatrix}. \quad (10.1)$$

$$\mathbf{a} = \begin{Vmatrix} s & u & 0 \\ u & -s & 0 \\ 0 & 0 & 0 \end{Vmatrix},$$

$$\mathbf{b} - \mathcal{G}_0^{b(a)}\mathbf{I} = \mathcal{G}_1^{b(a)}\begin{Vmatrix} s & u & 0 \\ u & -s & 0 \\ 0 & 0 & 0 \end{Vmatrix} + \mathcal{G}_2^{b(a)}\begin{Vmatrix} u^2+s^2 & 0 & 0 \\ 0 & u^2+s^2 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \quad (10.2)$$

$$\mathbf{a} = \begin{Vmatrix} 0 & 0 & u \\ 0 & 0 & s \\ u & s & 0 \end{Vmatrix},$$

$$\mathbf{b} - \mathcal{G}_0^{b(a)}\mathbf{I} = \mathcal{G}_1^{b(a)}\begin{Vmatrix} 0 & 0 & u \\ 0 & 0 & s \\ u & s & 0 \end{Vmatrix} + \mathcal{G}_2^{b(a)}\begin{Vmatrix} u^2 & us & 0 \\ us & s^2 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \quad (10.3)$$

[3] [1945, **2**, pp. A-261-A-262].

Now in a linear theory $\mathcal{G}_0^{b(a)} = KI_a$, $\mathcal{G}_1^{b(a)} = L$, $\mathcal{G}_2^{b(a)} = 0$, where K and L are constants. Hence *in the case* (10.1) *departures from classical results are generally of second order in* **a**, *and the influences of* $\mathcal{G}_0^{b(a)}$, $\mathcal{G}_1^{b(a)}$, *and* $\mathcal{G}_2^{b(a)}$ *are not distinguishable.* In the cases (10.2) and (10.3), however, appear two striking new phenomena.

(1) *Kelvin effect*: Even though $I_a = 0$, the coefficient $\mathcal{G}_0^{b(a)}$ gives rise to a spherical portion $\mathcal{G}_0^{b(a)}\mathbf{I}$ in **b**. This effect is lacking in a linear theory, but it occurs in a quasi-linear theory.

(2) *Poynting effect*: The off-diagonal components of **a** and the coefficient $\mathcal{G}_2^{b(a)}$ give rise to deviatoric diagonal components in **b**; this effect is lacking in a quasi-linear theory.

Both effects are of the *second* or higher order in **a**. Now in the cases (10.2) and (10.3), since $I_a = 0$ the result predicted by a classical linearized theory depends upon $\mathcal{G}_1^{b(a)}$ alone, while the Kelvin and Poynting effects are independent of $\mathcal{G}_1^{b(a)}$. Furthermore, the fact that $I_a = 0$ reduces the expansion for $\mathcal{G}_1^{b(a)}$ to $\mathcal{G}_1^{b(a)} = \mathcal{G}_{1000}^{b(a)} + \mathcal{G}_{1010}^{b(a)} II_a + \cdots$, so that the terms of second order in **a** do not affect the magnitude of the classical effect. *In the cases* (10.2) *and* (10.3), *an experiment measuring only a classical effect can yield no information about* $\mathcal{G}_0^{b(a)}$ *or* $\mathcal{G}_2^{b(a)}$; *in particular, such an experiment can never confirm the sufficiency or insufficiency of a quasi-linear theory. The coefficients* $\mathcal{G}_0^{b(a)}$, $\mathcal{G}_2^{b(a)}$ *give rise to* second *or higher order effects which are independent of the classical coefficient* $\mathcal{G}_1^{b(a)}$, *and an experiment measuring only these effects can yield no information about* $\mathcal{G}_1^{b(a)}$. *Quantitative departures from the classical measurements are* third *or higher order effects.* More generally it is easy to see that the magnitude of the classical effect is determined by the terms of odd order in **a**, while the magnitude of the Kelvin and Poynting effects is determined by the terms of even order. These extremely important conclusions will be illustrated many times (§§42, 45, 54, 71, 72). While they apply only to the specific examples (10.1), (10.2), (10.3), these examples include all the situations usually employed in testing a continuum theory by experiment.

11. "Retaining the non-linear terms." If indeed it were sufficient to "retain the non-linear terms," the foregoing analysis, yielding general yet explicit formulae for the approximations of any desired order,[1] would suffice. It seems hardly to be expected, however, that a right theory can be constructed from the mere notion of isotropy, without recourse to the principles of kinematics and of mechanics. Before the formulae developed above become relevant, two basic questions must be answered. (1) Does a relation of type (6.3) properly describe the concepts of springiness and fluidity? Is one tensorial independent variable sufficient? (2) What tensors shall be chosen for **b** and **a**? Do the variables used in the classical theories really embody exactly the concepts they are intended to represent, or do they merely approximate them? In the next chapter, therefore, the tools necessary for working at these questions will be assembled.

[1] CAUCHY's method of isotropic tensors was first applied to the construction of admissible non-linear expressions by BOUSSINESQ [1868, **2**, Note 1]. Cf. also [1900, **1**, §14] [1901, **2**, §§4–9]. It was used by LEVY [1869, **2**] to obtain elaborate linear expressions (cf. [1869, **1**]).

Chapter III. General theory of continuous bodies

12. Material and spatial coordinates. At some fixed time t_0 let the points of the continuum be assigned co-ordinates X^α in any Euclidean co-ordinate system, and let the metric tensor be $G_{\alpha\beta}$; at any other time t let them be assigned co-ordinates x^i in any independently selected Euclidean system,[1] and let the metric tensor be g_{ij} :

$$dS_0^2 = G_{\alpha\beta}dX^\alpha dX^\beta, \qquad ds^2 = g_{ij}dx^i dx^j. \tag{12.1}$$

Quantities which transform as tensors with respect to transformations of the X^α will be denoted by Latin majuscules with Greek miniscule indices: A, B^α, $C^{\alpha\beta}$, $C^\alpha{}_\beta$, $\cdots$. In a rectangular Cartesian system we write[2] $X = X^1$, $Y = X^2$, $Z = X^3$. Quantities which transform as tensors with respect to transformations of the x^i will be denoted by Latin miniscules[3] with Latin miniscule indices: a, b^i, c^{ij}, $c^i{}_j$, $\cdots$. In a rectangular Cartesian system we write $x = x^1$, $y = x^2$, $z = x^3$. The X^α are called *material co-ordinates*,[4] the x^i *spatial co-ordinates*.[5] We

[1] This completely general scheme, in which the choice of the spatial co-ordinate system is independent of the choice of the material system, is introduced by Murnaghan [1937, **1**, §1]. L. Brillouin [1928, **5**, §4] [1925, **1**, §2] [1938, **2**, Ch. X, §VI] instead requires that the two systems be the same: $g_{ij}(x^k) = G_{ij}(X^\alpha(x^k, t))$, thus regarding the actual motion (12.2) as a transformation of co-ordinates. Hence certain quantities which are relative invariants in Brillouin's scheme in Murnaghan's are absolute. Consider e.g. $(24.1)_3$ below, which when the X^L are taken as the X^α becomes $\rho J = \rho_0$. With respect to transformations of either spatial or material co-ordinates alone, all three quantities ρ, J, ρ_0 are absolute scalars; if ρ be regarded as the transformed value of ρ_0, however, then these two quantities must be the values of a scalar density in the x^i and X^α systems, respectively. Cf. also [1950, **12** §§2–3], and Note 4 below.

[2] This notation is that used by Euler [1762, **1**] [1770, **1**, §100] in introducing these variables; it was at first followed by Lagrange [1762, **2**, §44], who later [1788, **1**, Part II, Sect. 11, ¶4] employed instead the letters a, b, c now in common use.

[3] Observe that, *e.g.*, the $c^i{}_j$ transform as components of a mixed tensor field with respect to transformations of the x^i but as *scalars* with respect to transformations of the X^α; the $x^i{}_{,\alpha}$ form a contravariant vector field with respect to x^i, a covariant vector field with respect to X^α, etc. Cf. Note 1.

[4] Hill [1881, **1**, §IV] introduced material co-ordinates which are not necessarily initial co-ordinates of the particles, but merely any substantially constant independent functions; his scheme is elaborated by Deuker [1941, **8**, §§II–IV] and Oldroyd [1950, **3**, §§2–3]. Oldroyd bases his dynamical proposals upon an identification principle, in which the spatial and the material co-ordinates are made to coincide at a given instant and thus the numerical values of the components of a material tensor coincide with those of the corresponding components of an associated spatial tensor. Thus *e.g.* he shows that under these circumstances **d** coincides with **E**, a result equivalent to (22.15). Lodge [1951, **13**] insists

shall refer also to the *particle* X^α, the *place*[6] x^i. The *motion*[7] of the continuum carries various particles through various places in the course of time. It is embodied in the mappings

$$x^i = x^i(X^\alpha, t), \qquad X^\alpha = X^\alpha(x^i, t), \tag{12.2}$$

assumed to be single-valued and continuously differentiable with respect to each of their variables as many times as may be desired, except possibly at certain singular surfaces, curves, or points.

The *displacement gradients* are

$$x^i{}_{,\alpha} \equiv \frac{\partial x^i}{\partial X_\alpha}, \qquad X^\alpha{}_{,i} \equiv \frac{\partial X^\alpha}{\partial x^i}. \tag{12.3}$$

It is easy to see that these quantities enjoy the tensorial invariance denoted by their indices.

13. A variable reference configuration. Subject to the usual continuity and differentiability assumptions, let a reference configuration X^L and associated metric be assigned in an arbitrary way:

$$X^L = X^L(X^\alpha, t), \qquad dS^2 = G_{LM}dX^L dX^M. \tag{13.1}$$

Quantities which transform as tensors with respect to transformations of the X^L will be denoted by Latin majuscules with Latin majuscule indices A, B^L, C^{LM}, $C^L{}_M$, $\cdots$. The reference configuration may be one which the medium has actually assumed at some time t_r, $t_0 \leqq t_r \leqq t$, or it may be any other configuration satisfying the condition that the metric $(13.1)_2$ be Euclidean. In the classical treatments the reference configuration is taken as the initial configuration, i.e. $t_r = t_0$. The present distinction is introduced so as to permit the distortion of the medium to be measured with respect to a *varying* reference configuration if desirable; that is, *while the variables X^α and t are by hypothesis independent, so that dS_0^2 is necessarily constant for each particle, dS^2 may be chosen as a virtually arbitrary function of time for each particle*, as indicated by (13.1). Since

$$X^L{}_{,i}x^i{}_{,M} = \delta^L{}_M, \qquad x^i{}_{,L}X^L{}_{,j} = \delta^i{}_j, \tag{13.2}$$

upon the rather obvious fact that it is possible to use the X^α and t as the only independent variables in any given problem of the mechanics of continua. GRAHAM [1949, **14**] discusses a "natural frame" X^L intermediate between the x^i and the X^α, requiring that in a volume element "the number of atoms included never alters," but gives no means of finding such a frame. A more general concept including all these possibilities is given in §13.

[5] The current erroneous German terminology refers to the spatial co-ordinates, which were introduced by D'ALEMBERT [1752, **1**, §43], as EULERIAN, to the material co-ordinates, which were introduced by EULER [1762, **1**], as LAGRANGIAN..

[6] This happy term is introduced by LODGE [1951, **13**, §2].

[7] It is often preferable to regard t as simply a parameter, not necessarily the time.

the $x^i{}_{,L}$ may be considered as functions of the $X^M{}_{,j}$, and conversely. By differentiation we obtain

$$\frac{\partial X^L{}_{,i}}{\partial x^j{}_{,M}} = -X^L{}_{,j} X^M{}_{,i}, \qquad \frac{\partial x^i{}_{,L}}{\partial X^M{}_{,j}} = -x^i{}_{,M}\, x^j{}_{,L},$$
$$x_{j,M} = -X_{S,j}\frac{\partial x^{i,S}}{\partial X^{M,i}}, \qquad X_{L,i} = -X_{K,L}\frac{\partial X^{N,k}}{\partial x^{i,N}}. \tag{13.3}$$

Let J be that absolute scalar which reduces to the Jacobian $\partial x/\partial X$ when both co-ordinate systems are rectangular Cartesian:

$$J \equiv \frac{\sqrt{g}}{\sqrt{G}} \det x^i{}_{,L}.$$

In consequence of the regularity assumptions already stated, we have $J \neq 0, \infty$. Euler's relation[1] between the elements of volume dv and dV and Nanson's relation[2] between the elements of surface da_i and dA_L are respectively

$$dv = J dV, \qquad da_i = J X^L{}_{,i} dA_L. \tag{13.5}$$

From (13.4) follow Boussinesq's identity[3] and the Euler-C. Neumann identities,[4] respectively

$$\frac{\partial J}{\partial x^i{}_{,L}} = X^L{}_{,i} J, \qquad (J X^L{}_{,i})_{,L} = 0, \qquad (J^{-1} x^i{}_{,L})_{,i} = 0. \tag{13.6}$$

14. Measures of deformation I. Cauchy's and Green's deformation tensors. From (12.2) and $(13.1)_1$ it follows that $x^i = x^i(X^L, t)$, $X^L = X^L(x^i, t)$; hence

$$dx^i = x^i{}_{,L} dX^L, \qquad dX^L = X^L{}_{,i} dx^i, \tag{14.1}$$

and consequently

$$ds^2 = C_{LM} dX^L dX^M, \qquad dS^2 = c_{ij} dx^i dx^j, \tag{14.2}$$

where

$$C_{LM} \equiv g_{ij} x^i{}_{,L} x^j{}_{,M}, \qquad c_{ij} \equiv G_{LM} X^L{}_{,i} X^M{}_{,j}. \tag{14.3}$$

The symmetric tensors **C** and **c** are the *Cauchy-Green deformation tensors.*[1]

[1] [1762, **1**] [1770, **1**, §§112–118].
[2] [1878, **1**].
[3] [1872, **1**, §I, eq. (2)].
[4] While $(13.6)_2$ was first given by C. Neumann [1860, **1**, eq. (25)], it is equivalent to Euler's [1770, **1**, §§26, 49] solution of $A^L{}_{,L} = 0$ in the form $A^L = \epsilon^{LMN} B_{,M} C_{,N}$.

[1] Cauchy considered only the case of a fixed reference configuration. He introduced **c** explicity [1827, **2**, eq. (10) (11)] and **C** by implication [1841, **1**, §I, eq. (15)]. **C** had previously been introduced explicitly by Green (1839) [1841, **2**, pp. 295–296].

If both systems of co-ordinates be rectangular Cartesian, then in terms of the *displacements*[2] $u \equiv x - X$, $v \equiv y - Y$, $w \equiv z - Z$ we have

$$\begin{aligned} C_{AA} &= \left(1 + \frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2, \\ C_{AB} &= \left(1 + \frac{\partial u}{\partial X}\right)\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\left(1 + \frac{\partial v}{\partial Y}\right) + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}, \cdots, \\ c_{xx} &= \left(1 - \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2, \\ c_{xy} &= -\left(1 - \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\left(1 - \frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \cdots \end{aligned} \tag{14.4}$$

For a rigid displacement it is necessary and sufficient that $C^L{}_M = \delta^L{}_M$, or, equivalently, $c^i{}_j = \delta^i{}_j$.

The principal directions of **C** are carried by the mapping $x^i = x^i(X^L, t)$ into the principal directions of **c**, and in general these are the only mutually orthogonal directions which are carried into mutually orthogonal directions. If a vector of length L_i tangent to a principal axis of **C** be carried into a vector of length $L_i + \Delta L_i$, then $(1 + \Delta L_i/L_i)^2 = C_i$, where C_i is the proper value of **C** corresponding to the principal axis. Similarly, if a vector tangent to a principal axis of **c** be now of length $l_i + \Delta l_i$ but originally of length l_i, then $(1 + \Delta l_i/l_i)^{-2} = c_i$, where c_i is the proper value of **c** corresponding to this principal axis. Hence $C_i > 0$, $c_i > 0$. Now in fact $\Delta l_i/l_i = \Delta L_i/L_i$, the common value being the *principal extension* δ_i corresponding to the given principal direction. Equivalently, $c_i = C_i^{-1}$, a result which can be established as follows.[3] Since

$$\begin{aligned} C^L{}_M &= G^{LN} x^i{}_{,N}\, g_{ij} x^j{}_{,M} = x^{i,L} x_{i,M}, \\ c^i{}_j &= g^{ik} X^L{}_{,k} G_{LM} X^M{}_{,j} = X^{L,i} X_{L,j}, \end{aligned} \tag{14.5}$$

by (13.2) it is easy to see that[4]

$$(C^{-1})^L{}_M = X^{L,i} X_{M,i}, \qquad (C^{-1})^i{}_j = x^{i,L} x_{j,L}. \tag{14.6}$$

Since the matrices **AB** and **BA** have the same proper values, by inspection of

[2] KIRCHHOFF [1852, **1**, p. 762] most aptly remarked "durch Einführung der Verschiebungen gewinnt man nichts, wenn diese nicht unendlich klein sind, im Gegentheil verlieren dadurch die Formeln an Kürze und Übersichtlichkeit."

[3] These results are CAUCHY's [1841, **1**, p. 353]; the proof sketched here is adapted from [1937, **1**, Appendix]. Beautiful figures representing deformation ellipsoids by a rubber or cloth sheet stretched across an initially circular or rectangular frame are given by WEISSENBERG [1935, **2**, pp. 85–87] [1949, **31**]. BONVICINI [1932, **10**] [1935, **3**] discusses the separation of large pure strain from rotation.

[4] FINGER [1894, **2**, eq. (12) (31)] introduced $III_c\, \mathbf{C}^{-1}$ and $\mathbf{c}^{-1}$.

$(14.5)_4$ and $(14.6)_1$ it follows[5] that $c_i = (C^{-1})_i$. But $(C^{-1})_i = C_i^{-1}$. Hence $c_i = C_i^{-1}$. Q.E.D. Thus $C_i = (1 + \delta_i)^2$, $c_i = (1 + \delta_i)^{-2}$.

By (6.4) and the result just established we have[6]

$$I_c = \frac{II_C}{III_C}, \qquad II_c = \frac{I_C}{III_C}, \qquad III_c = \frac{1}{III_C}.$$

Since $(c^{-1})_i = c_i^{-1}$ and $(C^{-1})_i = C_i^{-1}$, similar relations connect the invariants of $\mathbf{c}^{-1}$ and $\mathbf{C}^{-1}$. These formulae yield the extremely important result that while in general a scalar function of $\mathbf{C}$ is not also a scalar function of $\mathbf{c}$ only, nevertheless *an analytic isotropic scalar function of any one of* $\mathbf{c}$, $\mathbf{c}^{-1}$, $\mathbf{C}$, $\mathbf{C}^{-1}$ *may be regarded equivalently as an analytic isotropic scalar function of any of the others.*

From $(14.5)_2$ and $(13.3)_3$ I perceive the first of the mnemonic and not inelegant forms

$$C_L{}^M = -\frac{\partial x^{i,M}}{\partial X^{L,i}}, \qquad c_i{}^j = -\frac{\partial X^{L,j}}{\partial x^{i,L}}, \qquad (C^{-1})^L{}_M = -\frac{\partial X^{L,i}}{\partial x^{i,M}}, \qquad (c^{-1})^i{}_j = -\frac{\partial x^{i,L}}{\partial X^{L,j}}. \tag{14.8}$$

Similarly

$$\frac{\partial C_{LM}}{\partial x^i{}_{,N}} = \delta^N{}_L x_{i,M} + \delta^N{}_M x_{i,L}, \qquad \frac{\partial (C^{-1})^{LM}}{\partial X^N{}_{,i}} = \delta^M{}_N X^{L,i} + \delta^L{}_N X^{M,i}, \tag{14.9}$$

there being analogues for $\mathbf{c}$ and $\mathbf{c}^{-1}$.

By $(6.4)_7$, $(14.5)_1$, and $(13.4)_2$ it follows that

$$III_C = \det C^L{}_M = \det g_{ij} \det G^{LN} (\det x^i{}_{,L})^2 = J^2. \tag{14.10}$$

Hence by $(13.5)_1$ we obtain[7]

$$dv = \sqrt{III_C}\, dV, \qquad dV = \sqrt{III_c}\, dv. \tag{14.11}$$

The general problem of comparing the Cauchy tensors $\mathbf{c}$ and $\mathbf{c}^*$ measuring the deformation of an actual configuration x^i from two *different* reference configurations X^L and X^{L*} is difficult. Consider, however, the special case in which, referred to a single rectangular Cartesian frame, we have $X^{L*} = KX^L$, so that the starred configuration is a uniform extension of the other, with linear coefficient K. Then by $(14.3)_2$ we have $\mathbf{c}^* = K^2\, \mathbf{c}$; hence in particular

$$I_{c^*} = K^2 I_c, \qquad II_{c^*} = K^4 II_c, \qquad III_{c^*} = K^6 III_c. \tag{14.12}$$

A deformation is *isochoric* if $dv = dV$; necessary and sufficient conditions[8]

[5] [1948, **10**, §3].
[6] [1948, **10**, 3]. The result of [1937, **1**, Appendix] is equivalent.
[7] $(14.11)_2$ was given by CAUCHY [1827, **2**, eq. (28)].
[8] [1948, **7**, §1], [1948, **10**, §6].

are $III_c = III_{c^{-1}} = III_C = III_{C^{-1}} = J = 1$. In an isochoric deformation (14.7) becomes

$$I_c = I_{C^{-1}} = II_C = II_{c^{-1}}, \qquad II_c = II_{C^{-1}} = I_C = I_{c^{-1}}. \tag{14.13}$$

Further, since $III_c = c_1c_2c_3 = 1$, we have $I_c = c_1 + c_2 + (c_1c_2)^{-1}$, whence since $c_i > 0$ follows the first of the inequalities

$$I_c, II_c, I_{c^{-1}}, II_{c^{-1}} \geqq 3. \tag{14.14}$$

If we write $III_c = 1$ in terms of the extension ratios we obtain $(1 + \delta_1)(1 + \delta_2)(1 + \delta_3) = 1$, or, in an evident notation, $I_\delta + II_\delta + III_\delta = 0$. Hence[9]

$$\begin{aligned} I_{c^{-1}} - 3 &= I_\delta^2 - 4II_\delta - 2III_\delta, \\ II_{c^{-1}} - 3 &= 2I_\delta^2 - II_\delta + 2I_\delta II_\delta - 7III_\delta + II_\delta^2 - 2I_\delta III_\delta. \end{aligned} \tag{14.15}$$

Consequently as $\delta_i \to 0$ we have $I_{c^{-1}} - 3 = O(\delta^2)$, $II_{c^{-1}} - 3 = O(\delta^2)$. These results are to be contrasted with the general case, when all that can be concluded is that $I_{c^{-1}}$, $I_{c^{-1}}$, $III_{c^{-1}}$ are positive and differ from 3 by a quantity which is $O(\delta)$.

15. Measures of deformation II. The Green-St. Venant and Alamansi-Hamel strain tensors. The *strain tensors*[1] **E** and **e**, defined by

$$2E^L{}_M \equiv C^L{}_M - \delta^L{}_M, \qquad 2e^i{}_j \equiv \delta^i{}_j - c^i{}_j, \tag{15.1}$$

are convenient for approximations, because their vanishing is a necessary and sufficient condition for a rigid displacement, and consequently, since their physical components[2] are dimensionless, a nearly rigid displacement may be specified by a series expansion in **E** or **e** in which all terms beyond some specified order are neglected.[3] The principal strains E_i and e_i are related to the principal ex-

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[9] [1951, **2**, §21].

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[1] **e** for infinitesimal strains (§19) was introduced by CAUCHY [1827, **2**, eq. (41)]; **E** for finite strains by GREEN [1841, **2**, p. 297] and ST. VENANT [1844, **1**] [1847, **1**, §2]; **e** for finite strains by ALMANSI [1911, **2**, §2] and HAMEL [1912, **1**, §363]. The present tensorial treatment is adapted from [1937, **1**], which generalizes [1925, **1**, §4] [1938, **2**, Ch. X, §VII]. I cannot understand the claim of SWAINGER [1949, **46**] that the tensors **E** and **e** "should not be interpreted physically," since their meaning in terms of change of length and co-ordinate angle is purely geometrical. MOUFANG [1947, **19**] gives a method of calculating approximately a function $f(I_e, II_e, III_e)$ such that $\mathbf{e}$-$\mathbf{I}f$ shall represent a change of shape without change of volume, but RICHTER [1948, **53**] claims the analysis is faulty. CISOTTI [1944, **16**] characterizes deformations in which $\mathbf{E} = f\mathbf{I}$.

[2] [1952, **3**].

[3] Approximate formulae for the $E^L{}_M$ in terms of the displacements are given in [1928, **5**, §4] [1925, **1**, §2] [1931, **1**].

tensions δ_i by

$$2E_i = C_i - 1 = (1 + \delta_i)^2 - 1, \qquad 2e_i = 1 - c_i = 1 - (1 + \delta_i)^{-2}, \quad (15.2)$$

and hence $E_i \geqq -\frac{1}{2}$, $e_i \leqq \frac{1}{2}$. From (15.2) and (14.7) follow equations connecting the invariants of these tensors:[4]

$$\begin{aligned} (dv/dV)^2 &= 1 + 2I_{\mathrm{E}} + 4II_{\mathrm{E}} + 8III_{\mathrm{E}} = III_{\mathrm{C}} = III_{\mathrm{c}^{-1}} = III_{\mathrm{c}^{-1}} \\ &= (1 - 2I_{\mathrm{e}} + 4II_{\mathrm{e}} - 8III_{\mathrm{e}})^{-1}, \quad (15.3) \\ III_{\mathrm{c}}I_{\mathrm{c}^{-1}} &= 3 - 4I_{\mathrm{e}} + 4II_{\mathrm{e}}\,, \; III_{\mathrm{c}}II_{\mathrm{c}^{-1}} = 3 - 2I_{\mathrm{e}}\,. \end{aligned}$$

Equivalent to (14.12) are[5]

$$\begin{aligned} I_{\mathrm{e}} &= \tfrac{3}{2}(1 - K^2) + K^2 I_{\mathrm{e}*}\,, II_{\mathrm{e}} = \tfrac{3}{4}(1 - K^2)^2 + K^2(1 - K^2)I_{\mathrm{e}*} + K^4 II_{\mathrm{e}*}\,, \\ III_{\mathrm{e}} &= \tfrac{1}{8}(1 - K^2)^3 + \tfrac{1}{4}K^2(1 - K^2)^2 I_{\mathrm{e}*} + \tfrac{1}{2}K^4(1 - K^2)II_{\mathrm{e}*} + K^6 III_{\mathrm{e}*}\,. \end{aligned} \quad (15.4)$$

16. Measures of deformation III. Hencky's logarithmic measure. Hencky[1] proposes to take $\log(1 + \delta_i)$ as a measure of principal strain. While Hencky himself does not give a systematic treatment, we may introduce the tensors **H** and **h**:

$$\begin{aligned} \mathbf{H} &\equiv \tfrac{1}{2}\log \mathbf{C} = \tfrac{1}{2}\log(\mathbf{I} + 2\mathbf{E}) = \mathbf{E} - \mathbf{E}^2 + \tfrac{4}{3}\mathbf{E}^3 - \cdots, \\ \mathbf{h} &\equiv -\tfrac{1}{2}\log \mathbf{c} = -\tfrac{1}{2}\log(\mathbf{I} - 2\mathbf{e}) = \mathbf{e} + \mathbf{e}^2 + \tfrac{4}{3}\mathbf{e}^3 + \cdots, \quad (16.1) \end{aligned}$$

whence follows $h_i = H_i = \log(1 + \delta_i)$, and hence

$$I_{\mathrm{h}} = I_{\mathrm{H}} = \log\sqrt{III_{\mathrm{c}}} = \log\frac{dv}{dV}. \quad (16.2)$$

Thus the deviators of **h** and **H** represent change of shape without change of volume. The off-diagonal components of **h** and **H** are not proportional to the logarithms of the corresponding components of **c** and **C**, however, but are more complicated infinite series. The tensors **h** and **H** are accordingly difficult to use in practice except in the simplest situations.[2]

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[4] [1911, **4**, §2], [1912, **1**, §§364–365] [1937, **1**, Appendix].

[5] Corresponding relations for **E** and **E*** are given in [1938, **2**, eq. (x.104)].

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[1] [1928, **2**, §1] [1929, **1**, §1] [1929, **2**, §2]. Cf. [1935, **2**, pp. 59–60] [1939, **8**, Ch. 1, ¶15] [1941, **1**, p. 127] [1948, **13**, §2] [1948, **1**, p. 127] [1949, **24**]. According to [1897, **1**, §I], this measure was used by A. Imbert (1880) to describe the simple extension of rubber. Cf. [1909, **3**, Pt. 1, §1].

[2] In [1948, **32**] strain is measured by the quantities H_i, ... , $C_{12}/\sqrt{C_1C_2}$..., which are awkward to use because they do not form components of any differential invariant, and indeed the consequent proposals in this paper are not admissible for isotropic media as intended (cf. §56[5]).

17. Equivalence of the various measures of deformation. Various other measures of deformation have been proposed.[1] It is important to realize that since each of the several material tensors **C**, **E**, **H**, etc. is an isotropic function of any one of the others, an exact description of strain in terms of any one is equivalent to a description in terms of any other[2]; only when an approximation is to be made may the choice of a particular measure become important. A similar statement holds for the spatial tensors **c**, **e**, **h**, etc. *But the spatial description in terms of* **c**, **e**, **h**, $\cdots$ *is not generally equivalent to the material description in terms of* **C**, **E**, **H**, $\cdots$, for while the proper values of any one of the former set of tensors are functions of the proper values of any one of the latter set, the principal spatial axes generally fail to coincide with the principal material axes. *Only in two special cases are the two descriptions equivalent: for a function which is independent of the axes of strain,*[3] *or for a pure strain unaccompanied by rotation.*

18. Conditions of compatibility. The converse problem of calculating a single-valued displacement when one of the tensors **C**, **c**, **E**, **e**, etc. is given arbitrarily does not generally admit a solution. Indeed, in order that the metrics (14.2) be Euclidean, as it is assumed at the outset they are, it is necessary and sufficient that the Riemann tensors[1] based upon **C** and upon **c** must vanish identically.[2] The six independent partial differential equations obtained in this way are St. Venant's *conditions of compatability.*[3] Since the validity of these equations has

[1] E.g. [1891, **1**, §1]. BIOT [1939, **4**, §1] [1939, **5**, p. 118] [1939, **6**, p. 108] [1940, **4**, §1] employs a symmetric tensor **B** with positive proper values such that $(\mathbf{I} + \mathbf{B})^2 = \mathbf{C}$. MURNAGHAN [1941, **1**, pp. 127–128] points out that BIOT's result can be given a form both simpler and more accurate in terms of a tensor **k** such that $\mathbf{k}^2 = \mathbf{c}^{-1}$ (cf. §41[1]); this tensor, whose proper values are $1 + \delta_i$, is employed by RICHTER [1948, **13**, §2]. BIOT's results appear to be equivalent to those of BIEZENO & HENCKY [1928, **3**, pp. 569–578] [1932, **13**], generalizing those of SOUTHWELL [1913, **3**] (cf. §41[1], §50). MOONEY [1948, **28**, pp. 435–436] proposes to use $\bar{\mathbf{k}} \equiv (III_c)^{1/6}\mathbf{k}$ so that $III_{\bar{\mathbf{k}}} = 1$. SWAINGER [1947, **21**] [1948, **46–47**] [1950, **6**] bases his theory of elasticity, plasticity, and strength upon a new definition of strain, but nowhere in the maze of quotations and cross-references to his published and unpublished work have I been able to find an equation expressing this strain in terms of the displacement. There are frequent references to "elastic" and "plastic" strain and to strain "due" to other strain, etc., in thorough confusion of geometrical quantities with the dynamical circumstances the author regards as producing them. It seems unlikely also that SWAINGER's proposals are tensorially admissible, but this point cannot be decided until they are first unequivocally stated. Cf. §56[5]. In a reply to a criticism [1950, **20**] he states that the theory shown to be inadequate is not his own, but his further remarks, which contain a profusion of symbols, quotation marks, and assertions, but a paucity of equations, serve only to deepen the mystery. OLDROYD [1950, **10**, §6] prefers $(III_c)^{-\frac{1}{3}}\mathbf{c}$.

[2] [1948, **1**, §3].

[3] E.g. for the strain energy of an elastically isotropic body.

[1] E.g. [1927, **1**, Ch. III, §18].

[2] [1911, **6**] [1928, **1**, §14].

[3] [1864, **1**, §32] [1871, **4**, §I] [1876, **3**, Vorl. 27, §4] [1886, **1**, Note at end] [1889, **4**] [1896, **1**, §13] [1906, **1**] [1907, **4**, Ch. I]. These equations had been given earlier by KIRCHHOFF [1859,

recently been challenged, quite incorrectly, by an eminent physicist,[4] I recapitulate their well-known significance. The conditions of compatibility are a necessary and sufficient condition that the space of which the continuum is a part be locally Euclidean. They are supposed to hold everywhere except possibly upon certain singular points, curves, or surfaces, where the material may divide or join. That they may fail to be satisfied upon these singular loci follows because here J becomes zero or infinite, so that for these points the first step in the theory of differential invariants cannot be taken and consequently the Riemann tensor even if it exist must fail of its usual significance. The fact that the conditions of compatibility may be meaningless or not satisfied at a junction or dislocation in a given deformation does not prevent the space in which the deformed material is imbedded from being Euclidean, and consequently admissible for the later imposition of the laws of Newtonian mechanics.[5]

19. Infinitesimal displacements, infinitesimal displacement gradients, and infinitesimal strains. The classical theory of elasticity is called the "infinitesimal theory," because (in addition, of course, to a dynamical law) it employs three purely geometrical hypotheses which permit the neglect of all terms except those of lowest order in certain expressions. These hypotheses will now be stated precisely.

If $| \delta_i | \ll 1$ the *strain* is said to be *infinitesimal*. In an infinitesimal strain it follows by (15.2) that $| E_i | \ll 1$, $| e_i | \ll 1$, so that squares and higher powers of $\mathbf{E}$ and $\mathbf{e}$ may be neglected in comparison to $\mathbf{E}$ and $\mathbf{e}$ themselves, and also $e_i \approx E_i \approx \delta_i$. Even in very great extension e_i remains fairly small, while even in very great compression E_i remains fairly small. Thus e.g. from (15.2) when $\delta_i = 1$ we have $E_i = 3\frac{1}{2}$, $e_i = \frac{3}{8}$, and when $\delta_i = -\frac{1}{2}$ we have $E_i = -\frac{3}{8}$,

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1, §§1–2], but without a clear statement of their meaning. For the case of infinitesimal displacements referred to curvilinear coordinates, see [1937, **2**] [1944, **11**]. Since for finite strains the explicit equations are easy to derive but so complicated as to be quite useless, they furnish a popular literary subject: [1902, **6**] [1905, **1**, **3**] [1911, **6**] [1914, **2**, §3] [1930, **3**, §2] [1942, **3**, p. 60] [1943, **1**, Ch. 1, ¶20] [1944, **10**] [1948, **26**, eq. (8) (9)] [1950, **10**, §5] [1950, **12**, §4] [1950, **8**, §4] [1950, **19**] [1951, **4**].

4 [1948, **14**, §§1–2]. In fact the first and last paragraphs of §2 contradict each other. The error does not invalidate the remainder of the author's analysis, since he makes no use of it This paper contains a spirited but not altogether fair attack upon the classical theory of elasticity, as well as an interesting tentative toward a more general theory (§82). The exposition is difficult to follow because of the author's habitual confusion of purely kinematical quantities, which exist in any continuous medium in motion, with physical phenomena in whose description he may or may not wish to employ them. Particularly unfortunate is his use of "strain" as somehow related to the dynamical action of cutting, and as an attribute of a single configuration rather than a relation between two configurations.

5 As illustrations consider an initially straight rod bent and joined so as to form a ring, or a ring cut and straightened out. The conditions of compatibility cannot be neglected except in an approximative procedure where the resulting error is demonstrably insignificant.

$e_i = 3\frac{1}{2}$. For an infinitesimal strain (15.3) reduces to

$$I_{c^{-1}} \approx 3 + 2I_{\tilde{e}}, \qquad II_{c^{-1}} \approx 3 + 4I_{\tilde{e}}, \qquad III_{c^{-1}} \approx 1 + 2\,I_{\tilde{e}}, \quad (19.1)$$

the last of which may be written also as[1] $dv/dV = 1 + I_{\tilde{e}}$. Also $I_{\tilde{e}} \approx I_{e} \approx I_{E}$, $II_{\tilde{e}} \approx II_{e} \approx II_{E}$, $III_{\tilde{e}} \approx III_{e} \approx III_{E}$.

Let both x^i and X^L refer to the same rectangular Cartesian system. If either of the equivalent sets of conditions

$$\begin{aligned} &\left|\frac{\partial u}{\partial X}\right| = \left|\frac{\partial x}{\partial X} - 1\right| \ll 1, \qquad \left|\frac{\partial u}{\partial Y}\right| = \left|\frac{\partial x}{\partial Y}\right| \ll 1, \cdots, \\ &\left|\frac{\partial u}{\partial x}\right| = \left|\frac{\partial X}{\partial x} - 1\right| \ll 1, \qquad \left|\frac{\partial u}{\partial y}\right| = \left|\frac{\partial X}{\partial x}\right| \ll 1, \cdots \end{aligned} \quad (19.2)$$

hold, the *displacement gradients* are said to be *infinitesimal.* From inspection of $(15.1)_2$ and (14.4) it is plain that when the displacement gradients are infinitesimal the strain components e_{ij} may be approximated by Cauchy's[2] linear expressions

$$e_{ij} \approx \tilde{e}_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad (19.3)$$

which are somewhat misleadingly called the *infinitesimal strain components,* and also that $\tilde{e}_{xx} \approx e_{xx} \approx h_{xx} \approx E_{XX}$; $\tilde{e}_{xy} \approx e_{xy} \approx h_{xy} \approx E_{XY}, \cdots$, and $I_{\tilde{e}} \approx I_{\tilde{e}}$, $II_{\tilde{e}} \approx II_{\tilde{e}}$, $III_{\tilde{e}} \approx III_{\tilde{e}}$. That is, in the case of infinitesimal displacement gradients all distinctions between spatial and material *strain measures* may be neglected.

If the displacement gradients be infinitesimal, plainly the strain is infinitesimal, but the converse is false.[3] Consider, for example, the rigid rotation $u = -2X = 2x$, $v = -2Y = 2y$, $w = 0$. Since $\partial u/\partial x = 2$, $\partial v/\partial y = 2$, the conditions (19.2) are not satisfied; the linear formulae (19.3) yield $\tilde{e}_{xx} = \tilde{e}_{yy} = 2$, while by $(15.1)_2$ for the true strain we have $e_{xx} = e_{yy} \cdots = 0$, as is obvious. Hence there is extensive literature concerning deformations where the strain is infinitesimal but the rotations are large (cf. §49), which are easily produced in bodies one of whose dimensions is very small with respect to another.[4]

The general problem of calculating the strain tensor $\mathbf{E}$ for the deformation resultant from the composition of two others with strain tensors ${}^1\mathbf{E}$ and ${}^2\mathbf{E}$, or,

[1] [1827, **2**, eq. (33)].

[2] [1827, **2**, eq. (41)].

[3] This distinction was observed by ST. VENANT [1844, **1**] [1847, **1**]. Cf. [1917, **2**, §3]. By using an expression for a general finite rigid displacement [1932, **12**], CISOTTI [1944, **16**] is able to put into general form the argument exemplified in the text above.

[4] ST. VENANT [1844, **1**] [1847, **1**] noted the following examples: a thin sheet may be bent back so that its two ends touch, a long slender shaft may be twisted through several diameters. In both cases the true strain components $E^L{}_M$ or $e^i{}_j$ may be very nearly zero everywhere, but at some points the linear expressions (19.3) become large and fail to approximate the strain components correctly. Cf. [1892, **4**, §2] [1939, **9**, §4, footnote].

equivalently, of comparing the strains of a given configuration with respect to different reference configurations, is very difficult. One simple special case is solved by (14.12) or (15.4). Another case when a simple explicit result can be written down is that in which the displacement gradients in the second deformation are infinitesimal. Let the actual configuration $x^i(X^L)$ be written $x^i = y^i + u^i$, where all three quantities are referred to the same rectangular Cartesian system. Then $u^i(X^L)$ is the displacement vector in the second deformation. From the usual regularity hypotheses we have $X^L = X^L(u^i)$ and we can form $\partial u^i/\partial y^j$, so that

$$x^i{}_{,L} = y^i{}_{,L} + \frac{\partial u^i}{\partial y^i} y^j{}_{,L}. \tag{19.4}$$

Supposing the displacement gradients $\partial u^i/\partial y^j$ to be infinitesimal, from $(14.5)_2$ and $(15.1)_1$ we have

$$\begin{aligned} E^L{}_M &= \frac{1}{2}\left(y^{i,L} + \frac{\partial u^i}{\partial y^j} y^{j,L}\right)\left(y_{i,M} + \frac{\partial u^i}{\partial y^k} y_{k,M}\right) - \frac{1}{2}\delta^L{}_M \\ &\approx {}^1E^L{}_M + \frac{1}{2}\frac{\partial u^i}{\partial y^j} y^{j,L} y_{i,M} + \frac{1}{2}\frac{\partial u^i}{\partial y^k} y^{i,L} y_{k,M}. \end{aligned} \tag{19.5}$$

Writing this result in tensor form, valid in all co-ordinate systems, from (19.3) we have

$$E^L{}_M = {}^1E^L{}_M + {}^2\tilde{e}^i{}_j y^{j,L} y_{i,M}, \tag{19.6}$$

where ${}^2\tilde{\mathbf{e}}$ is the infinitesimal strain tensor for the second deformation. When the gradients of the first displacement are also infinitesimal, (19.6) reduces to the classical superposition theorem $\tilde{\mathbf{e}} = {}^1\tilde{\mathbf{e}} + {}^2\tilde{\mathbf{e}}$.

Let $f(x, y, z)$ be some function of the spatial variables. Then by the mean value theorem we have

$$\begin{aligned} f(x, y, z) &= f(X + u, Y + v, Z + w), \\ &= f(X, Y, Z) + u\frac{\partial f'}{\partial x} + v\frac{\partial f'}{\partial y} + w\frac{df'}{\partial z}, \end{aligned} \tag{19.7}$$

where primes denote mean values. If

$$\left|\frac{u\dfrac{\partial f}{\partial x} + v\dfrac{\partial f}{\partial y} + w\dfrac{\partial f}{\partial z}}{f(X, Y, Z)}\right| \ll 1 \tag{19.8}$$

when $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ are evaluated at any possible mean value $X + u'$, $Y + v'$, $Z + w'$, then the *displacement* u, v, w is said to be *infinitesimal with respect to the function* f. For any function f with partial derivatives at X, Y, Z a range of infinitesimal displacements exists. In an infinitesimal displacement we have $f(x, y, z) \approx f(X, Y, Z)$, so that the distinction between material and spatial *co-ordinates* may be discarded. A displacement infinitesimal with respect to one function is generally not so with respect to another. A displacement which is simply called *infinitesimal* is implied to be so with respect to all functions which are used in connection with the problem. In the classical infinitesimal theory of elasticity these functions are: the displacements themselves, the stress components (§26), the extraneous force densities, and the surface loads.[5] If the displacement gradients be infinitesimal, it follows by (19.2) that (19.8) is necessarily satisfied by the displacements u, v, w; hence $u(x, y, z) \approx u(X, Y, Z)$, and for a general function f (19.8) becomes simply $| u\partial f/\partial X + v\partial f/\partial Y + w\partial f/\partial Z | \ll f(X, Y, Z)$.

20. Velocity and acceleration. If D/Dt denote time differentiation with the material variables X^α held constant, let a superposed dot denote the *material derivative*:[1]

$$\dot{b} \equiv \frac{Db}{Dt}, \qquad \dot{x}^i \equiv \frac{Dx^i}{Dt}, \qquad \dot{b}^i \equiv \frac{Db^i}{Dt} + \Gamma^i_{jk} b^j \dot{x}^k, \cdots ; \dot{B}^{\alpha \cdots \beta} \equiv \frac{DB^{\alpha \cdots \beta}}{Dt}, \tag{20.1}$$

where the Γ_j^{ik} are Christoffel symbols. $\dot{x}^i$ is the *velocity*[2] and $\ddot{x}^i$ is the *acceleration*. A spatial expression including all the cases of (20.1) is

$$\dot{(\;)} = \frac{\partial}{\partial t}(\;) + (\;)_{,i}\dot{x}^i, \tag{21.1}$$

where $\partial/\partial t$ denotes the time derivative when the spatial variables x^i are held constant. In particular,[3]

$$\ddot{x}^i = \frac{\partial \dot{x}^i}{\partial t} + \dot{x}^i{}_{,j}\dot{x}^j. \tag{20.3}$$

$\dot{x}^i{}_{,j}$ stands for $\overline{\dot{x}^i}_{,j}$; the reverse order of differentiation would be indicated by $\dot{\overline{x^i{}_{,j}}}$, bars being used in place of parentheses or brackets.

[5] Current terminology is misleading. In the theories of rods, plates, and shells, for example, the term *finite displacement* is often used to mean simply a displacement of the order of magnitude of the smallest dimension of the body, but nevertheless infinitesimal in the sense above; problems concerning such displacements may be in principle within the range of applicability of the classical theory of elasticity, although not correctly described by certain approximate theories of rods, plates, and shells.

[1] STOKES's notation [1845, **1**, §5] for the material derivative is D/Dt. In a rectangular Cartesian spatial frame we have $\dot{F}^{\cdots} = DF^{\cdots}/Dt$.

[2] The velocity was introduced as a primitive concept by D'ALEMBERT [1752, **1**, §43]. DEUKER [1941, **8**, §IV] employs a symmetrical system in which the velocity of the particles relative to the space (our $\dot{x}^i$) and the velocity of the space relative to the particles, viz. $\underset{.}{X}^\alpha \equiv \partial X^\alpha/\partial t\,|_{x^i = \text{const.}}$, are counterparts. Differentiating $(12.2)_2$ yields $0 = \underset{.}{X}^\alpha + X^\alpha{}_{,i}\dot{x}^i$, so that $\underset{.}{X}^\alpha = -X^\alpha{}_{,i}\dot{x}^i$, $\dot{x}^i = -\,x^i{}_{,\alpha}\underset{.}{X}^\alpha$.

[3] [1757, **2**, §XIX].

The material derivative of a quantity at a place is the rate of change of the quantity as apparent to an observer situate upon the moving particle instantaneously occupying the place. Since any function of the X^α alone is a scalar with respect to transformations of the spatial variables, by $(20.1)_1$ it follows that the material derivative of such a function is zero, and the operation "$_{,\alpha}$" always commutes with the operation "$\dot{\overline{\ }}$". By (20.2), if the spatial co-ordinate system be instantaneously stationary ($\partial g_{ij}/\partial t = 0$), we have

$$\dot{g}_{ij} = 0. \tag{20.4}$$

21. Rate of deformation. Vorticity. Since $\dot{\overline{dX^\alpha}} = 0$ and $\dot{\overline{x^i_{,\alpha}}} = \dot{x}^i_{,\alpha}$, we have

$$\dot{\overline{dx^i}} = \dot{\overline{x^i_{,\alpha}\, dX^\alpha}} = \dot{x}^i_{,\alpha}\, dX^\alpha = \dot{x}^i_{,j}\, dx^j. \tag{21.1}$$

By differentiating $x^i_{,\beta} X^\beta_{,j} = \delta^i_j$ we obtain a formula which after multiplication by $X^\alpha_{,i}$ yields

$$\dot{\overline{X^\alpha_{,j}}} = -\dot{x}^i_{,\beta} X^\beta_{,j} X^\alpha_{,i} = -\dot{x}^i_{,j} X^\alpha_{,i}. \tag{21.2}$$

Computing the material derivative of the squared element of arc length, by (20.4) we obtain

$$\dot{\overline{ds^2}} = \dot{\overline{g_{ij}\, dx^i dx^j}} = g_{ij}[\dot{\overline{dx^i}} dx^j + dx^i \dot{\overline{dx^j}}], \tag{21.3}$$

whence by (21.1) follows Beltrami's equation[1]

$$\dot{\overline{ds^2}} = 2 d_{ij}\, dx^i dx^j, \qquad d_{ij} \equiv \tfrac{1}{2}[\dot{x}_{i,j} + \dot{x}_{j,i}]. \tag{21.4}$$

$\mathbf{d}$ is the *rate of deformation tensor*, and the quantities d_{ij}, introduced and interpreted by Euler,[2] serve as measures of the local and instantaneous rate at which the shape of the medium is changing. The physical components of $\mathbf{d}$ are of dimension T^{-1}, where T is a unit of time. A necessary and sufficient condition that the instantaneous motion be one of which a rigid body is susceptible[3] is $\mathbf{d} = 0$. The conditions $\mathbf{e} = 0$, $\mathbf{E} = 0$, compare two discrete configurations, while $\mathbf{d} = 0$ refers to the instantaneous change of a given configuration. There is a formal analogy between the linear expressions $(21.4)_2$ and (19.3); the latter, however, yield only an approximate measure of the strain within limited circumstances, while the former are exact measures of the rate of deformation valid without exception.

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[1] [1871, **1**, §4].

[2] [1770, **1**, §§9–12]. A material expression for $\mathbf{d}$ is given in [1903, **11**]. HENCKY [1949, **40**] introduces "projective" rates of strain and rotation, defined respectively as the symmetric and skew parts of $\dot{x}_{i,j} + c_i x_j$, where $\mathbf{c}$ is an unknown vector. The kinematical signifiance and value of these quantities is difficult to appreciate.

[3] [1770, **1**, §13]. The generalization in differential geometry is known as KILLING's equation [1892, **1**, p. 167].

The *vorticity tensor* **w** is defined by

$$w_{ij} \equiv \tfrac{1}{2}(\dot{x}_{i,j} - \dot{x}_{j,i}), \tag{21.5}$$

so that $\dot{x}_{i,j} = d_{ij} + w_{ij}$. The corresponding axial vector **w**, given by $2w^i \equiv -\epsilon^{ijk}\dot{x}_{j,k}$, which was introduced by Lagrange and Cauchy,[4] was shown by Cauchy and Stokes[5] to represent a local instantaneous rate of rotation of the medium.

Let dS^2 and ds^2 be the squared elements of arc associated with a particle at times t and $t + \Delta t$, respectively. By Taylor's theorem and $(21.4)_1$ we have

$$ds^2 - dS^2 = \dot{\overline{ds^2}}\Delta t + \frac{1}{2!}\ddot{\overline{ds^2}}\Delta t^2 + \cdots = [2d_{ij}\Delta t + \cdots]dx^i dx^j. \tag{21.6}$$

Then by $(15.1)_2$ and $(14.2)_2$ it follows that

$$e_{ij} = d_{ij}\Delta t + \cdots. \tag{21.7}$$

The general term of this series is calculated by Dupont[6], who uses it to obtain approximative formulae for the strain in terms of the curvilinear displacements $u^i = \dot{x}^i\Delta t + \cdots$. (Cf. §15[3]).

22. Rate of strain. Differentiating $(14.2)_2$ and employing $(21.1)_3$, we have

$$\dot{\overline{dS^2}} = \dot{c}_{ij}dx^i dx^j + c_{ij}[\dot{x}^i{}_{,k}dx^k dx^j + dx^i\dot{x}^j{}_{,k}dx^k]. \tag{22.1}$$

Thus if Eckart's[1] *reference rate tensor* **r** be defined by

$$\dot{\overline{dS^2}} = 2r_{ij}dx^i dx^j, \tag{22.2}$$

we obtain

$$2r_{ij} = \dot{c}_{ij} + c_{kj}\dot{x}^k{}_{,i} + c_{ik}\dot{x}^k{}_{,j}. \tag{22.3}$$

The reference rate tensor **r** is a measure of the rate at which the reference configuration is being altered. Since this configuration is at the disposal of the observer, the quantities r_{ij} are partially arbitrary; since the Riemann tensor based upon **c** must vanish (§18), however, it is necessary that the r_{ij} satisfy six quite complicated partial differential equations. When the reference configuration for each particle is fixed in time, as is always assumed in the classical treatments, then $\mathbf{r} = 0$.

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[4] [1762, **2**, §XLII] [1827, **6**, Part 2, Sect. 1, §4].

[5] [1841, **1**, Th. IV] [1845, **1**, §2].

[6] [1931, **16**, §13].

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[1] Eckart [1948, **14**, §3] introduces the unhappy term "anelasticity tensor", suggesting the physical phenomenon in whose description he proposes to employ this purely kinematical quantity.

Supposing the spatial frame be steady, from (22.3), (20.4), and $(15.1)_2$ we calculate the *spatial rate of strain tensor*[2]:

$$\dot{e}_{ij} = d_{ij} - r_{ij} - [e_{kj}\dot{x}^k{}_{,i} + e_{ik}\dot{x}^k{}_{,j}]. \tag{22.4}$$

This result shows an inconvenient feature of **e**: in a strained medium, the values of the e_{ij} are changed by a rigid rotation, for while indeed $ds^2 - dS_0{}^2$, and hence also the quantity $e_{ij}dx^i dx^j$, remains constant for each particle, the individual components e_{ij} change because to an observer stationed upon a particle the spatial axes themselves seem to rotate as places move by.

The corresponding formula for $\dot{\mathbf{h}}$ is complicated, but since $h_i = \log(1 + \delta_i)$ we have $\dot{h}_i = \dot{\delta}_i/(1 + \delta_i)$ if the principal spatial axes of strain be instantaneously stationary[3]; hence if these axes coincide with the principal axes of **d**, then $\mathbf{w} = 0$ and (22.4) yields $\dot{e}_i = d_i(1 - 2e_i)$ when $\mathbf{r} = 0$, whence by $(15.2)_2$ we have $\dot{h}_i = d_i$.

Multiplying (22.3) by $(c^{-1})^{ij}$, we obtain $2(c^{-1})^{ij}r_{ij} = (c^{-1})^{ij}\dot{c}_{ij} + 2\dot{x}^i{}_{,i}$, whence follows Eckart's formula

$$\begin{aligned} I_{\mathrm{d}} = \dot{x}^i{}_{,i} = (c^{-1})^{ij}r_{ij} - \overline{\log\sqrt{III_{\mathrm{c}}}}^{\bullet} &= (c^{-1})^{ij}r_{ij} + \overline{\log\frac{dv}{dV}}^{\bullet}, \\ (c^{-1})^{ij}r_{ij} &= I_{\mathrm{d}} - \dot{I}_{\mathrm{h}} = I_{\mathrm{d}} - \dot{I}_{\mathrm{H}}. \end{aligned} \tag{22.5}$$

If for the moment we select the initial configuration as that of reference ($\mathbf{r} = 0$), (22.5) reduces to *Euler's expansion formula*[4]

$$I_{\mathrm{d}} = \dot{x}^i{}_{,i} = \overline{\log\frac{dv}{dV_0}}^{\bullet} = \overline{\log dv}^{\bullet}. \tag{22.6}$$

With the aid of this special case we may simplify (22.5):

$$(c^{-1})^{ij}r_{ij} = \overline{\log dV}^{\bullet}. \tag{22.7}$$

A corresponding material analysis is more elaborate. First, by (22.2) and $(14.1)_1$ we have

$$\overline{dS^2}^{\bullet} = 2R_{LM}\, dX^L dX^M, \qquad R_{LM} \equiv r_{ij}x^i{}_{,L}x^j{}_{,M}. \tag{22.8}$$

Let $\dot{X}^L$ be the velocity of the reference configuration with respect to the particles:

$$\dot{X}^L \equiv \left.\frac{\partial X^L}{\partial t}\right|_{X^\alpha = \mathrm{const.}}. \tag{22.9}$$

Now suppose $\dot{G}_{LM} = (\partial G_{LM}/\partial X^k)\dot{X}^k = 0$; i.e., the reference frame is steady

[2] The case $\mathbf{r} = 0$ is given in [1929, **2**, §2] [1937, **1**, p. 243].
[3] [1948, **1**, §7].
[4] [1757, **2**, §XV].

with respect to the particles. By differentiation of $(13.1)_2$ we obtain

$$\begin{aligned}\dot{\overline{dS^2}} &= G_{LM}\dot{\overline{X^L{}_{,\alpha}X^M{}_{,\beta}}}\,dX^\alpha dX^\beta,\\ &= G_{LM}(\dot{X}^L{}_{,\alpha}X^M{}_{,\beta} + X^L{}_{,\alpha}\dot{X}^M{}_{,\beta})dX^\alpha dX^\beta,\\ &= (\dot{X}_{L,M} + \dot{X}_{M,L})dX^L dX^M.\end{aligned} \tag{22.10}$$

Comparing this result with $(22.8)_1$ yields

$$R_{LM} = \tfrac{1}{2}(\dot{X}_{L,M} + \dot{X}_{M,L}). \tag{22.11}$$

The formal similarity between (22.11) and $(21.4)_2$ shows that the material reference rate is the rate of deformation of the reference configuration.

Supposing now that the spatial frame be steady, by differentiating $(14.3)_1$ and employing (20.4) we have

$$\begin{aligned}\dot{C}_{LM} &= \dot{\overline{g_{ij}\,x^i{}_{,\alpha}\,x^j{}_{,\beta}\,X^\alpha{}_{,L}\,X^\beta{}_{,M}}}\,,\\ &= g_{ij}\,(\dot{x}^i{}_{,\alpha}\,x^j{}_{,\beta}\,X^\alpha{}_{,L}\,X^\beta{}_{,M} + x^i{}_{,\alpha}\,\dot{x}^j{}_{,\beta}\,X^\alpha{}_{,L}\,X^\beta{}_{,M})\\ &\qquad + g_{ij}\,x^i{}_{,\alpha}\,x^j{}_{,\beta}\,(\dot{\overline{X^\alpha{}_{,L}}}\,X^\beta{}_{,M} + X^\alpha{}_{,L}\,\dot{\overline{X^\beta{}_{,M}}});\end{aligned} \tag{22.12}$$

but from $(21.2)_2$ it is plain that $\dot{\overline{X^\alpha{}_{,L}}} = -\,X^\alpha{}_{,P}\dot{X}^P{}_{,L}$, so that by $(21.4)_2$ and $(14.5)_2$ we have

$$\begin{aligned}\dot{C}_{LM} &= g_{ij}(\dot{x}^i{}_{,k}x^k{}_{,L}x^j{}_{,M} + \dot{x}^j{}_{,k}x^k{}_{,M}x^i{}_{,L})\\ &\qquad - g_{ij}(x^i{}_{,L}x^j{}_{,P}\dot{X}^P{}_{,M} + x^i{}_{,M}\,x^j{}_{,P}\dot{X}^P{}_{,L}),\\ &= 2d_{ij}x^i{}_{,L}x^j{}_{,M} - C_{LP}\dot{X}^P{}_{,M} - C_{MP}\dot{X}^P{}_{,L}\,.\end{aligned} \tag{22.13}$$

Thus $\dot{\mathbf{C}}$ depends not only upon the rate of deformation $\mathbf{d}$ and the material reference rate $\mathbf{R}$, but also upon the rate of rotation of the reference configuration $\dot{X}_{L,M} - \dot{X}_{M,L}$ and the existing deformation $\mathbf{C}$. Cf. (22.3). In the special case when $\mathbf{R} = 0$, (22.13) reduces to the Cosserats' formula[5]

$$\dot{E}_{\alpha\beta} = d_{ij}x^i{}_{,\alpha}x^j{}_{,\beta} \tag{22.14}$$

for the *material rate of strain tensor* $\dot{\mathbf{E}}$.

From (22.4) and (22.14) it is apparent that even in the case when strain is measured with respect to a fixed configuration ($\mathbf{r} = 0$) the two rates of strain are not all the same as the rate of deformation, nor directly related to one another. If the *present* configuration be taken as that of reference and $x^1 = X^1, \cdots$, so

[5] [1896, **1**, §15, eq. (2)]. Cf. [1904, **1**, Part I, Ch. I, §1, eq. (30)]. In DEUKER's scheme (§20[2]) [1941, **8**, §VI] we easily express $\dot{E}_{\alpha\beta}$ in terms of the velocity of the space relative to the body: $-2\dot{E}_{\alpha\beta} = \underset{\cdot}{X}_{\alpha,\beta} + \underset{\cdot}{X}_{\beta,\alpha}$, so that the material rate of strain is the negative of the rate of deformation of the space relative to the body. A material equivalent of the criterion $\mathbf{d} = 0$ for rigid motion is thus $\underset{\cdot}{X}_{\alpha,\beta} + \underset{\cdot}{X}_{\beta,\alpha} = 0$.

that $\mathbf{e} = \mathbf{E} = 0$, then indeed (22.4) and (22.14) yield

$$\dot{e}_{11} = d_{11} = \dot{E}_{11}, \cdots : \tag{22.15}$$

thus the *rate of deformation is the rate of strain if the medium be unstrained.* In the case of infinitesimal displacement gradients (§19), $(22.15)_2$ holds approximately, but $(22.15)_1$ is valid only subject to the proviso that the vorticity $\mathbf{w}$ be of at least no greater order of magnitude than the rate of deformation $\mathbf{d}$. The exact formulae (22.4) and (22.14) must be employed in any general theory where it is necessary to differentiate the strain tensors.

23. Digression: Mechanical phenomena for whose description the various kinematical quantities are appropriate. While the foregoing purely kinematical analysis is valid for any continuous medium, since our aim is to construct mechanical theories we resume certain mechanical attributes for whose description the various kinematical tensors may be relevant.[1] *Spring* in a body is the quality of resuming its initial shape, once released from deforming forces. Thus the dynamical response of a perfectly elastic solid might well be expected to depend upon the tensors $\mathbf{E}$ and $\mathbf{e}$ measuring its strain from a preferred initial configuration, and to be independent of all time rates. *Fluidity* in a body is the quality of yielding to any permanent deformation which is effected sufficiently slowly. Thus the dynamical response of a fluid might well be expected to depend upon the tensor $\mathbf{d}$ measuring its rate of deformation with respect to its present configuration, and to be independent of all quantities such as $\mathbf{E}$, $\mathbf{e}$, $\dot{\mathbf{E}}$, etc., which are functions also of some previous configuration. *Softness* in a body is the quality of partial yielding to deformation, sometimes qualified by some measure of spring. Thus the dynamical response of a plastic substance might well be expected to depend upon both its rate of deformation $\mathbf{d}$ and its strain $\mathbf{e}$ and rate of strain $\dot{\mathbf{e}}$ from some configuration intermediate between the initial and actual configurations; through this latter the reference rate tensor $\mathbf{r}$ might be involved.

24. Conservation of mass in simple media. Let $\rho(X^\alpha,t)$ be the *density,* and let $\rho_0(X^\alpha)$ be its value at the initial instant t_0. Its dimensions are ML^{-3}, where M is a unit of mass and L is a unit of length. The *specific volume* is $v \equiv \rho^{-1}$. A set of particles $\mathcal{V}_0$ of positive *mass* $\int_{\mathcal{V}_0} \rho_0 dV_0 = \int_{\mathcal{V}} \rho dv$ is a *body.* In regions where the density is continuous the *principle of conservation of mass*[1] may be expressed

[1] Cf. [1949, **30,** §4].

[1] The mathematically inclined reader will observe that this principle is but a partial restatement of the hypothesis of continuity of motion (§12), plus the introduction of *a new physical unit* M. Cf. [1952, **4,** §7]. The equation $(24.1)_1$ is the definition of density in terms of an arbitrary function ρ_0. Cf. the remarks of HILBERT [1907, **1,** pp. 100–108].

in the equivalent forms[2]

$$\rho dv = \rho_0 dV_0, \qquad \frac{dv}{dV_0} = \frac{v}{v_0}, \qquad \rho J = \tilde{\rho}, \qquad \dot{\overline{\rho dv}} = 0, \tag{24.1}$$

where $\tilde{\rho}$ is the density of the reference configuration. From (22.6) we now deduce several forms of Euler's *continuity equation*[3]:

$$\dot{x}^i{}_{,i} = -\dot{\overline{\log \rho}} = \dot{\overline{\log v}}, \qquad \frac{\partial \rho}{\partial t} + (\rho \dot{x}^i)_{,i} = 0, \tag{24.2}$$

while $(22.5)_2$ becomes[4]

$$\dot{\overline{\log (v\sqrt{III_c})}} = (c^{-1})^{ij} r_{ij}. \tag{24.3}$$

Thus III_c and v remain independent until the reference rate $\mathbf{r}$ be specified. If $\mathbf{r} = 0$ then $v\sqrt{III_c} = \text{const.}$, $\rho\sqrt{III_{c^{-1}}} = \text{const.}$ for each particle. For isochoric motions, (24.1) and (24.2) take the forms

$$dv = dV = dV_0\,, \qquad v = v_0\,, \qquad \rho = \rho_0 = \tilde{\rho}\,, \qquad \dot{x}^i{}_{,i} = I_d = 0. \tag{24.4}$$

The *kinetic energy* $\mathfrak{K}$ of a body $\mathcal{V}$ and its rate of change[5] are given respectively by

$$\mathfrak{K} \equiv \frac{1}{2}\int_{\mathcal{V}} \rho \dot{x}^i \dot{x}_i \, dv, \qquad \dot{\mathfrak{K}} = \int_{\mathcal{V}} \rho \dot{x}^i \ddot{x}_i \, dv. \tag{24.5}$$

25. Heterogeneous media. In order to treat motions of media whose components may suffer chemical and physical changes, Stefan[1] suggested the artifice of regarding each place as simultaneously occupied by a single particle $X_\Gamma{}^\alpha$ of each of several different continua. The *peculiar motion of the substance* Γ is of the form $x^i = f_\Gamma{}^i(X_\Gamma{}^\alpha, t)$. Greek majuscule indices (subscript or superscript), not indicating tensorial character, are reserved for the identification of quantities associated with these motions, all sums being assumed to run from $\Gamma = 1$ to $\Gamma = N$, where N is the total number of substances which actually occur anywhere in the medium, or may arise from any possible chemical or physical change. The *peculiar velocity* $\dot{x}_\Gamma{}^i$ *of the substance* Γ is given by

$$\dot{x}_\Gamma{}^i \equiv \left.\frac{\partial x^i}{\partial t}\right|_{X_\Gamma{}^\alpha = \text{const.}} \tag{25.1}$$

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[2] [1770, **1**, §§111–112, 123–129].
[3] [1757, **2**, §§XVI–XVII].
[4] [1948, **14**, §4].
[5] [1851, **1**, §49].

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[1] [1871, **5**]. Cf. [1907, **1**, pp. 43–47] [1907, **1a**, pp. 42–45]. Reynolds [1903, **19**, §35] employed a similar notion in order to obtain a mathematical resolution of a given motion, as in turbulent flow.

The peculiar motions are connected by postulated relations representing the physical principles of chemical combination of atoms to form molecules, the atoms themselves being regarded as indestructible. While we freely employ the language of atoms and molecules, the mathematical reader will constantly recognize it as simply a conceptual guide, since all our equations presuppose the medium to be the superposition of several continua. This structure was developed by Jaumann and Lohr,[2] but we shall outline here the elegant presentation of Eckart.[3]

For simplicity,[4] let the N substances Γ constituting the mixture consist of M pure substances of atomic weights $\mathfrak{M}_\Gamma$, $\Gamma = 1, 2, \cdots M$, and $N - M$ compounds of molecular weights $\mathfrak{M}_\Gamma$, $\Gamma = M + 1, \cdots, N$, which may be formed from the pure substances. Then

$$\mathfrak{M}_\Gamma = n_\Gamma{}^\Delta \mathfrak{M}_\Delta, \tag{25.2}$$

where the $n_\Gamma{}^\Delta$ are given non-negative integers subject to the restrictions $n_\Gamma{}^\Delta = \delta_\Gamma{}^\Delta$ if $\Gamma \leq M$, while $n_\Gamma{}^\Delta = 0$ if $\Delta > M$. The case when two of the substances are but different phases of the same chemical compound is not excluded. It is assumed that each substance has its own *peculiar density* $\overset{\Gamma}{\rho}$. Then the *total density* $\rho \equiv \sum_{\Gamma=1}^{N} \overset{\Gamma}{\rho}$, and the *molal concentration* c_Γ *of the substances* Γ is defined by

$$\overset{\Gamma}{c} \equiv \frac{\overset{\Gamma}{\rho}}{\rho \mathfrak{M}_\Gamma}. \tag{25.3}$$

Hence $\mathfrak{M}_\Gamma \overset{\Gamma}{c} = 1$.

The *rate of production* $\overset{\Gamma}{e}$ *of the substance* Γ is defined by[5]

$$\rho \overset{\Gamma}{e} \equiv \frac{\partial}{\partial t} (\rho \overset{\Gamma}{c}) + (\rho \overset{\Gamma}{c} \dot{x}_\Gamma{}^i)_{,i} \qquad (\Gamma \text{ unsummed}). \tag{25.4}$$

The indestructibility of atoms is now expressed by the postulate

$$n_\Delta{}^\Gamma e^\Delta = 0. \tag{25.5}$$

Multiplying (25.5) by $\mathfrak{M}_\Gamma$, from (25.2) we obtain $\mathfrak{M}_\Delta e^\Delta = 0$, a statement of the conservation of total mass in the mixture.

The *mean velocity* $\dot{x}^i$ is defined as that whose momentum equals the resultant momentum of the mixture: $\rho \dot{x}^i \equiv \overset{\Gamma}{\rho} \dot{x}_\Gamma{}^i$, or $\dot{x}^i = \mathfrak{M}_\Gamma \overset{\Gamma}{c} \dot{x}_\Gamma{}^i$. It then follows by

[2] [1911, **5**, §§I, VIII] [1917, **5**, §§5, 7–8, 12, 15]. Cf. §32[2].

[3] [1940, **2**, p. 271]. Cf. [1938, **10**, §III] [1939, **10**] [1941, **9**, §3] [1942, **13**, §§12–14] [1942, **14**, §§4–5] [1951, **10**, §§7–13].

[4] This framework is easily adjusted to describe ions in solution, atoms excited at different energy levels, or any other sort of heterogeneity governed by simple principles of dissociation and recombination.

[5] These quantities were introduced by Reynolds [1903, **19**, §§13, 21, 36]. Cf. [1911, **5**, §§IV, VIII] [1913, **1**, §24c].

(25.5) and (25.4) that

$$(\rho\dot{x}^i)_{,i} = (\rho\mathfrak{M}_\Gamma c^\Gamma \dot{x}_\Gamma{}^i)_{,i} = [\rho\mathfrak{M}_\Gamma e^\Gamma - \frac{\partial}{\partial t}(\rho\mathfrak{M}_\Gamma c^\Gamma)] = -\frac{\partial\rho}{\partial t}; \qquad (25.6)$$

that is, the mean velocity $\dot{x}^i$ and the total density ρ satisfy the ordinary Eulerian continuity equation (24.2).

Let $u_\Gamma{}^i \equiv \dot{x}_\Gamma{}^i - \dot{x}$ define the *diffusion velocities* $u_\Gamma{}^i$. The resultant momentum of the diffusion velocities is zero: $\rho^\Gamma u_\Gamma{}^i = 0$. If the $u_\Gamma{}^i$ be introduced into (25.4), the resulting expression is easily transformed into $\rho\dot{c}^\Gamma = \rho e^\Gamma - (\rho c^\Gamma u_\Gamma{}^i)_{,i}$ (Γ unsummed). A possible solution is $e^\Gamma = 0$, $u_\Gamma{}^i = 0$, $c^\Gamma = c^\Gamma(X^\alpha)$, representing an *inert mixture*.

26. Conservation of momentum. In the phenomenological mechanics of continua the concept of stress may be introduced either through mechanical or through thermodynamical principles; for the latter method, see §33. For the several alternative mechanical postulates, we refer to the excellent article of Hellinger.[1] Here we adopt the *stress principle* of Euler and Cauchy,[2] which postulates that upon any imagined closed surface with unit normal $\mathbf{n}$ within a body there exists a distribution of *stress vectors* $\mathbf{t}_{(n)}$ whose resultant and moment are equivalent respectively to those of the actual forces of material continuity exerted by the material outside upon that inside.[3] The postulated *momentum principle*[4] leads first to the existence of a *stress tensor* $\mathbf{t}$ such that[5]

$$t_{(n)}{}^i = t^{ij}n_j, \qquad t_{(n)}{}^i\,da = t^{ij}\,da_j, \qquad (26.1)$$

and second to *Cauchy's laws of motion*[6]

$$t^{ij}{}_{,j} + \rho f^i = \rho\ddot{x}^i, \qquad t^{ij} = t^{ji}, \qquad (26.2)$$

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[1] [1914, **1**, §§1–5]. Cf. [1913, **1**, §21]. To the best of my knowledge the only subsequent addition to this subject is MURHAGHAN's elegant formulation of the principle of virtual displacements [1937, **1**, §2]. Axiomatic treatment of general mechanics was introduced by HAMEL, [1908, **1**, Ch. 1], who, following a suggestion of KLEIN, pointed out that while indeed the real content of mass-point mechanics follows as a special case from continuum mechanics, the converse is not true at all (although the contrary impression may be gained from most physics texts), and thus that any study of the foundations of mechanics should be directed toward deformable continua.

[2] [1757, **1**, §§VI–IX] [1823, **1**] [1827, **1**, pp. 60–61]. EULER stated the concept clearly, but considered only the case when the stress vector is normal to the surface.

[3] POINCARÉ [1892, **4**, §38] objected that 1°, it is not evident that the assumption that a stress vector exists is correct, 2°, one should prove that the stress vector for a given direction is unique. These objections necessarily presuppose some prior concept of a body, such as a molecular model.

[4] A single integral formula containing as special cases the momentum equations and several others is given by CISOTTI [1940, **11**, §3].

[5] [1827, **1**, eq. (20)].

[6] [1827, **1**, Th. II]; [1827, **3**] [1828, **1**, eq. (25)].

where $\mathbf{f}$ is the *extraneous force vector*,[7] supposed to be a given function of position and time, and possibly also of velocity. The physical components of $\mathbf{t}$ are of the dimensions $\mathsf{ML}^{-1}\mathsf{T}^{-2}$.

Boussinesq's[8] material form of Cauchy's first law follows at once from $(13.6)_1$ and $(24.1)_3$:

$$t^{ij}{}_{,L}\frac{\partial J}{\partial x^i{}_{,L}} + \tilde{\rho}f^i = \tilde{\rho}\ddot{x}^i. \tag{26.3}$$

When the indicated differentiations are actually carried out, the first term on the left becomes a sum of three Jacobians[9]:

$$\tfrac{1}{2}\epsilon^{LMN}\epsilon_{jkl}t^{ij}{}_{,L}x^k{}_{,M}x^l{}_{,N} + \tilde{\rho}f^i = \tilde{\rho}\ddot{x}^i. \tag{26.4}$$

Kirchhoff[10] introduced the double vector T^{iL}:

$$T^{iL} \equiv JX^L{}_{,j}t^{ij}, \qquad t^{ij} = J^{-1}T^{iL}x^j{}_{,L}. \tag{26.5}$$

From (26.1) and $(13.5)_2$ we have then

$$t_{(n)}{}^i\, da = T^{iL}\, dA_L. \tag{26.6}$$

Since

$$T^{iL}{}_{,L} = (JX^L{}_{,j})_{,L}t^{ij} + JX^L{}_{,j}t^{ij}{}_{,k}x^k{}_{,L}, \tag{26.7}$$

by $(13.6)_3$, (26.2), and $(24.1)_3$ follows Kirchhoff's form of Cauchy's laws:

$$T^{iL}{}_{,L} + \tilde{\rho}f^i = \tilde{\rho}\ddot{x}^i, \qquad T^{iL}x^j{}_{,L} = T^{jL}x^i{}_{,L}. \tag{26.8}$$

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[7] The possibility of *extraneous moments* $\mathbf{l}$ per unit volume was noted by MAXWELL [1873, **3**, §641] (cf. [1891, **2**] [1902, **1**]). In the thermodynamical method of DUHEM [1904, **1**, Part I, Ch. II, §§I–V] an effective $\mathbf{l}$ appears when the potential of the mutual forces is not Newtonian. The COSSERATS [1909, **1**, §53] introduced a moment stress tensor $\mathbf{m}$ whose action must be added to the moments exerted by the force stress $\mathbf{t}$ upon any imagined closed surface in order to yield a system equivalent to the mutual forces (cf. [1913, **1**, §21a]). In place of $(26.2)_2$ there results $m^{ij}{}_{,j} + \epsilon^{ijk}t_{kj} + \rho l^i = 0$, while $(26.2)_1$ remains unchanged. In the present memoir we shall suppose $\mathbf{m} = 0, \mathbf{l} = 0$, so that the stress tensor $\mathbf{t}$ is symmetric.

[8] [1872, **1**, §I, eq. (3)]. The result follows by inspection from a general transformation of CLEBSCH [1857, **1**, §2]. Cf. [1896, **1**, §17].

[9] The special case $\mathbf{t} = -p\mathbf{I}$ was given by EULER [1770, **1**, §119]; the general result in virtually the same form is given by the COSSERATS [1896, **1**, §43, eq. (117)].

[10] [1852, **1**, pp. 763–764]. Cf. [1860, **1**, §§2, 4–5] [1896, **1**, §15, eq. (33)]. KIRCHHOFF remarked that the matrix $\| T^{iL} \|$ is not generally symmetric, as indeed is manifest from the present notation. Although POINCARÉ [1892, **4**, §40] gave a clear explanation of this fact, based upon (26.6) and the observation that if the dA_L be taken as the transformed value of the da_i then three orthogonal components da_i are not generally carried into orthogonal components dA_L , nevertheless his elaborate, confusing, and unnecessarily restricted manner of stating [1892, **4**, §35] that for infinitesimal displacement gradients $\| T^{iL} \|$ is approximately symmetric gave rise to an unnecessary discussion of this "paradox" and an incorrect notion that it is connected with the presence of initial stress [1896, **1**, §26] [1924, **6–7**].

Kirchhoff[11] introduced also the tensor

$$T^{LM} \equiv X^L{}_{,i}T^{iM} = JX^L{}_{,j}X^M{}_{,j}t^{ij}, \qquad t^{ij} = J^{-1}x^i{}_{,L}x^j{}_{,M}T^{LM}. \tag{26.9}$$

From (26.8) follows Signorini's[12] form of Cauchy's laws:

$$(x^i{}_{,L}T^{LM})_{,M} + \bar{\rho}f^i = \bar{\rho}\ddot{x}^i, \qquad T^{LM} = T^{ML}. \tag{26.10}$$

Let p be any scalar function. Then the resolution

$$t^i{}_j = -p\delta^i{}_j + v^i{}_j \tag{26.11}$$

separates the stress into a hydrostatic tension $-p\delta^i{}_j$ and an *extra stress* $v^i{}_j$. Here the function p is purposely left undefined, so that (26.11) defines a different $\mathbf{v}$ for each p. In §30 we discuss the definition of p, here vaguely called the *pressure*. In general, it is *not* equal to the *mean pressure* $\bar{p}$ given by

$$\bar{p} \equiv -\tfrac{1}{3}t^i{}_i = -\tfrac{1}{3}I_t. \tag{26.12}$$

In terms of p and $\mathbf{v}$ Cauchy's laws (26.2) become

$$v_i{}^j{}_{,j} - p_{,i} + \rho f_i = \rho\ddot{x}^i, \qquad v_{ij} = v_{ji}. \tag{26.13}$$

In heterogeneous media the stress may be regarded as a gross variable of the mean motion only, or it may be decomposed into a sum of partial stresses.[13]

27. Conservation of energy in simple media. The theory of energy is characterized by two basic concepts. First, the kinetic energy $\mathfrak{K}$ of a body $\mathcal{V}$ is regarded as a part of its *total energy* $\mathfrak{K} + \mathfrak{E}$; the remainder is the *internal energy* $\mathfrak{E}$, which may be expressed in terms of an internal *energy density* ϵ : $\mathfrak{E} \equiv \int_{\mathcal{V}} \rho\epsilon\, dv$. Second, the material rate of change of the total energy $\mathfrak{K} + \mathfrak{E}$ is the sum of the rate at which mechanical work is done upon the volume and the rate at which thermal energy enters or leaves the volume (by heat conduction, radiation, etc.). This second concept, *the interconvertibility of heat and mechanical work*, definitely

[11] [1852, **1**, p. 767]. Cf. [1896, **1**, §15, eq. (31)]. From (26.9) it follows that if $(13.1)_1$ be regarded as a transformation of co-ordinates, then the t^{ij} are the components in the x^i system of the *tensor density* whose components in the X^L system are T^{LM}; cf. [1928, **5**, §11] [1925, **1**, §7] [1938, **2**, Ch. X, §X] [1944, **14**].

[12] [1930, **4**, §4] [1943, **1**, Ch. II, §4] [1943, **12**]. Other material forms are given in [1930, **4**, §5] [1930, **8**, §2] [1948, **26**, eq. (4)] [1948, **34**, §7]. The attractive result of Deuker [1941, **8**, eq. (8.7)] is unfortunately false; correct application of Green's transformation to his eq. (8.2) leads not to his eq. (8.3) but eventually to our $(26.10)_1$. No interpretation I have been able to conjecture for the undefined symbols in eq. (3) of [1948, **40**] or eq. 5 of [1949, **38**] renders these supposed material equations correct.

[13] [1903, **19**, §38] [1911, **5**, §V].

known to Carnot (by 1832),[1] stated rather vaguely by Mohr, Seguin, and J. R. Meyer,[2] and first published in an unequivocal form and verified experimentally by Joule,[3] for its general mathematical expression requires the introduction of a new vector **q**, the *heat flux*, analogous to the stress tensor **t** in that it is a gross macroscopic variable representing the energetic equivalent upon any imagined closed surface of all non-mechanical transfer of energy across it. More precisely, we postulate that

$$\dot{\overline{\mathfrak{K} + \mathfrak{E}}} = \int_{\mathcal{V}} \rho f^i \dot{x}_i \, dv + \oint_{\mathcal{S}} t^{ij} \dot{x}_j \, da_i - \oint_{\mathcal{S}} q^i \, da_i . \tag{27.1}$$

In accord with the theory of stress (§26), the rate of doing mechanical work has been divided into two portions, the first being that arising from the extraneous force **f**, and the second equal to that arising from the forces of material continuity, expressed in terms of the equivalent stress vector $\mathbf{t}_{(n)}$ acting upon the bounding surface. Hence follows a differential equation which after simplification by (26.2) reduces to the Fourier-Kirchhoff-C. Neumann[4] energy equation[5]

$$\rho \dot{\epsilon} = t^i{}_j \, d^j{}_i - q^i{}_{,i} . \tag{27.2}$$

The scalar $\Phi(\mathbf{t}, \mathbf{d}) = t^i{}_j \, d^j{}_i$ is the *stress power*,[6] the rate at which internal mechanical work is being done per unit time and per unit volume of the present configuration. By (26.5) and $(13.5)_1$ we have[7] $t^i{}_j \, d^j{}_i \, dv = T^{iL} \dot{x}^i{}_{,L} \, dV$; if the strain **E**

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[1] In CARNOT's notes [1878, **2**] the principle is clearly stated, the mechanical equivalent of heat is calculated upon the basis of then existing data, and crucial experiments, including the central porous plug experiment later conceived independently and executed by JOULE and KELVIN, are projected. The contention of CLAUSIUS, still reproduced in text books, that the result in CARNOT's celebrated treatise [1824, **1**] though themselves correct are based upon an incorrect axiom, is quite false; cf. the excellent article of CALLENDER [1910, **2**, §§16, 20, 22]. Professor LAMER, who has kindly let me use his note [1949, **50**] in MS, feels that a proper interpretation of CARNOT's words in [1824, **1**] reveals that in 1824 he was already possessed of the "first law of thermodynamics," the best history of whose discovery is given by PARTINGTON [1949, **47**, Part II, §§10–12]; cf. also [1864, **2**] [1868, **3**, Introd. and Ch. I] [1876, **1**, lects. II, III, and intro. to 2nd ed.] [1882, **2**] [1929, **8**] [1937, **5**, §11].

[2] [1837, **1**]; [1839, **2**, Ch. VII, §1]; [1842, **1**].

[3] [1843, **2**] [1845, **2–3**] [1847, **2**].

[4] FOURIER's analysis [1833, **1**, eq. (3)] is based upon the notion of heat as an indestructible substance, and hence valid only for inviscid incompressible fluids. KIRCHHOFF [1868, **1**, §1] considered only small motions of a perfect gas. A fairly general case of (27.2) is given by C. NEUMANN [1894, **4**, §4]. Cf. also [1901, **1**, Part 1, Ch. 1, §7] [1903, **13**] [1933, **9**] [1942, **1**].

[5] A somewhat different treatment of the subject is given by WEISSENBERG [1931, **2**, §AcII] [1935, **2**, pp. 52, 138–141], who writes $\dot{\epsilon}$ in the form $\dot{\overline{vF}} + \dot{\overline{vG}} - \dot{x}^i \ddot{x}_i$, calling F and G the "free" and "bound" energies, respectively.

[6] [1851, **1**, §49].

[7] [1852, **1**, p. 771].

be referred to a fixed configuration, from (26.9) and (22.14) follows[8] $t^i{}_j d^j{}_i \, dv = T^{\alpha\beta} \dot{E}_{\alpha\beta} \, dV_0$.

28. Parenthesis: Classical thermodynamics.[1] Classical thermodynamics deals with the special situation when $t^i{}_j = -\pi\delta^i{}_j$, where $\pi = \pi(t)$. Putting this specialization into (27.2) and using $(22.4)_2$, then integrating over a body $\mathcal{V}$ whose volume is $\mathfrak{V}$, we obtain the classical formula

$$d\mathfrak{E} = -\pi \, d\mathfrak{V} + \delta\mathfrak{Q}, \tag{28.1}$$

where

$$d\mathfrak{E} \equiv \dot{\mathfrak{E}} \, dt, \qquad d\mathfrak{V} \equiv \dot{\mathfrak{V}} \, dt, \qquad \delta\mathfrak{Q} \equiv -\left[\int_{\mathcal{V}} q^i{}_{,i} \, dv\right] dt. \tag{28.2}$$

Characteristic of the subject is the further *assumption*[2] that $\mathfrak{E} = \mathfrak{E}(\pi, \mathfrak{V})$. Then $\delta\mathfrak{Q}$ becomes a differential form in two variables, for which there exist an infinite number of integrating factors. Calling one of these $1/\theta$, we have

$$\frac{\delta\mathfrak{Q}}{\theta} = \frac{d\mathfrak{E} + \pi \, d\mathfrak{V}}{\theta} = d\mathfrak{H}, \quad \text{say.} \tag{28.3}$$

It is possible to show[3] that $\theta(\pi, \mathfrak{V})$ and $\mathfrak{H}(\pi, \mathfrak{V})$ may be characterized in an essentially unique fashion; the former is the *temperature*, the latter the *entropy* of the body $\mathcal{V}$.

The foregoing remarks and the great mass of formal consequences of (28.3) which constitute the bulk of treatises on thermodynamics are valid only subject to the assumption $\mathfrak{E} = \mathfrak{E}(\pi, \mathfrak{V})$. A change of energy describable throughout a period of time by such a relation is called a (grossly) *reversible process*.[4] A more elaborate change $\mathfrak{E} = \mathfrak{E}(t)$ in which such a relation does *not* hold is called a (grossly) *irreversible process*. While the main formal structure of classical thermodynamics is not valid for irreversible processes, the second postulate of the subject states that if $\mathfrak{E} = \mathfrak{E}(\pi, \mathfrak{V})$ be valid for the *end states* π_1, $\mathfrak{V}_1$ and π_2, $\mathfrak{V}_2$ of

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[8] [1896, **1**, §15, eq. (30)]. Generalizations are given in [1909, **1**, §§51–52, 54–55].

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[1] The basic papers [1834, **1**] [1849, **1**] [1850, **2**] [1853, **2**] [1853, **3**] [1854, **1**] [1862, **1**] [1865, **1**] are elaborate, and fall short of the minimum requirements of mathematical clarity and rigor. See also [1950, **22**], which derives from 1854–5 or earlier. An axiomatic treatment was given by Carathéodory [1909, **4**] [1925, **3**]; this still controversial subject is discussed further in [1921, **1**] [1924, **5**] [1925, **4**] [1926, **2**] [1939, **21**] [1944, **12**] [1948, **44–45**], and Dr. Slud informs me that he has discovered a new approach. The history of this theory is outlined in [1910, **2**, §§20–23] [1949, **47**, Part II, §§22–37].

[2] This assumption is often disguised in certain remarks about perpetual motion.

[3] [1940, **1**].

[4] The rational student must cleave the stinging fog of pseudo-philosophical mysticism which hides this statement in the usual physical treatments.

an irreversible process, then

$$\Delta\mathfrak{H} \equiv \mathfrak{H}(\pi_2, \mathfrak{B}_2) - \mathfrak{H}(\pi_2, \mathfrak{B}_1) > \int_{t_1}^{t_2} \frac{\delta\mathfrak{Q}}{\theta}. \tag{28.4}$$

By (28.3), in the case of a reversible process inequality is replaced by equality in (28.4). Since for any change in $\mathfrak{E}$ either (28.3) or (28.4) is valid, we must have $\dot{\mathfrak{H}} \geq \delta\mathfrak{Q}/\theta$; since $\theta = \theta(\pi, \mathfrak{B})$, at any one instant θ is constant over $\mathcal{V}$; hence by $(28.2)_3$ we may put (28.3) and (28.4) together into the form of the Clausius-Duhem inequality[5]

$$\dot{\mathfrak{H}} + \oint_{\mathcal{S}} \frac{q^i}{\theta}\, da_i \geq 0, \tag{28.5}$$

where equality holds for reversible processes, inequality for irreversible ones.

Since the entropy appears now as a variable of central importance, it is more convenient, following Gibbs,[6] to take $\mathfrak{H}$, $\mathfrak{B}$ instead of π, $\mathfrak{B}$ as the primitive variables of the subject, to begin by postulating

$$\mathfrak{E} = \mathfrak{E}(\mathfrak{B}, \mathfrak{H}), \tag{28.6}$$

and to *define* π and θ by

$$\pi \equiv -\left(\frac{\partial\mathfrak{E}}{\partial\mathfrak{B}}\right)_{\mathfrak{H}}, \qquad \theta \equiv \left(\frac{\partial\mathfrak{E}}{\partial\mathfrak{H}}\right)_{\mathfrak{B}} \tag{28.7}$$

The resulting formal structure is identical with that initiated in the first paragraph above, and hence all the variables and formulae are susceptible of the same physical interpretation as in the conventional treatments.

Specifying the functional form of (28.6) specifies the medium (*e.g.*, perfect gas).

29. Thermodynamics of deformation. I. Basic postulates and definitions for homogeneous fluids. To obtain the structure of thermodynamics it is necessary to add assumptions beyond the bare conservation of energy (27.2). *For a gas in equilibrium* it is well established experimentally that a relation of the form $f(\pi, \mathfrak{B}, \theta) = 0$ exists, as was proposed by Euler.[1] Hence it is an easy step to assume $\mathfrak{E} = \mathfrak{E}(\mathfrak{B}, \mathfrak{H})$, thence proceeding as in §28. As far as physical interpretation is concerned, the results of classical thermodynamics are to be applied only

[5] Clausius [1854, **1**, p. 152] [1862, **1**, §1] [1865, **1**, §§1, 14–17] gave the case q = 0; the general form is apparently Duhem's [1901, **1**, part 1, Ch. 1, §6].

[6] [1873, **1**, p. 2, footnote] [1873, **2**, p. 31]. This manner of presenting the foundations of thermodynamics was employed also by Hilbert [1907, **1a**, pp. 435–438].

[1] [1745, **1**, Ch. I, laws 3, 4, 5]. I have found no earlier statement of the general principle or of the special case $\pi\mathfrak{B} = K\theta$, although experiments showing that for gases $\pi\mathfrak{B}$ = const. when θ = const. and π/θ = const. when $\mathfrak{B}$ = const. were published by Boyle (1662) and by Amontons (1699), respectively.

to the comparison of terminal states of a process which *starts from equilibrium and ends in equilibrium.*[2] Thus classical thermodynamics is totally inapplicable to the description of conditions *during* a grossly irreversible change.[3]

Although during a grossly irreversible process the system variables $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{V}$ are no longer functionally related, it is nevertheless quite possible that an equation of state subsists for the *local* variables ϵ, η, v. The entire existing theory of energy changes in grossly irreversible processes is confined[4] to the case of such *locally reversible systems.* By analogy[5] to (28.6) and (28.5) we lay down as postulates for homogeneous fluids

$$\epsilon = \epsilon(v, \eta), \qquad \dot{\mathfrak{H}} + \oint \frac{q^i}{\theta}\, da_i \geq 0, \tag{29.1}$$

defining the *thermodynamic pressure* π, *temperature* θ, and *entropy* $\mathfrak{H}$ by

$$\pi \equiv -\left(\frac{\partial \epsilon}{\partial v}\right)_\eta, \qquad \theta \equiv \left(\frac{\partial \epsilon}{\partial \eta}\right)_v, \qquad \mathfrak{H}(\mathfrak{V}, t) \equiv \int_{\mathcal{V}} \rho\eta\, dv, \tag{29.2}$$

$\mathcal{V}$ being any body. Specifying the functional form[6] of the relation $(29.1)_1$ now only partly specifies the medium (e.g. perfect gas) which may well have other physical properties (e.g. viscosity) which affect its motion without being represented in $(29.1)_1$ at all. In the ϵ-η-v space $(29.1)_1$ defines an *energy surface*[7] for

[2] In order to apply the results deduced from the mathematical theory of reversible processes, a fictitious and paradoxical physical "quasi-static process," in which the system is imagined changing so slowly as to be in equilibrium at all times, is often mentioned.

[3] E.g., while a gas is actually passing through a porous plug.

[4] According to the accepted results of the kinetic theory, a relation of the type $(29.1)_1$ holds only for gases sufficiently near to equilibrium. However, results derived from so incomplete a theory by methods so deficient in logic as those commonly employed in the kinetic theory should be warily regarded: "Attamen errores non sunt Artis sed Artificum."

[5] For the basic postulates (29.1) we have no direct experimental evidence. Cf. [1903, **1**, ¶127]. It is sometimes stated that local conditions change sufficiently slowly that the classical structure of §28 should be applicable to a very small volume, but it is difficult to assign a physical (i.e. dimensionless) meaning to this vague statement; more convincing, perhaps, is the fact that the extensively cultivated theory of gas dynamics, in which formulae equivalent to special cases of (29.1) are always adopted, at least up to the present has never been shown to yield a result in contradiction with experience in a situation to which it can reasonably be applied. There is also the council of despair: if we do not assume (29.1), there remain insufficient equations to solve any except the most degenerate problems of fluid mechanics (problems of incompressible fluids).

[6] Since the quantities occurring in $(29.1)_1$ are of independent dimensions: dim $\epsilon = \mathbf{L}^2\mathbf{T}^{-2}$, dim $\eta = \mathbf{L}^2\mathbf{T}^{-2}\Theta^{-1}$, dim $v = \mathbf{M}\mathbf{L}^{-3}$, in a change of units any one may be multiplied by an arbitrary quantity without affecting the values of either of the others. Thus the only properties of the energy surface $(29.1)_1$ which can represent physical (i.e. dimensionless) properties of the material must be *affine invariants* (cf. [1873, **2**, pp. 34–35] [1938, **2**, Ch. I, §§VIII-IX]). E.g., the postulate that "absolute" zeros of temperature and pressure exist may be put in the following form: only functions ϵ such that π and θ have finite greatest lower bounds P and T are admissible as energy functions. Referring θ and π to scales whose zeros are the absolute zeros then amounts to replacing ϵ by its affine equivalent $\epsilon + Pv - \eta T$.

[7] [1873, **2**, pp. 33–34].

the particular material, along any curve on which by (29.2) we have

$$\theta\, d\eta = d\epsilon + \pi\, dv. \tag{29.3}$$

In particular, $\theta\dot{\eta} = \dot{\epsilon} + \pi\dot{v}$, for adding $(29.1)_1$ as a postulate is equivalent to stating that the path of each particle in any possible motion of the medium is mapped onto some curve upon the energy surface.[8]

30. Thermodynamics of deformation. II. Pressure, dissipated power. Writing Φ for $\Phi(\mathbf{v}, \mathbf{d}) \equiv v^i{}_j\, d_j{}^i$ and putting (26.11), (29.3), and (24.2) into the energy equation (27.2), for compressible fluids we obtain

$$\rho\theta\dot{\eta} = (\pi - p)\dot{x}^k{}_{,k} + \Phi - q^i{}_{,i}\,, \tag{30.1}$$

while for incompressible fluids follows the simpler result

$$\rho\theta\dot{\eta} = \Phi - q^i{}_{,i}\,. \tag{30.2}$$

Since this last equation does not contain the arbitrary scalar pressure p at all, it implies that for incompressible substances no internal work is done by any hydrostatic stress $-p\mathbf{I}$, while any work done by the extra stress $\mathbf{v}$ is necessarily converted into one or another form of thermal energy. From (30.1) it appears that by adding the definition

$$p \equiv \pi \tag{30.3}$$

for compressible fluids we can simultaneously simplify the theory in two ways: (1) the stress tensor $\mathbf{t}$ is uniquely resolved into a portion $-\pi\mathbf{I}$ doing work which is always mechanically recoverable and a portion $\mathbf{v}$ doing work which is always dissipated, and (2) an energy equation of the same *form*, viz. (30.2), holds both for compressible and for incompressible fluids.[1] For incompressible fluids the specific volume is a given constant $v = v_0$, while the pressure p is a primitive unknown. For compressible fluids v is taken as a primitive unknown, but the pressure p is always taken as the thermodynamic pressure π. In the case of the incompressible fluids the pressure p occurs in none of the equations governing the motion except the dynamical equation $(26.13)_1$, so that it may be taken as any convenient scalar part of the stress,[2] there being no need to relate it to any other variables. For incompressible fluids it is only in comparison of a solution with the results of an experiment that the nature of the decomposition (26.11) need be examined, but here the greatest care must be taken to insure that what

[8] For an incompressible substance the energy surface degenerates to a curve in a certain plane $v =$ const., the definition $(29.2)_1$ of the thermodynamic pressure fails altogether, and (29.3) becomes simply $\theta d\eta = d\epsilon$.

[1] By an analysis in the kinetic theory of gases KOHLER [1950, **2**] obtains an entropy equation different from (30.2), a result which must be thoroughly investigated before it can be accepted. Cf. §29[4].

[2] [1948, **5**, §11] [1948, **7**, §3].

is measured by a pressure-measuring instrument actually corresponds to what is called p in the equations. For incompressible fluids it would be permissible, but does not appear to be advantageous, to *define* p as the mean pressure $\bar{p}$, given by (26.12); for compressible fluids it is not generally possible to have (30.2) and $p = \bar{p}$ simultaneously[3]; thus in general $p \neq \bar{p}$. The mathematical structure established above is valid for compressible and incompressible fluids alike, although the meanings of the variables in the two cases are somewhat different.[4]

Subject to this understanding, Φ in all cases is the rate at which mechanical work is converted into thermal energy, and may accordingly be called the *dissipated power*. The stress power $\Phi(\mathbf{t}, \mathbf{d})$ is thus resolved into a portion representing power used up in changing shape and power stored as internal energy arising from change in volume: $t^i{}_j d^j{}_i = \Phi(\mathbf{t}, \mathbf{d}) = -p\, \dot{\overline{\log v}} + \Phi$. If the pressure p in the relation (26.11) were given by any other definition than (30.3) for a compressible substance, the resulting function Φ would not have this simple meaning. By the results of §9, in the classical theory of viscous fluids Rayleigh's dissipation function[5] is $\frac{1}{2}\Phi$. In general, however, even in the case when a potential Γ for $\mathbf{v}$ exists, it is not simply related to Φ.

31. Thermodynamics of deformation. III. Consequences of the Clausius-Duhem inequality. From (30.2) follows

$$\dot{\mathfrak{H}} + \oint_{\mathcal{S}} \frac{q^i}{\theta}\, da_i = \int_{\mathcal{V}} \left[\frac{\Phi}{\theta} - \frac{q^i\theta_{,i}}{\theta^2}\right] dv \tag{31.1}$$

for any fluid body $\mathcal{V}$. Hence by $(29.1)_2$ $\Phi/\theta - q^i\theta_{,i}/\theta^2 \geq 0$. If the terms may be taken as independent, it must follow that[1]

$$q^i\theta_{,i} \leq 0, \qquad \Phi \geq 0. \tag{31.2}$$

These statements are the mathematical expressions of two familiar irreversibilities: (1), heat never flows against a temperature gradient, and, (2), deformation absorbs energy, but cannot release it. The inequalities constitute important restrictions upon the possible defining equations for various spe-

[3] The contrary impression is given by many treatments of fluid dynamics. LAMB, as always, is correct, although he handles the matter in a rather circuitous fashion [1932, **1**, §325 footnote, §358]. The distinction between π and $\bar{p}$ is noted by ZAREMBA [1903, **7**, pp. 385, 390, 392]. Cf. §61.

[4] Cf. the remarks of DUHEM [1901, **1**, Part I, Ch. I, §8] and HILBERT [1907, **1**, pp. 220–222].

[5] We do not follow the literature in calling $\frac{1}{2}\Phi$ the *dissipation function* in general, preferring a different usage because the function Φ always exists and always represents dissipation of energy, while the interest of RAYLEIGH's function (in the rare circumstances when such a function exists) is not that it represents dissipation but that it is a potential. Cf. §9[1].

[1] We assume $\theta \geq 0$ (cf. §29[6]).

cial types of fluids. As the simplest example, consider Fourier's law[2] of heat conduction: $q_i = -\kappa\theta_{,i}$. From $(31.2)_1$ it follows that this law is physically admissible only if $\kappa \geq 0$. Conversely, if we start with (31.2) we may *derive* the Clausius-Duhem inequality $(29.1)_2$ for fluids; thus that inequality is a direct consequence of simple physical observation expressed within the structure of thermodynamics.

The Clausius-Duhem postulate $(29.1)_2$ refers only to the *total* entropy; comparison of its consequence (31.2) with (30.2) shows that it is quite possible for the specific entropy η of a particle to decrease, as is remarked by Meissner.[3]

32. Thermodynamics of deformation. IV. Heterogeneous fluids. Gibbs's theory of the equilibrium of heterogeneous substances[1] was generalized to apply to media in motion by Jaumann and Lohr[2]; we sketch here the recent presentation of Eckart.[3] By analogy to $(29.1)_1$, the first postulate is

$$\epsilon = \epsilon(\eta, \rho^1, \rho^2, \cdots, \rho^N). \tag{32.1}$$

Entropy and energy are thus regarded as variables describing the mean motion, and the system is thermodynamically defined when the relation between them is given, just as in the case of a simple fluid; account is taken of the heterogeneity only by permitting the form of this relationship to depend upon the outcome, but not the process, of the peculiar motions. By (25.3), (32.1) becomes

$$\epsilon = \epsilon(\eta, \rho c^1, \rho c^2, \cdots, \rho c^N),$$

and it is in fact more convenient to write $\epsilon = \epsilon(\eta, \rho, c^1, \cdots, c^N)$, although one of the c^Γ may be eliminated ($\mathfrak{M}_\Gamma c^\Gamma = 1$). Pressure π and temperature θ are now defined by $(29.2)_1$ and $(29.2)_2$ as properties of the mean motion, with the understanding that the concentrations c^Γ are held constant during the differentiations; the *potential* μ_Γ *of the substance* Γ is defined similarly by[4] $\mu_\Gamma \equiv \partial\epsilon/\partial c^\Gamma$, and in place of (29.3) we now obtain $\theta\, d\eta = d\epsilon + \pi\, dv - \mu_\Gamma\, dc^\Gamma$, where the differentials are taken along any path on the energy surface (32.1), and in particular $\theta\dot{\eta} = \dot{\epsilon} + \pi\dot{v} - \mu_\Gamma \dot{c}^\Gamma$.

Now diffusion can change the total energy of a mass of the material even when

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[2] [1822, **1**, §127].

[3] [1938, **10**, §II].

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[1] [1875, **1**].

[2] [1911, **5**, §§IV–VI]; [1917, **5**] [1924, **5**]; cf. [1913, **1**, §24c]. JAUMANN pointed out that in an isolated spinning cylinder containing a non-uniform solution, diffusion will change the moment of inertia, so that it is impossible that both the kinetic energy and the moment of momentum can be conserved. Deciding in favor of conservation of energy, he chose to modify the Newtonian laws of momentum and the principle of conservation of mass. It is preferable to regard both moment of momentum and *total* energy $\mathfrak{K} + \mathfrak{E}$ as conserved, since by (32.1) the internal energy changes when the concentrations change.

[3] [1940, **2**, p. 272]. Cf. [1938, **10**, §III] [1941, **9**, §3] [1942, **13**, §§15–16] [1942, **14**, §§7–8].

[4] [1875, **1**, p. 63].

the mean motion and the heat flux vanish. To describe this effect Eckart adds to the right hand side of (27.1) the surface integral[5] of $-\rho^{\Gamma}\mu_{\Gamma}u_{\Gamma}^{i}$. The resulting energy equation now assumes the alternative forms

$$\begin{aligned} \rho\dot{\epsilon} &= t^{i}{}_{j}\, d^{j}{}_{i} - q^{i}{}_{,i} - (\rho c^{\Gamma}\mu_{\Gamma}u_{\Gamma}{}^{i})_{,i}\,, \\ \rho\theta\dot{\eta} &= \Phi - q^{i}{}_{,i} - \rho[\delta_{\Gamma}e^{\Gamma} + \mu_{\Gamma,i}u_{\Gamma}{}^{i}c^{\Gamma}], \end{aligned} \tag{32.2}$$

generalizing (27.2) and (30.2); in $(32.2)_2$ the quantity δ_{Γ} is the difference in potential between the substance Γ and its equivalent in free elements: $\delta_{\Gamma} \equiv \mu_{\Gamma} - n_{\Gamma}{}^{\Delta}\mu_{\Delta}$ (note that $\delta_{\Gamma} = 0$ if $\Gamma \leq M$).

Since entropy, heat flux, and temperature are variables describing the mean motion, the postulate $(29.1)_2$ may be retained for heterogeneous media without modification. In place of (31.2) we now obtain

$$\frac{\Phi}{\theta} - \frac{q^{i}\theta_{,i}}{\theta^{2}} - \frac{\rho}{\theta}\,[\delta_{\Gamma}e^{\Gamma} + \mu_{\Gamma,i}u_{\Gamma}{}^{i}c^{\Gamma}] \geq 0. \tag{32.3}$$

The defining equations and thermodynamic equations of motion for any substance must be such that (32.3) is satisfied in any possible motion. If the four terms may be regarded as independent, we obtain $\mu_{\Gamma,i}u_{\Gamma}{}^{i}c^{\Gamma} \leq 0$, $\delta_{\Gamma}e^{\Gamma} \leq 0$ in addition to (31.2). Since it is possible that at a particular point only one substance happens to be present, we conclude that $\mu_{\Gamma,i}u_{\Gamma}{}^{i} \leq 0$; that is, the diffusion current for the substance Γ always carries a particle of the substance Γ relative to the mean motion toward a region of lower potential. Similarly, in the case when all the potential differences δ_{Γ} but one vanish, if $\delta_{\Gamma} > 0$ the compound Γ must be dissociating, while if $\delta_{\Gamma} < 0$ it must be forming; thus when but a single compound substance can be created, the reaction proceeds so as to reduce its potential difference to zero.

A basically different approach is indicated by Leaf,[6] who regards each substance Γ of an inert mixture as an independent thermodynamic system with energy ϵ_{Γ}, entropy η_{Γ}, and an equation of state of type $\epsilon_{\Gamma} = \epsilon_{\Gamma}(\eta_{\Gamma}, \rho^{\Gamma})$. The partial energy ϵ_{Γ} is in effect defined by an equation of the form (27.2) in terms of the partial stress and partial heat flux for the substance Γ, and the partial entropy η_{Γ} satisfies an analogue of $(29.1)_2$. There are three different connections between the several substances: (1) there is an equation of state (32.1) for the mean motion, where $\rho\epsilon \equiv \rho^{\Gamma}\epsilon_{\Gamma}$, $\rho\eta \equiv \rho^{\Gamma}\eta_{\Gamma}$; (2) the temperature at a given point

[5] While no objection can be raised against this term, ECKART does not positively demonstrate that it correctly describes the desired mechanism, and his proposal differs from those of JAUMANN [1911, **5**, §IV] and LOHR [1917, **5**, eqq. (108), (109)], and a still different diffusion term occurs in the energy equation stated without derivation by HIRSCHFELDER & CURTISS [1949, **25**, App. A]. Cf. also [1948, **45**]. Earlier REYNOLDS [1903, **19**, §39] had given the equation governing interchanges of purely mechanical energy in a system of several components.

[6] [1946, **18**]. A continuum theory of diffusion involving differences instead of derivatives of density, similar to that derived by MAXWELL from the kinetic theory of gases, was formulated by STEFAN [1871, **5**].

and time is the same for all the substances: $\theta_\Gamma \equiv \partial\epsilon_\Gamma/\partial\eta_\Gamma = \theta_\Lambda \equiv \partial\epsilon_\Lambda/\partial\eta_\Lambda = \theta \equiv \partial\epsilon/\partial\eta$; (3) the partial pressures π_Γ are related to the chemical potentials μ_Γ by $\epsilon_\Gamma = \theta\eta_\Gamma - \pi_\Gamma/\rho^\Gamma + \mu_\Gamma$.

33. Thermodynamics of deformation. V. Elastic solids. There was at one time a tendency to regard all macroscopic phenomena as essentially thermodynamical and therefore to construct a system of energetics which included mechanics as a subsidiary part.[1] The simple equation of state $(29.1)_1$ is to be replaced by

$$\epsilon = \epsilon(\eta, P_1, P_2, \cdots, P_N), \tag{33.1}$$

where the P_J are "external parameters" characterizing the relation of the thermodynamic system to its surroundings. In the special case of the homogeneous fluid the single external parameter is v; another special case is the equation of state (32.1) for the heterogeneous fluid. The various proposals of this type, as far as pure mechanics is concerned, are summarized in the excellent article of Hellinger,[2] and we rest content here with a sketch of those pertaining to elastic bodies.

Green[3] in effect defined an elastic body as one for which $\rho_0\epsilon = f(X^\alpha, \mathbf{E})$, where the strain $\mathbf{E}$ is taken with respect to a fixed "natural state" X^α (§34 below). Thermodynamical aspects of Green's theory were clarified by Kelvin[4]; a similar method was used by Kirchhoff[5]; and Gibbs[6] proposed the more general defining equation of state

$$\rho_0\epsilon = f(\eta, X^\alpha, x^i{}_{,\beta}). \tag{33.2}$$

Developments based upon it are of two types: first, just as pressure is a defined concept in a fluid, so also stress can be subjected to a purely thermodynamic definition, and the conditions of equilibrium *derived* from a suitably general principle of virtual work, rather than postulated in the form of the momentum principle; and second, the existence of an elastic potential in certain circumstances may be demonstrated. These two procedures we now review.

A. *Thermodynamic theory of equilibrium.* By analogy to $(29.2)_1$, let auxiliary

[1] Such is the approach of DUHEM [1911, 1] and JAUMANN [1911, 5] [1918, 1]; the latter's work has been continued by LOHR [1917, 5] [1924, 5], who has shown recently [1940, 13] [1948, 43] that some phenomena for which a wave-mechanical explanation is customary may be described by thermodynamic means without introducing probability considerations.

[2] [1914, 1, §§7, 9, 15].

[3] [1839, 1, pp. 248–255] [1841, 2, pp. 298–300].

[4] [1856, 1, Chs. XIII, XIV] [1863, 1, §§61–67] [1867, 1, §673 and App. C, §§(c)–(d)]. An incomplete treatment is given in [1855, 1].

[5] [1850, 1, §1] [1852, 1, pp. 770–772].

[6] [1875, 1, pp. 184–190].

quantities F_i^{α} and θ be defined from (33.2) by

$$F_i^{\alpha} \equiv \rho_0 \frac{\partial \epsilon}{\partial x^i{}_{,\alpha}}, \qquad \theta \equiv \frac{\partial \epsilon}{\partial \eta}. \tag{33.3}$$

The postulated principle of virtual work[7] and principle of thermodynamic equilibrium are, respectively,

$$\delta \int_{\mathcal{V}} \rho\epsilon \, dv = \int_{\mathcal{V}} \rho f_i \, \delta x^i \, dv + \oint_{\mathcal{S}} t_{(n)i} \, \delta x^i \, da, \qquad \delta \mathfrak{H} = 0. \tag{33.4}$$

By executing the variations and taking account of $(24.1)_1$ and the definitions (33.3) we obtain

$$\int_{\mathcal{V}_0} \rho_0 \theta \, \delta\eta \, dV_0 + \oint_{\mathcal{S}_0} \left(F_i^{\alpha} \, dA_\alpha - t_{(n)i} \frac{da}{dA} \, dA \right) \delta x^i - \int_{\mathcal{V}_0} (F_i^{\alpha}{}_{,\alpha} + \rho_0 f_i) \, \delta x^i \, dV_0 = 0, \tag{33.5}$$

$$\int_{\mathcal{V}_0} \rho_0 \, \delta\eta \, dV_0 = 0.$$

Hence the local conditions of equilibrium are

$$\theta = \text{const.}, \qquad F_i^{\alpha} \, dA_\alpha = t_{(n)i} \, da, \qquad F_i^{\alpha}{}_{,\alpha} + \rho_0 f_i = 0. \tag{33.6}$$

The formal similarity of $(33.6)_2$ to (26.6) and of $(33.6)_3$ to $(26.8)_2$ with $\ddot{x}^i = 0$ enables us to identify F_i^{α} with Kirchhoff's mixed vector T_i^{α} and hence by taking $(26.5)_2$ as a definition to derive Cauchy's first law $(26.2)_1$ for the case of equilibrium. It does not appear to be possible to derive Cauchy's second law $(26.2)_2$, however, except when $(33.2)_2$ reduces to Green's form $\rho_0\epsilon = f(\eta, X^{\alpha}, \mathbf{E})$, when it becomes an immediate consequence of the definition $(33.3)_1$.

Despite the elegance of the foregoing derivation, its validity cannot be accepted. The equations of mechanics describe a wider range of phenomena than do the equations of thermodynamics. In particular, Cauchy's laws (26.2) are valid for viscous fluids and plastic bodies, for which there is certainly no equation of state of the type (33.2). The great weakness of the thermodynamic method is revealed by its treatment of viscosity, which always has to be dragged in by the heels by the assumption that $(33.4)_1$ is to be modified by the addition of a linear form[8] in $\delta\dot{x}^i$. Not only is such an assumption totally unmotivated in thermodynamics, but also it is extremely restrictive, for it can never lead to

[7] This method of deriving laws of motion is formally similar to the method of virtual displacements, but is founded upon a basically different idea. In the latter method the stress components are regarded as given variables, or else introduced as multipliers in a variational principle, while here they are defined as partial derivatives of a special function characterizing the medium. But for either method *the stress principle* (§26) *must be assumed.*

[8] [1901, **1**, Ch. I, §§1–5] [1903, **12–13**] [1904, **1**, Ch. II, §V].

viscous stresses which are non-linear functions of **d**, and in any event there are many types of ideal bodies in which the stress is not a simple superposition of purely elastic and purely viscous portions. But, worst of all, to obtain dynamical equations by the energetic method one must add d'Alembert's principle—at bottom indistinguishable from the momentum principle—as a new postulate. It would seem both more lucid and more simple to cleave to the momentum principle alone. The best construction we can put on the foregoing analysis is to say that it indicates the consistency of the mechanical theory of elasticity with thermodynamic principles.

B. *Elastic potentials.* A function $\Sigma = \Sigma(x^i{}_{,\alpha},\ldots)$ is said to be an *elastic potential* or *strain energy* if

$$\frac{\rho}{\rho_0}\dot{\Sigma} = t^i{}_j\, d^j{}_i. \tag{33.7}$$

The importance of this function will appear in Chapter IVA, where it will be shown that in consequence of its existence the stress can be proved to be given by a formula completely analogous to Gibbs's definition $(33.3)_1$.

A motion is *adiabatic* if no heat be added or taken away from any part of the material. Equivalently, $q^i{}_{,i} = 0$, so that the general energy equation (27.2) reduces to $\rho\dot{\epsilon} = t^i{}_j\, d^j{}_i$. Comparison with (33.7) shows that *a body with an equation of state* (33.2) *in adiabatic motion has an elastic potential* $\Sigma = \rho_0\epsilon$.

While it is generally stated that an elastic energy exists also for isothermal motions, the proof adduced is faulty. The fullest and clearest treatment, which was given by Voigt,[9] rests upon writing the "second law of thermodynamics" in the form

$$\rho\dot{\epsilon} = t^i{}_j\, d^j{}_i + \rho\theta\dot{\eta}. \tag{33.8}$$

Apparently Voigt proposed it simply as a natural extension of the classical relation (28.3) for pure fluids.[10] That relation, however, is derived either from the existence of an integrating factor for the form (28.1), or, alternatively, by taking (28.7) as definitions. The former method is not applicable to continuous media in general; the latter may be used if we adopt $(33.3)_1$ as the definition of stress ($F_i{}^\alpha = T_i{}^\alpha$), but such a procedure amounts to *assuming* stress-strain relations so as to transform the energy equation, while our reason for wishing an elastic potential is to be able to derive stress-strain relations.

Suppose, however, that somehow (33.8) has been derived. By (27.2) we obtain

$$\rho\theta\dot{\eta} = -q^i{}_{,i}. \tag{33.9}$$

[9] [1889, **2**, pp. 943–749] [1895, **2**, Pt. III, Ch. I, §§6–9] [1910, **1**, §§277, 381–382, 389, 392, 394]. Cf. [1907, **2**, §5c] [1938, **2**, Ch. X, §VIII] [1949, **39**, Cap. I, ¶¶1–2].

[10] At this point the standard texts (e.g. [1927, **3**, §62]) resort to verbal evasions.

Some writers regard this equation, which is equivalent to (33.8), as a general expression of the second law.[11] In any case, by introducing the *free energy*[12] $\phi \equiv \epsilon - \eta\theta$ we may put (33.8) into the form

$$\rho\dot{\phi} + \eta\dot{\theta} = t^i{}_j\, d^j{}_i\,. \tag{33.10}$$

Comparison with (33.7) shows that *a body with an equation of state* (33.2) *in an isothermal deformation* ($\dot{\theta} = 0$) *has an elastic potential* $\Sigma = \rho_0\phi$.

Mr. R. Toupin & Dr. J. Ericksen have pointed out another approach to the subject. Suppose (33.9) be assumed—say, as part of the definition of an elastic body, along with (33.2). Then we have (33.8), whence follows

$$\rho\left[\frac{\partial\epsilon}{\partial x^j{}_{,\alpha}}\, x^i{}_{,\alpha}\dot{x}^j{}_{,i} + \frac{\partial\epsilon}{\partial\eta}\,\dot{\eta}\right] = t^i{}_j\, d^j{}_i + \rho\theta\dot{\eta} \tag{33.11}$$

If now we *assume* both that $t^{ij} = t^{ji}$ and that $t^i{}_j$ and θ are independent of $\dot{x}^j{}_{,i}$ and $\dot{\eta}$, it follows that $T_i{}^\alpha = \rho_0\partial\epsilon/\partial x^i{}_{,\alpha}$ and $\theta = \partial\epsilon/\partial\eta$, without restriction to isothermal or adiabatic deformations. Cf. §44[4]. This method, however, is very near to assuming rather than proving the existence of stress-strain relations.

A satisfactory thermodynamical treatment of the foundations of elasticity theory remains to be discovered. For this reason I simply lay down (33.7) as a postulate for Chapter IVA.

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[11] Voigt showed that (33.9) reduces when $q_i = -\kappa_i{}^j\theta_{,j}$ and when the temperature changes are "small" to a generalization of the Fourier-Duhamel equation of heat conduction [1832, **1**, pp. 361–366].

[12] An equivalent function was introduced by Massieu [1869, **3–4**].

Chapter IV. Elasticity

IVA. Green's Method: The Natural State Theory

34. The idea of a natural state. The classical theory of elasticity formulates the intuitive notion of spring (§§2, 23) in a basic hypothesis: For a given body, there exists a fixed *natural state*[1], generally regarded as unstressed, such that the body when constrained into a form near to that in its natural state and then released from all external forces returns perfectly to this natural state independently of the manner in which the forces have been applied and removed. Thus the minimum energy required to effect a given deformation is stored by the body and is always available for recovery. Many authors[2] have insisted that the initial state is an arbitrary one, not characterized by any special properties. While the infinitesimal theory, which is essentially a theory of small stress differences (cf. §55), possesses this physically desirable attribute of being applicable to any initial state, in the classical finite strain theory the stress is a function of strain from some one configuration (§37), and the response of the body to strain from another configuration is necessarily different.

35. Cauchy's method and Green's method. Two possible methods of realizing this concept of springiness have been proposed. Cauchy's[1] method, which will be discussed in Chapter IVB, was to define a perfectly elastic body as one in which *the stress is a function of the strain.*[2] Green's[3] method was to define a perfectly elastic body as one in which the *internal energy* (*or a related quantity*) *is a function of the strain.* Within the range of infinitesimal strain both methods yield the same result for isotropic bodies,[4] but for finite strain Green's method

[1] E.g. Cauchy [1828, **1**, §II, p. 203] regarded the natural state as the state of a body in a vacuum at uniform temperature. Cf. [1829, **3**, Introd. p. 365] [1831, **1**, §14] [1845, **1**, §15] [1850, **1**, §1] [1852, **1**, p. 762] [1863, **2**, ¶2] [1896, **1**, §28].

[2] E.g. [1875, **1**, p. 185] [1938, **2**, Ch. X, §VIII].

[1] [1823, **1**] [1828, **1**, §11]. (Cauchy proposed also a molecular theory of elasticity (§3[2]).)

[2] St. Venant, prompted by Levy's theory of fluids (§60) and employing Navier's molecular notions, once [1869, **1**] proposed a theory in which the stress depends not merely upon the $x^i_{,\alpha}$ but also upon the $x^i_{,\alpha\beta}, \cdots, x^i_{,\alpha\beta\cdots\gamma}, \cdots$.

[3] [1839, **1**, p. 249] [1841, **2**, pp. 295–296].

[4] For infinitesimal strain of a general anisotropic body the method of Cauchy yields stress-strain relations containing 36 elastic moduli, which Cauchy [1829, **1**, eqq. (7) (8)] stated to be independent, while the method of Green [1841, **2**, p. 298] yields but 21 constants. The difference is not to be confused with that between the "multi-constant" and "rari-constant" theories; in the latter, based upon a molecular hypothesis, the number of moduli is further reduced to 15 by means of "Cauchy's relations" [1828, **3**, eqq. (36) (37)].

leads to a more definite theory, which has been studied by many authors,[5] employing such a variety of quantities and notations as to render the literature a trackless wilderness. Although the reader of certain recent papers might gain a contrary impression, since Kelvin's[6] improvement of Green's analysis the only significant addition to the general theory is Finger's[7] use of the spatial strain measure $\mathbf{c}^{-1}$ for isotropic media and consequent essential simplification of the stress-strain relations. Unfortunately the definitive expositions of the Cosserats[8] do not mention Finger's results, which have received neither the credit nor the attention they deserve. Fifty years after Finger's time they have been rediscovered by Rivlin and shown to be the instrument whereby the classical theory of finite elastic strain can give concrete results of the greatest importance (§42). Through the introduction of tensorial notations and methods, enabling pages of the earlier work to be not only presented but also easily and perspicuously derived in a few lines, L. Brillouin[9] has done great service, and Murnaghan has put the theory into its most elegant form. In the following sections we give a slight generalization of Murnaghan's first and best treatment.[10]

36. Definition of a perfectly elastic body according to the natural state theory. *The response of a body is perfectly elastic if for co-ordinates X^α referred to a certain natural state, there exists a* strain energy Σ *of the form*

$$\Sigma = \Sigma(X^\alpha, G_{\alpha\beta}, g_{ij}, \rho, X^\alpha{}_{,i}), \tag{36.1}$$

such that

$$\frac{\rho}{\rho_0}\dot{\Sigma} = t^i{}_j\, d^j{}_i. \tag{36.2}$$

Conditions under which the existence of a strain energy follows from the energy equation (27.2) and an hypothesis regarding the form of the internal energy ϵ have been discussed in §33; here we confine our attention to formal consequences of (36.1) and (36.2). The occurrence of the X^α in (36.1) permits the medium to be heterogeneous, so long as it be inert. By $(24.1)_3$ and $(14.10)_3$ we may eliminate ρ from (36.1):

$$\Sigma = \Sigma(X^\alpha, G_{\alpha\beta}, g_{ij}, X^\alpha{}_{,i}). \tag{36.3}$$

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[5] For references, see §§33, 39–41.

[6] [1863, **1**, §§51–67]. While KIRCHHOFF [1852, **1**, pp. 770–772] treated only a special case, his method was perfectly general and could have yielded KELVIN's results.

[7] [1894, **2**].

[8] [1896, **1**] [1909, **1**].

[9] [1928, **5**] [1925, **1**]. Cf. [1931, **15–17**]. The use of tensorial methods in the infinitesimal theory was indicated by RICCI & LEVI-CIVITA [1901, **7**, Ch. VI, §3]. Cf. [1924, **4**] [1938, **9**].

[10] [1937, **1**]. This paper, important for method despite its lack of new results, is translated into dyadic notation in [1949, **35**].

37. Proof that Σ is a function of C. In §§14–17 two methods for measuring strain were introduced: the material, employing any of the measures $\mathbf{C}$, $\mathbf{C}^{-1}$, $\mathbf{E}$, $\mathbf{H}$, etc., and the spatial, employing any of $\mathbf{c}$, $\mathbf{c}^{-1}$, $\mathbf{e}$, $\mathbf{h}$, etc. In general the two methods are not equivalent. Which then shall be used in elasticity? The classical authors[1] always employed $\mathbf{E}$ or $\mathbf{C}$, and Cellerier[2] demonstrated that this choice is actually correct. We give the proof as arranged by Murnaghan.[3] For a stationary co-ordinate system, by (20.4), $(21.2)_2$, $(21.4)_2$, and (21.5) we have from (36.3)

$$\dot{\Sigma} = \frac{\partial \Sigma}{\partial X^{\alpha}{}_{,i}} \overline{\dot{X^{\alpha}{}_{,i}}} = -\frac{\partial \Sigma}{\partial X^{\alpha}{}_{,i}} X^{\alpha,j} \dot{x}_{j,i} = -\frac{\partial \Sigma}{\partial X^{\alpha}{}_{,i}} X^{\alpha,j}(d_{ij} + w_{ij}). \quad (37.1)$$

By (36.2), $\dot{\Sigma} = 0$ if $\mathbf{d} = 0$; hence, since $\mathbf{w}$ is an arbitrary alternating tensor, from (37.1) it must follow

$$\frac{\partial \Sigma}{\partial X^{\alpha}{}_{,i}} X^{\alpha,j} = \frac{\partial \Sigma}{\partial X^{\alpha}{}_{,j}} X^{\alpha,i}. \quad (37.2)$$

Of the nine components of this tensor equation only three are independent and not automatically satisfied, and these form a complete system of linear homogeneous first order partial differential equations in nine independent variables, the commutator of any two being the third, so that their general solution is a function of $9 - 3 = 6$ independent solutions.[4] Now (37.2) is simply a condition that Σ shall be a solution χ of $\dot{\chi} = 0$ when $\mathbf{d} = 0$. By (22.14), the six quantities $E_{\alpha\beta}$ satisfy this condition, and hence furnish the required six independent solutions of (37.2). *Therefore* (36.3) *must reduce to*

$$\Sigma = \Sigma(X^{\alpha}, G_{\alpha\beta}, g_{ij}, C_{\alpha\beta}) = \Sigma(X^{\alpha}, G_{\alpha\beta}, g_{ij}, E_{\alpha\beta})$$
$$= \Sigma(X^{\alpha}, G_{\alpha\beta}, g_{ij}, (C^{-1})_{\alpha\beta}) = \cdots. \quad (37.3)$$

The derivatives occurring in $(37.3)_1$ and $(37.3)_2$ are material $(x^{i}{}_{,\alpha})$, while those occurring in $(37.3)_3$ are spatial $(X^{\alpha}{}_{,i})$.

It will henceforth be assumed that all scalar functions of symmetric tensors will be formally symmetrized; e.g., in $\Sigma(X^{\alpha}, G_{\alpha\beta}, g_{ij}, C_{\alpha\beta})$, the quantities $C_{\alpha\beta}$ wherever they occur will be replaced by the numerically equal quantities $\frac{1}{2}(C_{\alpha\beta} + C_{\beta\alpha})$. Hence $\partial\Sigma/\partial C_{\alpha\beta} = \partial\Sigma/\partial C_{\beta\alpha}$.

38. Elastically isotropic bodies. An elastic body will be said to be *isotropic* if Σ be an isotropic function of $\mathbf{C}$. By §6 we have then

$$\Sigma = \Sigma(X^{\alpha}, I_{\mathrm{C}}, II_{\mathrm{C}}, III_{\mathrm{C}}) = \Sigma(X^{\alpha}, I_{\mathrm{E}}, II_{\mathrm{E}}, III_{\mathrm{E}}) = \cdots \quad (38.1)$$

[1] E.g. [1841, **2**, p. 298] [1852, **1**, p. 769]. While Cauchy introduced **c** in his study of the geometry of finite strain, he did not propose a theory of elasticity intended to be valid for large strain.

[2] [1893, **3**, §4]. Cf. [1896, **1**, §26].

[3] [1937, **1**, §3].

[4] E.g. [1930, **2**, Ch. XV, §2].

By (14.7) it follows that

$$\Sigma = \Sigma(X^\alpha, I_c, II_c, III_c) = \Sigma(X^\alpha, I_e, II_e, III_e) = \cdots . \quad (38.2)$$

Thus *for isotropic bodies it is legitimate to regard the elastic energy alternatively as a function of any one of the various spatial tensors* **c**, **e**, **h**, $\cdots$.

$$\Sigma = \Sigma(X^\alpha, \mathbf{c}) = \Sigma(X^\alpha, \mathbf{c}^{-1}) = \Sigma(X^\alpha, \mathbf{e}) = \Sigma(X^\alpha, \mathbf{h}) = \cdots . \quad (38.3)$$

(Cf. §17).

39. Derivation of general stress-strain relations: I. Material forms. Putting $(37.3)_1$ in (36.2), by (22.14) we obtain

$$t^{ij} d_{ij} = \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial C_{\alpha\beta}} \dot{C}_{\alpha\beta} = 2 \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial C_{\alpha\beta}} x^i{}_{,\alpha} x^j{}_{,\beta} d_{ij} . \quad (39.1)$$

Since this relation must hold for an arbitrary **d**, it follows that

$$t^{ij} = 2 \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial C_{\alpha\beta}} x^i{}_{,\alpha} x^j{}_{,\beta} = \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial E_{\alpha\beta}} x^i{}_{,\alpha} x^j{}_{,\beta} = J^{-1} \frac{\partial \Sigma}{\partial E_{\alpha\beta}} x^i{}_{,\alpha} x^j{}_{,\beta} , \quad (39.2)$$

which is Boussinesq's[1] form of the stress-strain relations.[2] By (26.5) it is plain

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1 [1870, **1**] [1872, **1**, Note 3, p. 591]. Cf. [1896, **1**, §24, eq. (61)] [1937, **1**, §6] [1941, **1**, p. 125]

2 So far as I know, the fully general theory has been pursued in three directions only:

A. *Propagation of discontinuities.* HADAMARD [1901, **5**, §§7–8] [1903, **1**, §§264–267] states: (1) a plane acceleration wave (sound wave) traveling in a given direction can be the carrier of a discontinuity pointing in any one of three mutually orthogonal directions, to each of which there corresponds a definite speed of propagation; (2) if the present configuration of the body be a possible state of stable equilibrium, all three speeds of propagation are real; (3) the acceleration waves in general are neither transversal nor longitudinal. These results are considerably generalized by DUHEM [1903, **15–18**] [1904, **1**, Part IV], who shows also that no waves can propagate in a MEYER-VOIGT material (§81). JOUGUET [1920, **2–5**] [1920, **1**, 1st note] discusses the propagation of velocity waves (shock waves). The extension of HADAMARD's results to media in which there are stress-strain relations of arbitrary form claimed by FINZI [1942, **15**, §4] is not valid because the distinction between spatial and material tensors is not observed; in any case the reality of the directions of propagation is assumed, not proved. SIGNORINI has stated that HADAMARD's proof of the reality of the directions of propagation is not valid in the case of finite strain, and TOLOTTI [1943, **11**] derives a necessary and sufficient condition to be satisfied by the form of the strain energy in order that HADAMARD's results remain valid. Cf. also [1949, **22**, §6].

B. *Elastic stability.* [1903, **1**, ¶¶269–271] [1904, **1**, Part III].

C. *General integration theory.* SIGNORINI [1936, **4**, pp. 18–21] [1949, **39**, Cap. I, §§2–3] [1950, **15**] shows that for any form of strain energy if the extraneous and surface loads be written with a multiplicative factor k, then if these forces admit no axis of equilibrium the formal coefficients in a series for the displacement in powers of k may be determined uniquely; the coefficient of the first power is the displacement as predicted by the infinitesimal theory. If, however, the load system admit an axis of symmetry, he states that even the first coefficient is no longer uniquely determined, which he regards as a fault of the classical theory. It does not seem to me that there is any reason to expect even a formal that $(39.2)_1$ is equivalent to Kelvin's[3] form

$$T^i{}_\alpha = 2 \frac{\partial \Sigma}{\partial C_{\alpha\beta}} x^i{}_{,\beta}. \tag{39.3}$$

By (26.9) we then obtain the Cosserats'[4] form

$$T^\alpha{}_\beta = \frac{\partial \Sigma}{\partial E^\beta{}_\alpha}, \qquad t^{ij} = \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial E_{\beta\alpha}} x^i{}_{,\alpha} x^j{}_{,\beta}; \tag{39.4}$$

hence **T** as a function of **E** admits Σ as a potential (§6). From $(15.1)_1$ and $(14.9)_1$ we have

$$\frac{\partial \Sigma}{\partial x^i{}_{,\alpha}} = \frac{\partial \Sigma}{\partial E_{\beta\gamma}} \frac{\partial E_{\beta\gamma}}{\partial x^i{}_{,\alpha}} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial E_{\alpha\beta}} + \frac{\partial \Sigma}{\partial E_{\beta\alpha}}\right) x_{i,\beta}, \tag{39.5}$$

enabling us to reduce $(39.2)_3$ to C. Neumann's[5] form

$$t^i{}_j = J^{-1} x^i{}_{,\alpha} \frac{\partial \Sigma}{\partial x^j{}_{,\alpha}} = \frac{\rho}{\rho_0} x^i{}_{,\alpha} \frac{\partial \Sigma}{\partial x^j{}_{,\alpha}}, \tag{39.6}$$

which by (26.5) is equivalent to Kirchhoff's[6] form (cf. $(33.3)_1$):

$$T_i^\alpha = \frac{\partial \Sigma}{\partial x^i{}_{,\alpha}}. \tag{39.7}$$

These results presuppose the medium susceptible of any continuous motion. For an incompressible material,[7] however, since only isochoric deformations ($I_d = 0$) are admissible, an arbitrary hydrostatic pressure $p\delta^i{}_j$ can do no internal work (§30) and hence may be added to any stress $t^i{}_j$ without affecting the

power series in k to exist: rather, the infinitesimal theory might be expected to have an elaborate asymptotic character with respect to the general non-linear theory. The subject is considered further by TOLOTTI [1943, **10**].

[3] [1863, **1**, §62]. Four forms of the differential equations for the stresses in an elastic body are given by SUGAMOTO [1948, **52**]. His equations (L), (E), (L'), (E') correspond, respectively, to stress-strain relations in terms of **C**, **c**, $\mathbf{C}^{-1}$, $\mathbf{c}^{-1}$, and thus (E) and (E') are valid only for isotropic media. The stress-strain relations corresponding to (L), (E), (L') are, respectively, (39.3), (41.2), and $(40.3)_2$.

[4] [1896, **1**, §24, eq. (59)]. Cf. [1903, **13**] [1910, **4**, §1, eq. (6)] [1930, **4**, §6] [1939, **7**, §7]. GREEN [1841, **2**] and ST. VENANT [1863, **2**, ¶2] gave $(39.4)_1$ incorrectly with $t^i{}_j$ on the left. Cf. §49[2]. If the material and spatial coordinate systems be made to coincide at the instant under consideration, $(39.4)_2$ reduces to the deceivingly simple form $\rho_0 t_{ij} = 2\rho\, \partial\Sigma/\partial g_{ji}$ obtained by OLDROYD [1950, **10**, §2].

[5] [1860, **1**, eq. (21)]. Cf. [1872, **1**, Note 3, p. 594] [1894, **1**, eq. (39.6)] [1896, **1**, §24, eq. (62)] [1948, **7**, §7].

[6] [1852, **1**, p. 772]. Cf. [1875, **1**, p. 190] [1891, **1**, §3] [1896, **1**, §24, eq. (60)] [1909, **1**, §54] [1925, **1**, §13].

[7] OLDROYD [1950, **10**, §6] regards incompressibility as incompatible with the notion of perfect elasticity, preferring to define an "almost incompressible material" and to approach deformations without changes of volume by means of a limiting process.

energy balance equation (36.2) from which stress-strain relations are derived.[8] Thus e.g. we obtain in place of (39.6) Poincaré's[9] equation

$$t^i_{\ j} = x^i_{\ ,\alpha} \frac{\partial \Sigma}{\partial x^j_{\ ,\alpha}} - p\delta^i_{\ j}. \tag{39.8}$$

40. Derivation of general stress-strain relations. II. Spatial forms. The stress-strain relations derived in §39 contain only the material displacement gradients $x^i_{\ ,\alpha}$. To obtain stress-strain relations in terms of the spatial gradients $X^\alpha_{\ ,i}$, note first by $(13.3)_1$ that

$$\frac{\partial \Sigma}{\partial x^i_{\ ,\alpha}} = \frac{\partial \Sigma}{\partial X^\beta_{\ ,k}} \frac{\partial X^\beta_{\ ,k}}{\partial x^i_{\ ,\alpha}} = -\frac{\partial \Sigma}{\partial X^\beta_{\ ,k}} X^\beta_{\ ,i} X^\alpha_{\ ,k}, \tag{40.1}$$

whence by (39.7) and $(26.5)_2$ we obtain Hamel's[1] spatial form

$$t^i_{\ j} = -\frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial X^\beta_{\ ,i}} X^\beta_{\ ,j}; \tag{40.2}$$

hence by $(14.9)_2$ follows Murnaghan's[2] form

$$t^{ij} = -\frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial (C^{-1})^{\alpha\gamma}} \frac{\partial (C^{-1})^{\alpha\gamma}}{\partial X^\beta_{\ ,j}} X^{\beta,i} = -2\frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial (C^{-1})^{\alpha\beta}} X^{\alpha,i} X^{\beta,j}. \tag{40.3}$$

If the displacement gradients be infinitesimal (§19), it follows at once from any of the above equations or any of those of §39 that the stress-strain relations may be approximated by[3]

$$t^i_{\ j} \approx \frac{\partial \Sigma}{\partial \tilde{e}^j_{\ i}}, \tag{40.4}$$

the form used in the infinitesimal theory.

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[8] Cf. [1945, **2**, p. A-263, footnote].

[9] The first result of this type was derived by LAGRANGE [1762, **2**, Ch. XL] [1788, **1**, Part II, Sect. 11, ¶2] for incompressible perfect fluids, where $I_d = 0$ is added as a constraint upon a variational principle, and p is simply the "Lagrangian multiplier". The same method was applied to the elastic body by POINCARÉ [1889, **5**, §152] [1892, **4**, §33], the constraint now being any equation of the form $F(x^i_{\ ,\alpha}) = 0$, which includes the condition of incompressibility $J = 1$ as a special case. Cf. [1948, **7**, §8].

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[1] [1912, **1**, §369].

[2] [1937, **1**, §6].

[3] Since the general stress-strain relations (39.7), which, as shown above, are easily transformed into the forms $(39.2)_2$ and $(40.3)_2$ derived by MURNAGHAN, have been in the literature for nearly 100 years, MURNAGHAN's repeated statements [1937, **1**, Introd.] [1941, **2**, p. 127] that the approximation (40.4) characterizes the "classical" theory is somewhat misleading. In dealing with elasticity in his lectures, HADAMARD [1903, **1**, Ch. VI] presented the fully general theory following BOUSSINESQ, and HILBERT [1907, **1a**, pp. 120–173] gave an equally general treatment in the style of KIRCHHOFF and GIBBS.

41. Stress-strain relations for isotropically elastic bodies. As shown in §38, in an elastically isotropic body we may regard Σ as a function of any one of the spatial tensors **c**, **e**, **h**, $\cdots$, instead of employing one of the material tensors **C**, **E**, **H**, $\cdots$ required for anisotropic bodies (§37). From (40.2) we have then

$$t^{ij} = -\frac{\rho}{\rho_0} X^{\gamma,j} \frac{\partial \Sigma}{\partial c_{kl}} \frac{\partial c_{kl}}{\partial X^{\gamma}{}_{,i}}, \tag{41.1}$$

whence by an evident analogue of $(14.9)_1$ and by $(14.5)_4$ follows

$$t^i{}_j = -2\frac{\rho}{\rho_0} c^i{}_k \frac{\partial \Sigma}{\partial c^j{}_k} = \frac{\rho}{\rho_0}(\delta^i{}_k - 2e^i{}_k)\frac{\partial \Sigma}{\partial e^j{}_k}. \tag{41.2}$$

By $(16.1)_4$ we have also[1]

$$t^i{}_j = \frac{\rho}{\rho_0}\frac{\partial \Sigma}{\partial h^j{}_i}. \tag{41.3}$$

All these stress-strain relations are Murnaghan's[2]; the last shows that $\rho_0 \mathbf{t}/\rho$ as an isotropic function of **h** admits Σ as a potential.

Since the validity of these relations is limited to the isotropic case, they should be reduced to a more explicit form. From (6.11) we have

$$\frac{\partial \Sigma}{\partial c^j{}_k} = \frac{\partial \Sigma}{\partial I_c}\delta^k{}_j + \frac{\partial \Sigma}{\partial II_c}(I_c\delta^k{}_j - c^k{}_j) + \frac{\partial \Sigma}{\partial III_c} III_c (c^{-1})^k{}_j, \tag{41.4}$$

whence by $(41.2)_1$ follows

$$\mathfrak{G}_0^{\mathrm{t(c)}} = -\frac{2\rho}{\rho_0} III_c \frac{\partial \Sigma}{\partial III_c}, \qquad \mathfrak{G}_1^{\mathrm{t(c)}} = -\frac{2\rho}{\rho_0}\left(\frac{\partial \Sigma}{\partial I_c} + I_c \frac{\partial \Sigma}{\partial II_c}\right),$$
$$\mathfrak{G}_2^{\mathrm{t(c)}} = \frac{2\rho}{\rho_0}\frac{\partial \Sigma}{\partial II_c}. \tag{41.5}$$

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[1] According to MURNAGHAN (see §17[1] for refs.), the stress-strain relations implied by BIOT are $\partial\Sigma/\partial k^{j}{}_{i} = (1 + III_e)t^i{}_j - e^i{}_k t^k{}_j$, which MURNAGHAN derives as an approximation to the exact relation $\rho_0 t^i{}_j = \rho k^i{}_l\, \partial\Sigma/\partial k^j{}_l$, which is equivalent to $(41.2)_1$ since $\mathbf{k}^2 = \mathbf{c}^{-1}$; cf. [1943, **8**, §§4–6] and §50. Generalizing GRÜBLER's [1900, **5**, eq. (3a)] empirical formula $t(A + B\delta) = \delta$ for the extension of sandstone, SCHLECHTWEG [1931, **8**, §1] proposes stress-strain relations which are not tensorially admissible for isotropic media, but which if corrected would become essentially a special case of those implied by BIOT. SCHLECHTWEG later formulates a substitute hypothesis (§51).

[2] [1937, **1**, §3] [1941, **1**, pp. 127–128]. While for general forms of Σ the stress tensor t^{ij} given by (41.2) is not symmetric, MURNAGHAN indicates, as indeed is immediate from (41.5) or (41.6), that for the isotropic case the special form $(38.2)_1$ for Σ yields $t^{ij} = t^{ji}$.

By (14.7) and the fact that $\mathbf{C}$ and $\mathbf{c}^{-1}$ have the same invariants follow the elegant stress-strain relations of Finger[3]:

$$t^i_j = \frac{2\rho}{\rho_0}\left[\left(II_{c^{-1}}\frac{\partial\Sigma}{\partial II_{c^{-1}}} + III_{c^{-1}}\frac{\partial\Sigma}{\partial III_{c^{-1}}}\right)\delta^i_j + \frac{\partial\Sigma}{\partial I_{c^{-1}}}(c^{-1})^i_j - III_{c^{-1}}\frac{\partial\Sigma}{\partial II_{c^{-1}}}c^i_j\right]. \tag{41.6}$$

These stress-strain relations are admirably suited to the solution of specific problems of large elastic strain. Their only disadvantage is that checking of the end results with those of the infinitesimal theory is not immediate, but must be carried out through the substitutions (19.1) and

$$\frac{\partial\Sigma}{\partial I_{c^{-1}}} \approx \mu_E + \frac{\lambda_E + 2\mu_E}{2} I_{\tilde{e}}, \qquad \frac{\partial\Sigma}{\partial II_{c^{-1}}} \approx -\frac{\mu_E}{2}, \qquad \frac{\partial\Sigma}{\partial III_{c^{-1}}} \approx 0. \tag{41.7}$$

The result (41.6) is interesting in itself because it shows that when $\mathbf{t}$ is regarded, as it may be, as a linear function of $\mathbf{c}$ and $\mathbf{c}^{-1}$, the dependence of Σ upon $III_{c^{-1}}$, representing change of volume, affects the stress only through its hydrostatic part. Further, (41.6) shows that if the natural state be unstressed we must have

$$\left(\frac{\partial\Sigma}{\partial I_{c^{-1}}}\right)_0 + 2\left(\frac{\partial\Sigma}{\partial II_{c^{-1}}}\right)_0 + \left(\frac{\partial\Sigma}{\partial III_{c^{-1}}}\right)_0 = 0, \tag{41.8}$$

where subscript naughts indicate evaluation at the natural state $\mathbf{c} = \mathbf{c}^{-1} = \mathbf{I}$, $I_{c^{-1}} = II_{c^{-1}} = 3$, $III_{c^{-1}} = 1$; this formula expresses a condition upon the functional form of Σ.

From (41.3) and (6.11) we obtain Richter's[4] formulae

$$\mathcal{G}_0^{t(h)} = \frac{\rho}{\rho_0}\frac{\partial\Sigma}{\partial I_h}, \qquad \mathcal{G}_1^{t(h)} = \frac{2\rho}{\rho_0}\frac{\partial\Sigma}{\partial \overline{II}_h}, \qquad \mathcal{G}_2^{t(h)} = \frac{3\rho}{\rho_0}\frac{\partial\Sigma}{\partial \overline{III}_h}. \tag{41.9}$$

Hence follows[5] $\rho_0\mathcal{G}_T^{t(e)} = \rho\,\partial\Sigma/\partial\Lambda_T$, where $\Lambda_0 \equiv I_h$, $2\Lambda_1 \equiv \Lambda_0 - I_e$, $4\Lambda_2 \equiv 2\Lambda_1 - I_e^2 + 2II_e$. Either from this result and (15.2), or from (41.3) and (16.1), with the aid of $(24.1)_1$ and (14.11) it is easy to deduce the elegant and important formula of Kötter and Almansi[6] for the principal stresses t_i in terms of the principal extensions δ_i:

$$t_i(1 + \delta_j)(1 + \delta_k) = \frac{\partial\Sigma}{\partial\delta_i}, \qquad (i, j, k \neq) \tag{41.10}$$

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[3] [1894, **2**, eq. (35)] [1948, **10**, §4]. Finger gave also another form [eq. (47)] valid only in a pure strain.

[4] [1948, **13**, §3] [1949, **24**]. Richter calculates also the $\mathcal{G}_T^{t(k)}$ (cf. Note 1).

[5] [1911, **4**, §§2–3] [1933, **1**, §4].

[6] [1910, **4**, §1] [1911, **2**, §7]. Cf. [1894, **1**, eq. (35)] [1894, **2**, eq. (39)].

which shows that just as in the infinitesimal theory, so also for finite strains the principal stresses in an elastically isotropic body are the derivatives of the strain energy with respect to the corresponding extensions, provided the stress be reckoned with respect to area in the *undeformed* body. Equivalently, in terms of the principal strains[7] E_i and e_i we have $[(1 + 2E_1)(1 + 2E_2)(1 + 2E_3)]^{\frac{1}{2}} t_i = (1 + 2E_i)\, \partial\Sigma/\partial E_i$, $t_i = (1 - 2e_i)[(1 - 2e_1)(1 - 2e_2)(1 - 2e_3)]^{\frac{1}{2}}\, \partial\Sigma/\partial e_i$.

From Kötter and Almansi's formula (41.10) follows an elegant form for the condition of existence of a strain energy:

$$\frac{\partial}{\partial \delta_i}[t_i(1 + \delta_j)] = \frac{\partial}{\partial \delta_i}[t_j(1 + \delta_i)],$$

or (41.11)

$$t_i - t_j = \frac{\partial t_j}{\partial \log (1 + \delta_i)} - \frac{\partial t_i}{\partial \log (1 + \delta_j)}.$$

An equivalent formulation in terms of the coefficients (41.9) is[8]

$$2\frac{\partial \mathcal{G}_0^{t(h)}}{\partial II_h} = \mathcal{G}_1^{t(h)} + \frac{\partial \mathcal{G}_1^{t(h)}}{\partial I_h}, \qquad 3\frac{\partial \mathcal{G}_1^{t(h)}}{\partial III_h} = 2\frac{\partial \mathcal{G}_2^{t(h)}}{\partial II_h},$$
$$3\frac{\partial \mathcal{G}_0^{t(h)}}{\partial III_h} = \mathcal{G}_2^{t(h)} + \frac{\partial \mathcal{G}_2^{t(h)}}{\partial I_h}. \tag{41.12}$$

To obtain stress-strain relations valid in incompressible isotropically elastic bodies we put $III_{c^{-1}} = 1$ in (41.6) and add an arbitrary hydrostatic pressure $p\delta^i{}_j$, thus deriving Rivlin's[9] formula

$$t^i{}_j = -p\delta^i{}_j + 2\frac{\partial \Sigma}{\partial I_{c^{-1}}}(c^{-1})^i{}_j - 2\frac{\partial \Sigma}{\partial II_{c^{-1}}} c^i{}_j. \tag{41.13}$$

Upon these elegantly simple stress-strain relations rests the main progress in actual solution of problems of large strain (§42).

The second law of thermodynamics implies that $d\Sigma > 0$ in any loading process. I shall not attempt to define loading, but I shall simply observe that according to any reasonable definition an incompressible material subject to biaxial positive extension will be said to be loaded if either principal extension is increased. That is, if $\delta_1 > 0$, $\delta_2 > 0$, then a process in which $d\delta_1 > 0$, $d\delta_2 = 0$ is a loading process. By forming $d\Sigma$ in terms of the $d\delta_i$ it is possible to obtain thus

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7 [1930, **8**, §8] [1933, **1**, §4] [1949, **15**, §2].

8 The corresponding conditions for the $\mathcal{G}_T^{t(e)}$ are given by GOLDENBLAT [1950, **1**], who uses them to show how the form of the $\mathcal{G}_T^{t(e)}$ as functions of I_e, II_e, III_e may be obtained if all six quantities be experimentally determined functions of two parameters and if one further relation connecting the six be known. As a specimen of his method he derives MURNAGHAN'S stress-strain relations (52.2) from assumed empirical formulae.

9 [1948, **10**, §4].

the two conditions

$$\frac{\partial \Sigma}{\partial I_{c^{-1}}} + (1 + \delta_i)^2 \frac{\partial \Sigma}{\partial II_{c^{-1}}} > 0, \qquad i = 1, 2, \quad (41.13a)$$

as necessary that $d\Sigma > 0$ under the above conditions. Since all possible values of $I_{c^{-1}}$ and $II_{c^{-1}}$ may be reached with $\delta_i \geq 0$, it follows that (41.13a) are valid in all deformations. Hence one may derive the former of the conditions[10]

$$\frac{\partial \Sigma}{\partial I_{c^{-1}}} \geq 0, \qquad \frac{\partial \Sigma}{\partial II_{c^{-1}}} \geq 0. \qquad (41.14)$$

The latter is obviously valid when $I_{c^{-1}} = II_{c^{-1}} = 3$; thus far I have been unable to derive it in general, and I offer it here merely as a conjecture. Rivlin & Saunders' experiments[11] on various rubbers subjected to various deformations with values of δ_i up to 1.8 and values of $I_{c^{-1}}$ and $II_{c^{-1}}$ up to 30 yield 1.5 kg/cm^2 $\leq \partial\Sigma/\partial I_{c^{-1}} \leq$ 2.0 kg/cm^2, .06 kg/cm^2 $\leq \partial\Sigma/\partial II_{c^{-1}} \leq$ .30 kg/cm^2.

A basic difference between the relations employing material tensors $\mathbf{C}, \mathbf{C}^{-1}, \cdots$ and those employing spatial tensors $\mathbf{c}, \mathbf{c}^{-1}, \cdots$ has been observed by Signorini.[12] The former do not completely specify the present stress $\mathbf{t}$ in terms of any strain measure: while they yield the principal stresses, to determine also the principal axes of stress we must know the $x^i{}_{,\alpha}$ as well. This difficulty is inherent in the treatment of anisotropic media. In the isotropic case, however, we may employ the formulae of the present section, which show that *the principal axes of stress coincide with the principal axes of the spatial strain measures*, and herein lies the great advantage of these latter.[13]

42. Rivlin's exact solutions of the general equations. Rivlin has obtained in several important cases, especially for incompressible bodies, exact solutions of the general equations for an isotropically elastic solid with *arbitrary strain energy function*. Not all the results I shall now present are his, those which are his are derived in a briefer and easier way, but it was Rivlin who first perceived and demonstrated the possibility of obtaining such solutions, thereby opening a new field of elasticity theory and achieving one of the major advances in classical mechanics in this century. Of course the method of solution is inverse: a class of deformations is assumed and reduced, then stresses necessary to produce it, if possible, are found. The problem is made non-trivial by the demand that $\mathbf{f} = 0$. Since the tensor $\mathbf{c}^{-1}$ is used throughout this section, for ease of writing we drop the subscripts from its invariants I, II, III.

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[10] RIVLIN [1951, **2**, §16] derives the weaker condition $\partial\Sigma/\partial I_{c^{-1}} + \partial\Sigma/\partial II_{c^{-1}} \geq 0$ when $I_{c^{-1}} = II_{c^{-1}}$.

[11] [1951, **2**].

[12] [1942, **3**, pp. 66–67].

[13] Cf. [1935, **1**, §1] [1937, **1**, Introd.].

A. *Homogeneous strain in general.*[1] Let both the x^i and the X^α be referred to the same rectangular Cartesian system, and consider a deformation which can be written in matrix form $\mathbf{x} = \mathbf{AX}$, where $\mathbf{A}$ is a constant matrix and $\det \mathbf{A} \neq 0$. Using a prime to denote transposition, by $(14.6)_2$ and $(14.5)_2$ we have $\mathbf{c}^{-1} = \mathbf{AA}'$, $\mathbf{c} = (\mathbf{A}')^{-1}\mathbf{A}^{-1}$. Finger's stress-strain relations (41.6) yield

$$\frac{\mathbf{t}}{2} = \det \mathbf{A}\left(\operatorname{trace}\,[(\mathbf{A}')^{-1}\mathbf{A}^{-1}]\,\frac{\partial\Sigma}{\partial II} + \frac{\partial\Sigma}{\partial III}\right)\mathbf{I} + \frac{1}{\det \mathbf{A}}\frac{\partial\Sigma}{\partial I}\,\mathbf{AA}' - \det \mathbf{A}\,\frac{\partial\Sigma}{\partial II}\,(\mathbf{A}')^{-1}\mathbf{A}^{-1}. \tag{42.1}$$

Since all matrices on the right are constants and since the partial derivatives of Σ also are constants, the equilibrium equations with $\mathbf{f} = 0$ are satisfied.

When $\det \mathbf{A} = 1$, the deformation is isochoric. If further the body be incompressible, by Rivlin's stress-strain relations (41.13) we obtain

$$\mathbf{t} = -p\mathbf{I} + 2\frac{\partial\Sigma}{\partial I}\,\mathbf{AA}' - 2\frac{\partial\Sigma}{\partial II}\,(\mathbf{A}')^{-1}\mathbf{A}^{-1}. \tag{42.2}$$

The equilibrium equations with $\mathbf{f} = 0$ are satisfied if and only if $p = \text{const}$.

While the class of exact solutions just presented is fairly general, its implications are easiest seen through special cases.

B. *Pure extension.* Consider first the class of deformations $x = f(X)$, $y = g(Y)$, $z = h(Z)$. From $(14.3)_2$ it follows that the co-ordinate axes are principal axes of $\mathbf{c}$, hence also of $\mathbf{t}$, and we have $III = (f'g'h')^2$. For the deformation to be isochoric it is then necessary and sufficient that $f'g'h' = 1$, a case included in part A.

For a pure but not necessarily isochoric extension the matrix $\mathbf{A}$ becomes diagonal, with entries $1 + \delta_i$. In this case (42.1) yields the following expressions for the principal stresses t_i:

$$\frac{t_i}{2} = (1+\delta_i)\left\{\frac{1}{(1+\delta_j)(1+\delta_k)}\left[\frac{\partial\Sigma}{\partial I} + ([1+\delta_j]^2 + [1+\delta_k]^2)\frac{\partial\Sigma}{\partial II}\right] + (1+\delta_j)(1+\delta_k)\frac{\partial\Sigma}{\partial III}\right\}, \qquad (i, j, k \neq) \tag{42.3}$$

while (42.2) yields

$$t_i = -p + 2(1+\delta_i)^2\frac{\partial\Sigma}{\partial I} - 2(1+\delta_1)^2\frac{\partial\Sigma}{\partial II}. \tag{42.4}$$

C. *Hydrostatic pressure.* Put $1 + \delta_i = K$, $\mathbf{t} = -p\mathbf{I}$. Then (42.3) reduces to

$$-\frac{1}{2}p = \frac{1}{K}\frac{\partial\Sigma}{\partial I} + 2K\frac{\partial\Sigma}{\partial II} + K^3\frac{\partial\Sigma}{\partial III}. \tag{42.5}$$

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[1] [1948, **10**, §6].

No definite conclusion can be drawn from this result, except that since $K = (\rho/\rho_0)^{-\frac{1}{3}}$ it follows that (42.5) is an equation of state $p = p(\rho)$, and *any* such equation of state is compatible with the general theory of elasticity in the case of a body subject to hydrostatic pressure.

D. *Simple extension.* Put $\delta_2 = \delta_3$, $t_2 = t_3 = 0$ in (42.3), thus obtaining

$$0 = \frac{1}{(1 + \delta_1)(1 + \delta_2)} \frac{\partial \Sigma}{\partial I} + \left(\frac{1 + \delta_1}{1 + \delta_2} + \frac{1 + \delta_2}{1 + \delta_1}\right) \frac{\partial \Sigma}{\partial II} + (1 + \delta_1)(1 + \delta_2) \frac{\partial \Sigma}{\partial III}, \tag{42.6}$$

$$t_1 = (1 + \delta_1)\left\{\frac{1}{(1 + \delta_2)^2} \frac{\partial \Sigma}{\partial I} + 2 \frac{\partial \Sigma}{\partial II} + (1 + \delta_2)^2 \frac{\partial \Sigma}{\partial III}\right\}.$$

Since the three derivatives of Σ are in general functions of both δ_1 and δ_2, $(42.6)_1$ is a transcendental equation for δ_2 as a function of δ_1. Thus the amount of contraction $-\delta_2$ depends upon the amount of extension δ_1. Substitution of the approximations (19.1) and (41.7) valid in the infinitesimal theory yields the classical result

$$-\frac{\delta_2}{\delta_1} = \nu_E, \qquad \nu_E \equiv \frac{\lambda_E}{2(\lambda_E + \mu_E)}. \tag{42.7}$$

We shall call ν_E the *Poisson modulus* to distinguish it from the *Poisson ratio* $-\delta_2/\delta_1$, which it equals only in special circumstances. In general, since $(42.6)_1$ is an equation with essentially arbitrary coefficients, there is no indication of either existence or uniqueness of a real solution; that is, for some types of elastic body *it may be impossible to produce simple extension of a given amount by merely extensile loading, but suitable forces normal to the axis of extension must be supplied,* while for another type of body it may be possible that *more than one contraction ratio can correspond to a given extensile load, so that the equilibrium configuration actually assumed depends on the manner in which the load is applied.* This result raises also the question of the stability of the several possible contractions, when they exist. Once δ_2 has been obtained from $(42.6)_1$, it is to be put into $(42.6)_2$ to give t_1 as a function of δ_1. Substitution of the approximations (19.1) and (42.7) valid in the infinitesimal theory yields the classical result

$$\frac{t_1}{\delta_1} = E, \qquad E \equiv \frac{\mu_E(3\lambda_E + 2\mu_E)}{2(\lambda_E + \mu_E)}, \tag{42.8}$$

E being called *Young's modulus.*

For an isochoric simple extension the Poisson ratio is determined from the kinematical condition $(1 + \delta_1)(1 + \delta_2)^2 = 1$, or

$$\delta_2 = (1 + \delta_1)^{-\frac{1}{2}} - 1. \tag{42.9}$$

For an incompressible body in simple extension we may use (42.9) to obtain from (42.4) with $i = 2$

$$0 = -p + \frac{2}{1 + \delta_1} \frac{\partial \Sigma}{\partial I} - 2(1 + \delta_1) \frac{\partial \Sigma}{\partial II}, \tag{42.10}$$

whence, eliminating p from (42.4) with $i = 1$, we obtain for the stress $t_1(1 + \delta_2)^2$ referred to the initial area simply

$$t_1(1 + \delta_2)^2 = 2\left[(1 + \delta_1) - \frac{1}{(1 + \delta_1)^2}\right]\left[\frac{\partial \Sigma}{\partial I} + \frac{1}{1 + \delta_1} \frac{\partial \Sigma}{\partial II}\right]. \tag{42.11}$$

Since δ_2 may be eliminated by (42.9) wherever it occurs in the second factor on the right hand side of (42.11), it follows that t_1 is a function of δ_1 only. Hence *in an* incompressible *elastic body, a given simple extension of any magnitude can always be produced by a purely extensile load,* the magnitude of which is given by (42.11).

This example points the basic simplification which follows if an elastic body be incompressible.

1. *Of a whole class of deformations possible in a compressible body, only the simplest cases can occur at all in an incompressible body.* 2. *To produce these simple deformations in an incompressible body it is possible to leave free certain boundary surfaces which in a compressible body would have to be loaded in a special way.* 3. *The necessary stresses are expressed more simply and in terms of fewer parameters than for the corresponding problem in a compressible body.*

These same conclusions will be illustrated in the succeeding examples.

E. *Plane biaxial stress in an infinite slab.* Put $t_2 = 0$ in (42.3), but suppose δ_1 and δ_3 given and not necessarily equal. We have again $(42.6)_1$, which now becomes an implicit equation for δ_2 as a function of δ_1 and δ_3. In place of $(42.6)_2$, however, we have simply (42.3) with $i = 1, 3$, the only simplification being that δ_2 is to be eliminated by $(42.6)_1$.

For an incompressible body the problem becomes considerably simpler.[2] We have $1 + \delta_2 = (1 + \delta_1)^{-1}(1 + \delta_3)^{-1}$, and from (42.4) follows then

$$0 = -\tfrac{1}{2}p + \frac{1}{(1 + \delta_1)^2(1 + \delta_3)^2} \frac{\partial \Sigma}{\partial I} - (1 + \delta_1)^2(1 + \delta_3)^2 \frac{\partial \Sigma}{\partial II}, \tag{42.12}$$

whence

$$\tfrac{1}{2}t_i = \left[(1 + \delta_i)^2 - \frac{1}{(1 + \delta_i)^2(1 + \delta_j)^2}\right]\left[\frac{\partial \Sigma}{\partial I} + (1 + \delta_j)^2 \frac{\partial \Sigma}{\partial II}\right]. \qquad i \neq j, \quad i, j = 1, 3. \tag{42.13}$$

[2] [1951, **2**, §2].

Thus *in an incompressible slab a biaxial extension of any magnitude can always be produced by purely extensile loads*, the magnitudes of which are given by (42.13).

F. *Simple shear.* For this special case we have

$$\mathbf{A} = \begin{Vmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{42.14}$$

Hence (42.1) reduces to[3]

$$\begin{aligned} \tfrac{1}{2}\mathbf{t} = \left[\frac{\partial\Sigma}{\partial I} + (2 + K^2)\frac{\partial\Sigma}{\partial II} + \frac{\partial\Sigma}{\partial III}\right]\mathbf{I} + K\left(\frac{\partial\Sigma}{\partial I} + \frac{\partial\Sigma}{\partial II}\right)\begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \\ + K^2\frac{\partial\Sigma}{\partial I}\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} - K^2\frac{\partial\Sigma}{\partial II}\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \end{aligned} \tag{42.15}$$

This result gives information about the form of the strain energy function: if $t^x{}_x = t^z{}_z$, then $\partial\Sigma/\partial I = 0$; if $t^y{}_y = t^z{}_z$, then $\partial\Sigma/\partial II = 0$; if $t^x{}_x = t^y{}_y$, then $\partial\Sigma/\partial I = \partial\Sigma/\partial II$.

To calculate the tangential and normal tractions T and N acting upon the X = const. faces, we first observe that $\pm T = -t_x n_y + t_y n_y$, $\pm N = t_x n_x + t_y n_y$, where t_x, t_y are the components of the corresponding stress vector and n_x, n_y are direction cosines of the normal; but t_x, t_y follow from $(26.1)_1$, while $n_x = (1 + K^2)^{-\frac{1}{2}}$, $n_y = -K(1 + K^2)^{-\frac{1}{2}}$. Thus

$$\begin{aligned} \pm(1 + K^2)T &= K(t^x{}_x - t^y{}_y) + (1 - K^2)t^x{}_y, \\ \pm(1 + K^2)N &= t^x{}_x - 2Kt^x{}_y + K^2 t^y{}_y, \end{aligned} \tag{42.16}$$

whence follows

$$\begin{aligned} \pm(1 + K^2)T &= 2K\left(\frac{\partial\Sigma}{\partial I} + \frac{\partial\Sigma}{\partial II}\right) = t^x{}_y, \\ \pm(1 + K^2)N &= 2\left[\frac{\partial\Sigma}{\partial I} + (2 + K^2)\frac{\partial\Sigma}{\partial II} + (1 + K^2)\frac{\partial\Sigma}{\partial III}\right]. \end{aligned} \tag{42.17}$$

The tangential and normal tractions which act upon the Y = const. and Z = const. faces may be read off from (42.15). By (41.8) it is plain that $t^z{}_z$, $t^y{}_y$, and N are ultimately proportional to K^2 for small values of K.

From the foregoing results we see the striking differences between the in-

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[3] [1948, **10**, §13].

finitesimal theory of elasticity and the general theory. *In order to shear a block, shearing forces alone are insufficient; in addition, suitable normal forces, in first approximation proportional to the square of the shear angle, must be supplied to* all *the plane faces.* These normal forces may be separated into two portions. First, there is a *hydrostatic tension of amount* $t^z{}_z$, *ultimately proportional to* K^2. If this be not supplied, *the body will tend to change in volume; this is the* **Kelvin effect** *for elastic bodies* (cf. §10). Whether the change is an increase or decrease[4] depends upon the signs and magnitudes of the three partial derivatives of Σ. Second, on the faces normal to the plane of shearing there must be *additional normal tractions, ultimately proportional to* K^2; whether these be tensions or pressures again depends upon the signs of the coefficients. If these be not supplied, *the proportions of the body will tend to change; this is the* **Poynting effect** *for elastic bodies* (cf. §10). The tangential traction which must act on X = const. is $(1 + K^2)^{-1}$ times that acting on Y = const., the ratio growing smaller the more severe the shear. For a material obeying linearized stress-strain relations, but subjected to a displacement whose gradients are large (§49), the change of volume was predicted theoretically by Kelvin,[5] both effects by Poynting;[6] the latter demonstrated their physical existence in a series of classical experiments.[7] For general stress-strain relations, however, interpretation of the results of §10 shows that these same effects occur also in a shearing *strain*, when the strain *measure* reduces to the simple form (10.2) with $s = 0$ (cf. §45), and it is this more fundamental observation upon which rests the general terminology *Kelvin effect, Poynting effect,* used throughout this paper. The existence of these two phenomena point sharply the *difference of kind* between the infinitesimal theory and the general theory of elasticity, since according to the infinitesimal theory a block may be sheared by *equal* and *purely tangential* forces acting on the faces normal to the plane of shearing, the other faces being left free.

For simple shear of an incompressible[8] body (42.2) yields a result equivalent to that obtained by replacing $[\cdots]\,\mathbf{I}$ in (42.15) by $-p\mathbf{I}/2$. Hence by choice of p *any one pair of plane faces may be left free of normal tractions,* provided suitable loads be applied to the other faces. For example, the faces parallel to the plane of shearing may be left free by the choice $p = 0$. Then for the

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[4] The experiments of Freeman & Weissenberg [1948, **57**] show that when purely shearing forces are applied to a block of sponge rubber all dimensions *decrease*; they observe reductions in volume up to 40%. It is not certain, however, that sponge rubber can rightly be regarded as a homogeneous isotropic material.

[5] [1883, **1**, §679] [1890 reprint of [1877, **1**], §44 footnote].

[6] [1905, **2**, p. 338] [1909, **2**, §1] [1912, **4**, pp. 415–418].

[7] From his analysis of shearing (Note 6) he concluded that a wire when twisted would generally change in volume, in diameter, and in length, and then measured these phenomena in experiments on metal wires [1909, **2**, §II] [1912, **4**, pp. 397–412]. Then he separated the cross-stress effect from the volume effect by similar experiments on rubber cords [1913, **4**], where the change in volume, if any, was negligible, but the change in length more noteworthy.

[8] [1948, **10**, §12].

tractions on the X = const. faces by (42.16) we obtain $(42.17)_1$ and

$$\pm(1 + K^2)N = -2K^2\left[\frac{\partial\Sigma}{\partial I} + (2 + K^2)\frac{\partial\Sigma}{\partial II}\right]. \tag{42.18}$$

By (41.14) the quantity in brackets must be non-negative, so that N is always a *pressure*; from (42.15) it is plain that the normal force acting upon Y = const. must also be a pressure. If these pressures be not supplied, the specimen will tend to *broaden*, and hence also, since it is incompressible, to *shorten*. If a different pair of faces be left free of normal forces, a different stress system and different conclusions will result. The foregoing remarks illustrate the Poynting effect for incompressible elastic bodies; the Kelvin effect, of course, is not present, unless we please to apply this name to the formal occurrence of the arbitrary hydrostatic pressure p.

G. *Biaxial shear between parallel invariable plates.* When a block is sheared both in the x-y and the y-z planes with the planes y = const. remaining parallel to themselves and equidistant during the shear, we have

$$\mathbf{A} = \begin{Vmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & L & 1 \end{Vmatrix}. \tag{42.19}$$

Hence (42.1) yields

$$\begin{aligned} \tfrac{1}{2}\mathbf{t} = &\left[\frac{\partial\Sigma}{\partial I} + (2 + K^2 + L^2)\frac{\partial\Sigma}{\partial II} + \frac{\partial\Sigma}{\partial III}\right]\mathbf{I} \\ &+ K\frac{\partial\Sigma}{\partial I}\begin{Vmatrix} K & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + K\frac{\partial\Sigma}{\partial II}\begin{Vmatrix} 0 & 1 & 0 \\ 1 & -K & 0 \\ 0 & 0 & 0 \end{Vmatrix} + L\frac{\partial\Sigma}{\partial I}\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & L \end{Vmatrix} \\ &+ L\frac{\partial\Sigma}{\partial II}\begin{Vmatrix} 0 & 0 & 0 \\ 0 & -L & 1 \\ 0 & 1 & 0 \end{Vmatrix} + KL\frac{\partial\Sigma}{\partial I}\begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{Vmatrix}. \end{aligned} \tag{42.20}$$

Comparison with (42.15) shows that all of the stress system except for the last term occurring may be regarded as of the same *type* as would be obtained from superposing the stress systems corresponding to the two separate shears. The numerical values are not in general those obtained by superposition, since the partial derivatives depend upon both K and L; in fact, $I = 3 + K^2 + L^2 = II$, $III = 1$. A new effect, however, is revealed by the last term, an *interaction shear stress* whose plane is that of the two axes of shearing and whose magnitude for small shears is proportional to the product of the two shear angles. This shear stress may be regarded as arising from the repeated extension and tilting of the fibres initially normal to the shear planes y = const. The fact that I, II, and III depend upon the amount of the shear only through the quantity $K^2 + L^2$ makes it possible to express the result in a striking form in the case when all powers of K and L higher than the second are negligible. To this approximation we have the following *principle of superposition of large shears*: in order to produce a combined shear about two perpendicular axes, to the surface forces required to produce each shear separately must be added a shearing stress of magnitude $2KL\partial\Sigma/\partial I$ acting in the plane of the two axes. This secondary effect, of the same order of magnitude as the Kelvin and Poynting effects, is here observed for the first time.

For incompressible bodies $[\cdots]\mathbf{I}$ in (42.20) is to be replaced by $-p\mathbf{I}/2$. By choice of p any pair of parallel faces may be rendered free of normal traction. The same general conclusions hold as for compressible bodies.

H. *Torsion of a circular cylinder.*[9] Let both the x^i and the X^α be referred to a cylindrical co-ordinate system; for the former, write r, θ, z, while for the latter, write R, Θ, Z. Assume $r = R, \theta = \Theta + KZ, z = Z$, where K = const. = twist/unit length. From $(14.6)_2$ follows

$$\| (c^{-1})^i{}_j \| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 + r^2K & K \\ 0 & r^2K & 1 \end{Vmatrix}, \tag{42.21}$$

and hence $I = II = 3 + K^2r^2$, $III = 1$. We treat only the case of an incompressible material. Then from Rivlin's stress-strain relations (41.13) we obtain the stress system

$$\begin{gathered} t^\theta{}_\theta = t^r{}_r + 2r^2K^2 \frac{\partial\Sigma}{\partial I}, \qquad t^z{}_z = t^r{}_r - 2r^2K^2 \frac{\partial\Sigma}{\partial II}, \\ t^\theta{}_z = 2K\left(\frac{\partial\Sigma}{\partial I} + \frac{\partial\Sigma}{\partial II}\right), \qquad t^r{}_z = t^r{}_\theta = 0. \end{gathered} \tag{42.22}$$

The equilibrium equations with $\mathbf{f} = 0$ in the present instance reduce to

$$0 = \frac{\partial t^r{}_r}{\partial r} - 2rK^2 \frac{\partial\Sigma}{\partial I}, \qquad 0 = \frac{\partial t^r{}_r}{\partial\theta}, \qquad 0 = \frac{\partial t^r{}_r}{\partial z}. \tag{42.23}$$

Hence all stress components are functions of r only, and if the cylindrical mantle $r = a$ is to be free of traction we must have

$$t^r{}_r = 2K^2 \int_a^r r \frac{\partial\Sigma}{\partial I}\, dr. \tag{42.24}$$

[9] [1948, **10**, §14] [1949, **19**]. I am indebted to Mr. Rivlin for use of the latter paper in MS. Cf. also [1951, **11**, §8]. A shorter derivation is given by Green & Shield [1950, **11**, §5].

The stress system is now completely determined. From $(42.22)_3$ it is plain that as in the linear theory the twisting shear stress $\widehat{\theta z} = rt^{\theta}{}_z$ is proportional to the twist K for small twists, but the *Poynting effect appears in the form of the normal stress* $t^z{}_z$, *for small twists proportional to* K^2, *which must be supplied upon the plane ends of the cylinder in order to maintain the mantle free of traction.* The resultant moment M and normal traction N are easily calculated and simplified by some integrations by parts:

$$M = \int_0^a r\widehat{\theta z}\cdot 2\pi r\,dr = \frac{2\pi}{K}\left[a^2\Sigma\,|_{r=a} - \int_0^{a^2}\Sigma\,d\xi\right],$$
$$N = \int_0^a t^z{}_z\cdot 2\pi r\,dr = -\pi K^2\int_0^{a^2}\xi\left(\frac{\partial\Sigma}{\partial I} + 2\frac{\partial\Sigma}{\partial II}\right)d\xi, \qquad (42.25)$$

where $\xi \equiv r^2$. By (41.14) it follows that *the normal force to be supplied on the ends is always a pressure; if this force be lacking, the cylinder will tend to* elongate *when twisted.* This result embodies the Poynting effect in elasticity in its most striking form.

I. *Bending a block*[10]. Let the x^i be referred to a cylindrical system r, θ, z, while the X^α are rectangular Cartesian co-ordinates[11] X, Y, Z. Consider a deformation $r = f(X)$, $\theta = g(Y)$, $z = h(Z)$. For the co-ordinate systems selected $(14.6)_2$ reduces to $(c^{-1})^i{}_j = g_{jj}x^i{}_{,\alpha}x^j{}_{,\alpha}$ with α summed and j unsummed. Hence the spatial co-ordinate axes are the principal spatial axes of strain and stress, and $(c^{-1})^1{}_1 = f'^2$, $(c^{-1})^2{}_2 = f^2g'^2$, $(c^{-1})^3{}_3 = h'^2$. Consequently $III = (ff'g'h')^2$, so that if the deformation is to be isochoric we must have $ff' = A$, $g' = C$, $h' = (AC)^{-1} \equiv D$, where A, C, D are constants. Thus $r = (2Ax + B)^{\frac{1}{2}}$, $\theta = CY$, $z = DZ$, where two constants of integration have been set equal to zero to center the deformed block with respect to the spatial co-ordinates. Hence

$$\|(c^{-1})^i{}_j\| = \left\|\begin{matrix}\frac{A^2}{r^2} & 0 & 0\\ 0 & C^2r^2 & 0\\ 0 & 0 & D^2\end{matrix}\right\|, \qquad \|c^i{}_j\| = \left\|\begin{matrix}\frac{r^2}{A^2} & 0 & 0\\ 0 & \frac{1}{C^2r^2} & 0\\ 0 & 0 & \frac{1}{D^2}\end{matrix}\right\|, \qquad (42.26)$$

$$I = \frac{A^2}{r^2} + C^2r^2 + D^2, \qquad II = \frac{r^2}{A^2} + \frac{1}{C^2r^2} + \frac{1}{D^2},$$

and consequently $\Sigma = \Sigma(r)$. If $X = -a$ becomes $r = r_1$, $X = +a$ becomes $r = r_2$, and $Y = \pm b$ becomes $\theta = \pm\theta_0$, then we have $4Aa = r_2^2 - r_1^2$, $2B = r_1^2 + r_2^2$, $bC = \theta_0$, $\theta_0 D(r_2^2 - r_1^2) = 4ab$.

[10] [1949, **5**] [1949, **20**, §§14–16]. I am indebted to Mr. RIVLIN for use of the latter paper in MS.

[11] Here we see the great simplicity which results from MURNAGHAN's scheme of independently selected material and spatial co-ordinate systems.

For an incompressible material Rivlin's stress-strain relations (41.13) yield

$$
\begin{aligned}
t^r_{\ r} &= -p + 2\frac{A^2}{r^2}\frac{\partial\Sigma}{\partial I} - 2\frac{r^2}{A^2}\frac{\partial\Sigma}{\partial II},\\
t^\theta_{\ \theta} &= -p + 2C^2r^2\frac{\partial\Sigma}{\partial I} - \frac{2}{C^2r^2}\frac{\partial\Sigma}{\partial II},\\
t^z_{\ z} &= -p + 2D^2\frac{\partial\Sigma}{\partial I} - \frac{2}{D^2}\frac{\partial\Sigma}{\partial II}.
\end{aligned}
\tag{42.27}
$$

The equilibrium equations with $\mathbf{f} = 0$ reduce to

$$
\frac{\partial t^r_{\ r}}{\partial r} + \frac{t^r_{\ r} - t^\theta_{\ \theta}}{r} = 0, \qquad \frac{\partial t^\theta_{\ \theta}}{\partial \theta} = 0, \qquad \frac{\partial t^z_{\ z}}{\partial z} = 0. \tag{42.28}
$$

Hence $p = p(r)$; but since $\Sigma = \Sigma(r) = \Sigma(I, II)$, from $(42.21)_3$ and $(42.26)_4$ it is possible to reduce $(42.28)_1$ to the form $(\partial/\partial r)(t^r_{\ r} - \Sigma) = 0$. Hence (42.27) becomes finally

$$
\begin{aligned}
t^r_{\ r} &= \Sigma + K,\\
t^\theta_{\ \theta} &= 2\left(C^2r^2 - \frac{A^2}{r^2}\right)\left(\frac{\partial\Sigma}{\partial I} + D^2\frac{\partial\Sigma}{\partial II}\right) + \Sigma + K = r\frac{\partial\Sigma}{\partial r} + \Sigma + K,\\
t^z_{\ z} &= 2\left(D^2 - \frac{A^2}{r^2}\right)\left(\frac{\partial\Sigma}{\partial I} + r^2C^2\frac{\partial\Sigma}{\partial II}\right) + \Sigma + K,
\end{aligned}
\tag{42.29}
$$

where K is an arbitrary constant. By $(42.29)_3$ the normal force F per unit height on the plane ends is

$$
F = [r(\Sigma + K)]_{r_1}^{r_2}. \tag{42.30}
$$

If the bending is to be produced by terminal couples, we must have $\Sigma\,|_{r=r_1} = \Sigma\,|_{r=r_2} = -K$. By $(42.29)_1$, this same condition renders the curved surfaces $r = r_1$ and $r = r_2$ free of traction. Since $\Sigma = \Sigma(I, II)$, this amounts to a demand that $I\,|_{r=r_1} = I\,|_{r=r_2}$, $II\,|_{r=r_1} = II\,|_{r=r_2}$. By $(42.21)_3$ and $(42.21)_4$, this requirement is met if and only if $A = Cr_1r_2$, or, equivalently, $A^2 = r_1r_2/D$. Only two arbitrary constants, e.g. D and r_1, remain to specify the deformation uniquely. The couple per unit height required is

$$
M = \int_{r_1}^{r_2} rt^\theta_{\ \theta}\,dr = \tfrac{1}{2}(r_1^2 - r_2^2)K - \int_{r_1}^{r_2} r\Sigma\,dr. \tag{42.31}
$$

The neutral fibre for the entire deformation is obtained by solving $c^\theta_{\ \theta} = 1$, which yields $C^2r_0^2 = 1$, or $r_0 = (Dr_1r_2)^{\frac{1}{2}}$. The neutral fibre for the bending alone is obtained by putting $D = 1$, viz. $r_0 = (r_1r_2)^{\frac{1}{2}}$, just as in the infinitesimal theory. The normal stress $(42.29)_4$ shows that *in order to produce pure bending of an incompressible block with curved faces free of traction, suitable normal tractions on* *the faces parallel to the plane of bending must be supplied.* This phenomenon is the Poynting effect for bending.

J. *Other cases.* Another case solved by Rivlin[12] is that of a cylindrical tube extended, inflated, and twisted. In this deformation R, Θ, Z goes into $[(R^2 + K)/k]^{\frac{1}{2}}$, $\Theta + kDZ$, DZ, where K, k, and D are constants. Among the results is the universal formula, valid for any type of strain energy,

$$\frac{Na^2}{\left(\frac{M}{k}\right)_{k=0}} = 2\left(D - \frac{1}{D^2}\right), \tag{42.32}$$

for the normal fórce N required to produce the extension D in a solid rod of radius a. This formula gives a general solution, as far as incompressible materials are concerned, to the problem of determining the increase in torsional rigidity of a cylindrical rod subject to large extension, once the relation between N and D be known. Rivlin[13] has also made some progress toward determining the stresses produced when a cylindrical tube is extended, inflated, sheared about its axis, and sheared along its axis. In this deformation R, Θ, Z goes into $[(R^2 + K)/D]^{\frac{1}{2}}$, $\Theta + E$, $DZ + F$, where K and D are constants, $E = E(R)$, and $F = F(R)$. The two problems just mentioned correspond to the two natural ways of forcing a rubber tube over a glass one. Rivlin & Thomas[14] have outlined a method for solving numerically the problem of determining the strains in a sheet pierced by a hole and subjected to extension.

This class of problems has been taken up by Green & Shield.[15] First they find the stresses induced in a circular cylinder rotating about its axis. Next they solve the problem of a spherical shell subject to internal pressure p_1 and external pressure p_2, obtaining

$$p_2 - p_1 = -4\int_{R_1}^{R_2}\left(\frac{\partial\Sigma}{\partial I} + \frac{1}{Q^2}\frac{\partial\Sigma}{\partial II}\right)\frac{1 - Q^2}{R}\,dR, \tag{42.33}$$

where $Q \equiv r/R$, r being the initial and R the final radial distance of a typical particle.

In his latest work[16] Rivlin turns away from any attempt to determine or conjecture the form of the strain energy Σ. Instead, he compares with experimental data the exact solutions explained in this section. The quantities $\partial\Sigma/\partial I$ and $\partial\Sigma/\partial II$ are determined experimentally as functions of I and II in various circumstances. Consistency between the results of various experiments is a check of the excellent agreement between theory and experiment, while the tabulated values $\partial\Sigma/\partial I$ and $\partial\Sigma/\partial II$ give all that is required to obtain specific numerical predictions from any theoretical result.

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[12] [1949, **20**, §§3–8].
[13] [1949, **20**, §§9–13].
[14] [1951, **3**].
[15] [1950, **11**, §§6–7].
[16] [1951, **2–3**].

Added in proof. In an important paper just issued, Green & Shield[17] solve the torsion problem in a quite general way. Using the classical torsion function, they give the general solution for a small twist k of a cylinder of arbitrary cross-section and arbitrary strain energy subjected to extension of any amount. For an incompressible material they obtain in full generality the formula (42.32), derived by Rivlin for the circular cylinder. A similar result holds for compressible materials. They find also the general solution for small twist of a cylinder subject to large hydrostatic pressure.

43. Power series for the elastic energy. While Rivlin's work shows that in some cases (§42) solutions may be obtained without specialization of the form of Σ, the older authors, beginning with Green[1], often used a power series expansion. Assuming Σ to be analytic[2], for isotropic bodies we may write[3]

$$\begin{aligned}\Sigma &= \bar{\alpha} I_{\mathrm{E}} + \frac{\bar{\lambda}_{\mathrm{E}} + 2\bar{\mu}_{\mathrm{E}}}{2} I_{\mathrm{E}}^2 - 2\bar{\mu}_{\mathrm{E}} II_{\mathrm{E}} + \bar{l} I_{\mathrm{E}}^3 + \bar{m} I_{\mathrm{E}} II_{\mathrm{E}} + \bar{n} III_{\mathrm{E}} + \cdots, \\ &= \alpha I_{\mathrm{e}} + \frac{\lambda_{\mathrm{E}} + 2\mu_{\mathrm{E}}}{2} I_{\mathrm{e}}^2 - 2\mu_{\mathrm{E}} II_{\mathrm{e}} + l I_{\mathrm{e}}^3 + m I_{\mathrm{e}} II_{\mathrm{e}} + n III_{\mathrm{e}} + \cdots .\end{aligned} \tag{43.1}$$

Since by (14.7) and (15.3) either set of invariants can be expressed in terms of the other, it is easy to show that $\alpha = \bar{\alpha}$, $\lambda_{\mathrm{E}} = \bar{\lambda}_{\mathrm{E}}$, $\mu_{\mathrm{E}} = \bar{\mu}_{\mathrm{E}} + 2\bar{\alpha}$, $l = \bar{l} + 4\bar{\alpha}$, $m = \bar{m} - 4\bar{\lambda}_{\mathrm{E}} - 12\,\bar{\mu}_{\mathrm{E}} - 12\bar{\alpha}$, $n = \bar{n} + 12\bar{\mu}_{\mathrm{E}} + 12\bar{\alpha}$, $\cdots$. Putting $(43.1)_1$ into $(39.4)_1$, with the aid of (6.11) we obtain

$$\begin{aligned}T^{\alpha}{}_{\beta} = {} & [\bar{\alpha} + \bar{\lambda}_{\mathrm{E}} I_{\mathrm{E}} + (3\bar{l} + \bar{m}) I_{\mathrm{E}}^2 + (\bar{m} + \bar{n}) II_{\mathrm{E}} + \cdots]\,\delta^{\alpha}{}_{\beta} \\ & + [2\bar{\mu}_{\mathrm{E}} - (\bar{m} + \bar{n}) I_{\mathrm{E}} + \cdots] E^{\alpha}{}_{\beta} + [\bar{n} + \cdots] E^{\alpha}{}_{\gamma} E^{\gamma}{}_{\beta},\end{aligned} \tag{43.2}$$

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[17] [1951, **15**, §§ 3–4].

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[1] [1839, **1**, p. 249] [1841, **2**, p. 298]. References to second order theories based on (43.1) are given in §49[10,12,14,16].

[2] For sufficiently small strains the validity of the HOOKE-CAUCHY linear expression follows indeed from (43.1), but this mathematical fact can in no way replace experimental test of the linear relationship, since the analyticity of Σ is an *assumption* which is much more difficult to test experimentally than is the linear relation. Claiming to find experimental deviation from the linear relation for very small simple extensions, certain authors (e.g. [1902, **7**, §5, ¶3]) have replaced it by various other formulae, e.g. $T = Ee^m$, $m \neq 1$. A history and bibliography of such work from 1729 to 1893 is given by MEHMKE [1897, **1**, §II]. ICHINOSE [1941, **13**] after careful measurements concludes that no such deviations exist for steel, brass, and copper. He suggests that the contrary conclusion of some earlier experimenters may be explained by their failure to separate the effect of secondary deformations.

[3] The coefficients l, m, n are MURNAGHAN's [1937, **1**, p. 250]. The coefficients $\bar{l}$, $\bar{m}$, $\bar{n}$ are related to the A, B, C of L. BRILLOUIN [1938, **2**, eq. (X. 79)] by $\bar{l} = A + B + C$, $\bar{m} = -2A - 3C$, $\bar{n} = 3C$.

while putting $(43.1)_2$ into $(41.2)_2$, with the aid of (6.11), $(24.1)_1$, and $(15.3)_5$ we obtain[4]

$$
\begin{aligned}
t^i{}_j &= \sqrt{1 - 2I_e + 4II_e - 8III_e}\{[\alpha + \lambda_E I_e + (3l + m)I_e^2 + (m + n)II_e + \cdots]\delta^i{}_j \\
&\quad + [2(\mu_E - \alpha) - (m + n + \lambda_E)I_e + \cdots]e^i{}_j + [-4\mu_E + n + \cdots]e^i{}_k e^k{}_j\}, \\
&= [\alpha + (\lambda_E - \alpha)I_e + (3l + m - \lambda_E - \tfrac{1}{2}\alpha)I_e^2 + (m + n - 2\alpha)II_e + \cdots]\delta^i{}_j \\
&\quad + [2(\mu_E - \alpha) - (m + n + 2\lambda_E + 2\mu_E - 2\alpha)I_e + \cdots]e^i{}_j \qquad (43.3) \\
&\quad + [-4\mu_E + n + \cdots]e^i{}_k e^k{}_j .
\end{aligned}
$$

These expansions, carried as far as terms of the second order in the strain components, reveal the complicated way in which the coefficients in the strain-energy expansion enter the stress-strain relations, particularly in the spatial formulation. In the case of infinitesimal displacement gradients (§19), both (43.2) and (43.3) reduce approximately to the Hooke-Cauchy law (1.1) when $\alpha = 0$.

In the natural state ($\mathbf{E} = 0$, $\mathbf{e} = 0$) it follows that $t^i{}_j = \alpha\delta^i{}_j$, $T^\alpha{}_\beta = \alpha\delta^\alpha{}_\beta$. For strain from an unstressed natural state we have $\alpha = \bar{\alpha} = 0$. The subject of initial stress in general will be discussed in §55. Here, however, we notice that if an elastically isotropic body be initially subject to a pressure $p = -\alpha$, from $(43.3)_2$ it follows that the apparent values[5] of the elastic moduli in a subsequent infinitesimal strain are $\bar{\lambda}'_E = \lambda'_E = \lambda_E + p = \bar{\lambda}_E + p$, $\bar{\mu}'_E = \mu'_E = \mu_E + p = \bar{\mu}_E - p$.

For incompressible materials it is more convenient to use Mooney's expansion[6]

$$
\Sigma = \Sigma_{IJ}(I_{c^{-1}} - 3)^I(II_{c^{-1}} - 3)^J , \qquad (43.4)
$$

since by (14.14) both variables are non-negative and by (14.15) a polynomial approximation of degree n contains all terms of degree $2n$ in the extension ratios δ_i.

44. Thermoelasticity. In order to describe the effect upon a body of loading combined with a change of temperature or entropy, nineteenth century authors appear to have been content simply to allow all moduli to depend upon θ, taking care to retain in the series expansion (43.1) the first term, which represents an initial thermal pressure. As a first approximation, they wrote $\alpha = \kappa\theta$, where κ is a constant. The molecular derivations of this result by Duhamel[1] and F. Neu-

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[4] [1937, **1**, p. 251].

[5] L. Brillouin [1938, **2**, Ch. X, §XIII] [1940, **8**, §6] attributes this result to Poincaré [1892, **4**]. Cf. §52[5].

[6] Mooney [1940, **7**, pp. 588–589] uses an unnecessarily elaborate formula in which moments of $\mathbf{c}$ and $\mathbf{c}^{-1}$ of all orders are employed. While, as he in effect points out later [1948, **28**, pp. 441–443], (43.4) becomes valid for compressible bodies also if Σ_{IJ} be allowed to depend upon v/v_0, the two advantages mentioned in the text no longer hold. The expansion (43.4) is given by Rivlin [1949, **19**].

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[1] [1838, **1**, pp. 451–456].

mann[2] are not acceptable; most of the continuum treatments[3] seem to consist in a mere assumption, while those which are based upon thermodynamical principles do not appear to be free from objection[4]. No slight modification of the classical theory of pure elasticity can lead to a general theory of thermoelastic deformations, since the classical proof of the existence of a strain energy (§33) positively requires that the deformation be isothermal or adiabatic.

L. Brillouin[5] has shown, however, that it is possible to calculate the relations between the isothermal or adiabatic moduli at different temperatures. Let $\mathbf{c}$ be the Cauchy deformation tensor with respect to an unstressed natural state, and let λ_{E0}, μ_{E0}, $\cdots$, be the corresponding elastic moduli. The heating will induce a uniform expansion[6] with linear coefficient K at hydrostatic tension α_θ^* and density ρ_θ, and for the Cauchy tensor $\mathbf{c}^*$ with respect to this state we shall have $\mathbf{c}^* = K^2\mathbf{c}$. Let the corresponding spatial strain tensors be $\mathbf{e}$ and $\mathbf{e}^*$, respectively. Then for any given deformation we have $(\rho/\rho_0)[\Sigma(\mathbf{e}) - \Sigma(0)] = (\rho/\rho_\theta)[\Sigma(\mathbf{e}^*) - \Sigma(0)]$, since each side of this equation when integrated over a portion of the body gives the energy stored in the deformation of that portion. From $(43.1)_2$ follows

$$\begin{aligned}
&\sqrt{1 - 2I_e + 4II_e - 8III_e} \\
&\qquad \cdot \left[\frac{\lambda_{E0} + 2\mu_{E0}}{2} I_e^{\,2} - 2\mu_{E0} II_e + l_0 I_e + m_0 I_e II_e + n_0 III_e + \cdots\right] \\
&\quad = \sqrt{1 - 2I_{e^*} + 4II_{e^*} - 8III_{e^*}}\left[\alpha_\theta I_{e^*} + \frac{\lambda_{E\theta} + 2\mu_{E\theta}}{2} I_{e^*}^{\,2}\right. \qquad (44.1) \\
&\qquad\qquad \left. - 2\mu_{E\theta} II_{e^*} + l_\theta I_{e^*}^{\,2} + m_\theta I_{e^*} II_{e^*} + n_\theta III_{e^*} + \cdots\right].
\end{aligned}$$

Substituting from (15.4), expanding all quantities in power series in K, and supposing these may be rearranged, we obtain[7]

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[2] [1843, **3**, §10] [1885, **1**, §§54–55].

[3] [1853, **1**, eq. (13)] [1855, **1**, pp. 294–298] [1879, **3**, §1] [1881, **2**, eq. (2)] [1907, **3**, §39] [1920, **1**, 2nd note] [1939, **9**, §4].

[4] The analysis of PLANCK [1880, **1**, pp. 8–21] is open to the objection raised against that of VOIGT in §33. CELLERIER [1893, **3**, §5] claims to derive $T_i^{\alpha} = \rho_0\, \partial\phi/\partial x^i{}_{,\alpha}$ for all types of deformations; I cannot follow the analysis, and the result conflicts with the manifestly correct $\Sigma = \rho_0\epsilon$ (§33, Part B) for adiabatic motion.

[5] [1938, **2**, Ch. X, §XII]. Further developments, and a discussion of the relation of these results to structural theories of solids, are given in [1938, **6–7**] [1939, **11**] [1940, **8**]. Cf. [1939, **12–13**].

[6] This fact is usually assumed; SIGNORINI [1943, **1**, Ch. III, ¶¶13–15] has given a proof of it.

[7] These results differ in appearance from those of L. BRILLOUIN [1938, **2**, eq. (X.106)] because instead of $(43.1)_2$ he employs $(43.1)_1$ with the notation indicated in §43[3]. BIRCH [1938, **5**] obtains the first terms in the series.

$$
\begin{aligned}
K\alpha_\theta &= (1 - K^2)\left[\frac{3\lambda_{E0} + 2\mu_{E0}}{2} + \frac{27l_0 + 9m_0 + n_0}{4}(1 - K^2) + \cdots\right],\\
\lambda_{E\theta} &= K\left[\lambda_{E0} + \frac{18l_0 + 7m_0 + n_0}{2}(1 - K^2) + \cdots\right],\\
\mu_{E\theta} &= K\left[\mu_{E0} - \frac{3m_0 + n_0}{4}(1 - K^2) + \cdots\right],\\
l_\theta &= K^3[l_0 + \cdots], \qquad m_\theta = K^3[m_0 + \cdots], \qquad n_\theta = K^3[n_0 + \cdots].
\end{aligned}
\tag{44.2}
$$

These formulae exhibit the effect of a small expansion, whether arising from thermal pressure or any other cause, upon the ordinary Lamé constants. They show that *by measuring the Lamé constants for small strains at two different temperatures, one of which corresponds to an unstressed state of the body, the other to a uniform hydrostatic tension* α_θ, *it is possible to determine the second order moduli* l_0, m_0, n_0. I do not know whether such a determination has been attempted or is even practicable.

In order for the free energy ϕ to depend not only upon the strain but also upon the temperature θ, it is necessary from purely dimensional considerations that it depend also upon a second temperature. Signorini[8] has pointed out that this *reference temperature*[9] θ_0 must be arbitrary, and hence that the theory when properly formulated must be independent of it[10]. Tolotti[11] shows how this fact can be used to relate the isothermal strain energy Σ to ϕ. Let $\mathbf{e}_i$ denote strain measured with respect to the natural state of the body at temperature θ_i for two natural states $i = 0, 1$. Then for an isotropic body the invariance requirement assumes the form

$$
\frac{d\phi}{d\theta_i} = \frac{\partial\phi}{\partial\theta_i} + \frac{\partial\phi}{\partial I_{e_i}}\frac{\partial I_{e_i}}{\partial\theta_i} + \frac{\partial\phi}{\partial II_{e_i}}\frac{\partial II_{e_i}}{\partial\theta_i} + \frac{\partial\phi}{\partial III_{e_i}}\frac{\partial III_{e_i}}{\partial\theta_i} = 0. \tag{44.3}
$$

Since this condition is a linear partial differential equation of first order in four variables, its general solution is an arbitrary function of three independent particular solutions. Now since, as mentioned above, the two natural states are related to one another by a pure extension, the formulae (15.4) with linear coefficient K_i can be applied with respect to each in turn; an equivalent statement is that the three independent functions

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[8] [1945, **7**, pp. 166–167]. For earlier remarks on thermoelasticity, see [1930, **8**, §5] [1936, **4**, pp. 15–16].

[9] The need for a reference temperature has been remarked by many authors, e.g. [1828, **1**, §II, p. 203] [1945, **7**, p. 163] [1948, **13**, §§1, 6]. For the manner in which θ_0 arises in a structural theory, cf. [1943, **5**, §6] [1948, **22**, p. 247].

[10] RICHTER [1948, **13**, §§5–6] derives a condition that the elastic energy be separable as in the infinitesimal theory into the sum of a part arising from changes of volume alone and a part arising from changes of shape alone. He shows that if this decomposition be possible, it is invariant under change of reference temperature.

[11] [1943, **13**, §§1–4].

$$K_i^2(3 - 2I_{e_i}), \qquad K_i^4(3 - 4I_{e_i} + 4II_{e_i}), \qquad K_i^6(1 - 2I_{e_i} + 4II_{e_i} - 8III_{e_i}) \tag{44.4}$$

are invariant with respect to change of θ_i. Now put $\theta_1 = \theta$ and let **e** refer to strain from the natural state at temperature θ. Then the functions I_e, II_e, III_e as given by (15.4), being linear combinations of (44.4), are invariant under change of θ_i. Hence the most general form for ϕ may be taken as

$$\phi(I_{e_0}, II_{e_0}, III_{e_0}, \theta, \theta_0) = f(I_e, II_e, III_e, \theta), \tag{44.5}$$

where the first three arguments of the function on the right are to be replaced by the right sides of (15.4).

Now as we have seen in Part B of §33, the strain energy Σ_{θ_0} for isothermal deformation when $\theta = \theta_0$ is given by

$$\Sigma_{\theta_0}(I_{e_0}, II_{e_0}, III_{e_0}) = \rho_0\phi(I_{e_0}, II_{e_0}, III_{e_0}, \theta_0, \theta_0) + \text{const.}, \tag{44.6}$$

where the constant depends upon θ_0. Put $\theta = \theta_0$ in (44.5); substitute the resulting expression for ϕ into (44.6); drop the subscript naught, and substitute the resulting expression for f into (44.5), thus obtaining

$$\phi(I_e, II_e, III_e, \theta, \theta_0) = \frac{1}{\rho}\Sigma(I_e, II_e, III_e) + f(\theta). \tag{44.7}$$

Tolotti shows further that

$$f(\theta) = -\int^{\theta} d\xi \int^{\xi} c_{p_0}\frac{d\theta_0}{\theta_0}, \tag{44.8}$$

where c_{p_0} is the specific heat at constant pressure in the natural state at temperature θ_0. Since the linear coefficient K in (15.4) may be expressed in terms of the density ρ_0 of the natural state, it follows that *if* ρ_0, c_{p_0} *and* Σ_{θ_0} *be known for a range of reference temperatures* θ_0, *then* ϕ *is determined to within a linear function of* θ for the same range of values of θ_0.

Tolotti[12] carries out a similar analysis for the case when the adiabatic rather than the isothermal strain energy is known.

IV B. Cauchy's Method: Reiner's Semi-empirical Theory

45. Reiner's theory of isotropic elastic bodies. Stating that physical materials nearly always take some permanent set in any deformation, Reiner proposes

[12] [1943, **13**, §6]. Cf. [1940, **8**, §§7–8].

to abandon the idea of a natural state and to measure strain with respect to whatever configuration be assumed by the body when relieved of deforming forces[1], supposing further that this relaxed configuration be unstressed. Since a portion of the work of deformation is thus not recoverable, energy considerations are not easy to apply. Reiner therefore employs Cauchy's method rather than Green's method (§36), and he proposes $\mathbf{t} = f(\mathbf{m})$ as the general law of elasticity[2], where $\mathbf{m}$ is any spatial measure of strain. From (6.7) follows[3]

$$t^i{}_j = \mathcal{G}_0^{\mathrm{t(m)}}\delta^i{}_j + \mathcal{G}_1^{\mathrm{t(m)}}m^i{}_j + \mathcal{G}_2^{\mathrm{t(m)}}m^i{}_k m^k{}_j\,. \tag{45.1}$$

Reiner proceeds to define various elastic "moduli" or "coefficients"[4] in terms of the values assumed by the $\mathcal{G}_{\mathrm{T}}^{\mathrm{t(m)}}$ and the $\mathcal{G}_{\mathrm{T}}^{\mathrm{m(t)}}$ in special situations. Since these moduli depend not only upon the material but also upon the strain and the particular measure of strain, and since unique values for the $\mathcal{G}_{\mathrm{T}}^{\mathrm{m(t)}}$ do not generally exist (cf. §7), their usefulness is dubious.

Reiner justly observes[5] that by retaining $\mathcal{G}_2^{\mathrm{t(m)}}$ we obtain a theory which (apart from the values of the various coefficients) is independent of the measure $\mathbf{m}$ of strain, providing that the strain in all cases be taken with respect to the same reference configuration. One or another of the tensors of §§14–17 has been recommended by one or another author in proposing a linear or quasi-linear relation of the type (9.2) or (9.3) (cf. §51). If such a relation be quasi-linear with respect to one measure, it will in general fail to be so with respect to others, except in the case of infinitesimal displacement gradients. In general considerations it is therefore essential that $\mathcal{G}_2^{\mathrm{t(m)}} \neq 0$.

Among other situations, Reiner considers the typical and highly illustrative case of a shearing stress u by putting $s = 0$, $\mathbf{a} = \mathbf{t}$, $\mathbf{b} = \mathbf{h}$ in (10.2). Since $I_{\mathrm{t}} = III_{\mathrm{t}} = 0$, $II_{\mathrm{t}} = u^2$, the Kelvin effect (§10) generally appears as a dilatation[6] I_{h} proportional to u^2; the Poynting effect is a cross-stress also proportional to u^2 (cf. §42). Application of the observations of §10 to the present case shows that *in extension or hydrostatic pressure quantitative departures from the results predicted by the linear theory will be observed sooner than in shearing or torsion;*

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[1] [1948, **1**, §§1–2]. This idea was suggested by THOMPSON [1933, **3**, §1.3 footnote] and by WEISSENBERG [1947, **8**].

[2] This semi-empirical viewpoint is shared by GLEYZAL [1949, **38**, §2], except that he prefers a material formulation; he states that "there exists for any material a stress-strain law of form $\mathbf{T} = f(\mathrm{C}, X^\alpha, t)$", where $\mathbf{T}$ is an undefined (and probably also incorrect, cf. §26[12]) material stress tensor. The expression for $\mathbf{t}$ as a tensorial power series in $\mathbf{E}$ proposed in [1931, **15**, §3] is incorrect because it fails to distinguish between spatial and material tensors; further developments based on this series expansion are given in [1942, **15**] [1943, **17**, §4] [1944, **17**, §1].

[3] [1945, **5**, §2] [1947, **9**, §2] contain a similar expression for infinitesimal strains produced by loading in the strain-hardening range (cf. §56[3]).

[4] [1948, **1**, §8].

[5] [1948, **1**, §3] [1951, **11**, §4].

[6] For incompressible materials dilatation is impossible; the KELVIN effect then appears only as a hydrostatic pressure (§42).

results of experiments on extension and hydrostatic pressure, however, cannot yield information sufficient to predict the response of the material to shearing or torsion[7], *in which appear the Kelvin and Poynting effects, which are totally absent in the classical theory and whose magnitudes are of second order, while quantitative departures from the classical results are only of third order.*

The foregoing discussion of special situations applies equally to the classical natural state theory (Chapter IV A), but for this theory Rivlin's explicit solutions (§42) are more informative.

46. Criticism of Reiner's theory. Reiner proposes that the coefficients $\mathcal{G}_{\mathrm{T}}^{\mathrm{t(m)}}$ be determined by experiment. The stresses and the configuration of a test body in the deformed state are to be observed, the body is to be allowed to relax to the reference configuration, and then the strain of the deformed body relative to the reference configuration is to be measured. The results of a variety of such experiments are to yield approximate values for the coefficients $\mathcal{G}_{\mathrm{T}}^{\mathrm{t(m)}}$ under these conditions. It seems doubtful that such a process could ever be carried out except in the most degenerate cases, for usually the stresses cannot themselves be measured directly, but rather must be calculated from the theory in terms of the applied surface forces and extraneous forces. Reiner's theory, however, will not permit the solution of a single boundary problem, for it is altogether indeterminate. Strains are calculated with respect to an unspecified relaxed configuration which in any given boundary problem is unknown. In effect three new unknown functions $X^L(X^\alpha)$ are introduced; an adequate theory of set should yield values for these functions as a part of each solution, but Reiner proposes no new differential equations. If generalized to anisotropic media, Reiner's theory consists of nothing more than the rather empty functional relationship $\mathbf{t} = f(\mathbf{m})$. Isotropy is an idealization whose only value is mathematical simplicity; while indeed many physical media may be represented as isotropic without sensible error, the basic laws of elasticity are surely independent of this accident, and an adequate theory cannot rest on isotropy alone.

Cauchy's method makes no use of the principles of mechanics and thermodynamics in general, nor of the particular fact that since a part of the energy is recoverable, at least a part of the stress should be derivable from this energy. As very well expressed by J. H. C. Thompson[1]: "Now, any molar (or large scale) theory of the properties of a strained and straining solid can be nothing else but the '*thermodynamics of a continuous medium*'. It must be emphasized that the stress-strain relations are not the basis but the result of such a theory. A stress-strain hypothesis, as it stands, can tell us very little about the properties of the solid it claims to represent, and it is not necessarily consistent with the first and second laws of thermodynamics or with the fact that the solid is to be regarded as continuous. Further, the mere statement of a stress-strain hypothesis

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[7] Cf. [1930, **8**, §10] [1948, **25**] [1951, **11**].

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[1] [1933, **3**, p. 350].

gives no suggestion as to the limits in which it is valid, whereas a thermodynamical theory can sometimes supply information concerning the character of the limits." In particular, in Reiner's theory there are many more coefficients of elasticity than in the natural state theory: for example, in (43.4) there are three coefficients of second order: l, m, n; while in Reiner's theory there are four: $\mathfrak{G}_{0200}^{t(m)}$, $\mathfrak{G}_{0010}^{t(m)}$, $\mathfrak{G}_{1100}^{t(m)}$, and $\mathfrak{G}_{2000}^{t(m)}$. This excess of coefficients (cf. §8) arises from the neglect of energy considerations and therefore probably is not correct (cf. §52).

The main value of Reiner's theory is its demonstration that *any* theory of isotropic bodies which is not quasi-linear will predict Kelvin and Poynting effects (§§10, 42, 45). *Hence the experimental detection of these phenomena, while evidence that a non-linear theory is required, does not substantiate any particular theory.* If, however, for a particular material the Poynting effect be *not* present when a particular measure of strain is used, then the appropriate theory may be quasi-linear with respect to this measure (§10).

47. Dimensional invariance. Whether we consider Cauchy's method or Green's, the stress $\mathbf{t}$ depends not only upon the value of a certain strain measure $\mathbf{m}$ but also upon the particular body and the temperature[1] θ. Characteristic of the theory of elasticity is the additional postulate that *each elastic material has one and only one independent dimensional modulus, the natural elasticity* μ_{En}, *whose physical dimensions are those of stress*, and which shall be defined in terms of some specified state of strain at some particular *reference temperature*[2] θ_0. Thus if we write

$$\mathbf{t} = f(\mathbf{m}, \mu_{En}, \theta, \theta_0), \tag{47.1}$$

where $\dim \mathbf{t} = \dim \mu_{En} = \mathbf{ML}^{-1}\mathbf{T}^{-2}$, $\dim \mathbf{m} = 0$, $\dim \theta = \dim \theta_0 = \mathbf{\Theta}$, we have included among the arguments of f all dimensional quantities upon which $\mathbf{t}$ may depend. If we write, equivalently,

$$\frac{\mathbf{t}}{\mu_{En}} = g(\mathbf{m}, \mu_{En}, \theta, \theta_0), \tag{47.2}$$

we have a relation connecting various dimensionless quantities with the three quantities μ_{En}, θ, θ_0; since these latter are composed of two independent dimensions, the relation (47.2) must reduce to one in which the dimensional quantities occur only in $3 - 2 = 1$ dimensionless ratio. Since such a ratio is θ/θ_0, we have finally

$$\mathbf{t} = \mu_{En}\, g\left(\mathbf{m}, \frac{\theta}{\theta_0}\right), \tag{47.3}$$

where the function g is dimensionless, as the most general admissible form of stress-strain relation in Cauchy's theory, and hence *a fortiori* in Green's. This

[1] We are here employing coefficients with respect to some particular thermodynamic condition (isothermal, adiabatic, etc.).

[2] Cf. §44[9].

result may be used to exhibit the dimensional quality of all expressions previously obtained. For the Lamé coefficients, for example, we have $\lambda_E = \mu_{En} f(\theta/\theta_0)$, $\mu_E = \mu_{En} g(\theta/\theta_0)$, where the functions f and g are dimensionless.

Let the value of $\mathbf{m}$ for a rigid deformation be $\mathbf{m}_0$, and suppose $\mathbf{m}' \equiv \mathbf{m} - \mathbf{m}_0$; then from (47.3) it is plain that a dimensionless criterion for neglect of second and higher order terms in the stress-strain relations is simply that the intensity of $\mathbf{m}'$ be small:

$$\sqrt{\overline{II}_{\mathbf{m}'}} << 1. \tag{47.4}$$

Note that $\overline{II}_{\mathbf{m}'}$ *is a function of strain alone: for a given material the validity of an approximate theory of elasticity depends only upon the particular deformation in question and is otherwise independent of the stress, thermodynamic state, etc.*

IV C. Approximations

48. The nature of approximate theories. For purposes of calculation it is usually convenient to introduce approximations. These are of three types: (I) Linearizations, (II) Other special assumptions regarding stress-strain relations, (III) Other special assumptions regarding the strain energy.

49. Theories of infinitesimal strain but large displacement gradients and rotations. For most structural materials, even in the largest purely elastic deformation the strain itself (with respect to the natural state) is infinitesimal, although the displacement gradients and rotations may be large (§19). This case corresponds to a wide range of important applications for systems one of whose dimensions is small with respect to the others: plates, shells, rods, etc. A thin plane sheet of spring steel may be bent into the shape of a circular cylinder, and a long slender shaft may be twisted through several revolutions, without any noticeable failure of *perfect* elasticity, although the displacements are by no means *linear* functions of the loads[1]. For these problems it is then permissible to linearize (43.2) or (43.3), but not permissible to linearize the expressions for $\mathbf{E}$ or $\mathbf{e}$ in terms of the $x^i{}_{,\alpha}$ or the $X^\alpha{}_{,i}$.

From (43.2) we thus obtain the stress-strain relations of the St. Venant-Kirchhoff theory[2]:

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[1] [1844, **1**] [1847, **1**, §1] [1892, **4**, §2] [1916, **1**] [1917, **2**, §4] [1942, **3**, p. 57].

[2] ST. VENANT [1844, **1**] [1847, **1**, §2] first described the relations $(49.1)_1$ in words. Later [1863, **2**, ¶2], however, he incorrectly put $\mathbf{t}$ rather than $\mathbf{T}$ upon the left. The resulting theory can be valid only when the displacement gradients are infinitesimal and $(49.1)_1$ reduces to (1.1). BRILL and BOUSSINESQ pointed out the error to ST. VENANT, who accepted

$$T^\alpha{}_\beta = \lambda_E E^\gamma{}_\gamma \delta^\alpha{}_\beta + 2\mu_E E^\alpha{}_\beta,$$
$$t^i{}_j = J^{-1}[\lambda_E E^\gamma{}_\gamma \delta^\alpha{}_\beta + 2\mu_E E^\alpha{}_\beta] x^i{}_{,\alpha} x_j{}^{,\beta}. \tag{49.1}$$

The corresponding strain energy has precisely the same form as that of the infinitesimal theory:

$$\Sigma = \frac{\lambda_E + 2\mu_E}{2} I_E{}^2 - 2\mu_E II_E, \tag{49.2}$$

and (49.1) follows from it without further approximation.

An exact spatial counterpart is proposed by Hencky[3]. Taking the strain energy as

$$\Sigma = \frac{\lambda_E + 2\mu_E}{2} I_h{}^2 - 2\mu_E II_h, \tag{49.3}$$

by (41.3) and (16.2) he derives

$$t^i{}_j = e^{-I_h}[\lambda_E h^k{}_k \delta^i{}_j + 2\mu_E h^i{}_j]. \tag{49.4}$$

Because of the difficulty of calculating the off-diagonal components of **h** in terms of the displacement gradients (§16), Hencky's theory is hard to use except in trivial cases[4].

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the criticism and adopted [1871, **2**, ¶7] the equations previously given by Boussinesq [1870, **1**], which follow by replacing J^{-1} and $x^i{}_{,\alpha} x_j{}^{,\beta}$ in $(49.1)_2$ by expressions involving **E** and then linearizing with respect to **E**. Stimulated by St. Venant's first notes, Kirchhoff [1852, **1**, p. 770] had previously derived the correct form (49.1). Apparently (49.2) formed the basis of the stability theory of Bryan [1888, **2**], whose erroneous confusion of material and spatial co-ordinates, noted by Bonvicini [1931, **13**], is repeated by García [1950, **24**, eq. (7)]. The incorrect form of St. Venant's theory is adopted in certain theories of finite (in the sense of §19) displacements of plates [1907, **3**, §24] [1910, **3**, §8] [1922, **1**, pp. 99–101] and shells [1939, **22**] [1945, **6**, §1] [1948, **49**], with the further assumption that only certain of the $x^i{}_{,\alpha}$ need be retained in the expressions for **E**. It is sometimes alleged in apology for these not self-consistent theories that they refer not to **t** but to stress taken with respect to the undeformed area, so that the relevant equilibrium equations are not $(26.2)_1$ but (26.8), but this remark is equally erroneous because the $T^{i\alpha}$ occurring in $T^{i\alpha}{}_{,\alpha} + \rho_0 f^i = 0$ may be confounded with the $T^{\alpha\beta}$ occurring in the permissible stress-strain relations $(49.1)_1$ only when the displacement gradients are infinitesimal, reducing $(49.1)_1$ to (1.1). Kirchhoff's theory was tacitly adopted by Trefftz [1930, **9**] [1933, **6**], who applied it to the general problem of elastic stability, as well as to the special cases of the elastica and of a bent and twisted beam. The formulation of Trefftz is applied in [1937, **7–9**] to the theory of plates and in [1939, **9**, §§8–10] to the theory of torsion and bending of shafts and the problem of stability. Cf. [1940, **5**, Ch. VIII, §1, ¶11].

[3] [1928, **2**, §§2–3 and p. 457] [1929, **1**, §1] [1929, **2**, §1]. The theory is applied to the cases of simple extension and simple shear [1928, **2**, §§4–5], of bodies under initial stress [1929, **5**], of hydrostatic pressure [1931, **5**], of wave propagation in bodies under initial hydrostatic pressure [1932, **15**], of extension of an incompressible material [1933, **4**].

[4] Thus in the few other cases to which it has been applied, some further approximation has been added. In [1939, **8**, Ch. 1, §15] the theory resulting from writing $\mathbf{h} \approx \mathbf{e} + \mathbf{e}^2$ in

An approximate spatial counterpart is proposed by Seth[5]:

$$t^i{}_j = \lambda_E e^k{}_k \delta^i{}_j + 2\mu_E e^i{}_j . \tag{49.5}$$

Since (49.5) is not exactly derivable from a non-negative strain energy[6], as indeed is evident from a glance at the full series $(43.3)_1$, Seth's theory has received much adverse criticism,[7] largely unjustified. For infinitesimal strains $I_E \approx I_e$, $II_E \approx II_e$, and (49.2) is the approximate strain energy, which is indeed non-negative when $3\lambda_E + 2\mu_E \geq 0$, $\mu_E > 0$.

The range of accurate validity of the foregoing three theories is the same: that in which $E_i \approx e_i \approx h_i \approx \delta_i$; i.e., say, $|\delta_i| \leqq 0.01$. Which theory is to be used is then but a matter of convenience, and Seth's seems to be preferable on this ground. Results calculated from the three theories should agree[8], to the approximation considered. In border cases, however, there are great differences: in a 100% extension $\delta_i = 1$ we have $E_i = 3\frac{1}{2}$, $h_i = \log 2$, $e_i = \frac{3}{8}$, so that in this case the St. Venant-Kirchhoff and Hencky theories are worthless but Seth's theory might possibly yield a crude approximation; while in a 50% compression $\delta_i = -\frac{1}{2}$ we have $E_i = -\frac{3}{8}$, $h_i = -\log 2$, $e_i = -3\frac{1}{2}$, so that for this case Hencky's and Seth's theories are worthless, but the St. Venant-Kirchhoff theory might possibly yield a crude approximation[9].

(49.4) is considered. Powers of the angle of shear higher than the second are discarded in the analysis of shearing in [1948, **39**]; the results, of course, are the same as would follow from (49.5) or from (49.1) and thus do not differ from those of POYNTING cited in §42[6].

[5] [1935, **1**, §5].

[6] Specifically, ZVOLINSKI & RIZ [1939, **14**], SIGNORINI [1942, **3**, p. 69], and RIVLIN [1948, **10**, §5] prove that (49.5) is exactly derivable from a strain energy if and only if $\lambda_E = -\mu_E$, or, equivalently, $\nu_E = \infty$. Cf. §50.

[7] The title of SETH's paper, "Finite strain . . . ," is deceiving; he adopts (49.5) as "the simplest tensor form that we can take." MURNAGHAN [1937, **1**, §3] is quite correct that "simplicity is not a sufficiently compelling reason," but (49.5) follows from his own result $(42.5)_2$ by linearization when the strain is infinitesimal, and he himself in the same paper adopts a quadratic approximation, no different in principle. ZVOLINSKI & RIZ also propose a quadratic approximation. Whether SETH expects his theory to be valid for large strains is not clear; in only one of his examples (that of simple extension, below) do large strains necessarily occur. While in generalizing his theory so as to apply to aelotropic media [1945, **9**] [1946, **15**] SETH justifies his linear law on substantially the grounds given above, nevertheless he subsequently [1946, **16**] [1948, **42**] employs it again in simple extension, where it cannot possibly be valid except in the range when (1.1) is valid. SETH's stress-strain relations are correctly employed in SYNGE & CHIEN's theory of finite deformation of plates and shells [1941, **7**] [1944, **5**], where the strain is explicitly stated to be infinitesimal.

[8] No comparison of the results in specific cases has been attempted, no doubt because of the discouraging complication of most of the results.

[9] Other linear stress-strain relations have been put forward. M. BRILLOUIN [1891, **1**] proposed a theory (for isotropic media) in which quantities such as $\partial u/\partial Y - \partial v/\partial X$ are taken as measures of the rotation, and thus to be retained; his strain energy is not a scalar, however. Substantially the same proposal is made by FROLA [1940, **12**], who does not trouble to distinguish between spatial and material co-ordinates. In answer to a criticism by CICALA [1941, **18**, footnote 3], FROLA [1942, **18**] states that his theory is intended only for the case

Some of the easy consequences of Seth's theory are interesting. In simple extension[10] (49.5) together with $(42.7)_2$ and $(42.8)_2$ yields the following values for the tensile stress and the Poisson contraction:

$$2t_1 = 2Ee_1 = E\left[1 - \frac{1}{(1+\delta_1)^2}\right], \quad \delta_2 = \frac{1+\delta_1}{\sqrt{1+\delta_1(2+\delta_1)(\nu_E+1)}} - 1. \quad (49.6)$$

Now if $|e_i| \ll 1$, as must be assumed to justify (49.5), the δ_i and e_i are of the same order of magnitude, so that within the range of validity of this theory we are justified in employing (49.6) only when it reduces to the classical linearizations $(42.7)_1$ and $(42.8)_1$. Suppose, however, we neglect this fact and interpret (49.6) as it stands. We find that as δ_1 runs from -1 to $+\infty$, t_1 increases monotonely from $-\infty$ to $+\frac{1}{2}E$, or, equivalently, the stress referred to the initial area, $t_1(1+\delta_1)^2$, increases monotonely from $-\infty$ to $\frac{1}{2}E/(\nu_E+1)$. As far as *general shape* is concerned, this stress-strain curve is as good as can be expected from any theory of purely elastic extension: *it distinguishes between pressure and tension, stating that no finite pressure can annul the length of the specimen, but to produce infinite length in tension only a finite limit stress is required.* The existence of the limit stress may be regarded as a qualitative suggestion of yield or rupture, as nearly as such a phenomenon can be indicated in a theory of pure elasticity. The contraction formula $(49.6)_2$, however, gives two obviously false results: first, if $\delta_1 < -1 + [\nu_E/(\nu_E+1)]^{\frac{1}{2}}$ then δ_2 is imaginary, so that large contractions are impossible, and, second, as $\delta_1 \to \infty$ we have $-\delta_2 \to 1 - (1+\nu_E)^{\frac{1}{2}}$, so that a specimen indefinitely extended does not indefinitely contract. The foregoing results indicate on the one hand that Seth's theory is a qualitative improvement over the classical linear theory, but on the other hand that it is utterly wrong for really large strains.

Turning next to the case of simple shear (42.14), from (49.5) we obtain

$$\mathrm{t} = -\frac{1}{2}\lambda_E K^2 \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} + \mu_E K \left\{ \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} - K \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \right\}. \quad (49.7)$$

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of infinitesimal displacement gradients, though the displacements themselves may be large. He in turn criticises the work of CICALA, who prefers the ST. VENANT-KIRCHHOFF theory. SWAINGER [1948, **47**] claims that the case of large rotations can be treated simply by referring infinitesimal strain to a frame oriented so that an arbitrarily selected particle has suffered no rotation. This procedure is always tacitly adopted in the classical infinitesimal theory, but is completely inadequate for treating genuinely large rotations, where there exist particles whose rotations *differ* by a large amount. Later [1950, **13**] he claims to get excellent agreement with experimental data from a theory in which all equations of the classical linear theory are referred to the deformed body and taken as valid for finite strain; what he offers as a derivation [§4] seems to me to be setting down the equations by fiat, and the equations themselves wholly false. GRAHAM [1949, **14**] proposes $T_{LM} = G(U_{L,M} + U_{M,L})$, where T and U are respectively stress and displacement referred to an unspecified "natural frame". Cf. §12[5].

[10] [1935, **1**, §7] [1946, **16**].

Comparison with the general solution (42.15) shows that even in Seth's theory, *retention of the non-linear terms in* **e** *gives rise to Kelvin and Poynting effects. Thus the mere experimental occurrence of these effects offers no information about the form of the strain energy function.* In §53 we shall find unexpected support for (49.7), a result which is somewhat better than the insecure foundation of Seth's theory would suggest. The example of Seth's theory shows how a slight generalization of the infinitesimal theory, even if not self-consistent, may yield non-linear results in strikingly better qualitative accord with experiment. In discussion of a different example Treloar[11] has justly observed that these effects are typical of the *kind* of phenomenon which occurs when *any* non-linear terms are retained[12]. As far as Seth's theory itself is concerned, in most of the numerous special solutions which have been worked out[13], the strain itself may be very small, so that the results should be quantitatively correct within a useful range. The employment of this theory in a special case should always be followed by a careful check of the resulting strains, so as to verify that they be at least sufficiently small that the strain energy be everywhere positive.

The most obvious extension of a linearized theory is a quadratic theory; that is, a theory in which the strain energy Σ is approximated by a cubic function of the strain. The first proposal of this type is Voigt's[14], in which the displacement gradients $x^i{}_{,\alpha}$ are so small that **E** may be replaced by the infinitesimal strain tensor **e**, yet the terms whose coefficients are $\bar{l}$, $\bar{m}$, $\bar{n}$, in $(43.1)_1$ must be retained. After pointing out the inconsistency of Voigt's theory[15], Finger[16] gave the equation

[11] [1949, **10**, Ch. 13, §4]. I am indebted to Messrs. TRELOAR and RIVLIN for use of this chapter in MS.

[12] Perhaps this observation accounts for the qualitative success of the inconsistent theories of plates and shells mentioned in Note 2.

[13] In [1935, **1**]: bending of a block, torsion of a right cylinder; [1936, **2**]: a spherical shell subject to uniform but not necessarily equal normal tractions on the inner and outer surfaces, a cylindrical shell in plane strain and subject to similar tractions, a cylindrical tube turned inside out, its curved surfaces being free of traction; [1938, **3**]: a uniform circular cylindrical tube from which a section bounded by axial planes is removed, and the resulting edges joined, a similar cylinder slit along an axial plane and forced apart; [1939, **2**]: a cylinder subject to radial and extraneous force, a spherical shell turned inside out; [1941, **10**]: elastic failure; [1941, **11**]: a rotating shaft; [1947, **18**]: flexure of a block.

[14] [1893, **1**, p. 535] [1895, **1**, p. 35] [1894, **3**] [1901, **9**].

[15] [1894, **1**, pp. 198–199]. Some terms neglected by VOIGT are retained by STERNBERG [1946, **17**]. The notion behind this almost certainly wrong theory, which is proposed also in [1949, **27**, §§4–6], is that there exists a range of infinitesimal displacements and infinitesimal displacement gradients, and hence (§19) *a fortiori* infinitesimal strain, in which deviations from linear stress-strain relations are observable. Such a case is possible only if Σ be not analytic or if l, m, n greatly exceed λ_E, μ_E; the latter alternative is hardly to be expected and has never been observed, while the former is not considered by the authors in question. Experiments appearing to indicate such deviations are probably explicable in one or both of two ways: (1) the experimenters neglect the distinction between material and spatial co-ordinates in a range beyond that in which such neglect is permissible (i.e. the displacement gradients are not infinitesimal), or (2) the effects of secondary strain are neglected. Cf. also §43².

[16] [1894, **1**, eqq. (47) (50)] [1894, **2**, eq. (42)]. He gives also the corresponding stress-strain

to which the cubic approximation $(43.1)_1$ reduces when all terms of order higher than three in the $x^i{}_{,\alpha}$ are discarded. So as to obtain a formula for the strain energy correct up to terms of third order in the extensions, Kötter[17] retained the terms whose coefficients are $\bar{l}$, $\bar{m}$, $\bar{n}$, in $(43.1)_1$. Murnaghan[18] in his first work considers the effect of retaining the terms whose coefficients are l, m, n, in $(43.1)_2$. L. Brillouin[19] considers at length the propagation of elastic waves in a medium where the terms whose coefficients are $\bar{l}$, $\bar{m}$, $\bar{n}$, in $(43.1)_1$ are retained.

50. Theories based on assumptions regarding the extensions. Following the notions of Brill and Boussinesq, St. Venant[1] proposed that the strain energy should be a quadratic function of the extensions δ_i rather than of the principal strains E_i or e_i :

$$\Sigma = \frac{\lambda_E + 2\mu_E}{2} I_\delta^{\,2} - 2\mu_E II_\delta. \tag{50.1}$$

He concluded that for infinitesimal strains (49.1) follows. Almansi[2] showed from (41.11) that the corresponding "Hooke's law"

$$t_i = \lambda_E I_\delta + 2\mu_E \delta_i \tag{50.2}$$

is exactly derivable from a strain energy if and only if $\nu_E = \frac{1}{3}$. Armanni[3] constructed an example showing that even in the resulting very simple non-linear theory it is possible for a state of non-vanishing strain in a simply-connected region devoid of singularities to be an equilibrium configuration when there are no surface or extraneous loads (cf. §54).

relations (43.2): [1894, **1**, eqq. (51) (54)], [1894, **2**, eq. (43)]. FINGER realizes that this neglect of 4th, 5th, and 6th order terms in the $x^i{}_{,\alpha}$ invalidates his theory for the case of large rotations. On the basis of a molecular argument [1894, **1**, pp. 232–244] he proposes three relations connecting $\bar{\alpha}$, $\bar{\lambda}_E$, $\bar{\mu}_E$, $\bar{l}$, $\bar{m}$, $\bar{n}$.

17 [1910, **4**, §1].

18 [1937, **1**, §§3–5]. MURNAGHAN's formulae are adopted as a basis for the theory of thin rods by HAY, who indicates [1942, **8**, §14] an experiment by means of which one combination of l, m, n might be measured.

19 [1938, **2**, Ch. XI] [1940, **8**, §§9–10, 15–16]. BRILLOUIN mentions that POYNTING's experiments (§42[7]) can yield numerical values for two combinations of $\bar{l}$, $\bar{m}$, $\bar{n}$, and presents further considerations suggesting a third, as well as certain inequalities [1938, **2**, Ch. XI, §XVII, and Ch. XII, §XI]. Formulae for $\bar{l}$, $\bar{m}$, $\bar{n}$ are derived from molecular hypotheses by HERPIN [1949, **29**].

1 [1871, **2**, ¶¶1–2]. The extensions and angles enter GREEN's first formulation [1839, **1**, p. 249]. RICHTER [1948, **13**] uses a tensor **k** whose principal components are the extensions δ_i (§17[1]).

2 [1911, **3**, §1]. Cf. [1948, **13**, §7]. SIGNORINI [1930, **8**, §9] shows that in order for $t_i = \lambda_E I_E + 2\mu_E E_i$ to be exactly derivable from a strain energy, it is necessary and sufficient that $\nu_E = \frac{1}{4}$, as in the classical "rari-constant" theory. Cf. §49[6].

3 [1915, **1**].

Southwell[4] in treating homogeneous strain proposed that (50.1) should be taken as the strain energy; by (41.10) we then obtain stress-strain relations of the form of the classical Hooke-Cauchy law, but in which the stress is referred to the undeformed area (cf. (41.10) et seq.):

$$t_i(1 + \delta_j)(1 + \delta_k) = \lambda_E I_\delta + 2\mu_E \delta_i . \qquad (i, j, k \neq). \tag{50.3}$$

These stress-strain relations are the starting point for the lengthy calculations of Biezeno & Hencky[5]; equivalent results are obtained by Biot[6] in a different way (cf. §41[1]). These theories, too elaborate to develop here, enjoy a certain popularity, despite the number of special assumptions regarding the order of magnitude of certain terms which must be made in order to derive their final equations. Since they are mere second approximations (cf. §3), it is not surprising that the analyses leading to similar results differ one from another, but are alike in being more elaborate than those of the exact theory.

Claiming that (50.2) is "experimentally established" for "finite displacements not exceeding some small value (the limit of proportionality)", Riz & Zvolinski[7] require that (50.1) shall be an approximate consequence of $(43.1)_2$ when terms of *third* and higher order in the δ_i may be neglected, but their analysis is faulty and indeed no such approximate reduction is possible[8]. Next[9] they propose to take the principal stresses referred to the undeformed area as quadratic functions of the principal extensions. By (41.11) and (41.10) follow

$$t_i(1 + \delta_j)(1 + \delta_k) = \lambda_E I_\delta + 2\mu_E \delta_i + A\delta_i^2 + BI_\delta^2 + C(\delta_i I_\delta - II_\delta), \qquad (i, j, k \neq)$$

$$\Sigma = \frac{\lambda_E + 2\mu_E}{2} I_\delta^2 - 2\mu_E I_\delta + \frac{A}{3}\overline{III}_\delta + \frac{B - C}{3} I_\delta^3 - CI_\delta II_\delta. \tag{50.4}$$

---

[4] [1913, **3**, eqq. (15) (16)]. He investigated the stability of a strip of thin plating subject to thrusts in its own plane, of a boiler flue, of a tubular strut. His equations form the foundation of the theory of cylindrical shells derived by DEAN [1925, **2**], who investigated the stability of a plane plate and of a tubular strut.

[5] [1928, **3**, pp. 569–578]. Their results are applied to a plate subject to edge stresses and extraneous forces, to the buckling of a girder clamped on one side, to a circular ring under external pressure.

[6] Cf. §41[1]. He gives the following applications: in [1939, **5**]: elastic stability, elastic waves in a material under hydrostatic pressure due to gravity; in [1939, **6**]: elastic stability; in [1940, **4**]: stresses in plates, torsion of a shaft subject to axial extension, a problem which is treated in [1939, **7**] and [1948, **18**] also. NEUBER [1943, **8**, §§4–6] applies this theory to the study of plane waves in a stressed anisotropic medium, and gives a general method of integrating the equations. An equivalent variational principle for the consideration of stability is given in [1946, **12**, §3]; cf. [1948, **20**]. An expression for the second variation of Σ similar to that obtainable from BIEZENO & HENCKY's or BIOT's approximations is given in [1933, **7**]. The planar theory constructed by KRYLOV [1946, **19**, §§1–3] apparently coincides with that of BIEZENO & HENCKY and BIOT.

[7] [1938, **1**, **11**].

[8] Specifically, they put $l = \frac{3}{2}(\lambda_E + 2\mu_E)$, $m = 3(\lambda_E - \mu_E)$, $n = 9\mu_E$ but if these values be put back into $(43.3)_2$, by using $(15.2)_3$ it is easy to show that the resulting expression does

Approximating the extensions by quadratic functions of the principal strains, Zvolinski & Riz now obtain the stress-strain relations of Murnaghan's quadratic theory (§49)[10]. Taking up the subject afresh,[11] Riz observes from (50.4) that if the original Riz & Zvolinski hypothesis be referred to stress taken with respect to the undeformed area, it leads to (50.3). He prefers, however, to retain A, B, and C in (50.4), and claims that experience indicates two relations between them[12].

There is no reason to accept the premises or the conclusions of these theories.

51. Semi-empirical quasi-linear theories. There have been several attempts to describe strain-hardening phenomena by supposing the material obeys some semi-empirical quasi-linear stress-strain relationship in loading, but follows the Hooke-Cauchy law in unloading[1]. Since the authors writing in this field seldom trouble to define their tensors, it is difficult to assign a precise meaning to their remarks except in the range of infinitesimal strains; the following summary of several proposals in a fashion which is tensorially legitimate for finite strain in isotropic media is therefore perhaps a misrepresentation.

A relation equivalent to Δ_t = const. was proposed as a law of strain-hardening by Huber[2] and tested experimentally by Roš & Eichinger[3]. More generally, R. Schmidt[4] proposed $\Delta_t = f(\Delta_e)$; if it be assumed further that the stress-strain

not reduce to (50.2). From REINER's theory (§45) follows that in order for stress-strain relations which are quadratic in **e** to reduce to (50.2) when terms of third and higher order in the δ_i are neglected, it is necessary and sufficient that $t^i{}_j = \lambda_E[I_e + \frac{3}{2}I_e^2 - 3II_e]\delta^i{}_j + 2\mu_E e^i{}_j + 3\mu_E e^i{}_k e^k{}_j$; these relations are not derivable from a strain energy, even as a quadratic approximation.

[9] [1939, **14**]. In both [1938, **1**] and [1939, **14**] they apply their theory to a study of the increase in torsional rigidity of a bar subjected to tension, using an identical method but obtaining different results in the two cases. Simultaneous flexure and extension are discussed in [1939, **16**] [1944, **18–19**] [1947, **24**]; bending of a bar by both a couple and a transverse force in [1942, **19**]; by two couples, [1947, **24**]. An elliptical cross-section is considered in [1939, **3**]; a general cross-section again in [1943, **16**].

[10] They claim that from their assumption it is easy to find the special values for l and n stated in Note 8 and also $m = -3(\lambda_E + 3\mu_E)$, but they do not state what this assumption is.

[11] [1947, **7**]. This paper is marred by serious misprints.

[12] He claims that the vanishing of the cross-stress in simple extension yields $A\nu_E^2 + B(1 - 2\nu_E) + 3\nu_E C(1 - \nu_E) = 0$, and that the conditions of equilibrium in simple shear require $A + C = \lambda_E + 3\mu_E$. In a later paper [1948, **50**], in which he attempts to generalize HENCKY's theory of plasticity, he refers to the former condition as $A\nu_E^2 + B(1 - 2\nu_E)^2 + C\nu_E(1 + \nu_E) = 0$. The work of RIZ & ZVOLINSKI is deservedly criticized by SETH [1950, **18**, §4].

[1] A means of determining stress-strain relations of this type from experimental data in which all quantities depend upon a single parameter is constructed by GOLDENBLAT [1949, **44**]. Cf. §41[8].

[2] [1904, **2**]. The above mathematical form was given by v. MISES [1913, **2**], who proposed this relation as a plastic yield condition.

[3] [1926, **1**] and subsequent papers.

[4] [1932, **9**, §II].

relations are quasi-linear, then by (9.5) this proposal is equivalent[5] to $\mathcal{G}_1^{t(e)} = f(\Delta_e)$. Nadai[6] prefers $\Delta_t = f(\Delta_h)$, from which follows a similar conclusion with regard to a quasi-linear relation between **t** and **h**. Schlechtweg[7] proposes a theory equivalent to a special case of Schmidt's: $I_e = AI_t$, $\Delta_e = B\Delta_t(1 + CI_t + D\Delta_t + E\sqrt{\Delta_t}I_t)^{-1}$, where $A, \cdots, E$ are coefficients. For anisotropic "elastoplastic" media Locatelli[8] proposes a relation which can be written

$$\mathbf{t} = \mathbf{t}^*[1 + \overline{II}_{t^*}f(\overline{II}_t)]^{-\frac{1}{2}}, \tag{51.1}$$

where $\mathbf{t}^*$ is the stress according to the linear theory: $t^{*i}{}_j \equiv c^i{}_j{}^k{}_l e^l{}_k$, while $f(u)$ is a non-negative monotone increasing function which approaches a constant as $u \to \infty$. Udeschini[9] prefers

$$\mathbf{t} = \mathbf{t}^*[1 + A(II_{t^*} - I_{t^*}{}^2/3)]^{-\frac{1}{2}}. \tag{51.2}$$

Gleyzal[10] calls a quasi-linear relation between **t** and **e** with $\Delta_t = f(\Delta_e)$, $I_t = g(I_e)$ "general stress-strain laws of elasticity and plasticity". He recommends $I_t = 3\kappa I_e$ "in view of experimental evidence"; from (9.5) then

$$\mathbf{t} = \frac{1}{3}\left(\kappa - \frac{f(\Delta_e)}{\Delta_e}\right)\mathbf{I} + \frac{f(\Delta_e)}{\Delta_e}\mathbf{e}; \tag{51.3}$$

since the coefficients now are functions of Δ_e only, for an arbitrary hydrostatic pressure p the relation between $\mathbf{t} - p\mathbf{I}$ and $\mathbf{e} - p\mathbf{I}/3\kappa$ is the same as that between **t** and **e** (cf. §6). Davis[11] generalizes Gleyzal's relations in the case of infinitesimal strain by relinquishing the quasi-linearity and instead letting $(2t_2 - t_1 - t_3)/(t_1 - t_3)$ be an experimentally determined function of the same combination of principal strain components; this proposal is not admissible for isotropic media[12] because it is not generally expressible in terms of invariants, but if it could be corrected it would amount to a choice of a special form for $\mathcal{G}_2^{t(e)}(I_e, II_e, III_e)$. Although the authors of several of the foregoing mutually contradictory theories justify them by statements about experiments[13], they appear to consist solely in a rather arid type of abstract speculation.

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[5] PHILIPPIDIS [1947, **12**] gives the mistaken impression that from $\Delta_t = f(\Delta_e)$ one may conclude that quasi-linear stress-strain relations exist.

[6] [1933, **5**, §II] [1937, **4**, eq. (31)].

[7] [1934, **4**, Part I, §3]. For an earlier proposal of SCHLECHTWEG, v. §41[1].

[8] In [1940, **14**] the theory is applied to simple extension, bending, and torsion; in [1941, **16**], to dislocations.

[9] [1941, **17**] [1944, **17**, §2].

[10] [1945, **2**].

[11] [1948, **25**].

[12] NADAI [1931, **11**, Ch. IV, eq. (14)] proposes $(2t_2 - t_1 - t_3)/(t_1 - t_3) = (2e_2 - e_1 - e_3)/(e_1 - e_3)$ in this connection; this special form is admissible because by (9.4) it is equivalent to $\mathcal{G}_2^{t(e)} = 0$, i.e., to the statement of quasi-linearity.

[13] GLEYZAL's experimental curves [1945, **2**, p. A-263] refer to the tensile test, the results of which can be fitted by any stress-strain relationship containing an arbitrary function of strain.

Weissenberg[14] proposes as a theory of "rheological phenomena" the general quasi-linear relationship between stress **t** and "recoverable" strain **e**, the latter being measured with respect to the relaxed state assumed after deformation has ceased (cf. §45). He claims that this law suffices to describe the results of a series of striking experiments on the rotational shearing of liquids, in which the Poynting effect appears in the form of strong curvature of the free surface. The application of a theory of elasticity to fluids is somewhat artificial, and Rivlin has shown that these same phenomena are predicted by a simple and natural theory of fluid dynamics (§72)[15]. Weissenberg's theory is subject *a fortiori* to all the serious objections raised against Reiner's theory (§46).

52. Special forms for the strain energy.[1] I. Murnaghan's "proper density" theory. Signorini[2] strongly criticises theories of the type discussed in §§50–51 because of their neglect of the requirement[3] $\Sigma \geq 0$. Although Murnaghan's

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[14] [1947, **8**]. In [1949, **31**, §3] WEISSENBERG gives more details of his experiments, and also proposes a generalization [p. 135] of his theory, employing rather vague undefined terms which I am unable to understand.

[15] OLDROYD [1950, **3**, Note at end] questions RIVLIN's discussion of the possible effects of elasticity of the fluid in these experiments.

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[1] Several proposed forms for the strain energy have already been noted: $(43.1)_1$, $(43.1)_2$, (49.2), (49.3), that referred to in §49[9], (50.1), $(50.4)_2$. Three more are now mentioned:

(A). OSEEN's theory [1929, **7**]. $\Sigma = (III_c)^{-\frac{1}{2}} [\lambda_E K + \mu_E L]$, where $2K \equiv (III_{c^{-1}})^{\frac{1}{2}} - (III_c)^{\frac{1}{2}} - 2$, $4L \equiv I_{c^{-1}} + I_c - 6$.

(B). DEUKER's theory [1941, **8**]. I am unable to understand the proposed strain energy or to follow the analysis. The end result is $T^\alpha{}_\beta = \lambda_E [I_E - (I_E{}^2 - 2\, II_E)] \delta^\alpha{}_\beta + 2\mu_E[E^\alpha{}_\beta - E^\alpha{}_\gamma E^\gamma{}_\beta]$, which cannot be derived exactly from a strain energy. The theory is applied to a study of elastic stability in general and [1943, **7**] to an elaborate investigation of the stability of thin shells, particularly those of circular cylindrical or spherical form.

(C). WEBER's theory [1948, **31**]. The objective is to obtain legitimate stress-strain relations of the type proposed empirically for strain-hardening (§51), but the analysis is in material variables, so that the results are basically different from those of others working in this field (the stress tensor is not defined, but since the stress-strain relations are $(39.4)_1$, $T^\alpha{}_\beta$ is the only correct possibility; the paper is confused by serious misprints.) WEBER finds that if $\Sigma = A_1(I_E/3) + A_2(3\Delta_E{}^2/2)$, then $3\mathcal{G}_0^{T(E)} = A_1' - I_E A'$, $\mathcal{G}_1^{T(E)} = \Delta_T/\Delta_E = A_2'$, $\mathcal{G}_2^{T(E)} = 0$, $I_T = A_1'$; the stress-strain relations are material analogues of GLEYZAL's equations (51.3), which cannot be derived from a strain energy.

[2] [1942, **3**, p. 65]. He particularly objects to SETH's theory (cf. §49[6] and [1948, **23**]).

[3] This requirement arises because the strain is measured with respect to an unstressed configuration. While in a theory of initial stress there must be some irreversibility relations (cf. [1949, **12**, §IV]), it is not clear what they should be, for a small amount of energy put into a stressed body can cause it to give out a much greater amount, as when a hemisphere initially strained into a nearly plane configuration if deformed but slightly further will then of itself turn inside out and thus release the initially stored energy as well as that recently added. We should have $d\Sigma > 0$ in a loading process, but a precise definition is lacking. Cf. (41.14). In the theory of infinitesimal strain Σ is a quadratic form in the $e^i{}_j$, and the conditions that this form shall be positive are $\mu_E \geq 0$, $3\lambda_E + 2\mu_E \geq 0$, or $-1 \leq \nu_E \leq \frac{1}{2}$. In proposing his form for the strain energy (cf. Note 1 (A)), OSEEN required that not

first treatment[4] is open to Signorini's objection that the higher order theories simply introduce more constants so as to yield a better empirical fit to experimental data, he soon proposes[5] in effect

$$\Sigma = \frac{\lambda_E + 2\mu_E}{2} I_e^2 - 2\mu_E II_e; \tag{52.1}$$

from $(43.3)_2$ follows

$$t^i{}_j = \sqrt{1 - 2I_e + 4II_e - 8III_e}\{\lambda_E I_e \delta^i{}_j + [2\mu_E - 2\lambda_E I_e]e^i{}_j - 4\mu_E e^i{}_k e^k{}_j\}. \tag{52.2}$$

No significant applications of this theory have been made.

53. Special forms for the strain energy. II. Signorini's exact quadratic theories. Signorini has long contended that progress in the theory of finite elastic strain depends upon the determination of the correct form of the strain energy and has sought one leading to the simplest possible stress-strain relations.

In his earlier work[1] he showed that in order for t_i to be exactly a quadratic function of E_i in an isotropic body it is necessary and sufficient that

$$\begin{aligned} \Sigma &= \mu_E - 2k - \sqrt{1 + 2I_{e^*} + 4II_{e^*} + 8III_{e^*}}\,\{(\mu_E - 2k)(1 - I_{e^*}) \\ &\qquad - \tfrac{1}{2}(\lambda_E - \mu_E + 6k)I_{e^*}^2 + 4kII_{e^*}\}, \\ t^i{}_j &= [\lambda_E I_{e^*} + \tfrac{1}{2}(\lambda_E - \mu_E + 6k)I_{e^*}^2 - 4kII_{e^*}]\delta^i{}_j \\ &\qquad + 2[\mu_E + (\lambda_E - \mu_E + 2k)I_{e^*}]e^{*i}{}_j + 4ke^{*i}{}_k e^{*k}{}_j, \end{aligned} \tag{53.1}$$

where $2\mathbf{e}^* \equiv \mathbf{c}^{-1} - \mathbf{I}$, $\mu_E > 0$, $3\lambda_E + 2\mu_E \geq 0$, $k \geq 0$.

Turning away from this theory, in his recent work[2] he shows that in order for $\mathbf{t}$ to be exactly a quadratic function of $\mathbf{e}$ it is necessary and sufficient that

$$\Sigma = \frac{-p + \alpha(1 - I_e) + \beta I_e^2 + cII_e}{\sqrt{1 - 2I_e + 4II_e - 8III_e}} - \gamma\rho. \tag{53.2}$$

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only shall Σ remain constant in a rigid motion, but also that it shall *not* remain so in an orthogonal transformation whose Jacobian is −1, and on this ground he criticised other forms for Σ.

[4] In his first treatment of hydrostatic pressure [1937, **1**, §4], MURNAGHAN in effect simply supposes the pressure to be a quadratic function of the change of volume, with coefficients to be determined by experiment.

[5] Tentatively in his treatment of simple extension [1937, **1**, §5] and definitely in [1941, **1**, pp. 129, 135]: "The main defect of the classical theory has been its neglect of the variation of density in the compressed body." He applies his theory to the cases of hydrostatic pressure [pp. 129–132] and of a hollow circular cylinder subject to internal pressure [pp. 132–135]. BIRCH [1938, **5**] adopts (52.1). Cf. §44[7].

[1] [1930, **8**, §9].

[2] [1942, **3**, pp. 67–68] [1945, **7**, pp. 164–167] [1949, **15**] [1949, **39**, Cap. II]. I am obliged to Professor SIGNORINI for use of the third of these papers in MS.

Taking the state at $\theta = \theta_0$ as unstressed, we obtain $p = 0$. The coefficients may be identified by noting that for infinitesimal isothermal strains from this unstressed state we obtain $2\beta - \alpha = \lambda_E$, $\alpha = \mu_E$ for the Lamé coefficients of the classical theory. Signorini gives special attention to the case $c = 0$, which leads to

$$\Sigma = \frac{\mu_E(1 - I_e) + \frac{\lambda_E + \mu_E}{2} I_e^2}{\sqrt{1 - 2I_e + 4II_e - 8III_e}} - \mu_E \tag{53.3}$$

and to the quasi-linear[3] stress-strain relations

$$t^i{}_j = \left[\lambda_E I_e + \frac{\lambda_E + \mu_E}{2} I_e^2\right] \delta^i{}_j + [2\mu_E - (\lambda_E + \mu_E) I_e] e^i{}_j . \tag{53.4}$$

Necessary and sufficient conditions that (53.3) be positive (subject, of course, to the condition $e_i \leq \frac{1}{2}$), are $\mu_E > 0$, $-\frac{5}{8} \leq \nu_E \leq \frac{1}{2}$.

In simple extension Signorini[4] obtains

$$T \equiv t_1(1 + \delta_2)^2 = (\lambda_E + \mu_E) \left\{ \frac{1 + 2\nu_E + e_1 - (e_1)^2}{\sqrt{1 + (e_1)^2 + 4\nu_E e_1}} - 1 - 2\nu_E + e_1 \right\} . \tag{53.5}$$

Necessary and sufficient conditions for the right-hand side to be a real continuous function of δ_1 for $-1 \leq \delta_1 \leq \infty$ are precisely the same as for $\Sigma > 0$. The curve of T against δ_1 is of the same general character as Seth's (49.6)[5], requiring an infinite pressure for $\delta_1 = -1$, having an asymptote $2T = (\lambda_E + \mu_E) \{\sqrt{5 + 8\nu_E} - 1 - 4\nu_E\}$ as $\delta_1 \to \infty$, and exhibiting positive curvature everywhere provided $\nu_E > 0$. If we expand both (49.6) and (53.5) in powers of δ_1, however, we find that even the second order terms do not agree.

For simple shear (42.14) we obtain from (53.4)

$$\mathbf{t} = -\tfrac{1}{2}\lambda_E K^2 \left[1 - \left(1 + \frac{\mu_E}{\lambda_E}\right)\frac{K^2}{4}\right] \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} + \mu_E K \left[1 + \left(1 + \frac{\lambda_E}{\mu_E}\right)\frac{K^2}{4}\right] \left\{ \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} - K \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \right\} . \tag{53.6}$$

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[3] The case $c \neq 0$ is considered by Tolotti [1942, **12**], who finds necessary and sufficient conditions in order that $\Sigma \geq 0$, and considers the cases of hydrostatic pressure and simple extension.

[4] [1942, **3**, pp. 69–71] [1949, **15**, §5]. This and other examples are presented in [1949, **39**, Cap. II, §2].

[5] One may contrast this result with the thoroughly unsatisfactory parabolic stress-strain curve with axis parallel to the axis of stress which was obtained by Murnaghan [1937, **1**, §5].

Signorini[6] solves the problem of bending of a block. Supposing that the curved cylindrical faces be unstressed, and that suitable surface tractions be applied to the four plane faces, he shows that for any prescribed angle of flexure and change of altitude there exists one and only one solution, which he exhibits.

Signorini[7] analyzes thermoelastic effects in his special theory, obtaining the following expressions[8] for the coefficients in (53.2):

$$p = \frac{\kappa^2 - 1}{8} [3\kappa^2(\lambda_\theta + \mu_\theta) + 9\lambda_\theta + 5\mu_\theta],$$

$$\alpha = \frac{\kappa^2}{2} [3\kappa^2(\lambda_\theta + \mu_\theta) - 3\lambda_\theta - \mu_\theta],$$

$$\beta = \frac{\kappa^4}{2} (\lambda_\theta + \mu_\theta), \qquad \gamma = \mu_\theta v_\theta + \int^\theta d\xi \int^\xi c_p(\zeta) \frac{d\zeta}{\zeta}, \tag{53.7}$$

where κ is the coefficient of expansion arising from a change of temperature from θ_0 to θ, λ_θ and μ_θ are the Lamé constants corresponding to the temperature θ, and c_p is the specific heat at constant pressure. He concludes also that in the limiting case of a solid which can suffer no thermal expansion, for each θ we shall have $d^2\mu_\theta/d\theta^2 < 0$, $d^2(9\lambda_\theta + 5\mu_\theta)/d\theta^2 < 0$, and consequently $d^2E_\theta/d\theta^2$, where E_θ is Young's modulus at temperature θ. These inequalities are in accord with experimental evidence.

Since his theory satisfies every qualitative requirement, Signorini expresses the hope that it may be borne out quantitatively as well. Some doubts must be expressed, however. Comparison of (53.6) with (49.7) shows that *in simple shear, the results of Seth's and Signorini's theories agree as far as terms of second order*. While this fact might be taken as indicating Seth's theory to be a little better than its derivation would suggest, comparison of (53.6) with Rivlin's general solution (42.15) reveals rather a common defect of *both* theories. While by (42.15) the normal stresses t^x_x, t^y_y, and t^z_z are generally governed by independent coefficients, in both Seth's and Signorini's theories we always have $t^x_x = t^z_z$ in a shear of any magnitude. In the case of Seth's theory, the cause is to be found in the *ad hoc* character of the stress-strain relations; in Signorini's case, from his putting $c = 0$ and thus rendering Σ independent of $I_{c^{-1}}$. The defect is made more serious by the experimental fact, mentioned in §41, that in rubber $\partial\Sigma/\partial I_{c^{-1}} \approx 5\partial\Sigma/\partial II_{c^{-1}}$.

Comparison of Seth's and Signorini's theories is interesting. Seth's theory, adequate for treating large rotations and displacements so long as the strain be very small, at moderate strains exhibits *some* of the qualities of the general theory. Such anomalies as were noted in the case of Seth's theory (§49) in the case of large strains cannot occur in Signorini's, which is perfectly consistent with the principles of mechanics, and thus will exemplify the *type* of behavior which

[6] [1949, **15**, §6] [1949, **39**, Cap. II, §3]. His result is not included in RIVLIN's general solution (42.29), because RIVLIN considers only incompressible materials.

[7] [1945, **7**, pp. 166–167] [1949, **39**, Cap. II, ¶¶6–7].

[8] TOLOTTI [1943, **12**] readily obtains (53.7) from an appropriate special case of (44.7).

can be expected from a compressible body capable of suffering large elastic strain of *any* magnitude. But in Signorini's theory, too, there is an *ad hoc* quality, for why should $\mathbf{t}$ be a quadratic function of $\mathbf{e}$, or of any other arbitrarily selected strain measure? This arbitrariness manifests itself in the case of simple shear, mentioned above, and may be expected to reduce the quantitative usefulness of Signorini's theory in general.

The work of Signorini, distinguished by careful treatment of mechanical questions and by painstaking and accurate analysis, is the only serious advance up to the present in specific problems of large strain of compressible bodies. I cannot help questioning, however, the underlying idea of guessing at some apparently simple form for Σ. It would be preferable to obtain *general* solutions, like those of Rivlin for incompressible bodies, valid for any type of strain energy, but for compressible bodies no such solutions have yet been found except for the trivial, though instructive, case of homogeneous strain.

54. Special forms for the strain energy. III. The Mooney-Rivlin theory for rubber. The foregoing approximations are directed mainly toward application to structural materials, where purely elastic behavior, if it exist at all, is limited to an extension range $|\delta_i| < 0.1$ (or often much less). Rubber, the first example of an elastic material which comes to mind, for many years defied all efforts to describe its behavior by a theory of elasticity. It is virtually incompressible[1], and its perfectly elastic range is often as great as $-\frac{1}{2} \leq \delta_i \leq 3$. Although in many deformations $I_{c^{-1}}$ and $II_{c^{-1}}$ assume values up to 30, nevertheless Mooney[2] has suggested using the two terms of lowest order in the series (43.4):

$$\Sigma = \tfrac{1}{2}\alpha(I_{c^{-1}} - 3) + \tfrac{1}{2}\beta(II_{c^{-1}} - 3), \qquad t^i{}_j = -p\delta^i{}_j + \alpha(c^{-1})^i{}_j - \beta c^i{}_j. \quad (54.1)$$

Necessary and sufficient for $\Sigma \geq 0$ are the conditions $\alpha \geq 0, \beta \geq 0$, which follow also from (41.14). The Mooney theory gives results in excellent agreement with measurements[3] on all sorts of extensile and shearing deformations of rubber with δ_i taking values as high as 1. Later experimental work of Rivlin & Saunders[4], in which the extension range and variety of deformation is much greater, has indicated that the form

$$\Sigma = \tfrac{1}{2}\alpha(I_{c^{-1}} - 3) + f(II_{c^{-1}} - 3) \quad (54.2)$$

fits all data known up to the present time with sufficient accuracy, but in any case the Mooney theory gives a good first approximation. Because of the very large values of the arguments $I_{c^{-1}} - 3$ and $II_{c^{-1}} - 3$, this fact indicates that the higher order coefficients in the series (43.4) are vastly inferior to the first

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[1] Experiments on the compressibility of rubber are described in [1948, **29**].

[2] [1940, **7**, eq. (14) and pp. 586–587].

[3] [1948, **24, 30**]. A theory of photoelasticity for interpreting experiments on large deformation of rubber is discussed in [1942, **11**] [1947, **10, 13**] [1949, **10**, Ch. VIII].

[4] [1951, **2**, esp. §19].

two; and, as noted in §41, we have roughly $\alpha = 5\beta$ for rubber. From $(6.11)_5$ and $(15.1)_2$ it follows that $(54.1)_2$ is equivalent to

$$\begin{aligned} \mathcal{G}_0^{t(e)} &= -p + \alpha(1 - 2I_e + 4II_e) - \beta, \\ \mathcal{G}_1^{t(e)} &= 2[\alpha(1 - I_e) + \beta], \qquad \mathcal{G}_2^{t(e)} = 4\alpha. \end{aligned} \tag{54.3}$$

Thus all the characteristic effects of non-linear elasticity, and in particular the Poynting effect[5], are present even if $\beta = 0$, but of course their quantity is more crudely specified than in the fully general theory. The expressions (54.3) indicate further that the Mooney theory, like Signorini's (§53), gives **t** as exactly a quadratic function of **e**. Hence in particular (§10), shear stress is proportional to shear strain, just as in the classical theory,[6] and by (42.15) we have $\mu_E = \alpha + \beta$.

The special case $\beta = 0$ in (54.1) is of particular interest, since Guth & James (1939)[7] and numerous others[8] have succeeded in deriving the resulting form for Σ in certain simple cases from a theory of long chain molecules[9]. The corresponding material[10] is called *neo-Hookean* by Rivlin, who began his researches

[5] REINER [1951, **7**] states that the distribution of pressure on the plane ends of a twisted cylinder given below in the text is not "in accordance with what is required by the POYNTING effect," claiming that his "coefficient of cross-elasticity" $\mathcal{G}_2^{t(e)}$ is neglected. The formula $(54.3)_8$ shows that the criticism is wholly ill-taken, since even in RIVLIN's neo-Hookean theory this coefficient is four times the shear modulus. From a more general standpoint the criticism is yet more inappropriate, since RIVLIN's beautiful solution (42.22) (42.24) is valid for any form of strain energy.

[6] MOONEY [1940, **7**, p. 582] begins his considerations with the experimental fact that the shearing stress and shearing strain are proportional even for very great strains; cf. [1948, **7**, §4]. WALL & TRELOAR [1942, **5**, pp. 482–488] [1943, **2**, p. 530] [1943, **3**, pp. 37–38] give a separate consideration to shearing, and conclude that "HOOKE's law" is satisfied; they do not observe the POYNTING effect.

[7] [1941, **5**, §3] [1942, **7**] [1943, **5**, §3] [1944, **8**].

[8] [1942, **5**, p. 487] [1943, **2–4**, **9**] [1944, **3**, §III, ¶¶3–4]. [1938, **4**] and [1942, **4**, p. 134] are less successful attempts. Comparison of theory with experiment is given in [1944, **4**] [1944, **3**, §IV]. For reviews of the structure theories of polymers, v. [1948, **22**, C II 1] [1949, **10**, Chs. III, IV].

[9] A first approximation is $\alpha = Nk\theta$, where k is BOLTZMANN's constant and N is the number of molecules per unit volume; a more accurate theoretical value of α is given in [1944, **8**, §3]. Thermoelastic experiments on rubber are described in [1948, **30**].

[10] Since $(54.1)_1$ can be written $\Sigma = \alpha I_E$ when $\beta = 0$, comparison with $(43.1)_1$ shows that the neo-Hookean theory is based upon the approximation of lowest possible order for incompressible bodies in a series expansion in **E**, and in this respect is the counterpart of the ST. VENANT-KIRCHHOFF theory (§49) for compressible bodies; because of the arbitrary hydrostatic pressure p, the condition $\alpha = 0$ is no longer as in the case of compressible bodies a statement that the natural state be unstressed. On the other hand, the last statement in §43 shows that if we define an approximation in terms of the extensions, then the MOONEY theory is the proper counterpart for incompressible bodies of the ST. VENANT-KIRCHHOFF theory for compressible bodies, since each retains all terms of degree 2 in the δ_i in the series expansion for Σ.

It is hard to see how so simple a theory could have been overlooked until 1939, but the explanation perhaps lies first in the still prevalent custom of avoiding tensorial methods, so that the distinction between material and spatial stress components, although in the with an extensive study of it[11]. The same form for the strain energy was proposed earlier by Kubo[12], whose work was not published until after Rivlin's had begun to appear. Rivlin[13] analyzes the possible states of pure homogeneous strain corresponding to given uniform surface tractions upon the surface of a cube. Even in this simplest of deformations in the simplest of the exact non-linear theories, there are ranges of loadings in which *the deformation is not unique* and thus the outcome of an experiment depends upon the manner in which the load is applied. (Cf. the complementary result of Armanni, §50).

Returning to the more general Mooney theory, Rivlin[14] notes that since $c_i \geq 0$, $\alpha \geq 0$, $\beta \geq 0$ it follows that for given $t^i{}_j$ the quantities c_i are yielded by $(54.1)_2$ as single-valued monotone decreasing functions of p; hence $III_c = c_1c_2c_3$ is such a function; and hence there is one and only one real value of p compatible with the condition $III_c = 1$. Thus not only does $\mathbf{c}$ or $\mathbf{c}^{-1}$ determine $\mathbf{t}$, but, conversely, $\mathbf{t}$ determines $\mathbf{c}$ and $\mathbf{c}^{-1}$. Rivlin concludes also that the equilibrium is always stable.

The general solutions given in §42 can be specialized to the present case at a glance. For simple extension we have from (42.11)

$$t_1(1 + \delta_2)^2 = \left[1 + \delta_1 - \frac{1}{(1 + \delta_1)^2}\right]\left[\alpha + \frac{\beta}{1 + \delta_1}\right], \tag{54.4}$$

showing that $E = 3(\alpha + \beta) = 3\mu_E$ as in the classical theory and that in extension the effect of β is small, though in great compressions its effect may predominate over that of α. The result (54.4) was discovered for the case $\beta = 0$ by Guth & James[15]. Unfortunately, unlike the extension curves of Seth and Signorini (§§49, 53), this one approaches no finite limit as $\delta_1 \to \infty$.

For torsion of a circular cylinder the normal tension $t^z{}_z$ on the plane ends,

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literature nearly 100 years, is not generally understood even now, and second in the far reaching effect of the constraint of incompressibility, which though discussed by POINCARÉ (§39[9]) has reappeared in the literature only recently.

[11] For the neo-Hookean material, RIVLIN considers the following problems: in [1948, **7**]: simple extension, simple shear, determination of the pressure when the stress is given; in [1948, **8**]: uniqueness of strain corresponding to given stress; in [1948, **9**]: simple torsion or combined torsion and extension of a hollow circular cylinder. Some of RIVLIN's work is reviewed in [1949, **10**, Ch. 7, §4 and Ch. 13]. Cf. §42.

[12] KUBO [1948, **55**] considers the following special situations: a spherical shell, a cylindrical shell, a medium in transverse vibration.

[13] [1948, **8**].

[14] [1948, **11**]. The case $\beta = 0$ is treated in [1948, **7**, §§5–6].

[15] [1941, **5**, eq. (7)]. Cf. [1948, **7**, §4]. According to GUTH & JAMES [1941, **6**] [1943, **5**, §6], (54.5) is an approximation valid only for relatively small extensions. For the general case they give $t_1(1 + \delta_2)^2 = \alpha[\mathcal{L}^{-1}(k[1 + \delta_1]) - (1 + \delta_1)^{-\frac{3}{2}} \mathcal{L}^{-1}(k[1 + \delta_1]^{-\frac{1}{2}})]$, where $\mathcal{L}(\rho) \equiv \coth \rho - \rho$. In [1943, **15**, §10], [1947, **14**] GUTH gives instead $t_1(1 + \delta_2)^2 = \alpha[\mathcal{L}^{-1}(k[1 + \delta_1]) - 3k(1 + \delta_1)^{-2}]$. For expositions of the GUTH-JAMES theory, see [1946, **13**], [1948, **22**, Ch II 1C], [1949, **18**]. Experiments on the extension of rubber are discussed in [1950, **16**], [1951, **2**, §§9–10].

given in general by (42.22) and (42.24), becomes[16] $t^z_{z\cdot} = -K^2[\frac{1}{2}\alpha(a^2 - r^2) + \beta r^2]$. While in the general case we were able to conclude only that the resultant force was always a pressure, for the Mooney material the local end stress is everywhere a pressure. Moreover, this result enables us to distinguish experimentally between the Mooney and neo-Hookean materials, for in the latter the normal pressure vanishes at the edge $r = a$, while in the former it attains the value $K^2\beta a^2$. In an experiment surely to be regarded as classical, Rivlin[17] verifies all predictions of the Mooney theory for torsion of a rubber cylinder through large angles of twist. The normal pressure, assumed to be proportional to the bulging of the rubber into small holes in the metal plate bonded to one end of the cylinder[18], does not vanish at the edge, thus showing with finality that the neo-Hookean theory is inadequate. From this experiment Rivlin concluded $\beta = 0.14\alpha$.

Added in proof. Using complex variable methods, Green & Shield[19] construct a general theory of the moderate torsion of a cylinder of Mooney material of any simply-connected cross-section, discarding powers of the twist higher than two. They outline the solution also for the case of a large extension followed by a moderate twist.

This section must close with a reference to the end of §42, where, based upon the numerous exact solutions obtained, was expressed the more recent view of Rivlin, diametrically opposed to that of Signorini, that it is futile and unnecessary to conjecture the form of Σ. The main value of the Mooney theory is that it enables one sometimes in cases too difficult to be solved for general Σ to exhibit a solution which may serve as a first approximation.

IV D. The Jaumann-Murnaghan Rate of Deformation Theory

55. Initial stress. Even in their most general form (§45), the foregoing theories of isotropic elasticity admit at most a uniform hydrostatic stress $t^i_{\ j} = \mathcal{G}_0^{t(e)}(0, 0, 0)\delta^i_{\ j}$ in the unstrained configuration[1]. Now there may be very con-

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[16] [1947, **5**, §III]. Solution for an anisotropic cylinder is attempted in [1950, **10**, §7], where the anomalous results obtained arise no doubt from the fact that stress-strain relations of the type of §41, valid only for isotropic bodies, are erroneously employed.

[17] [1947, **5**, §§IV–V]. More refined torsion experiments are given in [1951, **2**, Part C].

[18] Experimental difficulties prevented measuring the force required to plug the holes.

[19] [1951, **15**, §§ 5–11].

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[1] From the evident generalization of $(43.1)_1$ to anisotropic bodies it is plain that the only stress possible in the unstrained state is *uniform*. Such stresses were included by GREEN [1841, **2**, p. 298]. KELVIN's criticism [1888, **3**, §§4–6] [1904, **3**, Addition to lecture XV, §§4–6] of GREEN's theory of initial stress is only partly sound.

siderable initial or residual stresses in a physical body, and the configuration the body would assume should all these stresses be relaxed is not known. Indeed, in many cases relief of stress is not possible without destroying the continuity of the body or effecting some physical change within it.

There have been three main routes of attack upon the problem of initial stress[2]. The first lies wholly within the bounds of the classical linear theory of elasticity, attempting by some artifice to calculate a stress system, associated with a strain from an unstressed natural state, which is appropriate for representing the stresses naturally occurring in physical bodies as they come to hand. The strain in question need not be physically realizable by a purely elastic process. The example of a tube formed by cutting out a sector from an unstressed tube and forcibly rejoining the edges is included among Volterra's celebrated "dislocations,"[3] which typify this kind of work; such a body may be regarded as initially stressed, since its surfaces are free of load but the interior stress is not zero. The stress in any subsequent displacement with infinitesimal gradients may be calculated independently of the amount of initial stress, by the usual method of superposition.[4] As soon as the gradients of the displacement required to return the body to its ideal unstressed natural state become large, this entire method is closed off.

The second route, a natural extension of the foregoing, rests upon calculating stress-strain relations modified to take account of initial stress ${}^1\mathbf{t}$, which, however, is presumed known *a priori*. The correct form of the result was obtained twice by Cauchy, first by the method of stress-strain relations[5] and second from a molecular model,[6] but we shall derive it from the general form of Green's theory of the elastic energy.[7]

We consider the case $x^i = y^i + u^i$, where $y^i(X^\alpha)$ is an arbitrary initial deformation and where the gradients of the second displacement u^i with respect to y^i are infinitesimal, and all quantities are referred to the same Cartesian

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[2] Much of the literature, some parts of which are summarized by NEMÉNYI [1931, **4**], is cloudy.

[3] [1907, **4**].

[4] Thus JOUGUET [1924, **7**] speaks of an "état . . . quasi-naturel à tensions petites." "Small" stress, i.e. stress negligible with respect to λ_E and μ_E, is not sufficient, however, since not only the strain but also the rotation must be infinitesimal before superposition is justified.

[5] [1829, **1**, eqq. (36) (37)].

[6] [1829, **2**, eqq. (36) (37)].

[7] ST. VENANT's attempt [1863, **2**, ¶3] to derive CAUCHY's equations from GREEN's theory was faulty because of his erroneous formulation (cf. §49²) of the theory itself. PEARSON's criticism [1893, **2**, §§129–130] of ST. VENANT's derivation, however, appears to cast doubt on CAUCHY's result (cf. [1927, **3**, §75]), which in fact solves completely a problem to which a great deal of literature, much of it faulty, has been devoted: [1883, **3**, §§7–23] [1904, **3**, Addition to lecture XV, §§7–36] [1920, **1**, Note 2] [1921, **2**] [1924, **7**] [1928, **3**] [1929, **5**] [1932, **15**] [1939, **4**, §4] [1939, **5**, pp. 120–121] [1940, **4**, §4] [1942, **10**]. The derivation outlined in the text above is similar to one given by MURNAGHAN [1949, **41**, p. 333], who does not mention CAUCHY or any other source for the result itself.

frame. Let ${}^1\mathbf{E}$ be the strain tensor calculated from $y^i(X^\alpha)$, and let $\mathbf{E}$ be the total strain. Then $\mathbf{E}$ and ${}^1\mathbf{E}$ are related by (19.6). Supposing the body to be perfectly elastic with respect to the configuration X^α, by Boussinesq's stress-strain relations $(39.2)_2$ we have for the initial stress ${}^1\mathbf{t}$ and for the total stress $\mathbf{t}$

$$ {}^1t^{ij} = \frac{\rho_1}{\rho_0}\frac{\partial\Sigma}{\partial^1 E_{\alpha\beta}}\, y^i_{,\alpha} y^j_{,\beta}, \qquad t^{ij} = \frac{\rho}{\rho_0}\frac{\partial\Sigma}{\partial E_{\alpha\beta}}\, x^i_{,\alpha} x^j_{,\beta}. \tag{55.1} $$

Now by $(19.1)_3$ follows

$$ \frac{\rho}{\rho_0} = \frac{\rho_1}{\rho_0}\cdot\frac{\rho}{\rho_1} \approx \frac{\rho_1}{\rho_0}(1 - I_{\tilde{e}}), \tag{55.2} $$

where $\tilde{\mathbf{e}}$ is the infinitesimal strain tensor for the second deformation. Also

$$ \begin{aligned} \frac{\partial\Sigma}{\partial E_{\alpha\beta}} = f(\mathrm{E}) &\approx f({}^1E_{\gamma\delta} + \tilde{e}^i{}_j\, y_{i,\gamma} y^j{}_{,\delta}), \\ &\approx f({}^1E_{\gamma\delta}) + \frac{\partial f}{\partial^1 E_{\epsilon\zeta}}\, \tilde{e}^i{}_j\, y_{i,\epsilon} y^j{}_{,\zeta}, \\ &= \frac{\partial\Sigma^1}{\partial^1 E_{\alpha\beta}} + \frac{\partial^2\Sigma}{\partial^1 E_{\alpha\beta}\,\partial^1 E_{\epsilon\zeta}}\, \tilde{e}^i{}_j\, y_{i,\epsilon} y^j{}_{,\zeta}. \end{aligned} \tag{55.3} $$

Putting (55.2), (55.3), and (19.4) into $(55.1)_2$, by dropping all terms of order higher than one in the displacement gradients $u^i{}_{,j}$ and by employing $(55.1)_1$ we obtain Cauchy's result, which when written in tensor form, valid in all coordinate systems, is

$$ \begin{aligned} t^i{}_j &= {}^1t^i{}_j(1 - I_{\tilde{e}}) + {}^1t^{mj}u^i{}_{,m} + {}^1t^{in}u^j{}_{,n} + C^i{}_j{}^k{}_l\,\tilde{e}^l{}_k, \\ C^i{}_j{}^k{}_l &\equiv \frac{\rho_1}{\rho_0}\frac{\partial^2\Sigma}{\partial^1 E_{\alpha\beta}\,\partial^1 E_{\gamma\delta}}\, y^i{}_{,\alpha}\, y_{j,\beta}\, y^k{}_{,\gamma}\, y_{l,\delta}, \end{aligned} \tag{55.4} $$

where $u^i{}_{,m}$ is the covariant derivative of u^i with respect to y^m.

This important result is not sufficiently understood.[8] It gives a precise mathematical form to the common vague statement that an isotropic body subjected to large strain loses its isotropy: even if the displacement gradients $u^i{}_{,m}$ be small enough that (55.4) reduces to an apparent superposition formula $\mathbf{t} = {}^1\mathbf{t} + {}^2\mathbf{t}$, where ${}^2t^i{}_j \equiv C^i{}_j{}^k{}_l\,\tilde{e}^l{}_k$, if Σ be an isotropic function of $\mathbf{E}$ then neither $\mathbf{t}$ nor ${}^2\mathbf{t}$ can be an isotropic function of $\tilde{\mathbf{e}}$ unless the $y^i{}_{,\alpha}$ be infinitesimal also, or, in other words, *if an isotropic elastic body be subjected to a large strain followed by a small one, its response to the small one, considered by itself, is linear but not isotropic, and its tetradic $C^i{}_j{}^k{}_l$ of elastic coefficients $(55.4)_2$ for such small linear strains depends not only upon the elastic moduli but also upon the gradients $y^i{}_{,\alpha}$ of the large*

[8] LOVE [1927, 3, §75] describes the last term in (55.4) so vaguely that one easily gains the wholly false impression that for an isotropic body it is given by (1.1).

initial deformation. If, for example, the strain energy have the very simple form (49.2), then $(55.4)_2$ yields

$$C^{i\,k}_{j\,l} = \frac{\rho_1}{\rho_0}\,[\lambda_E\,(c^{-1})^i_{\;j}(c^{-1})^k_{\;l} + 2\mu_E(c^{-1})^i_{\;l}(c^{-1})^k_{\;j}], \tag{55.5}$$

where $\mathbf{c}^{-1}$ is calculated from $y^i(X^\alpha)$. By measuring the $C^{i\,k}_{j\,l}$ experimentally it might be possible to determine the unstressed natural state by integration. In any case the elegant formula (55.4) is difficult to use, since not only must ${}^1\mathbf{t}$ be known, but also there is no real simplification for isotropic bodies. The generalization to the case of large strain of an initially stressed body is hopelessly difficult.

Thus even if the theories based upon stress-strain relations be correct in principle, they are not properly applicable to many practical cases, since they employ strain measures we can never calculate because we do not know the ideal unstressed configuration with respect to which they are to be taken. A theory in which strains can be measured from any given initial configuration (cf. §34), and in which moreover the notion of isotropy can be put to real use in cases when it is relevant, is therefore desirable if it can be formulated correctly. The third route of attack has been therefore to abandon the idea of stress-strain relations, either partially[9] or wholly. An attempt in this direction is described in the succeeding sections.

56. Defining relations for the Jaumann-Murnaghan rate of deformation theory. Murnaghan[1] suggests therefore that the whole idea of a natural state be abandoned, as representing a sort of ideal behavior which is irrelevant: while indeed it is usually employed in motivation of the infinitesimal theory, the Hooke-Cauchy law (1.1) is also a first approximation to a quite different concept. Starting with a body subject at time t_0 to any state of stress ${}^1\mathbf{t}$, Murnaghan regards the present configuration as "built up from a succession of infinitesimal strains by a method of integration," the stress increments being computed from the strain increments in the limit by a relation having the form of the ordinary Hooke-Cauchy law (1.1). Although he does not give any general formulae,

[9] A purely formal definition of a "source" of initial stress is given in [1931, **3**]; while stress-strain relations are abandoned, the equations proposed are formal analogues of those of linear elasticity theory.

[1] [1944, **6, 13**] [1945, **11**] [1949, **6**, §3]. I am indebted to Professor MURNAGHAN for use of the last paper in MS. It is rather difficult to determine precisely what is assumed and what is not in MURNAGHAN's latest papers. Thus in [1949, **41**] he refers to his work on large hydrostatic pressure rather perplexingly as "some results of a comparison of the exact formula [i.e. BOUSSINESQ's formula $(39.2)_2$] with experiment". I am not sure that the contents of §§56–58 represents MURNAGHAN's own views; it should perhaps be regarded rather as an alternative development of equivalent results.

limiting his attention to special cases, a realization of his proposal is[2]

$$\overset{\circ}{t}{}^i{}_j = A^i{}_j{}^k{}_l\, d^l{}_k\,, \qquad A^{ijkl} = A^{jikl}. \tag{56.1}$$

More complicated equations of a similar type were proposed by Jaumann,[3] whose profound researches have not received the attention they deserve. We thus obtain a *rate of deformation theory*[4] in which any stress components $t^i{}_j$ satisfying Cauchy's equations (26.2) are admissible as initial values for the solutions of the system (56.1). The coefficients A^{ijkl} may be supposed to be functions of **t**. A body is now *elastically isotropic* if $\overset{\circ}{\mathbf{t}}$ be an isotropic function of **d**:

$$\overset{\circ}{t}{}^i{}_j = \lambda_{\mathrm{E}}\, d^k{}_k \delta^i{}_j + 2\mu_{\mathrm{E}}\, d^i{}_j\,. \tag{56.2}$$

The Hooke-Cauchy law (1.1) results as a first approximation[5] when the initial state is unstressed: $\mathbf{t} \approx \overset{\circ}{\mathbf{t}}\, dt$, $\tilde{\mathbf{e}} \approx \mathbf{d}\, dt$.

Suppose more generally that $\overset{\circ}{\mathbf{t}}$ be an isotropic but not necessarily linear function of **d**:

$$\overset{\circ}{t}{}^i{}_j = \mathcal{G}_0^{\overset{\circ}{\mathrm{t}}(\mathrm{d})}\delta^i{}_j + \mathcal{G}_1^{\overset{\circ}{\mathrm{t}}(\mathrm{d})} d^i{}_j + \mathcal{G}_2^{\overset{\circ}{\mathrm{t}}(\mathrm{d})} d^i{}_k\, d^k{}_j\,. \tag{56.3}$$

As in the classical theory, suppose that there are *only two dimensional moduli* (§47), *a natural elasticity* $\mu_{\mathrm{E}n}$ *of dimension* $\mathbf{ML}^{-1}\mathbf{T}^{-2}$ *and a reference temperature*

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[2] Hencky [1921, **2**, §2] has indicated another approach to rate of strain theories. Let a relation for a measure **m** of strain in terms of the stress **t** be given. To eliminate the influence of the natural state, differentiate materially, obtaining a power series for $\dot{\mathbf{m}}$ in terms of **t** and **t**. Eliminate $\dot{\mathbf{m}}$ by a kinematical formula giving $\dot{\mathbf{m}}$ in terms of **m**, **d**, and possibly **w**; then eliminate **m** wherever it occurs by means of the stress-strain relations. We now have a power series in **t**, $\overset{\circ}{\mathbf{t}}$, and possibly also $x^i{}_{,L}$. If all but the linear terms in this power series be neglected, (56.1) results.

[3] [1911, **5**, §IX]. In our notation Jaumann's equation (38) is $\dot{\Psi}^i{}_j + r^i{}_j{}^k{}_l \tilde{d}^l{}_k = 0$, where $(\alpha + 3\beta)\,\tilde{d}^l{}_k \equiv \alpha\, d^l{}_k + \beta d^m{}_n \delta^l{}_k$; his equation (43) is $t^i{}_j = \Psi^l{}_k \eta^k{}_l{}^m{}_n r^n{}_m{}^i{}_j - \frac{1}{2}\Psi^l{}_k \eta^k{}_l{}^m{}_n \Psi^n_m \delta^i{}_j$, where α, β, $r^i{}_j{}^k{}_l$, $\eta^k{}_l{}^m{}_n$ are material constants, and $\Psi^i{}_j$ is to be eliminated between the two equations. By a method of power series expansion, Jaumann shows that hi; theory yields (1.1) as a linear approximation, and concludes that the consequences of his theory and of the classical infinitesimal theory are the same within the range of experimentally observable phenomena. Cf. [1917, **5**, §§21–22] [1918, **1**, §§96–100].

[4] Murnaghan employs the misleading terms "local action" and "action at a distance" for the present and the natural state theories, respectively.

[5] Handelman, Lin, & Prager [1947, **9**, §§2–4] propose relations similar to an inverse of (56.2) for infinitesimal strains produced by loading in the strain-hardening range, after concluding that finite relations of the type (45.1) for loading in general cannot be combined with (1.1) for unloading. Equations similar to (56.2) but tensorially not admissible for isotropic media are proposed in [1948, **32**]. If I correctly understand the work of Swainger [1947, **21**, §4.1] [1949, **37**, §4.1], he puts each component of a certain undefined strain rate (§17[1]) separately proportional to the corresponding component of the stress rate $\overset{\circ}{\mathbf{t}}$; relations of this type are inadmissible for isotropic media unless the factors of proportionality for the separate components are the same. Cf. §60[2]. A pair of elaborate stress rate equations containing his "projective" rates (§21[2]) are proposed by Hencky [1949, **40**] as "universal equations of rheology".

θ_0 *of dimension* Θ. Let the coefficients $\mathcal{G}_{\Gamma}^{\dot{\mathrm{t}}(\mathrm{d})}$ be functions[6] of I_d, II_d, III_d, I_t, II_t, III_t, $\mu_{\mathrm{E}n}$, θ, θ_0. Since dim $\mathcal{G}_0^{\dot{\mathrm{t}}(\mathrm{d})} = \mathbf{ML^{-1}T^{-3}}$, it follows that $\mathcal{G}_0^{\dot{\mathrm{t}}(\mathrm{d})}/I_\mathrm{d}\mu_{\mathrm{E}n}$ is a dimensionless function of 9 quantities composed of three fundamental dimensions $\mathbf{ML^{-1}T^{-2}}$, $\mathbf{T}$, and Θ, and hence must reduce to a function of $9 - 3 = 6$ dimensionless ratios of these quantities:

$$\mathcal{G}_0^{\dot{\mathrm{t}}(\mathrm{d})} = \mu_{\mathrm{E}n} I_\mathrm{d}\, g_0^{\dot{\mathrm{t}}(\mathrm{d})}\left(\frac{I_\mathrm{t}}{\mu_{\mathrm{E}n}}, \frac{II_\mathrm{t}}{\mu_{\mathrm{E}n}^2}, \frac{III_\mathrm{t}}{\mu_{\mathrm{E}n}^3}, \frac{II_\mathrm{d}}{I_\mathrm{d}^2}, \frac{III_\mathrm{d}}{I_\mathrm{d}^3}, \frac{\theta}{\theta_0}\right),$$

say. Similar reasoning applied to $\mathcal{G}_1^{\dot{\mathrm{t}}(\mathrm{d})}$ and $\mathcal{G}_2^{\dot{\mathrm{t}}(\mathrm{d})}$ yields $\mathcal{G}_1^{\dot{\mathrm{t}}(\mathrm{d})} = \mu_{\mathrm{E}n} g_1^{\dot{\mathrm{t}}(\mathrm{d})}$, $I_\mathrm{d}\mathcal{G}_2^{\dot{\mathrm{t}}(\mathrm{d})} = \mu_{\mathrm{E}n} g_2^{\dot{\mathrm{t}}(\mathrm{d})}$, where the $g_\Gamma^{\dot{\mathrm{t}}(\mathrm{d})}$ are dimensionless. In order to prevent the stress rate from becoming singular when $\mathbf{d} = 0$ or $\mathbf{t} = 0$ it is now necessary to put $\mathcal{G}_2^{\dot{\mathrm{t}}(\mathrm{d})} = 0$ and to suppose $\mathcal{G}_0^{\dot{\mathrm{t}}(\mathrm{d})}$ and $\mathcal{G}_1^{\dot{\mathrm{t}}(\mathrm{d})}$ to be independent of II_d and III_d. Thus we obtain a quasi-linear theory[7] characterized by

$$\dot{t}^i{}_j = \mu_{\mathrm{E}n} I_\mathrm{d}\, g_0^{\dot{\mathrm{t}}(\mathrm{d})} \delta^i{}_j + \mu_{\mathrm{E}n}\, g_1^{\dot{\mathrm{t}}(\mathrm{d})} d^i{}_j, \tag{56.5}$$

where $g_0^{\dot{\mathrm{t}}(\mathrm{d})}$ and $g_1^{\dot{\mathrm{t}}(\mathrm{d})}$ are dimensionless functions of $I_\mathrm{t}/\mu_{\mathrm{E}n}$, $II_\mathrm{t}/\mu_{\mathrm{E}n}^2$, $III_\mathrm{t}/\mu_{\mathrm{E}n}^3$, θ/θ_0. If $\sqrt{II_\mathrm{t}}/\mu_{\mathrm{E}n} \ll 1$, (56.5) may be approximated by

$$\dot{t}^i{}_j = (\lambda_\mathrm{E} + \beta_0 I_\mathrm{t}) I_\mathrm{d} \delta^i{}_j + 2\mu_\mathrm{E}\left(1 + \beta_1 \frac{I_t}{\mu_{\mathrm{E}n}}\right) d^i{}_j, \tag{56.6}$$

where $\lambda_\mathrm{E}/\mu_{\mathrm{E}n}$, $\mu_\mathrm{E}/\mu_{\mathrm{E}n}$, β_0, β_1 are dimensionless functions of θ/θ_0.

57. Murnaghan's formula for hydrostatic pressure. By putting $t^i{}_j = -p\delta^i{}_j$, $3d^i{}_j = d^k{}_k\delta^i{}_j = \dot{\overline{\log v}}\, \delta^i{}_j$ into (56.5) we obtain

$$-v\frac{dp}{dv} = f(p), \tag{57.1}$$

derived by Murnaghan in a different way.[1] The linear approximation (56.6) yields $f(p) = a(p + A)$, where a and A are constants. In this case by integrating (57.1) we obtain

$$p = A\left\{\left(\frac{v_0}{v}\right)^a - 1\right\}, \tag{57.2}$$

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[6] This assumption is not the most general possible, since $\Phi(\mathbf{t}, \mathbf{d})$, $\Phi(\mathbf{t}^2, \mathbf{d})$, $\Phi(\mathbf{t}, \mathbf{d}^2)$, and $\Phi(\mathbf{t}^2, \mathbf{d}^2)$ might well be added to the list.

[7] The present theory is one of pure elasticity, since there is no "natural time" or "time of relaxation" (§62). In fact, (56.2) results from letting the relaxation time t_n approach ∞ in the equations $\dot{t}^i{}_j = \lambda_\mathrm{E} I_\mathrm{d}\delta^i{}_j + 2\mu_\mathrm{E}\, d^i{}_j + t_n^{-1}\{t^i{}_j - p\delta^i{}_j) - \alpha(p - \bar{p})\delta^i{}_j\}$, which were proposed by ZAREMBA [1903, **7**, eq. (28)] as a generalization of MAXWELL's theory of relaxation (§81).

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[1] [1944, **6**, p. 246] [1945, **11**] [1949, **6**, §3]. GLEYZAL [1949, **38**, §3] shows that from the assumption $p\delta^i{}_j x^j{}_{,\beta} X^\alpha{}_{,i} = A\tilde{E}^\alpha{}_\beta + B\tilde{E}^\gamma{}_\gamma\delta^\alpha{}_\beta$, where for $\tilde{\mathbf{E}}$ various rather arbitrarily selected combinations of $\mathbf{C}$, $\dot{\mathbf{C}}$, and their time integrals are selected, result various laws similar to (57.2). Simple extension is treated in the same tentative spirit.

where v_0 is the value of v when $p = 0$. *This formula distinguishes between hydrostatic pressure and hydrostatic tension:* for no finite value of p can v become zero, but if $p = -A$ then v becomes infinite, so that A is an upper bound for the rupture or yield stress. By choice of A and a, Murnaghan[2] fits within 1% the results of Bridgman's experiments on the compression of sodium up to pressures of 10^5 atm. If $a = 2$, (57.2) becomes equivalent to Laplace's law for the density of the earth.

Mr. R. Toupin & Dr. J. Ericksen have remarked that for any motion whatever the relation (57.2) is a consequence of (56.6), provided only that p be replaced by $\bar{p}$.

Comparing (57.1) with its counterpart (42.5) reveals a characteristic difference between the present theory and the classical natural state theory: when f is specified, (57.1) determines the equation of state $v = v(p)$ only to within an arbitrary constant, while the functional form of Σ completely determines the hydrostatic equation of state. Moreover, the striking special case (57.2) arises naturally by approximating (57.1), while, although it is quite compatible with the classical finite strain theory, it does not appear to result from (42.5) by any otherwise natural choice of the form of Σ.

58. Simple extension. To treat simple extension Murnaghan[1] appears to abandon the notions of §56 and instead to fall back upon a special case of (55.4), together with further assumptions which I do not understand. I shall derive from (56.5) a result equivalent to his by considering the special motion $x = Xe^{-k\sigma t}, y = Ye^{-k\sigma t}, z = Ze^{kt}$. Then $\dot{x} = -k\sigma x, \dot{y} = -k\sigma y, \dot{z} = kz, d^x{}_x = d^y{}_y = -k\sigma, d^z{}_z = k = \dot{z}/z = \dot{\overline{\log z}}, I_{\mathrm{d}} = (1 - 2\sigma)k = (1 - 2\sigma)\dot{\overline{\log z}}$. If the inertial forces may be neglected, this motion becomes dynamically possible for a stress $t^x{}_x = t^y{}_y = 0, t^z{}_z = I_{\mathrm{t}}$. The former condition put into (56.5) yields $(1 - 2\sigma)\mathfrak{g}_0^{\mathfrak{t}(\mathrm{d})} = \mathfrak{g}_1^{\mathfrak{t}(\mathrm{d})}$. If $\mathfrak{g}_0^{\mathfrak{t}(\mathrm{d})}$ and $\mathfrak{g}_1^{\mathfrak{t}(\mathrm{d})}$ be expanded as in (56.6), the first terms yield the classical value $\sigma = \nu_{\mathrm{E}}$ for Poisson's ratio, while the next terms yield $\beta_1 = \nu_{\mathrm{E}}\beta_0$. From (56.5) follows $\dot{t}^z{}_z = \lambda_{\mathrm{E}}(1 + \nu_{\mathrm{E}})\mathfrak{g}_1^{\mathfrak{t}(\mathrm{d})}\dot{\overline{\log z}}$ and hence

$$\log (1 + \delta_1) = \log \frac{z}{Z} = \frac{1}{\lambda_{\mathrm{E}}(1 + \nu_{\mathrm{E}})} \int_{{}^1t^z{}_z}^{t^z{}_z} \frac{d\xi}{\mathfrak{g}_1^{\mathfrak{t}(\mathrm{d})}\left(\frac{\xi}{\mu_{\mathrm{E}}}, 0, 0, \frac{\theta}{\theta_0}\right)}, \tag{58.1}$$

where ${}^1t^z{}_z$ is the initial uniaxial stress. Murnaghan considers the special case when $\mathfrak{g}_1^{\mathfrak{t}(\mathrm{d})}$ is a quadratic function of $t^z{}_z$, obtaining

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[2] [1949, **6**, introd.]. Cf. [1944, **6**], where by putting $a = 2$ and choosing but the single parameter A a less good fit for the compression of lithium in the same range is obtained. There remains, however, an element of curve-fitting, since Murnaghan makes no use of the relation $aA = \lambda_{\mathrm{E}} + \frac{2}{3}\mu_{\mathrm{E}}$, whereby a and A are related to the ordinary Lamé coefficients measurable in infinitesimal strains.

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[1] [1944, **13**] [1945, **11**] [1949, **6**, §4].

$$(1 + \delta_1)^a = \frac{1 + \dfrac{t^z{}_z}{A}}{1 - \dfrac{t^z{}_z}{B}}, \tag{58.2}$$

where a, A, and B are constants. If we adopt this approximation, we fail to make use of the principal advantage of the new theory, for now ${}^1t^z{}_z = 0$ as in the classical theory. While there is an upper bound $t^z{}_z = B$ for the yield stress, corresponding to $\delta_1 = \infty$, unfortunately $\delta_1 = -1$ if $t^z{}_z = -A$, indicating that a finite pressure may reduce the volume to zero.

A great deal of work remains to be done before the rate of deformation theory can be properly understood and evaluated.

Chapter V. FLUID DYNAMICS

V A. Stokes's Principle. Viscosity

59. Stokes's principle. A partial statement of the simplest concept of fluidity (§§1, 23) is embodied in *Stokes's principle*:[1] "That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned." The expression of Stokes's principle in the present notation is $\mathbf{v} = f(\mathbf{d})$. In particular, for a given value of $\mathbf{d}$, $\mathbf{v}$ is independent of $\mathbf{w}$, $\mathbf{e}$, $\dot{\mathbf{e}}$, etc.

A fluid is *isotropic* if $\mathbf{v}$ be an isotropic function of $\mathbf{d}$; by (6.7), then,

$$v^i{}_j = \mathcal{G}_0^{\mathrm{v(d)}}\delta^i{}_j + \mathcal{G}_1^{\mathrm{v(d)}}d^i{}_j + \mathcal{G}_2^{\mathrm{v(d)}}d^i{}_k d^k{}_j\,. \tag{59.1}$$

This formula is given by Reiner,[2] to whom we owe the introduction of tensorial methods in general fluid dynamics. Putting $\lambda_{\mathrm{V}} \equiv \mathcal{G}_{0\,100}^{\mathrm{v(d)}}$, $2\mu_{\mathrm{V}} \equiv \mathcal{G}_{1000}^{\mathrm{v(d)}}$, by linearizing[3] this basic equation we obtain the Newton-Cauchy-Poisson law(1.2)[4].

In a rectilinear shearing flow $\dot{x} = ky$, $\dot{y} = 0$, $\dot{z} = 0$, since $I_{\mathrm{d}} = III_{\mathrm{d}} = 0$ Stokes's principle (59.1) yields a power series expansion for $t^x{}_y$ as an *odd* function of $d^x{}_y$. Series of this type, sometimes incorrectly containing even powers, are common in the engineering literature[5].

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[1] [1845, **1**, §1].

[2] [1945, **1**, §4]. Five of the six terms of degree three in the $d^i{}_j$ are given in [1931, **14**, Ch. III].

[3] STOKES [1845, **1**, §3] justified this linearization on the basis of a molecular argument.

[4] Materials satisfying STOKES's principle but not satisfying the linearization (1.2) are often called *non-Newtonian fluids*, paint being a familiar example; cf. [1943, **6**, pp. 134–138].

[5] ST. VENANT [1869, **2**] attributes such a series to DUPUIT. That only odd powers are admissible is observed by REINER [1929, **4**, p. 17] [1943, **6**, lect. X, pp. 139, 150]. GIRAULT [1931, **14**, Ch. III] claims that when a flow is reversed all stresses should change in sign, and thus that no terms of even degree in the $d^i{}_j$ should occur in the general series expansion for $\mathbf{v}$. Premise and conclusion are alike false. In a shearing flow, for example, when the flow is reversed the shearing forces should indeed change sign, but if the cross-viscous forces should change sign it would imply the absurd conclusion that if pressure is required to maintain a flow from left to right, tension is required for a flow from right to left. In §72 below we shall see that in such a shearing flow $t^x{}_x$ and $t^y{}_y$ are necessarily *even* functions of $d^x{}_y$. GIRAULT's theory thus fails to reveal the POYNTING effect altogether. VIGUIER applies it to some special situations; in some of these [1949, **13**] [1950, **7**] the terms incorrectly an-

60. Generalizations: Boussinesq's principle and Levy's principle[1]. While great efforts have been expended in generalizing the linear theory of elasticity, little attention of this sort has been given to fluid dynamics. Kleitz[2] proposed $t^1{}_1 - p = f(d^1{}_1)$, $t^1{}_2 = f(d^1{}_2)$, $\cdots$, a type of relation tensorially admissible only when f is a linear function. Boussinesq[3] was the first to show that the Newton-Cauchy-Poisson law (1.2) is only the first term in an infinite series. His concept of fluidity, which we may call *Boussinesq's principle*, was somewhat more general than Stokes's: *the extra stress* $\mathbf{v}$ *is a function of both the rate of deformation and the vorticity, and in a rigid rotation*[4] *it reduces to zero*:

$$\mathbf{v} = \mathbf{v}(\mathbf{d}, \mathbf{w}), \qquad \mathbf{v}(0, \mathbf{w}) = 0. \tag{60.1}$$

Levy[5], however, reasoning from Navier's molecular notions, introduced higher derivatives of the velocity; while he obtained only terms of the type $\mu_k d^i{}_{j,}{}^{l_1}{}_{,l_1}{}^{\cdots,l_k}{}_{,l_k}$, there is no reason for limiting oneself to such linear expressions, and we shall prefer to state *Levy's principle* in the form

$$v^i{}_j = f(\dot{x}^{i_1}{}_{,i_2}, \dot{x}^{i_1}{}_{,i_2 i_3}, \cdots, \dot{x}^{i_1}{}_{,i_2 i_3 \cdots i_k}, \cdots). \tag{60.2}$$

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nulled happen to be zero for the particular case considered, but for the most part [1947, **17**, 26–27] [1948, **54**] [1949, **48**] [1951, **9**] the results are misleadingly incomplete. A supposed derivation of the corresponding dynamical equations is given in [1949, **54**].

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[1] Cf. [1882, **1**, pp. 79–80] [1932, **2**, Part II, §1.7; Part III, §§1.1, 1.3].

[2] [1866, **3**] [1872, **2**]. KLEITZ's equations thus necessarily reduce to the form (1.2); he states that the coefficients are to be variable, but the manner of their variation is not given in the published abstract of his paper.

[3] [1868, **2**, Note I.] BOUSSINESQ wrote out the second order terms resulting from (60.1) (cf. §11[1]). The first equations employed by SAKADI [1941, **12**] in his discussion of certain flows of incompressible fluids coincide with the second approximation to STOKES's principle; then for the same reason as GIRAULT (§59[5]) he abandons these equations and proposes, without any explanation, a more complicated type of stress. In his next work [1942, **16**] he withdraws his objection to his former results, but proposes instead (1) equations equivalent to the second approximation to $(60.1)_1$, and (2) equations equivalent to the second approximation in GIRAULT's theory.

[4] Proceeding by analogy to his linearized theory of elasticity (§49), SETH [1944, **9**] "for the sake of simplicity" (cf. §49[7]) proposes $t^i{}_j = -p\delta^i{}_j - \frac{2}{3}\mu_V s^k{}_k \delta^i{}_j + 2\mu_V s^i{}_j$, where s is a formal analogue of $\mathbf{e}$: i.e., $2s^i{}_j \equiv \dot{x}^i{}_{,j} + \dot{x}_j{}^{,i} - \dot{x}^{k,i}\dot{x}_{k,j} = 2d^i{}_j - d^i{}_k d^k{}_j - d^i{}_k w^k{}_j - d_j{}^k w_k{}^i - w^i{}_k w_j{}^k$. Thus according to SETH a rigid rotation of a fluid induces viscous stresses. Cf. §77. But here indeed we see the absurdity which can result from pursuing "simplicity" and a false analogy to elasticity (§3) in fluid dynamics, for in forming his tensor s SETH has abused the most elementary physical principles by adding together terms of different physical dimensions! Only in the present state of development of science could such a blunder be published *twice* in three years: the "formula cardinal completa y exacta . . . para deformaciones finitas" of GARCIA [1947, **22**, eq. (26)] [1948, **48**] [1949, **49**, eq. (21)] differs from SETH's but in a change of sign.

[5] [1869, **2**]. Earlier [1867, **3**, pp. 240–241] he proposed $F(\dot{x}, \partial\dot{x}/\partial y)$ for the shearing stress in a rectilinear shearing flow, which he reduced to $F(\dot{x})\partial\dot{x}/\partial y$.

In Chapter VD we discuss a fluid still more general than that described by Levy's principle, and at that time also we summarize contributions from the kinetic theory of gases.

61. Possible relations among the coefficients. Mean pressure and thermodynamic pressure. Incompressible fluids. Returning to the consequences of Stokes's principle (59.1) in the isotropic case, consider the dissipated power Φ to be written in the form (8.2). By $(31.2)_2$, we must have $\Phi \geqq 0$. Now in the classical theory Φ is reduced to a quadratic form; for this form to be non-negative the Duhem-Stokes[1] conditions $\mu_V \geqq 0$, $3\lambda_V + 2\mu_V \geqq 0$ are necessary and sufficient. If the next approximation be a cubic form, it is impossible[2] that $\Phi \geqq 0$ for all values of **d**. But it is not necessary that $\Phi \geqq 0$ automatically; all that is required is that $\Phi \geqq 0$ in any actual motion, a condition which can be expressed at most partially in terms of inequalities to be satisfied by the coefficients, but rather must distinguish those of the various solutions of the non-linear equations which are admissible[3] (cf. §72[3]).

From (8.3) it follows that there are possible stresses, e.g.

$$v^i{}_j = A[(-I_d{}^2 + 2II_d)\delta^i{}_j + I_d d^i{}_j],$$

which are not zero yet do no work in any deformation. So as to exclude such anomalies I propose

$$-\mathcal{G}^{v(d)}_{0200} + 2\mathcal{G}^{v(d)}_{0010} + \mathcal{G}^{v(d)}_{1100} = 0 \tag{61.1}$$

for the cubic terms in Φ, and analogous restrictions for the higher order coefficients.

From (26.11), (26.12), and $(6.10)_3$ we have

$$3(p - \bar{p}) = 3\mathcal{G}_0^{v(d)} + \mathcal{G}_1^{v(d)} I_d + \mathcal{G}_2^{v(d)}(I_d{}^2 - II_d). \tag{61.2}$$

We must now distinguish between compressible and incompressible fluids. For the former, p is a perfectly definite quantity defined by (30.3) and $(29.2)_1$. From (61.2) it follows then that $p \neq \bar{p}$ in general; a necessary and sufficient condition that $p = \bar{p}$ in every motion would be

$$3\mathcal{G}^{v(d)}_{0\mathsf{IJK}} + \mathcal{G}^{v(d)}_{1,\mathsf{I}-1,\mathsf{J},\mathsf{K}} + \mathcal{G}^{v(d)}_{2,\mathsf{I}-1,\mathsf{J},\mathsf{K}} - 2\mathcal{G}^{v(d)}_{2,\mathsf{I},\mathsf{J}-1,\mathsf{K}} = 0. \tag{61.3}$$

The case $\mathsf{I} = 1$, $\mathsf{J} = 0$, $\mathsf{K} = 0$ yields $3\mathcal{G}^{v(d)}_{0100} + \mathcal{G}^{v(d)}_{1000} = 0$, which is (cf. §59) the "Stokes relation"[4] $3\lambda_V + 2\mu_V = 0$. Since both theory and experiment now speak

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[1] [1901, **1**, Part I, Ch. 1, §3]; [Note, pp. 136–137 of 1901 reprint of [1851, **1**]].

[2] The case of third order terms in the strain energy in elasticity (§53) is quite different, because the strain components have finite bounds (§15).

[3] Analogously, in gas dynamics the differential equations admit solutions representing either compression or rarefaction shocks, and the CLAUSIUS inequality is used to show that only the former are admissible.

[4] [1845, **1**, §3]. This relation was implied also by ST. VENANT [1843, **1**, §6].

against the Stokes relation (see the appendix to this section), we shall not adopt its generalization (61.3).

For an *incompressible* fluid the pressure p was purposely left undefined (§30). Now for an incompressible fluid p does not enter any of the basic equations except Cauchy's first law $(26.13)_1$, where, by (59.1), it enters only in the combination $-p + \mathcal{G}_0^{v(d)}$. This latter quantity may therefore be taken as a basic unknown instead of p. There will never be any need to use the form of $\mathcal{G}_0^{v(d)}$ if the boundary conditions be phrased in terms of stresses or of $-p + \mathcal{G}_0^{v(d)}$. In other words (cf. §39), *for an incompressible fluid there is no loss in generality*[5] *in taking* $\mathcal{G}_0^{v(d)} = 0$. Since $I_d = 0$ we thus obtain Rivlin's formula[6]

$$v^i{}_j = \mathcal{G}_1^{v(d)}(II_d, III_d)d^i{}_j + \mathcal{G}_2^{v(d)}(II_d, III_d)d^i{}_k d^k{}_j. \tag{61.4}$$

In place of (61.2) we have

$$3(p - \bar{p}) = \overline{II}_d \, \mathcal{G}_2^{v(d)}(II_d, III_d). \tag{61.5}$$

Thus $p = \bar{p}$ in all motions if and only if $\mathcal{G}_2^{v(d)} = 0$; that is, the general equality of pressure and mean pressure[7] for an isotropic incompressible fluid is equivalent to a statement that the relation between $\mathbf{v}$ and $\mathbf{d}$ is quasi-linear. In particular, $p = \bar{p}$ in the classical theory based upon the linear relation (1.2). If, as we shall see in §72 to be very plausible, it can be established that $\mathcal{G}_2^{v(d)} > 0$, it will follow then, since $\overline{II}_d > 0$ at any point where $\mathbf{d} \neq 0$, that the equality of p and $\bar{p}$ is an accident of the linear theory: *in the general theory of incompressible fluids the hydrostatic pressure exceeds the mean pressure at every point where the fluid is suffering deformation.*

The term "pressure" is often taken loosely. In this memoir we have given it several definite theoretical meanings, distinguished by different symbols. The results of theory and experiment should be compared with caution, for the measurement called "pressure" by the experimenter depends upon the nature of the measuring instrument; often it is a particular component of the stress tensor, and in a fluid suffering deformation cannot be identified with any of the quantities called pressure in the theory.

61A. Appendix. The Stokes relation. We summarize here the long controversy regarding the STOKES relation $3\lambda_V + 2\mu_V = 0$ in the classical theory of viscous fluids.

(a.) *Theory.* POISSON [1831, **1**, ¶¶60–64] stated that λ_V and μ_V must be independent; cf. [1901, **1**, Part I, Ch. I, §3]. In proposing that $3\lambda_V + 2\mu_V = 0$ STOKES [1845, **1**, §3] remarked "in most cases in which it would be interesting to apply the theory of the friction of fluids

[5] [1948, **5**, §11].

[6] [1947, 4] [1948, **5**, §9].

[7] It is possible, of course, to *define* p as $\bar{p}$ for incompressible fluids in general, but then $\mathcal{G}_0^{v(d)} \neq 0$ and we no longer have the simple formula (61.4) for the viscous stress.

the density is either constant or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same and in the third nearly the same whether $3\lambda_V + 2\mu_V$ were equal to zero or not." His point is that if $I_d = 0$ everywhere the value of λ_V does not affect the motion at all, and that if $I_d \approx 0$ it does not make much difference what the value of λ_V may be. Later [Note, pp. 136–137 of 1901 reprint of [1851, **1**]] he stated that he had never put great faith in this relation.

That the effect of λ_V is of the same order as that of μ_V in the damping of sound waves was shown by STEFAN [1866, **4**] and KIRCHHOFF [1868, **1**]; cf. [1894, **5**, §349] [1903, **4**, §§2, 24–25]. This fact suggests a critical experimental test, discussed in part (b) below. The STOKES relation implies the anomalous result that a spherical mass of fluid may perform symmetrical oscillations in perpetuity, without frictional loss. Various related reversibility phenomena, both for simple fluids and for mixtures, are discussed by MEIXNER [1942, **17**]. ECKART [1948, **35**] takes up RAYLEIGH's phenomenon of acoustical streaming, in which a circulation is set up in a vessel of fluid by the oscillation of a source of sound upon one of the boundaries; he derives second order acoustical equations to represent this phenomenon, and obtains a formula connecting the ratio λ_V/μ_V with the velocity on the acoustical axis. To Professor TSIEN I owe the observation that ECKART's neglect of heat conduction renders his results quantitatively incorrect, though easily amended (cf. §82, last sentence).

To controvert the STOKES relation REINER [1946, **3**, §§3–6] [1951, **6**, §§2–5] gives an elaborate analysis of the simple extension of a viscous fluid. Cf. also [1949, **30**, §§5, 7]. The core of his argument lies in his claim that in the initial stages of an extension experiment the thermodynamic pressure $\pi = p$ has not yet come up to the mean pressure $\bar{p}$. In the classical theory (61.2) reduces to $3(p - \bar{p}) = (3\lambda_V + 2\mu_V)I_d$. Thus if $p \neq \bar{p}$ and $3\lambda_V + 2\mu_V \to 0$ it must follow that $I_d \to \infty$, so that the initial stage is instantaneous, contrary to experience. If $I_d \to 0$, however, we must have $\lambda_V \to \infty$ if $p \neq \bar{p}$. Thus REINER contends that $\lambda_V = \infty$ and $I_d = 0$ are equivalent. This whole argument is difficult to follow since λ_V is in first approximation a *constant* of the fluid, while I_d is a *variable* of the flow. At points where $I_d = 0$ we have $p = \bar{p}$ for *any* fluid obeying the classical theory, while if $3\lambda_V + 2\mu_V = 0$ for a *particular* fluid then $p = \bar{p}$ at *all* points where I_d is finite.

One often hears that the STOKES relation is proved in the kinetic theory of gases. It is more accurate to say that this result is implicitly *assumed* by that theory (cf. [1929, **6**] [1932, **6**, pp. 54–56, 151–153]). MAXWELL's definitions of stress and temperature [1867, **2**, eqq. (63), (102)] are $t_{ij} \equiv -\rho\langle c_i c_j\rangle$, $3R\theta \equiv \langle c^i c_i\rangle$, where $\langle\cdots\rangle$ denotes an average value and c_i is the relative velocity of the molecules. Hence by definition $\bar{p} = R\rho\theta$. While indeed this result is not in itself an equation of state (cf. [1950, **5**, §23]) since $\bar{p}$ is not yet identified with the thermodynamic pressure p, nevertheless it shows that $\bar{p}$ like p is determined by the thermodynamic state alone. Suppose now the NEWTON-CAUCHY-POISSON law (1.2) somehow to have been derived. Then $3(p - \bar{p}) = (3\lambda_V + 2\mu_V)d^k{}_k$. Since the left side of this result is a function of state only, it must be independent of the value of $d^k{}_k$; hence, so also must be the right side; and hence each side must vanish separately. Hence $p = \bar{p}$, $3\lambda_V + 2\mu_V = 0$: *the only viscous fluid consistent with Maxwell's definitions of stress and temperature is a perfect gas in which the Stokes relation holds* ([1952, **1**, §§2–3]).

We now pass in review those efforts in the kinetic theory which have yielded a value other than $-\frac{2}{3}\mu_V$ for λ_V. Those of them which purport to fall within the systematic, deductive part of the theory are analysed in greater detail in [1952, **1**, §§4–8]. By taking into account the transfer of momentum at collisions, ENSKOG [1922, **2**, pp. 18–19] [1939, **1**, §16.50] constructs a theory of dense gases obeying an equation of state which approximates that of VAN DER WAALS; he obtains $\lambda_V/\mu_V = (-\frac{2}{3} + \alpha)/(1 + \alpha)$, where $\alpha \equiv .6012b^2\rho^2\chi^2(1 + .4b\rho\chi)^{-2}$, b being the usual VAN DER WAALS constant and χ a parameter such that $\chi = 0$ if $\rho = 0$, $\chi = \infty$ if $\rho = \rho_{\min}$, and he remarks [1945, **10**] that hence $-\frac{2}{3} \leqq \lambda_V/\mu_V \leqq .649$. From heuristic molecular statements BUSEMANN [1931, **12**, §7] obtains $\lambda_V = (1 - \gamma)\mu_V$, where γ is the ratio of specific heats; this result is plainly false, since it yields $\lambda_V < 0$ for all gases, while experimental data (see part (b)) yields positive values for λ_V almost always. KOHLER [1946, **14**] [1947, **15**] claims that PIDDUCK's model of a gas molecule as a rough perfectly elastic sphere leads to the formula $\lambda_V/\mu_V = (10K)^{-1} - 9/20$, where $K \equiv 4I/m\sigma^2$, I being the moment of inertia of the molecule, m its mass, and σ its diameter; hence, since $0 \leqq K \leqq \frac{2}{3}$ it follows that $\infty > \lambda_V/\mu_V \geqq -\frac{3}{10}$; more generally, if f_r denote the number of rotational degrees of freedom of the molecule, KOHLER [1949, **21**, Teil II] obtains $(3\lambda_V + 2\mu_V)/\mu_V = 2f_r p\tau_r/(3 + f_r)$ where τ_r is a "relaxation time," hence concluding that $\lambda_V/\mu_V \geqq -2/(3 + f_r)$. For a molecule with vibrational degrees of freedom, KOHLER obtains $(3\lambda_V + 2\mu_V)/\mu_V = (\gamma - 1)pk$, where k is a complicated coefficient; for a molecule with both types of internal degrees of freedom, the two values of λ_V are to be added together. It has been shown [1952, **1**, §5], however, that KOHLER's theory in essence rests upon purely phenomenological assumptions which if not impossible are at least quite implausible. By heuristic arguments of momentum transfer SKUDRZYK [1948, **56**, 10] claims to show that in a plane sound wave of velocity u we have $t_{xx} = -p + 2\mu_V \partial u/\partial x$, a statement equivalent to $\lambda_V = 0$, but, since he does not modify MAXWELL's definitions of stress or temperature, his result is contradicted by the theorem of the paragraph preceding. PREDVODITELEV [1948, **27**, eq. (26a)] derives new hydrodynamical equations from intuitive continuum and molecular notions; his result implies BUSEMANN's relation, but at the same time he obtains a modified form of the acceleration, in which a third coefficient appears. A formula for $3\lambda_V + 2\mu_V$ for a gas composed of polyatomic molecules or molecules with internal quantum numbers is given by CHANG & UHLENBECK [1948, **4**, §IVA]. A formula for λ_V for water is derived from molecular assumptions in [1947, **20**]. Formulae for both λ_V and μ_V are derived from a kinetic theory of monatomic liquids by KIRKWOOD, BUFF, & GREEN [1949, **42**]; in this theory the STOKES relation never holds.

Of course STOKES's point that the value of λ_V may not be very important remains valid, for in many flow phenomena compressibility effects are negligible.

(b.) *Experiment.* Despite the flimsy theory upon which it rests, the STOKES relation stood virtually unquestioned for more than fifty years. The need for experimental tests was first pointed out by M. v. SMOLUCHOWSKI [1903, **4**, §§2, 24–25], who noted the effect of the value of λ_V on the absorption of sound, expansion of a gas into a vacuum, etc.; cf. [1907, **5**, §30].

The experiments of NEKLEPAJEV [1911, **10**], made at the suggestion of LEBEDEW, indicate for air double the absorption of sound predicted from the theory employing the STOKES relation, and of the abundant subsequent measurements very few, even with the generous allowance for experimental error customary in the ultrasonic field, can be forced into agreement with the result universally miscalled the "STOKES-KIRCHHOFF formula," for whose derivation the STOKES relation, as well as various other assumptions, is required. The experimental data is summarized in [1949, **51**, Kap. 3bd] [1949, **52**] [1951, **8**]; the matter is complicated by the inaccuracy of many reported values. Even for monatomic gases the data must be rather leniently interpreted if it is to be taken as confirming the STOKES relation. While the earlier authors gave elaborate molecular explanations for this discrepancy, MANDELSTAM & LEONTOVIČ [1936, **5**] recall the influence of the value of λ_V, from the experimental data concluding that in most fluids λ_V/μ_V is very large (about 90 for benzene), and that in fact it is λ_V rather than μ_V which accounts for major part of the absorption of sound; similarly, according to TISZA [1942, **2**] the absorption data for CO_2 and NO_2 indicates a value of about 10^3 for λ_V/μ_V; for water this ratio appears to lie between 1 and 3. The subject is not closed, however, since the "STOKES-KIRCHHOFF formula," even if corrected by insertion of a part arising from λ_V, is a mere linearization, valid only at frequencies ω such that $\mu_V\omega[2 + (\lambda_V/\mu_V)]/\rho c_0^2 \ll 1$. Elsewhere [1953, **1**] I discuss this matter more fully.

Using ECKART's results on acoustical streaming (part (a) above), LIEBERMANN [1949, **16**] obtains measured values of λ_V/μ_V lying between 1.3 and 200 for twelve liquids, the numerical results being in rough agreement with those obtained from sound absorption data. From experimental data on traction REINER [1949, **23**, §6] concludes that $\lambda_V/\mu_V \approx .07$ for concrete. Various methods of measuring λ_V are discussed and illustrated for the case of solid CO_2 by BOSWORTH [1949, **45**].

In nearly all cases experimentally investigated λ_V is *positive*, and for many fluids it is orders of magnitude greater than μ_V.

62. Viscosity. Natural time. In the conceptual experiment of Maxwell[1] a viscous fluid is confined between infinite parallel plane boundaries in constant relative motion at a speed $\dot{x}$; then the walls experience a retarding force according to the hypothesis of Newton[2]:

$$\text{force} = \mu_V \frac{\dot{x}}{\text{distance between walls}} \text{ (area in contact)}, \tag{62.1}$$

μ_V being a factor of proportionality, independent of the other magnitudes in (62.1) but dependent upon the particular fluid. We are not concerned with the exactness of (62.1); it is sufficient that a crude experiment indicates it to be a fair approximation, for our only use for it is to indicate that *a characterizing property of fluids, not less important than Stokes's principle, is the existence of a* viscosity μ_V *of dimension* $\mathbf{ML^{-1}T^{-1}}$.

This viscosity is a manifest function of temperature θ. However, a relation $\mu_V = \mu_V(\theta)$ is not possible dimensionally unless it involve at least one other dimensional quantity. We propose $\mu_V/\mu_{V_n} = f(\theta/\theta_0, \cdots)$, where μ_{V_n} and θ_0 are material moduli of dimensions $\mathbf{ML^{-1}T^{-1}}$ and $\mathbf{\Theta}$, respectively, and the dots stand for other possible dimensionless scalars. Now viscosity depends upon pressure as well as temperature. A relation of form $\mu_V/\mu_{V_n} = f(\theta/\theta_0, p/\mu_{E_n}, \cdots)$ where $\dim \mu_{E_n} = \mathbf{ML^{-1}T^{-2}}$, is dimensionally admissible, but the existence of a natural elasticity μ_{E_n} implies a measure of springiness. If μ_{E_n} and μ_{V_n} be material constants, so also is $t_n \equiv \mu_{V_n}/\mu_{E_n}$, but $\dim t_n = \mathbf{T}$. Thus follows the important conclusion that *any body endowed both with viscosity and with elasticity unavoidably possesses also a material constant of the dimension of time.* Such a modulus t_n we shall call a *natural time* of the substance[3].

In Chapter VB we treat the concept of a fluid devoid of a natural time, in Chapter VC the more general case in which a natural time exists.

In either case, for θ_0 we shall always take the *boiling point of the fluid*, so that θ_0 becomes *a property of the fluid*, not an arbitrary reference point.

[1] [1866, **2**, pp. 7–8] [1871, **3**, pp. 277–278].

[2] [1687, **1**, Lib. II, Sect. IX].

[3] MAXWELL [1867, **2**, pp. 69–71] noted that μ_V/p is a time of relaxation for the stresses in a gas, provided the pressure p be kept constant; this relaxation time is associated with a definite phenomenon, relaxation of stress at constant pressure, and is a property jointly of the substance and of its circumstances. The natural time t_n above, however, is a material constant and thus a property of the *substance alone*, hence indicating a time-dependent response of the substance in general.

V B. The Stokesian Fluid

63. A fluid without a natural time. The simplest sort of fluid is a body which offers no response whatever to its history, or, one might say, a body without a memory[1]. Such a material is described partially by Stokes's principle (§59); a natural time t_n, however, is evidence of a sort of memory, and if we require further that there be no such time, from §62 we conclude that the viscosity coefficient μ_V must either be independent of p or, more generally, must be dependent both upon p and upon some other scalar possessed of the same dimensions but not a constant. The simplest such scalar is the mean pressure[2] $\bar{p}$ defined by (26.12). Thus $\mu_V/\mu_{V_n} = f(\theta/\theta_0, p/\bar{p})$. In a fluid of this type, if $p = \bar{p}$ in all circumstances then the viscosity is independent of pressure[3].

64. Definition of the Stokesian fluid.[1] *A Stokesian fluid is a continuous medium obeying Stokes's principle in the specific form*

$$\mathbf{t} = \mathbf{t}(\mathbf{d}, \mu_{V_n}, \theta, \theta_0, p, \bar{p}), \tag{64.1}$$

where μ_{V_n} and θ_0 are material constants of dimensions

$$\dim \mu_{V_n} = \mathsf{ML^{-1}T^{-1}}, \qquad \dim \theta_0 = \Theta, \tag{64.2}$$

and where further

$$\mathbf{t}(0, \mu_{V_n}, \theta, \theta_0, p, \bar{p}) = -p\mathbf{I}. \tag{64.3}$$

The restriction (64.3) is a statement that in equilibrium the Stokesian fluid obeys the ordinary law of hydrostatics.

65. The consequences of invariance requirements.[1] Since there are 6 inde-

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[1] According to CAUCHY [1828, **1**, §III], "l'élasticité disparait entièrement". Cf. §1[3]. As it is put by JEFFREYS [1931, **9**, Ch. IX], "We may say that an elastic solid has a memory; a fluid has none."

[2] Equations (for a Maxwellian material) in which both p and $\bar{p}$ occur were first proposed by ZAREMBA [1903, **7**, eq. (28)].

[3] According to a celebrated result of MAXWELL (e.g. [1939, **1**, §§6.2, 7.41, 9.7]) in the kinetic theory of gases, μ_V is independent of density. Since in the kinetic theory of monatomic gases it is assumed that $p = \bar{p}$ (cf. §61A, (a)), the above result shows the consistency of the Stokesian with the kinetic theory in this regard. For the dependence of viscosity on pressure in a dense gas, see [1932, **6**, §35]; in a liquid, [1926, **3**] [1946, **11**, Ch. IV, §2].

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[1] [1949, **7**, §4] [1950, **14**, §5]. In [1947, **6**, §XI] is a preliminary study.

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[1] [1949, **7**, §6] [1950, **14**, §7] [1952, **2**].

pendent components of **d**, by (64.1) it follows that any one component of **t** is functionally related to 11 quantities composed of the 3 fundamental dimensions T, $\mathsf{ML^{-1}T^{-1}}$ and Θ. Such a relation must reduce to a dimensionless equation connecting $12 - 3 = 9$ dimensionless ratios. Six such ratios are given by the components of $\mu_{\mathrm{V}n}\mathbf{d}/p$; another is the appropriate component of $\mathbf{t}/p$; and for the remaining two we may select $p/\bar{p}$ and θ/θ_0. Hence (64.1) must reduce to

$$\mathbf{t} = pf\left(\frac{\mu_{\mathrm{V}n}\,\mathbf{d}}{p}, \frac{p}{\bar{p}}, \frac{\theta}{\theta_0}\right), \tag{65.1}$$

where f is a dimensionless function. This equation gives the most general form of stress possible in a Stokesian fluid.

For an isotropic fluid the result (65.1) implies that the general power series (59.1) must reduce to

$$\begin{aligned} v^i_{\ j} &= p\,\overset{\mathrm{v(d)}}{\mathfrak{g}_0}\,\delta^i_{\ j} + \mu_{\mathrm{V}n}\overset{\mathrm{v(d)}}{\mathfrak{g}_1}\,d^i_{\ j} + \frac{\mu_{\mathrm{V}n}^{\ 2}}{p}\overset{\mathrm{v(d)}}{\mathfrak{g}_2}\,d^i_{\ k}d^k_{\ j}, \\ \overset{\mathrm{v(d)}}{\mathfrak{g}_{\mathrm{T}}} &= \overset{\mathrm{v(d)}}{\mathfrak{g}_{\mathrm{TIJK}}}\left(\frac{\mu_{\mathrm{V}n}}{p}\right)^{\mathrm{I+2J+3K}} I_{\mathrm{d}}{}^{\mathrm{I}}II_{\mathrm{d}}{}^{\mathrm{J}}III_{\mathrm{d}}{}^{\mathrm{K}}, \end{aligned} \tag{65.2}$$

where $\overset{\mathrm{v(d)}}{\mathfrak{g}_{\mathrm{TIJK}}}$ is a dimensionless function of $p/\bar{p}$, θ/θ_0, and dimensionless material constants only, and where $\overset{\mathrm{v(d)}}{\mathfrak{g}_{0000}} = 0$. In this expansion we have the complete and yet perfectly definite mathematical realization of the isotropic Stokesian fluid. The terms of first and second order in **d** are

$$\begin{aligned} v^i_{\ j} = \mu_{\mathrm{V}n}&[\overset{\mathrm{v(d)}}{\mathfrak{g}_{0100}}I_{\mathrm{d}}\delta^i_{\ j} + \overset{\mathrm{v(d)}}{\mathfrak{g}_{1000}}\,d^i_{\ j}] \\ &+ \frac{\mu_{\mathrm{V}n}^{\ 2}}{p}[\overset{\mathrm{v(d)}}{\mathfrak{g}_{0200}}I_{\mathrm{d}}{}^2\delta^i_{\ j} + \overset{\mathrm{v(d)}}{\mathfrak{g}_{0010}}II_{\mathrm{d}}\delta^i_{\ j} + \overset{\mathrm{v(d)}}{\mathfrak{g}_{1100}}I_{\mathrm{d}}\,d^i_{\ j} + \overset{\mathrm{v(d)}}{\mathfrak{g}_{2000}}\,d^i_{\ k}\,d^k_{\ j}]. \end{aligned} \tag{65.3}$$

The Newton-Cauchy-Poisson law (1.2) results from linearization of (65.2), providing we write $\lambda_{\mathrm{V}} = \mu_{\mathrm{V}n}\overset{\mathrm{v(d)}}{\mathfrak{g}_{0100}}$, $2\mu_{\mathrm{V}} = \mu_{\mathrm{V}n}\overset{\mathrm{v(d)}}{\mathfrak{g}_{1000}}$.

66. Dynamical similarity.[1] The classical theory of viscous fluids has a remarkable characteristic: although (1.2) defines a fluid by its response to rate of deformation, in none of the parameters (Mach number, Reynolds number, etc.[2]) governing local dynamic similarity in the resulting theory does any typical rate d or any other quantity of the same dimension occur. This anomaly is an accident of linearization. From (65.1) it follows that for local dynamic similarity not only the usual parameters of the classical theory but also the number $\mathfrak{T}$

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[1] [1949, **7**, §7] [1950, **14**, §8].

[2] E.g. [1932, **2**, Part I, Ch. III, §1.3].

given by

$$\mathfrak{T} \equiv \frac{\mu_{\rm v}\, d}{p}, \tag{66.1}$$

where d is a typical rate of deformation, vorticity, frequency, or reciprocal of a time, must be considered. In addition, of course, the numbers $\overset{\rm v(d)}{\mathfrak{g}}_{\rm TIJK}/\overset{\rm v(d)}{\mathfrak{g}}_{1000}$ are new similarity parameters, but these do not introduce any new dimensional quantities. Thus in the similarity theory of the Stokesian fluid time rates play the dominant part we should expect from the notion of fluidity, and moreover this part is essentially the same no matter how many terms in the series for the viscous stresses are retained, so long as they go past the linear terms.

Now consider the whole list of quantities from which the complete set of similarity parameters is composed. The quantities ρ, $(\partial p/\partial\rho)_\eta$, $\overset{\rm v(d)}{\mathfrak{g}}_{\rm TIJK}$, etc. represent properties of the *fluid*, while l (characteristic length), d, p, and θ are properties of *the flow*. Thus, in the fully general Stokesian theory, *to ascertain local dynamic similarity four and only four properties* (l, d, p, θ) *of the* flow *need be observed*; *in the classical linearization one of these, d, may be neglected.*

Under the not unreasonable hypothesis that $|\, \overset{\rm v(d)}{\mathfrak{g}}_{\rm TIJK}/\overset{\rm v(d)}{\mathfrak{g}}_{1000}\,|$ is of the order of 1, a criterion for neglect of all higher order terms and consequent validity of the Newton-Cauchy-Poisson linearization (1.2) is

$$\mathfrak{T} \ll 1, \tag{66.2}$$

whence it appears that $\mathfrak{T}$ may be called the *truncation number*. That is, the *classical theory becomes less adequate the larger the viscosity, the larger the rate of deformation or vorticity, and the lower the pressure.* For water at 1 atm. the rate of deformation must be about 7×10^5 sec^{-1} in order that $\mathfrak{T} \approx 1$, so that at normal pressures the higher order terms need be retained only for deformations much more rapid than those usually encountered[3]. By ascent into the atmosphere, however, $\mathfrak{T}$ can be made arbitrarily large, and thus the higher order terms in the theory of the Stokesian fluid are particularly appropriate to problems of high altitude aerodynamics (cf. Chapter VD).

A glance at (66.1) and (47.4) reveals a profound difference of kind between the elastic body and the fluid[4]. To determine the adequacy of the infinitesimal approximation in elasticity we need only consider the geometry of the particular deformation: if the displacements and displacement gradients be in-

[3] In this connection it should be noticed, however, that near a point where the velocity gradient becomes infinite the solutions given by the classical theory of viscous fluids cannot be used with confidence, even if they be perfectly adequate elsewhere.

[4] Several authors [1874, **2**, p. 109] [1888, **1**, §466] [1929, **9**, Kap. 7, §4] [1931, **14**, Ch. I] [1933, **3**, p. 359] [1945, **1**, p. 355] in pursuing a false analogy between elasticity and fluid dynamics have spoken of "small velocities", "small relative velocities", "small rates of strain", etc., in the present connection. Since none of these quantities is dimensionless, the statements are meaningless, except perhaps as a loose reference to a limit process.

finitesimal, the infinitesimal theory is applicable, and no elastic moduli of the body or dimensional characteristics of the deformation need be observed. To determine the adequacy of the classical approximation for a viscous fluid, however, we must know the viscosity of the fluid and must observe both the pressure and a typical rate or time. While this basic difference of kind is obscured by the classical linearizations, in the general theory it is at once apparent even formally from the contrast of the second order terms in (43.3) and (65.3).

V C. The Reiner-Rivlin Fluid with a Natural Time

67. The natural time in the Reiner-Rivlin theory. The theories of fluid dynamics of Reiner[1] and Rivlin[2] begin with Stokes's principle (§59) and hence in the isotropic case with (59.1). Reiner considers the fluid to be compressible, while Rivlin treats only incompressible fluids and hence employs (61.4). According to Reiner, the $\mathcal{G}_{\mathrm{T}}^{\mathrm{v(d)}}$ "will generally be functions of the hydrostatic pressure or of the density of the material," while according to Rivlin the $\mathcal{G}_{\mathrm{TIJK}}^{\mathrm{v(d)}}$ are "constants characterizing the particular fluid considered." Both authors imply also that the coefficients depend upon the temperature.

Now dim $\mathcal{G}_{2000}^{\mathrm{v(d)}}/\mathcal{G}_{1000}^{\mathrm{v(d)}} = \mathbf{T}$, and thus if Rivlin's statement can be taken literally it follows that *this theory employs a natural time* t_n. Neither Reiner nor Rivlin remarks upon the dimensions of any of the terms, but I believe it a fair interpretation of the quotations above to conclude that both[3] imply the existence of material constants having the same dimensions as the coefficients $\mathcal{G}_{\mathrm{TIJK}}^{\mathrm{v(d)}}$. For a fluid with a natural time, $\mu_{\mathrm{V}n}/t_n$ is also a material constant, and since dim $\mu_{\mathrm{V}n}/t_n = \mathbf{M}\mathbf{L}^{-1}\mathbf{T}^{-2}$, we may say that *the Reiner-Rivlin theory concerns fluids having a natural elasticity* (cf. §62), so that it may account for Schwedoff's[4] "stiffness" or "rigidity" of such substances as gelatine.

Any material constant endowed with the dimension $\mathbf{T}$ may be interpreted as a natural time t_n. It is desirable, however, to have a precise definition. We shall always take

$$t_n \equiv \frac{\mathcal{G}_{2000}^{\mathrm{v(d)}}}{\mathcal{G}_{1000}^{\mathrm{v(d)}}} = \frac{\mathcal{G}_{2000}^{\mathrm{v(d)}}}{2\mu_{\mathrm{V}}}, \tag{67.1}$$

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[1] [1945, **1**, §§2, 4]. We give no account of the theory of [1945, **1**, §3], which proceeds by a false analogy to the classical theory of finite elastic strain.

[2] [1947, **4**] [1948, **5**, §§8–11].

[3] Mr. Rivlin has confirmed this interpretation orally (1949).

[4] [1889, **3**] [1900, **2-3**]; cf. [1932, **2**, Part I, Ch. III, §9.2]. This effect is sometimes partially described by stating that the "apparent viscosity" depends upon the rate of shear.

where it is understood that the right hand side is to be evaluated under specified thermodynamic conditions—say when $p = 0$ and $\theta = c\theta_0$, where $c = \frac{1}{2}$ for liquids, $c = 2$ for gases[5]. Thus t_n will always be *a material constant, a property of the fluid alone*, rather than of the fluid and its circumstances. That it is possible to find such a natural time is characteristic of the Reiner-Rivlin theory.

68. Definition of the Reiner-Rivlin fluid. The *Reiner-Rivlin fluid* may be defined as follows: *In the definition of the Stokesian fluid* (§64), *add a* natural time t_n (dim $t_n = \mathbf{T}$) *to the list of quantities upon which the stress* $\mathbf{t}$ *may depend.*

69. The consequences of invariance requirements. By a dimensional analysis similar to that of §65, we may now reduce the Reiner-Rivlin theory to a definite form. In place of (65.1) follows

$$\mathbf{t} = pf(t_n\mathbf{d},\, pt_n/\mu_{\mathrm{V}n},\, p/\bar{p},\, \theta/\theta_0), \tag{69.1}$$

while in the case of isotropy in place of (65.2) follows

$$\begin{aligned} v^i_{\ j} &= \mu_{\mathrm{V}n}\left[\frac{1}{t_n}\overset{\mathrm{v(d)}}{g_0}\delta^i_{\ j} + \overset{\mathrm{v(d)}}{g_1}\, d^i_{\ j} + t_n\overset{\mathrm{v(d)}}{g_2}\, d^i_{\ k}\, d^k_{\ j}\right], \\ \overset{\mathrm{v(d)}}{g_\Gamma} &= \overset{\mathrm{v(d)}}{g_{\Gamma\mathrm{IJK}}}\, t_n^{\,\mathrm{I+2J+3K}}\, I_{\mathrm{d}}^{\ \mathrm{I}} II_{\mathrm{d}}^{\ \mathrm{J}} III_{\mathrm{d}}^{\ \mathrm{K}}, \end{aligned} \tag{69.2}$$

where the $\overset{\mathrm{v(d)}}{g_{\Gamma\mathrm{IJK}}}$ are dimensionless functions of $p/\bar{p}$, θ/θ_0, $pt_n/\mu_{\mathrm{V}n}$, and dimensionless material constants only; here too $\overset{\mathrm{v(d)}}{g_{0000}} = 0$. The Newton-Cauchy-Poisson law (1.2) results from linearization of (69.2), providing we write $\lambda_{\mathrm{V}} \equiv \mu_{\mathrm{V}n}\overset{\mathrm{v(d)}}{g_{0100}}$, $2\mu_{\mathrm{V}} \equiv \mu_{\mathrm{V}}\overset{\mathrm{v(d)}}{g_{1000}}$. *Hence the classical theory of viscous fluids is the limiting case* $t_n \to 0$ *of the Reiner-Rivlin theory.* The coefficients $\overset{\mathrm{v(d)}}{g_{\Gamma\mathrm{IJK}}}$ may depend upon the pressure even if $p = \bar{p}$, while for the Stokesian fluid the equality of the two pressures implies that the coefficients $\overset{\mathrm{v(d)}}{g_{\Gamma\mathrm{IJK}}}$ become independent of both.

While the Reiner-Rivlin fluid generalizes the Stokesian fluid, in one respect its theory is simpler. It is natural in solving specific cases to take the dimensionless coefficients $\overset{\mathrm{v(d)}}{g_\Gamma}$ in first approximation as constants. By (69.2) this approximation makes the viscous stress $\mathbf{v}$ independent of the static pressure p, while for the Stokesian fluid any approximation going beyond the classical linearization (1.2) must necessarily take the dependence of $\mathbf{v}$ upon p into account: indeed, that dependence is one of the characteristic features of the Stokesian theory. The foregoing remark is illustrated in the examples given in §72 below.

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[5] Recall the convention of §62 that θ_0 is the boiling point. The choice $c = 1$ for gases and liquids alike is appealing, but impracticable because of the jump in viscosity coefficients at a change of phase.

70. Dynamical similarity. The meaning of the natural time. The analysis of §66 is easily modified to show that now it is the number

$$\mathfrak{R}_R \equiv t_n d, \tag{70.1}$$

rather than (or in addition to) the number $\mathfrak{T}$, which must be added to the list of parameters governing dynamical similarity. A criterion for the validity of the classical theory thus becomes $\mathfrak{R}_R \ll 1$, or $t_n^{-1} \ll d$: *the classical theory becomes inadequate as the vorticity or the rate of deformation approaches the natural frequency t_n^{-1} of the fluid.*

To illustrate the meaning of the natural time t_n, consider[1] a simple shearing flow $\dot{x} = ky$, $\dot{y} = 0$, $\dot{z} = 0$. According to (59.1) we have $t^x{}_y = \frac{1}{2}k\mathcal{G}_1^{\text{v(d)}}$, $t^x{}_x = t^y{}_y = -p + \mathcal{G}_0^{\text{v(d)}} + \frac{1}{4}k^2\mathcal{G}_2^{\text{v(d)}}$. Now define an instantaneous reference configuration (§13) as that occupied by the medium a time t_n in the past: $X \equiv x - kt_n y$, $Y \equiv y$, $Z \equiv z$. Computing the strain with respect to this reference configuration we have $e^x{}_y = \frac{1}{2}kt_n$, $e^y{}_y = -\frac{1}{2}k^2t_n{}^2$. If the material were an isotropic relaxing elastic body obeying Weissenberg's quasi-linear theory (§51) we should obtain $t^x{}_y = \frac{1}{2}kt_n\mathcal{G}_1^{\text{t(e)}}$, $t^x{}_x = \mathcal{G}_0^{\text{t(e)}}$, $t^y{}_y = \mathcal{G}_0^{\text{t(e)}} - \frac{1}{2}k^2t_n{}^2\mathcal{G}_1^{\text{t(e)}}$ By adding a suitable hydrostatic pressure and a tension $\text{t}^x{}_x$ normal to the motion we may reduce this state of stress to that above predicted by the Reiner-Rivlin theory, meanwhile identifying the viscosity coefficient $\mathcal{G}_1^{\text{v(d)}}$ with the product $t_n\mathcal{G}_1^{\text{t(e)}}$ of the elasticity by the natural time (cf. §62). If we may generalize from this example, *the Kelvin and Poynting effects of the Reiner-Rivlin theory appear to be similar to those arising from a quasi-linear but finite displacement-gradient theory of elasticity in which strain is measured with respect to the configuration assumed by the medium at the time $t - t_n$.* This conclusion survives the limiting process to the classical theory, indicated in §69, for in this limit the Kelvin and Poynting effects vanish.

From structural considerations concerning high-polymer solutions Rivlin[2] has calculated $\mathcal{G}_{2000}^{\text{v(d)}} = \alpha[\mathcal{G}_{1000}^{\text{v(d)}}]^2/k\theta N$, where k is Boltzmann's constant, N is the number of high-polymer molecules per unit volume of solution, and α is 4/5 for one model and 6/5 for another. Numerical estimates indicate that for such liquids Poynting effects are significant. By (67.1) we may write Rivlin's formula as an expression for the natural time:

$$t_n = \frac{2\alpha\mu_{\text{v}}}{k\theta N}, \tag{70.2}$$

where $\theta = \frac{1}{2}\theta_0$. In so far as this result applies, it indicates that Poynting effects, which arise in consequence of an appreciable natural time, are more noticeable

[1] This example is suggested by Burgers [1948, **12**, §3], but his analysis is somewhat different. A related idea was used by O.-E. Meyer [1874, **1**].

[2] [1948, **6**] [1949, **9**] [1950, **9**, §6]. I am indebted to Mr. Rivlin for use of the second of these papers in MS.

in highly viscous fluids. Decrease in temperature also increases the relative magnitude of the Poynting effects.

71. Reiner's treatment of dilatancy. Putting $\mathbf{t}$ for $\mathbf{a}$ and $\mathbf{d}$ for $\mathbf{b}$ and putting $s = 0$ in (10.2), Reiner[1] observes that in any fluid defined by Stokes's principle a simple shearing stress r gives rise to a change of volume ($I_d \neq 0$) if any of the coefficients $\mathcal{G}^{v(d)}_{00J0}$ be different from zero. This property, which here appears as an example of the Kelvin effect (§10), was called *dilatancy* by Reynolds[2]: "... a definite change of bulk, consequent on a definite change of shape ..."

Reiner gives several other examples of the Kelvin and Poynting effects in fluids, which follow at once from the cases listed in §10. He considers also the definition of various rheological coefficients[3], but since these depend upon the particular flow as well as the fluid and sometimes are not uniquely determined, their usefulness is doubtful.

72. Rivlin's general solutions. A variety of Poynting effects in the Reiner-Rivlin fluid are revealed by some examples constructed by Rivlin[1]. All are cases of isochoric flow ($I_d = 0$), and in Rivlin's original treatment only incompressible fluids are considered, where (§61) we may put $\mathcal{G}^{v(d)}_0 = 0$. The results then constitute or imply exact general solutions. We extend them here to the case of compressible fluids, taking care to note, however, that in general they then will fail to be compatible with the energy equation, and hence are to be regarded only as indications of tendency, not actual solutions.

In this section for ease of writing we drop the subscript d from I_d, II_d, and III_d and the superscript v(d) from $\mathcal{G}^{v(d)}_\Gamma$. In all cases possible dependence of $\mathcal{G}_\Gamma$ on θ/θ_0, $p/\bar{p}$, or $pt_n/\mu v_n$ will be neglected.

Since $I = 0$ in all the following examples, the series expansions (8.1) begin $\mathcal{G}_\Gamma = \mathcal{G}_{\Gamma 000} + \mathcal{G}_{\Gamma 010} II + \cdots$. Hence in a second order theory, all terms of third

[1] [1945, **1**, §7]; also [1949, **30**, §§6, 9] [1951, **6**, §7]. REINER calls a fluid in which $\mathfrak{g}^{v(d)}_2 = 0$ and moreover the $\mathfrak{g}^{v(d)}_\Gamma$ are independent of $III_d - \frac{1}{3}II_d I_d + 2I_d^2$ a *Reynolds liquid*. Now in the classical theory $\mathfrak{g}^{v(d)}_2 = 0$, and the standard viscometric tests measure only the effects predicted by the classical theory. Therefore, if we neglect to observe the POYNTING effect in the tests, we can agree with REINER [1946, **3**, §1] that "if no other tests were available, the existence of any viscous liquid more general than the REYNOLDS liquid could not be detected." It would seem more reasonable, however, to detect the existence of more general fluids by quantitative measurements of POYNTING effects in the classical experiments (cf. §72).

[2] [1885, **2**] [1887, **1**]. REYNOLDS explained dilatancy by means of a model composed of rigid close-packed spheres, and elaborated this notion into a general theory of matter [1903, **19**]; cf. [1913, **5**].

[3] [1945, **1**, §§6–7] [1946, **3**, §4].

[1] [1948, **5**, §§14–16] [1949, **4**].

and higher order being neglected, the coefficients $\mathfrak{G}_1$ and $\mathfrak{G}_2$ reduce to the constants $\mathfrak{G}_{1000}$ and $\mathfrak{G}_{2000}$, while $\mathfrak{G}_0$ is proportional to II.

A. *Rectilinear shearing flow.* Suppose $\dot{x} = f(y)$, $\dot{y} = 0$, $\dot{z} = 0$. Then $\mathbf{d}$ has the form $(10.2)_1$ with $s = 0$, $u = \frac{1}{2}f'$, and hence by $(10.2)_2$ we have

$$\mathfrak{t} = (-p + \mathfrak{G}_0)\mathbf{I} + \tfrac{1}{2}f'\mathfrak{G}_1 \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + \tfrac{1}{4}f'^2\mathfrak{G}_2 \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \qquad (72.1)$$

where $\mathfrak{G}_\Gamma = \mathfrak{G}_\Gamma(0, -\frac{1}{4}f'^2, 0) = \mathfrak{G}_\Gamma(y)$. A particular consequence of (72.1) is that the specific resistance $t^x{}_y$ is necessarily an *odd* function of f', the rate of shearing[2]. Since $\ddot{\mathbf{x}} = 0$, if we put $\mathbf{f} = 0$ the dynamical equations $(26.2)_1$ reduce to

$$-\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\frac{1}{2}f'\mathfrak{G}_1\right) = 0, \qquad \frac{\partial}{\partial y}\left(-p + \mathfrak{G}_0 + \frac{1}{4}f'^2\mathfrak{G}_2\right) = 0. \qquad (72.2)$$

Hence

$$\begin{aligned} t^x{}_y &= \tfrac{1}{2}f'\mathfrak{G}_1(0, -\tfrac{1}{4}f'^2, 0) = Cy + D, \\ t^x{}_x &= t^y{}_y = -p + \mathfrak{G}_0 + \tfrac{1}{4}f'^2\mathfrak{G}_2 = -Cx + E, \end{aligned} \qquad (72.3)$$

where C, D, E are constants. The former of these equations is an ordinary differential equation for the velocity profile f. In the case $C = 0$, it becomes a transcendental equation for f' in terms of the arbitrary constant D, which can then be interpreted as the uniform resistance per unit area, and since any polynomial approximation to $(72.3)_1$ is of odd degree, it follows that in general *there are an odd number of velocity profiles, all linear but of different slopes f', yielding the same resistance.* In the case of a second order theory only the classical $f = 2Dy/\mathfrak{G}_{1000} = Dy/\mu_V$ is possible, but in a third order theory there may be one or three profiles, depending upon the values of D, $\mathfrak{G}_{1000}$, and $\mathfrak{G}_{1010}$. In the case when $C = 0$ the normal stresses $t^x{}_x$ and $t^y{}_y$ in the plane of shearing assume an equal independent constant value E, which differs from the normal stress $t^z{}_z$ normal to the plane of shearing by the amount $\frac{1}{4}f'^2\mathfrak{G}_2$.

The results just given constitute a simple and striking example of the general conclusions of §10. *To produce a shearing flow between two parallel infinite plates, shearing forces alone are insufficient; in addition, stresses normal to the plates,* exceeding the normal stress on the planes of flow by the amount $\frac{1}{4}f'^2\mathfrak{G}_2(0,$

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[2] There is an extensive literature devoted to the analysis of the flow in tube and rotation viscometers under the assumption that $t^x{}_y$ is a power series in $d^x{}_y$: e.g. [1927, **4–5**] [1928, **6**] [1929, **4**] [1930, **7, 10–11**] [1931, **6–7, 10**] [1932, **3**, §§1–2] [1934, **2**] [1943, **6**, pp. 148–150] [1948, **22**, A I 5B] [1949, **43**, Lect. XI]. These one-dimensional treatments of course cannot reveal the POYNTING effect. A proper second order theory of these instruments is constructed by REINER from the solutions given in parts B and C of this section. I am obliged to him for use of his paper [1952, **5**] in MS.

$-\frac{1}{4}f'^2, 0)$, *must be applied.* If $\mathcal{G}_2(0, -\frac{1}{4}f'^2, 0) > 0$, this excess is a tension, so that *in its absence the plates would tend to draw together when sheared.* This phenomenon is an example of the Poynting effect. For a given specific resistance, *the various possible velocity profiles, odd in number, require different normal forces.* Finally, the *resistance* as a function of f', which is the classical subject of measurement, is independent of $\mathcal{G}_2$, and *its departure from the classical value is an effect of* third *order in the shearing rate* f', while the magnitude of the Poynting effect for a given profile f depends *only* upon $\mathcal{G}_2$, the effect itself being of *second* order.

For a Stokesian fluid[3] a similar analysis leads to (72.1), but the coefficients $\mathcal{G}_\Gamma$ are necessarily functions of p as well as of y. It is still possible to obtain a solution in which p = const., however, and for this special case the result (72.3) with $C = 0$ still holds. The difference between the Reiner-Rivlin and Stokesian theories becomes plain when $\mathcal{G}_2$ is replaced by its power series expansion in (72.3). For the former, by $(69.2)_2$ we have

$$\frac{1}{2} f' t_n g_{10J0} (-1)^J \left(\frac{1}{2} f' t_n\right)^{2J} = \frac{Dt_n}{\mu_{\mathrm{V}n}}, \tag{72.4}$$

while for the latter by $(65.2)_2$ we have

$$\frac{1}{2} \frac{f'\mu_{\mathrm{V}n}}{p} g_{10J0} (-1)^J \left(\frac{1}{2} \frac{f'\mu_{\mathrm{V}n}}{p}\right)^{2J} = \frac{D}{p}. \tag{72.5}$$

For a given specific resistance D, in the Reiner-Rivlin theory the product $t_n f'$ is determined, and hence the permissible velocity profiles f' are independent of the pressure. In the Stokesian theory, however, it is $f'\mu_{\mathrm{V}n}/p$ which is determined by D, and hence the *velocity profile depends upon the pressure.* If, alternatively, we think of f' as given, then from (72.4) and (72.5) we obtain formulae for the resistance in the two theories. For the Reiner-Rivlin theory we have[4]

$$D = \mu_{\mathrm{V}} f' - \mu_{\mathrm{V}n} g_{1010} \frac{f'^3 t_n^2}{8} + \cdots, \tag{72.6}$$

while for the Stokesian theory we have

$$D = \mu_{\mathrm{V}} f' - g_{1010} \frac{\mu_{\mathrm{V}n}^3 f'^3}{8p^2} + \cdots. \tag{72.7}$$

Thus in the Stokesian theory *the third order correction to the specific resistance is inversely proportional to the square of the pressure.* This result indicates the appropriateness of the Stokesian theory for low pressure phenomena in gases.

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[3] [1950, **14**, §12]. This type of motion in a rarefied gas is discussed by SCHAMBERG [1947, **3**, Ch. VI], using BURNETT's equations (§76) and his own boundary conditions (§79).

[4] The numerical value -0.72×10^{-3} c.g.s. for the coefficient of the term in f'^3 is obtained from experimental data by VIGUIER [1949, **13**].

B. *Poiseuille flow.*[5] In cylindrical co-ordinates r, ϕ, z, suppose $\dot{r} = 0$, $\dot{\phi} = 0$, $\dot{z} = v(r)$. Then $\mathbf{d}$ is of the form $(10.3)_1$ with $u = \frac{1}{2}v'$, $s = 0$, so that by $(10.3)_2$ we have

$$\mathbf{t} = (-p + \mathcal{G}_0)\mathbf{I} + \tfrac{1}{2}v'\mathcal{G}_1 \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{Vmatrix} + \tfrac{1}{4}v'^2\mathcal{G}_2 \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{72.8}$$

where $\mathcal{G}_\Gamma = \mathcal{G}_\Gamma(0, -\frac{1}{4}v'^2, 0) = \mathcal{G}_\Gamma(r)$. The dynamical equations $(26.2)_1$ reduce to

$$\begin{gathered} -\frac{\partial p}{\partial r} + \mathcal{G}_0' + \frac{1}{r}(\tfrac{1}{4}rv'^2\,\mathcal{G}_2)' = 0, \\ \frac{\partial p}{\partial \theta} = 0, \qquad -\frac{\partial p}{\partial z} + \frac{1}{r}(\tfrac{1}{2}rv'\,\mathcal{G}_1)' = 0. \end{gathered} \tag{72.9}$$

Hence follows easily

$$\begin{gathered} \frac{\partial p}{\partial z} = -C, \qquad v'\mathcal{G}_1(0, -\tfrac{1}{4}v'^2, 0) = -Cr, \\ -t^\theta_\theta = p - \mathcal{G}_0 = -Cz + \tfrac{1}{4}v'^2\,\mathcal{G}_2 + \tfrac{1}{4}C^2 \int r\frac{\mathcal{G}_2}{\mathcal{G}_1^2}\,dr, \\ t^r_r = t^z_z = Cz - \tfrac{1}{4}C^2 \int r\frac{\mathcal{G}_2}{\mathcal{G}_1^2}\,dr. \end{gathered} \tag{72.10}$$

The downstream pressure gradient is the arbitrary constant C, while $(72.10)_2$ is a differential equation for the velocity profile v. As in the shearing flow of part A, an odd number of velocity slopes v' are possible for a given pressure gradient. The Poynting effect now appears as the excess of the radial and downstream normal stresses over the classical value Cz, an excess which, apart from an arbitrary constant, if $\mathcal{G}_2(0, m, 0) > 0$ when $m < 0$ is *always a pressure and always greatest upon the periphery*. Reiner[6] remarks that if this extra pressure be wanting, the stream issuing from a tube viscometer will tend to swell, a phenomenon which has been observed experimentally[7] and which suggests that indeed $\mathcal{G}_{2000} > 0$. The total mass flux through a circular section of radius a, supposing $v = 0$ at the periphery, is

$$\pi\rho C \int_0^a (r^3/\mathcal{G}_1)\,dr.$$

[5] For a generalized POISEUILLE flow $\dot{x} = f(y, z)$, $\dot{y} = 0$, $\dot{z} = 0$ a method of solving a class of dynamical equations including those of the REINER-RIVLIN theory as well as those for some types of plastic materials is given by OLDROYD [1949, **34**] [1951, **12**].

[6] [1952, **5**, §4].

[7] [1943, **19**].

For the second approximation it follows by the general theory of §10 that the classical effects—in this case, the parabolic velocity distribution and the Hagen-Poiseuille efflux formula—are unchanged in magnitude. The excess pressure in $(72.10)_6$ becomes now simply $C^2r^2\mathcal{G}_{2000}/32\mu_V^2$. The magnitude of the exit swelling mentioned above should then increase in first approximation as the square of the pressure gradient. Reiner suggests that insertion of a peripheral pressure gauge in the usual tube viscometer should make it possible to check this result and to measure $\mathcal{G}_{2000}$.

C. *Couette flow.*[8] Suppose $\dot{r} = 0$, $\dot{\theta} = \omega(r)$, $\dot{z} = 0$. Then the matrix of physical components of **d** is of the form $(10.2)_1$ with $u = \frac{1}{2}r\omega'$, $s = 0$, so that by $(10.2)_2$ for the physical components $\widehat{ij}$ of stress we have

$$\| \widehat{ij} \| = (-p + \mathcal{G}_0)\mathbf{I} + \tfrac{1}{2}r\omega'\mathcal{G}_1 \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + \tfrac{1}{4}r^2\omega'^2\mathcal{G}_2 \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \tag{72.11}$$

where $\mathcal{G}_\Gamma = \mathcal{G}_\Gamma(0, -\frac{1}{4}r^2\omega'^2, 0) = \mathcal{G}_\Gamma(r)$. Supposing gravity to act along the z-axis, we may put the dynamical equations $(26.2)_1$ into the form

$$\begin{gathered} \frac{\partial}{\partial r}[-p + \mathcal{G}_0 + \tfrac{1}{4}r^2\omega'^2\,\mathcal{G}_2] = -\rho r\omega^2, \\ -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\partial}{\partial r}[\tfrac{1}{2}r\omega'\mathcal{G}_1] + \omega'\mathcal{G}_1 = 0, \\ -\frac{\partial p}{\partial z} + \rho g = 0. \end{gathered} \tag{72.12}$$

A symmetrical solution is easily obtained:

$$\begin{aligned} \widehat{r\theta} &= \tfrac{1}{2}r\omega'\mathcal{G}_1(0, -\tfrac{1}{4}r^2\omega'^2, 0) = \frac{C}{r^2}, \qquad \rho = \rho(r), \\ \widehat{zz} &= -p + \mathcal{G}_0 - \rho gz - \tfrac{1}{4}r^2\omega'^2\mathcal{G}_2 + \int_0^r \rho r\omega^2\,dr + D, \\ \widehat{rr} &= \widehat{\theta\theta} = -\rho gz + \int_0^r \rho r\omega^2\,dr + D. \end{aligned} \tag{72.13}$$

The arbitrary constant $2\pi C$ is the resultant couple per unit length required to produce the motion. Corresponding to any such couple, there are an odd num-

[8] RIVLIN's analysis does not carry over the Stokesian fluid. COUETTE flow of a rarefied gas is discussed by SCHAMBERG [1947, **3**, Ch. VII], using BURNETT's equations (§76) and his own boundary conditions (§79).

ber of possible angular velocity slopes ω'. The Poynting effect now appears in its most striking form for fluids: to produce the motion normal stresses, which if $\mathcal{G}_2(0, m, 0) > 0$ when $m < 0$ are *pressures*, must be supplied to the planes perpendicular to the axis. *If these pressures be wanting, the fluid surface will tend to rise, whatever be the speeds and senses of rotation.* A more explicit result can be obtained in the second approximation, since by the general analysis in §10 it follows that all classical results—in this case, the single Couette velocity distribution $\omega = Ar^{-2} + B$—remain valid. Neglecting the terms representing the effects of gravity and centrifugal forces, in this case we reduce (72.13) to the form

$$C = 2\mu_V \frac{r_1^2 r_2^2 [\omega(r_1) - \omega(r_2)]}{r_1^2 - r_2^2}, \qquad \widehat{rr} = D,$$
$$\widehat{zz} = -\mathcal{G}_{2000} \frac{r_1^2 r_2^2 [\omega(r_1) - \omega(r_2)]^2}{r^4(r_1^2 - r_2^2)} + D, \tag{72.14}$$

where the arbitrary constant D can be adjusted so as to let $\widehat{zz}$ equilibrate the atmospheric pressure on some specified ring $r = r_0$ between the inner cylinder $r = r_1$ and the outer cylinder $r = r_2$. Thus if the surface be left free, it may be expected to slope upward or downward toward the center according as $\mathcal{G}_{2000} > 0$ or $\mathcal{G}_{2000} < 0$, *the effect being greatest at the inner cylinder*, whatever be the speeds and senses of rotation. Garner & Nisson[9] first observed that in an actual Couette flow with a free surface *the liquid climbs up the inner cylinder*, and the phenomenon is positively demonstrated in a series of striking experiments by Weissenberg[10]. Hence we may conclude[11] that in the actual fluids tested $\mathcal{G}_{2000} > 0$.

D. *Parallel plate viscometer*[12]. A motion in which $\dot{r} = 0$, $\dot{\theta} = \omega(z)$, $\dot{z} = 0$ approximates that occurring in a parallel plate viscometer, provided the fluid be sufficiently viscous that the effect of gravity is negligible. The matrix of physical

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[9] [1946, **20**].

[10] [1947, **8**] [1949, **31**]; cf. also [1949, **11**]. Earlier REINER [1945, **1**, p. 360] had concluded the possibility of such phenomena from his theory, but had dismissed them as "never . . . observed". WEISSENBERG himself attempts an explanation by his theory of elasticity (§51), and REINER [1952, **5**] proposes experiments to decide whether the effect arises from elastic or viscous properties. To me the issue seems not to exist: the theory of elasticity is applicable only to substances in which the stress does not subside until recovery of strain from some specified reference configuration, but the physical fluids used in the experiments show no such tendency whatever, and an attempt to represent phenomena appearing in fluids by a theory of elasticity seems unnecessary, irrelevant, and inappropriate. There is, however, a sort of reconciliation of the two views in the fact that the REINER-RIVLIN fluid possesses a natural elasticity, say $\mu_V^2/\mathcal{G}_{2000}$ (§§67, 70).

[11] REINER [1952, **5**, §3] suggests a modified rotating cylinder viscometer in which there is a rigid lid with pressure gauges inserted to measure $\widehat{zz}$. It is doubtful if the theory given here applies to such an instrument, however, since the adherence of the fluid to the lid will seriously disturb the flow.

[12] The analysis generalizes that of RIVLIN [1948, **5**, §15] and REINER [1952, **5**, §5].

components of $\mathbf{d}$ is then of the form $(10.3)_1$ with $u = 0$, $s = \frac{1}{2}r\omega'$, and hence by $(10.3)_2$ we have

$$\| \widehat{ij} \| = (-p + \mathcal{G}_0)\mathbf{I} + \tfrac{1}{2}r\omega'\mathcal{G}_1 \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} + \tfrac{1}{4}r^2\omega'^2\mathcal{G}_2 \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{72.15}$$

where $\mathcal{G}_\Gamma = \mathcal{G}_\Gamma(0, -\frac{1}{4}r^2\omega'^2, 0)$. If inertial forces may be neglected, the dynamical equations $(20.6)_1$ reduce to

$$\begin{aligned} &\frac{\partial}{\partial r}(-p + \mathcal{G}_0) - \tfrac{1}{4}r\omega'^2\mathcal{G}_2 = 0, \\ &-\frac{\partial p}{\partial \theta} + \tfrac{1}{2}r^2\frac{\partial}{\partial z}(\omega'\mathcal{G}_1) = 0, \\ &\frac{\partial}{\partial z}(-p + \mathcal{G}_0) + \tfrac{1}{4}r^2\frac{\partial}{\partial z}(\omega'^2\mathcal{G}_2) = 0. \end{aligned} \tag{72.16}$$

If we seek a solution in which $\partial p/\partial\theta = 0$, conditions of integrability for this system are

$$\begin{gathered} \frac{1}{r}\frac{\partial}{\partial r}\left[-r^2\frac{\partial}{\partial z}(\omega'^2\mathcal{G}_2)\right] = \frac{\partial}{\partial z}[\omega'^2\mathcal{G}_2], \qquad \frac{\partial^2}{\partial z^2}(\omega'\mathcal{G}_1) = 0, \\ \frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial z}(\omega'\mathcal{G}_1)\right] = 0; \end{gathered} \tag{72.17}$$

hence

$$\begin{aligned} r\omega'\mathcal{G}_1(0, -\tfrac{1}{4}r^2\omega'^2, 0) &= \frac{Cz}{r} + f(r), \\ \omega'^2\mathcal{G}_2(0, -\tfrac{1}{4}r^2\omega'^2, 0) &= r^{-\frac{1}{2}}g(z) + h(r). \end{aligned} \tag{72.18}$$

If the axis $r = 0$ lie within the body of fluid, we must put $C = 0$, $g(z) = 0$. The conditions (72.18) are then satisfied if and only if $\omega' = \text{const.}$, and we have

$$\widehat{\theta z} = \tfrac{1}{2}\omega' r\mathcal{G}_1, \qquad \widehat{zz} = \tfrac{1}{4}\omega'^2\left[r^2\mathcal{G}_2 + \int_a^r r\mathcal{G}_2\,dr\right], \tag{72.19}$$

where the constant of integration has been adjusted so that $\widehat{rr} = 0$ on the cylindrical boundary $r = a$. It is easy to show that the resultant couple L and force F required to produce the motion are given by

$$L = \pi\omega'\int_0^a r^3\mathcal{G}_1\,dr, \qquad F = \frac{\pi}{4}\omega'^2\int_0^a r^3\mathcal{G}_2\,dr; \tag{72.20}$$

thus the latter is always a *tension* if $\mathcal{G}_2 > 0$, so that in the absence of this force *the plates of the viscometer will tend to draw together.*

For the second order theory $(72.19)_2$ becomes simply

$$\widehat{zz} = \tfrac{1}{8}\omega'^2 \mathcal{G}_{2000}(3r^2 - a^2). \qquad (72.21)$$

Hence *the stress which must be exerted upon the plates of the viscometer in order to produce the motion is distributed parabolically, being a tension if* $r > a/\sqrt{3}$, *zero when* $r = a/\sqrt{3}$ *and a pressure when* $r < a/\sqrt{3}$. Reiner[13] states that in a viscometer with stand pipes inserted into the stationary plate, the zero point of the pressure is clearly observed.

E. *General remarks.* The four examples just given illustrate the general conclusions of §10. In all cases the classical effect is governed by the coefficient $\mathcal{G}_1$ alone, and its value is an odd function of the classical parameter. The Poynting effect is governed by the coefficient $\mathcal{G}_2$ alone, and its value is an even function of the classical parameter. This separation of the two effects makes it possible by (67.1) to obtain from each experiment a value for the natural time of the fluid. For example, from (72.20) we have for the second approximation theory

$$t_n = \frac{4F}{\omega' L}. \qquad (72.21)$$

V D. *The Maxwellian Fluid*

73. Slip flow. At altitudes of 20 to 60 miles the atmosphere may still be regarded for most purposes as a continuum, but not as a simple viscous fluid represented by the Navier-Stokes equations. For aerodynamics in this region Tsien[1] introduced the term *slip flow* because the classical condition of adherence to a solid boundary is no longer satisfied[2]. At these low pressures the effects described by the higher order terms in the theory of the Stokesian fluid (Chapter *VB*) become significant, but also a number of others are of equal importance.

[13] [1952, **5**, §5].

[1] [1946, **1**, p. 654]. In the present review are mentioned only those efforts in the kinetic theory which have led to stress formulae different from (1.2). A more detailed history is given in [1951, **1**, §18].

[2] For a résumé of the controversy regarding slip in the classical theory, see [1901, **1**, 4^{me} partie, Ch. 5] [1932, **2**, Part II, §§ 1.2, 1.3, 1.7, 3.2, 7.3]. The possible effect of slip on a viscometric measurement in a non-Newtonian fluid is discussed in [1931, **6**, **7**] [1932, **3**, **11**]. A corresponding thermal discontinuity is observed in [1898, **2**, §I] [1911, **8**]; a molecular theory for it is given in [1913, **6**, §4].

One of these is *thermal transpiration*, discovered by Reynolds[3], in which a temperature gradient in a fluid at rest is of itself sufficient to produce non-equilibrated stresses and resultant motion. Stress formulae appropriate to this phenomenon were first obtained from the kinetic theory of gases by Maxwell[4], on the basis of his celebrated hypothesis of molecular forces varying inversely as the fifth power of the distance. The method of calculating expressions for the stress and heat flux by approximate integration of the Boltzmann equation was introduced by M. Brillouin[5] and developed by Hilbert[6], Enskog[7], Lennard-Jones[8], Rocard[9], Burnett[10], and Chapman & Cowling[11]; while Brillouin

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[3] [1879, **2**, §2]; cf. [1949, **47**, Part III, §6]. Since his similarity laws for this and related phenomena involve both the density of the gas and the dimensions of the boundaries, REYNOLDS [§§4–6] concluded that they cannot result from a continuum theory and thus afford proof of the molecular nature of gases. The conclusion is incorrect. In the one-dimensional case considered by REYNOLDS, the boundary conditions and the dimensions of the boundaries determine the quantities which occur in the similarity parameters of the continuum theory.

[4] [1879, **1**, §4]. A different kinetic theory treatment was given by REYNOLDS [1879, **2**, Sects. VI-VIII]. Elaborating MAXWELL's earlier ideas, O.-E. MEYER [1865, **3**, §8] for the case of rectilinear shearing flow had already obtained the power series $t^x{}_y = \mu\text{v}[\partial\dot{x}/\partial y + \frac{2}{5}\lambda^2\partial^3\dot{x}/\partial y^3 + \cdots]$, where λ is the mean free path; his method is generalized in [1949, **53**]; and by an application of Maxwellian notions, BUTCHER [1876, **2**, pp. 103–111] [1882, **1**, pp. 79–80] had obtained the dynamical equations $[kl + (k + \frac{1}{3}\nu)\partial/\partial t]\dot{x}^i{}_{,ji} - \nu\dot{x}_{i'}{}^i{}_{,j} + [l + \partial/\partial t]\rho(f_i - \ddot{x}_i) = 0$, which for steady motions or for $l = \infty$ reduce to the classical ones. REYNOLDS [1901, **8**] [1932, **2**, Part II, §1.7] at one time (1883) claimed that the ordinary NAVIER-STOKES equations are inconsistent with the condition of adherence at a solid boundary at the commencement of the motion, and by some obscure arguments of mixed molecular and phenomenological nature proposed equations which (if I have correctly understood his unexplained symbols) contain a term $- l^2\dot{x}^{i,j}{}_{,j}$ added to the acceleration, l being a constant length, and which cannot be derived from any form of symmetric stress tensor. The dynamical equations proposed on the basis of molecular notions by PREDVODITELEV [1948, **27**, eq. (26a)] differ from the usual ones in that the acceleration $\ddot{x}^i$ is replaced by $(1 - \beta)\ddot{x}^i + \beta[\partial\dot{x}^i/\partial t - \dot{x}^i\dot{x}^k{}_{,k}]$, where β is a coefficient. A similar equation for incompressible liquids is derived from mixed molecular and phenomenological notions by FRENKEL [1946, **11**, Ch. IV, §10]. According to the molecular calculations of ABODY-ANDERLIK [1949, **33**], the NEWTON-MAXWELL formula (62.1) should be replaced by $\rho t^x{}_y = \mu\text{v}d(\rho\dot{x})/dy$, but this result does not satisfy the requirement of Galilean invariance.

[5] [1900, **1**].

[6] [1912, **3**] [1912, **2**, Ch. 21]. HILBERT's notion is to consider "λ-series" solutions $\lambda^i f_i$, where λ is an arbitrary parameter, essentially the mean free path. If valid at all, the results are appropriate to a slightly rarefied gas. Cf. Note 7.

[7] [1917, **4**, Chs. II, VI] [1939, **1**, App. A]. The basic idea is similar to HILBERT's, but while in HILBERT's method the successive approximations are uniquely determined at each stage, in ENSKOG's a considerable arbitrariness is introduced, to be eliminated later by a formal procedure whose mathematical and physical meaning is obscure. ENSKOG's results were derived by CHAPMAN [1916, **2**] [1917, **3**] in a different way.

[8] [1923, **1**, §§9–12]. The incompleteness of these results is noted by ROCARD [1927, **2**, §11].

[9] [1924, **3**] [1927, **2**, §§10-16] [1932, **6**, Ch. VIII]. Cf. [1932, **2**, Part IV, §1.1].

[10] [1936, **1**]. BURNETT's stress formulae contain those of MAXWELL, ENSKOG, and LENNARD-JONES as special cases.

[11] [1939, **1**, §§15.3–15.41]. Numerical values of the higher order coefficients are obtained

was content to obtain the general form of the terms (as in a pure continuum theory) by constructing a series expansion for the distribution function in terms of isotropic tensors, the later authors actually calculate both the functional form and the numerical values of the coefficients[12]. A new method of integration has been discovered by H. Grad[13]. It is the expressions of Burnett and Chapman-Cowling which Tsien recommends as a basis for slip flow aerodynamics[14]. Until quite recently the only attempt to use continuum methods to describe phenomena of this type was Korteweg's[15] theory of motions in which very large density gradients occur. It is possible, however, to formulate a continuum definition of a fluid which quickly yields a general yet perfectly definite expression for both stress and heat-flux, including all of Brillouin's, Korteweg's, Burnett's and Chapman & Cowling's formulae as special cases, and which is easily carried systematically to any desired degree of approximation.

74. Definition of the Maxwellian fluid. Brillouin's principle.[1] It is the phenomenon of thermal transpiration (§73) which is the key to a general continuum theory of fluids, for it shows that an effect ordinarily associated with heat flux only can of itself produce a stress. Now in reality stress and heat flux are closely analogous. From a molecular point of view both are averages of purely mechanical actions—the one representing the average momentum transfer across an imaginary surface, the other the average energy transfer. From a phenomenological point of view also the stress and heat flux represent but different aspects of the same principle. The stress is a tensor whose divergence is added to the extraneous force in order to represent a resultant force equivalent to the unspecified mutual forces and so to balance the momentum equation $(26.2)_1$. The heat flux is a vector whose divergence is added to the stress power in order to represent a resultant power equivalent to that supplied by unspecified exchanges

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in a shorter way by KOHLER [1950, **2**] who also derives one relation among the coefficients valid independently of the particular law of intermolecular forces [1950, **4**].

[12] MOTT-SMITH [1951, **16**] has used a different type of approximate solution to describe a plane shock wave; I am indebted to him for use of this paper in MS since 1948. Cf. [1948, **51**]. BURNETT's equations are applied to the calculation of the thickness of weak shocks in [1948, **36**]. Methods of obtaining solutions of the form $\lambda^{-i} f_i$, and hence appropriate to very rarefied gases, have been given by BOLZA, BORN, & v. KÁRMÁN [1913, **6**, §2], JAFFÉ [1930, **6**, §4], and J. KELLER [1948, **21**]. Some special situations are treated in [1949, **28**].

[13] [1949, **36**, §§4–5] [1950, **5**, §§31–32]. The method consists in a systematic calculation of relations among all the moments of the distribution function, while the HILBERT-ENSKOG method treats the first and second moments only.

[14] A systematic kinetic theory of liquids has been constructed by BORN & GREEN [1946, **2**] [1947, **1–2**] [1948, **16–17**] [1949, **26**] and by KIRKWOOD [1946, **21**] [1949, **42**].

[15] [1901, **2**, §7] [1932, **2**, Part IV, §1.3]. To discuss thermal phenomena in gases VERSCHAFFELT [1948, **37–38**] in effect proposes to add a thermal pressure $\alpha \log (\theta/\theta_0)\delta^i_j$ to the stress. The coefficient α is to be a function of θ/θ_0 only; hence such a fluid must possess a natural time (§62).

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[1] [1949, **8**, §§2–4] [1951, **1**, §§19–21].

of mutual energies without performance of mechanical work, and so to balance the energy equation (27.2). A phenomenological definition of a fluid specifies the dependence of these two quantities upon other gross variables. The close analogy suggests that both *stress and heat flux should depend upon the same gross variables.* The counterpart of thermal transpiration for example, will then be a heat flux arising in a deformation at uniform temperature. The difference between heat flux and stress is to be expressed only by their *different physical dimensions and different tensorial order*—but this difference is far reaching.

Hence we are led to the following definition: *A* Maxwellian fluid *is a continuous medium such that*:

(a) *It possesses three and only three dimensionally independent material constants, a* natural viscosity $\mu_{\mathrm{V}n}$, *a* natural conductivity κ_n, *and a* reference temperature θ_0 :

$$\dim \mu_{\mathrm{V}n} = \frac{\mathsf{M}}{\mathsf{LT}}, \qquad \dim \kappa_n = \frac{\mathsf{ML}}{\mathsf{T}^3\Theta}, \qquad \dim \theta_0 = \Theta. \tag{74.1}$$

(b) *Both the stress* $\mathbf{t}$ *and the heat flux* $\mathbf{q}$ *may be considered functions of the following variables only*:

$$\begin{aligned} &\mu_{\mathrm{V}n},\ \kappa_n,\ \theta_0,\ \bar{p},\ p,\ \theta,\ p_{,i},\ \theta_{,i},\ \dot{x}^i{}_{,j},\ f^i{}_{,j},\ p_{,ij}, \\ &\qquad \theta_{,ij},\ \cdots,\ \dot{x}^{i_1}{}_{,i_2\cdots i_n}, f^{i_1}{}_{,i_2\cdots i_n}, p_{,i_1\cdots i_n}, \theta_{,i_1\cdots i_n},\ \cdots, \end{aligned} \tag{74.2}$$

and they are analytic functions of all the vectors and tensors listed.

(c) *If all the vectors and tensors in the above list vanish, then* $\mathbf{t} = -p\mathbf{I}$ *and* $\mathbf{q} = 0$.

Since the list (74.2) contains the vorticity and the velocity gradients of all orders, the present theory generalizes both Boussinesq's and Levy's principles (§60) (cf. §77). Thermodynamic gradients and extraneous force gradients of all orders are included also, as so to assure full generality.

Now M. Brillouin's method of integrating the Boltzmann equation of the kinetic theory amounts to the assumption that the distribution function may be expanded in a series of powers of the vectors and tensors in the list (74.2), an assumption which he justified *a posteriori* by arguments similar to those given above[2]. The part (b) of the definition may therefore be called *Brillouin's principle*[3]. The Maxwellian fluid, like the Stokesian fluid, is a material altogether without elasticity or "memory" effects, for no natural time is permitted and the stress is independent of any previous configuration; it generalizes the Stokesian fluid, but does not include the Reiner-Rivlin fluid.

It is convenient to introduce also a *fluid constant* $R_n \equiv c\kappa_n/\mu_{\mathrm{V}n}$, where c is a

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[2] [1900, **1**, §37].

[3] To Dr. MOTT-SMITH I owe the remark that M. BRILLOUIN's method is essentially a continuum theory method, and that my method of deriving expressions for the heat flux and stress consists in part of applying BRILLOUIN's treatment directly to the continuous medium, without the unnecessary intervention of kinetic theory paraphernalia.

dimensionless constant; R_n is of the same dimensions as the gas constant R in the perfect gas law $p = R\rho\theta$, viz., $\dim R_n = \mathbf{L}^2\mathbf{T}^{-2}\mathbf{\Theta}^{-1} = \dim \eta$. From the quantities (74.2) we may construct scalars of all four fundamental dimensions:

$$\dim \frac{\mu_{Vn}}{p} = \mathbf{T}, \qquad \dim \frac{\mu_{Vn}(R_n\theta)^{\frac{1}{2}}}{p} = \mathbf{L},$$
$$\dim \frac{\mu_{Vn}{}^3(R_n\theta)^{\frac{1}{2}}}{p^2} = \mathbf{M}, \qquad \dim \theta_0 = \mathbf{\Theta}. \tag{74.3}$$

75. Expressions for the stress and heat flux. While the complexity of the present theory necessitates a rather elaborate treatment as far as tensorial form is concerned, the dimensional method of §65 can be applied with only slight changes. It is possible to prove[1] as a mathematical consequence of the definition given in §74 that *both stress and heat flux are power series in the natural viscosity* μ_{Vn}.

Let the coefficient of $(\mu_{Vn})^p$ in the expansion for $t^i{}_j$ be written ${}^{(p)}t^i{}_j$. Then after considerable analysis it is possible to show that for an isotropic fluid[2] the first three of these coefficients must be of the form

$$
\begin{aligned}
{}^{(0)}t^i{}_j &= -p\delta^i{}_j, \qquad {}^{(1)}t^i{}_j = Ad^k{}_k\delta^i{}_j + Bd^i{}_j,\\
{}^{(2)}t^i{}_j &= \frac{R_n\theta}{p^3}[D_1 p^{,k}p_{,k}\delta^i{}_j + 2C_1 p^{,i}p_{,j}] + \frac{R_n}{p^2}[D_n p^{,k}\theta_{,k}\delta^i{}_j + C_2(p^{,i}\theta_{,j} + \theta^{,i}p_{,j})]\\
&\quad + \frac{R_n}{p\theta}[D_3\theta^{,k}\theta_{,k}\delta^i{}_j + 2C_3\theta^{,i}\theta_{,j}] + \frac{R_n\theta}{p^2}[D_4 p^{,k}{}_{,k}\delta^i{}_j + 2C_4 p^{,i}{}_{,j}]\\
&\quad + \frac{1}{p}[D_5 f^k{}_{,k}\delta^i{}_j + C_5(f^i{}_{,j} + f_j{}^{,i})] + \frac{R_n}{p}[D_6\theta^{,k}{}_{,k}\delta^i{}_j + 2C_6\theta^{,i}{}_{,j}]\\
&\quad + \frac{1}{p}[F_1(d^k{}_k)^2\delta^i{}_j + F_2 d^l{}_k d^k{}_l\delta^i{}_j + F_3 w^l{}_k w^k{}_l\delta^i{}_j + 2E_1 d^k{}_k d^i{}_j\\
&\quad + 2E_2 d^i{}_k d^k{}_j + 2E_3 w^i{}_k w^k{}_j + E_4(d^i{}_k w^k{}_j + w_k{}^i d^k{}_j)],
\end{aligned}
\tag{75.1}
$$

where $A, \cdots, E_4$ are dimensionless functions of $p/\bar{p}$, $\hat{\theta}/\theta_0$ only. If we put $\lambda_V \equiv A\mu_{Vn}$, $2\mu_V \equiv B\mu_{Vn}$, then ${}^{(0)}t^i{}_j + \mu_{Vn}{}^{(1)}t^i{}_j$ yields the Newton-Cauchy-Poisson

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[1] [1949, **8**, §§6, 8] [1951, **1**, §§23, 25].

[2] [1948, **3**]. For both the anisotropic and the isotropic cases I have given in [1948, **2**, §§19–20] full expressions for ${}^{(3)}t^i{}_j$ and ${}^{(4)}t^i{}_j$ for the slightly less general theory in which second and higher order velocity gradients are neglected. A full expression for ${}^{(3)}t^i{}_j$, for both the anisotropic and the isotropic cases, is given in [1949, **8**, §§6, 9] [1951, **1**, §§23, 26]. The terms whose coefficients are F_1, F_2, E_1, E_2, E_4 were given by BOUSSINESQ [1868, **2**, Note I] with coefficients unspecified in form. From the thermal equation of state $f(p, \rho, \theta) = 0$ it follows that the result of KORTEWEG [1901, **2**, §7], containing terms of the form $\rho^{,k}\rho_{,k}\delta^i{}_j$, $\rho^{,i}\rho_{,j}$, $\rho^{,k}{}_{,k}\delta^i{}_j$, and $\rho^{,i}{}_{,j}$, with unspecified coefficients, is included in $(75.1)_3$ as a special case.

law (1.2), so that as far as stress is concerned the isotropic Maxwellian fluid is indistinguishable in first approximation from the Stokesian, both theories reducing then to the classical one[3].

Let the coefficient of $(\mu v_n)^p$ in the expansion for q_i be written ${}^{(p)}q_i$. Then for the isotropic fluid it is possible to show that[4]

$$
\begin{aligned}
{}^{(0)}q_i &= 0, \qquad {}^{(1)}q_i = R_n P_1 \theta_{,i} + \frac{R_n \theta}{p} P_2 p_{,i}, \\
{}^{(2)}q_i &= \frac{R_n \theta}{p^2} [S_1 p_{,j} d^j{}_i + S_2 p_{,j} w^j{}_i + S_3 p_{,i} d^k{}_k] \\
&\quad + \frac{R_n}{p} [S_4 \theta_{,j} d^j{}_i + S_5 \theta_{,j} w^j{}_i + S_6 \theta_{,i} d^k{}_k] \\
&\quad + \frac{R_n \theta}{p} [U_1 \dot{x}_i{}^{,j}{}_{,j} + U_2 d^k{}_{k,i}],
\end{aligned}
\tag{75.2}
$$

where $P_1, \cdots, U_2$ are dimensionless functions of $p/\bar{p}$, θ/θ_0 only. If we put $-\kappa \equiv \mu v_n R_n P_1$, the first term in ${}^{(1)}q_i$ yields Fourier's law of heat conduction.

The forms of the various terms indicate many interesting phenomena, including thermal transpiration and its heat flux counterpart.

There is not space to write down the generalizations of these results to anisotropic fluids. If one does so, however, one finds a much closer analogy between stress and heat flux, as would be expected from Brillouin's principle (§74). It is the combination of the different tensorial orders of **t** and **q** with the notion of isotropy which forces those terms which appear in the one to be absent from the other, as indicated by (75.1) and (75.2).

76. Comparison with the equations of Burnett and Chapman-Cowling.[1] The idealization of the kinetic theory model of a gas results in the loss of but one of the ordinary coefficients of viscosity, as expressed by the Stokes relation (cf. §61A). In the second order theory, however, 13 of the 19 coefficients are lost. To reduce our formula for ${}^{(2)}t^i{}_j$ to Burnett's we first put $3D_i + 2C_i = 0$, $3F_i + 2E_i = 0$, so as to express the equality $p = \bar{p}$ which is assumed[2] in the kin-

[3] This conclusion does not hold for anisotropic fluids; cf. [1949, **8,** §9], [1951, **1,** §26].

[4] [1949, **8,** §10] [1951, **1,** §27]. In [1950, **14,** §§13–16] a theory of heat flux in the isotropic Stokesian fluid is worked out. On the supposition that thermal and mechanical reactions are to be associated exclusively with thermal and mechanical phenomena, respectively, it is shown that FOURIER's law is then the only possible law of heat conduction. An approximation to this result can be read off from (75.2), since the term representing FOURIER's law is the only one in which $\theta_{,i}$ is the only vector or tensor occurring.

[1] [1949, **8,** §§13–14] [1951, **1,** §§30–31]. In the corresponding discussion in [1948, **2**] [1948, **3,** §22] there are errors in sign.

[2] E.g. [1939, **1,** §§2.31–2.32, 2.41–2.42]. Cf. §61A.

etic theory. In addition we must assume the relations $C_1 = -C_4 = C_5 = -E_3 = \frac{1}{2}E_4$, whose physical meaning is not apparent.

Turning now to the heat flux, we find that while (75.3) contains two first order and eight second order coefficients, the equations of Chapman & Cowling contain but one and five, respectively[3]. To reduce our result to theirs, we must put $S_1 + 3S_3 = 0$, $S_5 = 3U_2 + U_1$, $P_2 = 0$, $S_2 = 0$. Now the first two of these restrictions express the linking of the magnitudes of various effects, as in the case of the stress above. The last two conditions indicate a more serious shortcoming of the kinetic theory result, however, for *two phenomena possible in a Maxwellian fluid are not predicted at all by Chapman & Cowling's heat flux formulae.* The first of these, which I propose to call the *Brillouin effect* since a term equivalent to that which reveals it occurs in M. Brillouin's formulae for the heat flux[4], shows that at sufficiently high temperatures and low pressures, a sufficiently large pressure gradient induces thermal flux even at uniform temperature. This effect is of first order in the viscosity (or conductivity), and, unless the coefficient P_2 be found very small experimentally, at high temperatures and low pressures may predominate over ordinary thermal conduction. The term whose coefficient is S_2 shows that just as a combination of rigid rotation and temperature gradient, so also a combination of rigid rotation and pressure gradient gives rise to a flow of energy.

From the foregoing results it appears that the kinetic theory, at least as ordinarily presented, becomes increasingly inadequate for the prediction of macroscopic phenomena the higher the approximation required[5].

77. The effect of vorticity.[1] Expressions for **t** involving the vorticity **w** were first obtained in the kinetic theory of M. Brillouin[2], who believed, following Stokes's principle (§59), that the stress should be independent of vorticity for a given rate of deformation, and thus several times emphasized his opinion that coefficients such as F_3, E_3, and E_4 would be found to be zero in the case of conservative intermolecular forces. Burnett's calculations[3], however, yield $E_3 = 1$, $F_3 = -\frac{2}{3}$ for Maxwellian molecules. This result is most improbable, for it implies the existence of a deviatoric stress proportional to $\mu_{\text{v}n}{}^2$ even in a

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[3] [1939, **1**, §15.4].

[4] BRILLOUIN [1900, **1**, §36] did not show that this effect is of first order in the viscosity, however, and in his equations this term is but one of some thirty and not distinguished in any way. Since BRILLOUIN had no means of ordering in powers of μ_n some of his remaining terms would occur in the ${}^{(3)}q_i$, ${}^{(4)}q_i$, $\cdots$ of the Maxwellian fluid, and several of the terms in our ${}^{(2)}q_i$ he failed to obtain.

[5] A more extended comparison of continuum and kinetic theory methods may be found in [1949, **8**, §15] [1951, **1**, §32].

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[1] [1948, **2**] [1948, **3**, §23] [1949, **8**, §13] [1951, **1**, §30].

[2] [1900, **1**, §§14 footnote, 23 end, 27 end, 34 end, 39 end].

[3] [1939, **1**, §15.41].

rigid body rotation[4] when $p_{,i} = \theta_{,i} = f_{i,j} = 0$. Adopting a modification of Boussinesq's principle $(60.1)_2$, we exclude such an effect by strengthening (c) of the definition of §74 to read:

(c)′ *If all the vectors and tensors in the list* (74.2) *with the exception* $\dot{x}^i_{\ ,j}$ *vanish, and if this latter reduce to a rigid rotation* ($\dot{x}^i_{\ ,j} = -\dot{x}_{j,}{}^i$), *then* $\mathbf{t} = -p\mathbf{I}$ *and* $\mathbf{q} = 0$. It follows then that $E_3 = F_3 = 0$.

78. Approach to a theory involving a natural time. The list of variables (74.2) includes velocity gradients $\dot{x}^{i_1}{}_{,i_2\cdots i_n}$ of all orders, and as the approximations are carried further terms in which there appear gradients of higher and higher order occur. Except for local time derivatives, the approximations are specified in terms of the same quantities which occur in a polynomial approximation to $\dot{x}^i(t + \Delta t)$ for a moving particle as a function of time. In a certain formal sense the theory may be said to approach in the limit one in which the succession of strains suffered by a particle in an interval of time influences its present response. The higher approximating formulae are so complicated as to render the theory quite useless for any type of calculation, and perhaps the type of behavior they suggest is represented both more accurately and more simply by a theory of fluids possessed of a natural time t_n and thus endowed at least with a natural elasticity (Chapter VC), possibly also with slight elasticity of shape (Chapter VI).

79. Boundary conditions. Several authors[1] have discussed the boundary conditions in rarefied gases from the standpoint of the kinetic theory, but a satisfactory result is yet to be obtained. Chang & Uhlenbeck[2], motivated by the desire to find boundary conditions in some sense intermediate between the adherence of the classical theory and the complete slip of the "Knudsen gas," have tentatively proposed not only the classical adherence condition $\Delta\dot{x}^i = 0$ but also the stronger condition $\dot{x}^i_{\ ,j}n^j = 0$, where $\mathbf{n}$ is the unit normal to the surface. The former condition is contrary to what experimental evidence is available (§73); the latter implies that all the frictional resistance arises from the second order terms. Under rather general hypotheses Grad[3] by his kinetic theory method derives a result equivalent to Navier's classical slip condition $\Delta\dot{x}^i = k\partial\dot{x}^i/\partial n$.

The question of boundary conditions is equally critical[4] in the Stokesian and

[4] Criticisms of Professors SYNGE and TSIEN are repeated and answered in [1951, **1**, §30].

[1] E.g. [1879, **2**, §§80, 83, 84] [1924, **3**] [1929, **6**] [1932, **6**, Ch. IX–X] [1947, **3**, Ch. V]. The validity of the results given in the last noted reference was questioned by UHLENBECK in a lecture at a meeting of the Am. Phys. Soc., New York, January 1948.

[2] [1948, **4**, §IVc].

[3] [1949, **36**, §6] [1950, **5**, §33].

[4] RIVLIN [1948, **5**, §13] apparently overlooks this fact. VIGUIER [1950, **7**] in treating a special case adds another boundary condition "pour simplifier".

Reiner-Rivlin theories. It does not arise in Rivlin's examples (§72) simply because these are developed by an inverse method, in which a motion is defined kinematically and then shown to be dynamically possible, provided certain surface stresses be supplied (but cf. §84 C).

80. Propagation of small disturbances in the Maxwellian fluid. At present experimental tests of the theory are necessarily limited to phenomena unaffected by the boundary conditions, such as the propagation of plane infinitesimal waves in a Maxwellian fluid of indefinite extent. Primakoff[1] and Tsien & Schamberg,[2] using for the coefficients special values calculated from the kinetic theory, find that while the damping is much increased as the pressure decreases, the speed of propagation is nearly unaltered. These conclusions depend upon the numerical values of the coefficients in an essential way; according to Chang & Uhlenbeck[3], when two errors in Chapman & Cowling's coefficients for the heat flux are corrected, the dispersion becomes considerable. Assuming a special force law for the molecules and special values for the coefficients, Mme Chang[4] concludes that the speed of sound in helium as predicted by the linearized Burnett equations exceeds the usual velocity by $.444\mathfrak{T}^2\%$, where d in the definition (66.1) of the number $\mathfrak{T}$ is taken as the frequency of a harmonically oscillating source. Mme Chang calculates that if $p = 10^{-2}$ atm. and $\theta = 300°$ K, a 3 mc. sound is propagated about 20 per cent faster than at standard conditions, and hence that the effect should be experimentally measurable. In any case these results have no quantitative value for gases like air because the Stokes relation is adopted in the kinetic theory (cf. §61), while the value of λ_V affects the magnitude of both absorption and dispersion in an essential way. On the other hand, recent measurements of Greenspan[5] indicate that in rarefied helium over a wide range of conditions the absorption and dispersion are very closely predicted by the *exact* solution of Kirchhoff's[6] complex frequency equation, derived from the Navier-Stokes equations, and of the four theoretical curves he compares with the experimental data that of Chang & Uhlenbeck gives by far the poorest fit.

The subject of ultrasonic absorption and dispersion in general is muddied by the roughness of the experimental data on the one hand, the disease of unnecessary and inappropriate theoretical approximations on the other. A review of the whole matter, together with the correct and complete results derivable from the linearized Navier-Stokes equations, will appear elsewhere[7].

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[1] [1942, **9**].
[2] [1946, **10**].
[3] [1948, **4**, §V].
[4] [1948, **41**, §IV].
[5] [1949, **55**] [1950, **23**].
[6] [1868, **1**].
[7] [1953, **1**].

Chapter VI. Superposition Theories

81. The classical superposition theories. Since elasticity and fluidity are two limiting cases of physical behavior (§2), it is natural to try to represent actual materials by a mathematical model possessing both elastic and fluid properties, and one way of formulating such models is by superposition. For simplicity, consider a case of simple infinitesimal extension. Then for the stress in an elastic body we have $T_{el} = 2\mu_E e_{el}$, while for a viscous fluid we have

$$T_{fl} = 2\mu_V\, d_{fl} = 2\mu_V\, \dot{e}_{fl}, \quad \text{or} \quad 2\mu_V\, e_{fl} = \int T_{fl}\, dt.$$

Superposing stresses, we obtain

$$T = T_{el} + T_{fl} = 2\mu_E e + 2\mu_V \dot{e}. \tag{81.1}$$

Introduced by O.-E. Meyer[1], the material defined by this equation was studied by Voigt[2] and has been named a *Voigt material* by v. Mises[3]; its theory has been thoroughly investigated by Duhem[4] and Thompson[5]. If $T = 0$ the material relaxes according to the law $e = e_0 \exp(-\mu_E t/\mu_V)$, so that *the natural time* $t_n \equiv \mu_V/\mu_E$ (§62) *is a relaxation time for the strain when the stress is removed.*

If instead we superpose strains, we obtain

$$e = e_{el} + e_{fl} = \frac{1}{2\mu_E} T + \frac{1}{2\mu_V} \int T\, dt, \qquad \dot{T} + \frac{\mu_E}{\mu_V} T = 2\mu_E \dot{e}. \tag{81.2}$$

In the material defined by this equation, which was introduced by Maxwell[6] and generalized by Natanson and Zaremba[7], *the natural time* $t_n \equiv \mu_V/\mu_E$ *is a relaxation time for the stress when the strain is kept constant.*

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[1] Meyer [1874, **1–2**] [1875, **2**] employed both molecular and continuum arguments.

[2] [1889, **1**] [1892, **2–3**] [1910, **1**, §§395–396].

[3] [1930, **1**].

[4] Duhem considered not only infinitesimal motions [1903, **14**] [1904, **1**, Part II] as did Meyer, Voigt, Thompson, etc., but also the case of finite elastic strain (with linear viscous damping) [1903, **12**, **15–17**] [1904, **1**, Part I, Ch. II; Part IV, Chs. II-III].

[5] [1933, **3**].

[6] Maxwell [1867, **1**, pp. 30–31] considered only the one-dimensional case used above for illustration. The occurrence of a relaxation time of this sort in the kinetic theory of gases is discussed in [1867, **1**, p. 69] [1902, **5**] [1946, **11**, Ch. IV, §3].

[7] [1901, **3–4**, 6] [1902, **2–3**] [1902, **4**, §§4–7] [1903, **2–3**, **5–10**]. These papers contain a heated controversy as to the proper form of the general equations, but both authors agree in using **d** rather than ė or Ė. Natanson's final equations are given in [1903, **5**, eqq. (1a) (2a)] [1903, **6**, eqq. (1a) (2a)]. Zaremba's final equations are [1903, **9**, eq. (32)], but [1903, **7**, eq. (28)] seem preferable. Zaremba discusses some solutions of his equations in [1937, **6**].

It is easy to form more and more complicated materials by further superpositions. The corresponding general equations in three dimensions may be written down by inspection so long as the strains be infinitesimal; in the finite case, however, the distinction between rate of deformation **d** and strain rate $\dot{\mathbf{E}}$ or $\dot{\mathbf{e}}$ occasions difficulty. Much attention[8] has been given to these theories, particularly in connection with infinitesimal oscillations. We make no attempt to review this field, but since Eckart's generalization of the Meyer-Voigt material contains a new idea as far as pure elasticity is concerned, we shall sketch its development.

82. Eckart's anelastic material. For an elastically isotropic[1] material Eckart[2] proposes the defining equation

$$\epsilon = \epsilon(\eta, v, \mathbf{c}), \tag{82.1}$$

where the Cauchy tensor **c** is taken with respect to a *varying* reference configuration whose rate of change **r** will be specified later. In the natural state theory (§36), v was eliminated by $(24.1)_2$ and $(14.11)_2$. The corresponding step here would complicate (82.1) by the introduction of an invariant of a different **c**, namely, that taken with respect to a fixed initial configuration; alternatively, v may be expressed in terms of $\tilde{v}$ and an invariant of the present **c**, but since $\tilde{v}$ is not a function of the X^α alone (as is v_0), nothing would be gained. Let pressure π and temperature θ be defined by (29.2), and let an *elastic stress* **s** be defined (cf. §33) by[3]

$$s^i{}_j \equiv -2\rho c^i{}_k \frac{\partial \epsilon}{\partial c^j{}_k}. \tag{82.2}$$

Then by (22.3) we have

$$\begin{aligned} \dot{\epsilon} &= \theta\dot{\eta} - v\pi\dot{x}^i{}_{,i} + \frac{\partial \epsilon}{\partial c^j{}_k}(2r^j{}_k - c^j{}_m\dot{x}^m{}_{,k} - c^m{}_k\dot{x}_m{}^{,j}), \\ &= \theta\dot{\eta} - v\pi\dot{x}^i{}_{,i} + vs^i{}_j(d^j{}_i - r^j{}_l(c^{-1})^l{}_i), \end{aligned} \tag{82.3}$$

[8] [1898, **1,** §2] [1917, **1**] [1918, **1,** §100] [1928, 4] [1929, **2,** §3] [1929, **3**] [1930, **5, 16**] [1935, **4,** §§6–7] [1938, **8**] [1939, **15**] [1941, **14–15**] [1943, **18**] [1945, **4**] [1947, **23, 25**] [1948, **33, 56**] [1949, **1–3, 32**] [1951, **13,** §4]. Comparative expositions are given in [1924, **1,** §§14.421–14.423] [1930, **1**] [1931, **2**] [1932, **4, 8**] [1933, **3,** Introd.] [1942, **6,** §18] [1943, **6,** Lect. VI] [1944, **7**] [1947, **16**] [1948, **22,** AI8 and Apps. II, III] [1949, **43,** Lect. XI].

[1] For anisotropic media we should use **C** in order to obtain a theory containing the classical natural state theory of finite strain as the special case $\mathbf{r} = 0$, $\mathbf{v} = \mathbf{s}$. The possible rotation of the reference configuration introduces a difficulty (§22) which I have not been able to overcome.

[2] [1948, **14,** §4]. For the case of strain referred to a fixed initial configuration, with superposed linear viscous damping, essentially this same analysis was given by Duhem [1904, **1,** Part I, Ch. II, §§I–IV]; cf. also [1931, **9,** pp. 77–79] [1933, **2,** §§8–9] [1933, **3,** §1.7].

[3] Eckart instead puts $s^i{}_j \equiv -2\rho\partial\epsilon/\partial c^j{}_i$, which also will yield (1.1) of the infinitesimal theory as a first approximation, but is not consistent with (41.2) of the classical theory of finite strain. This error complicates his subsequent equations.

so that by (27.2), (26.11), and (30.3) we obtain the elegant equation

$$\rho\theta\dot{\eta} = (v^i{}_j - s^i{}_j)d^j{}_i + s^i{}_j r^j{}_l (c^{-1})^l{}_i - q^i{}_{,i} \tag{82.4}$$

as a generalization of (30.2). Equivalently,

$$\rho\dot{\eta} + \left(\frac{q_i}{\theta}\right)_{,i} = -\frac{q^i\theta_{,i}}{\theta^2} + \frac{1}{\theta}(v^i{}_j - s^i{}_j)\, d^j{}_i + \frac{1}{\theta} s^i{}_j r^j{}_l\,(c^{-1})^l{}_i. \tag{82.5}$$

Thus the dissipated power is the sum of two portions (§30): the rate at which the extra stress **v** less the elastic stress **s** does work of deformation, and the rate at which the elastic stress **s** does work of relaxation. By $(29.1)_2$, the right side of (82.5) must be everywhere non-negative in any admissible motion.

We may now define particular types of bodies by specifying the vector **q** and both the tensors **v** − **s** and **r** as functions of other variables[4]. We shall not discuss possibilities for **q** (cf. §§74–75). It is reasonable to suppose that $\mathbf{v}' \equiv \mathbf{v} - \mathbf{s}$ represents viscous damping, so that by (59.1)

$$v^i{}_j - s^i{}_j = \mathcal{G}_0^{\mathrm{v}'(\mathrm{d})}\delta^i{}_j + \mathcal{G}_1^{\mathrm{v}'(\mathrm{d})}\, d^i{}_j + \mathcal{G}_2^{\mathrm{v}'(\mathrm{d})}\, d^i{}_k\, d^k{}_j\,, \tag{82.6}$$

where the $\mathcal{G}_\Gamma^{\mathrm{v}'(\mathrm{d})}$ are functions of I_d, II_d, III_d, scalar state variables, and scalar moduli only. Eckart suggests that the relaxation phenomena may well depend only on **s**, the elastic part of the stress:[5]

$$r^i{}_l(c^{-1})^l{}_j = \mathcal{G}_0^{\mathrm{r}(\mathrm{s})}\delta^i{}_j + \mathcal{G}_1^{\mathrm{r}(\mathrm{s})}s^i{}_j + \mathcal{G}_2^{\mathrm{r}(\mathrm{s})}s^i{}_k s^k{}_j\,, \tag{82.7}$$

where the $\mathcal{G}_\Gamma^{\mathrm{r}(\mathrm{s})}$ are functions of I_s, II_s, III_s, scalar state variables and scalar moduli only. Finally

$$s^i{}_j = \mathcal{G}_0^{\mathrm{s}(\mathrm{c})}\delta^i{}_j + \mathcal{G}_1^{\mathrm{s}(\mathrm{c})}c^i{}_j + \mathcal{G}_2^{\mathrm{s}(\mathrm{c})}c^i{}_k c^k{}_j\,, \tag{82.8}$$

where the $\mathcal{G}_\Gamma^{\mathrm{s}(\mathrm{c})}$ are of the special form (41.5). The various coefficients $\mathcal{G}$ are subject to the following restrictions. (1) When put into (82.5), the expressions (82.6), (82.8) must yield a non-negative right hand side in any admissible motion. (2) The appropriate conditions of compatibility (§22) upon **r** must be satisfied.

In place of (82.8) we may write the equivalent

$$s^i{}_j = \mathcal{G}_0^{\mathrm{s}(\mathrm{e})}\delta^i{}_j + \mathcal{G}_1^{\mathrm{s}(\mathrm{e})}e^i{}_j + \mathcal{G}_2^{\mathrm{s}(\mathrm{e})}e^i{}_k e^k{}_j\,, \tag{82.9}$$

---

[4] ECKART proposes $q_i = -\kappa\theta_{,i}$, $v^{ij} - s^{ij} = N^{ij}{}_{kl}d^{kl}$, $r^{ij} = M^{ij}{}_{kl}s^{kl}$, where $N^{ij}{}_{kl} = N^{ji}{}_{kl}$, $M^{ij}{}_{kl} = M^{ji}{}_{kl}$. Thus he considers a body of the following remarkable type: it is thermally linear and isotropic, its viscosity and relaxation are linear but (incorrectly) anisotropic (v. Note 1), while its elasticity is (incorrectly) non-linear (v. Note 3).

[5] ECKART actually proposes that **r** be an isotropic linear function of **s**.

a form which is convenient in case the strain from the relaxed configuration be very small (although the strain from the initial configuration may be very large), as for a fluid exhibiting a very slight springiness. Under these conditions the first requirement above can be satisfied in part by the linearized equations

$$s^i{}_j = \rho[\lambda_S I_e \delta^i{}_j + 2\mu_S e^i{}_j], \tag{82.10}$$

$$t^i{}_j = -p\delta^i{}_j + v^i{}_j = s^i{}_j + (-p + \lambda_V I_d)\delta^i{}_j + 2\mu_V d^i{}_j\,, \tag{82.11}$$

where $\mu_S \geqq 0$, $3\lambda_S + 2\mu_S \geqq 0$, $\mu_V \geqq 0$, $3\lambda_V + 2\mu_V \geqq 0$. Note that ρ is the present density (cf. §52). Now by hypothesis the quantities $s^i{}_j/\rho\mu_S$ are very small, and hence it may be permissible to linearize (82.7) also:

$$r^i{}_l(c^{-1})^l{}_j = (\alpha + \lambda_R I_s)\delta^i{}_j + 2\mu_R s^i{}_j\,, \tag{82.12}$$

where the condition $r^i{}_l(c^{-1})^l{}_j s^j{}_i \geqq 0$ requires that $\alpha = 0$, $3\lambda_R + 2\mu_R \geqq 0$, $\mu_R \geqq 0$. Eckart remarks that in experience great hydrostatic pressure effects very little permanent set; hence in the linearized theory[6] we shall put $3\lambda_R + 2\mu_R = 0$, so that $\mathbf{r} = 0$ if $\mathbf{s} = s\mathbf{I}$. Thus from (82.11)

$$r^i{}_l(c^{-1})^l{}_j = 4\rho\mu_S\mu_R[-\tfrac{1}{3}I_e\delta^i{}_j + e^i{}_j]. \tag{82.13}$$

The differential equations of the theory may now be obtained in two ways, just as in classical elasticity. (1) Since $\mathbf{e}$ must satisfy the St. Venant compatibility equations[7] (§18) with the X^L as independent variables, six partial differential equations each for $r^i{}_l(c^{-1})^l{}_j/4\rho\mu_S\mu_R$ and for $s^i{}_j/\rho$, analogous to the Beltrami-Michell equations[8], may be derived; and since the $d^i{}_j$ must satisfy the St. Venant equations with respect to the $\dot{x}^i$, six similar partial differential equations for $t^i{}_j - s^i{}_j + p\delta^i{}_j$ may be derived. To these must be added Cauchy's equations (26.2). Just as in classical elasticity, in this way we obtain an over-determined system. (2) From the definitions of $\mathbf{e}$ and $\mathbf{d}$ both (82.11) and (82.13) may be expressed in terms of the $x^i{}_{,L}$ and $\dot{x}^i{}_{,j}$, so that the compatibility conditions are automatically satisfied. The $x^i{}_{,L}$ may be expressed in terms of the $X^L{}_{,i}$. Putting (82.11) into $(26.2)_1$ yields three equations; putting (82.12) into (22.3) yields six equations. Thus nine equations for the six unknowns x^i and X^L are obtained. While in the classical theory of elasticity the analogous procedure yields a determinate system, in the present theory an over-determined system results again.

There is nothing new in the superposition of elastic and fluid properties, which is simply that of the Meyer-Voigt material (§81) extended to the case of finite deformations; in this respect Duhem's treatment is both correct and more thorough. The novel and valuable feature of Eckart's theory is that it

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[6] ECKART apparently concludes that the mere absence of the thermodynamic pressure from the expression resulting from substitution of (82.12) into (22.3) is sufficient to represent this effect.

[7] We assume here that the displacement gradients $x^i{}_{,L}$ are also infinitesimal (§19).

[8] [1892, **5**] [1900, **4**, pp. 111–114] [1927, **3**, §§92–93].

specifies the relaxation of the reference configuration.[9] Perhaps (82.7) is not the best possible hypothesis, but in any case it is a first step toward a concrete theory of relaxation based upon thermodynamic principles.

The relaxation moduli such as μ_R are of the dimension of reciprocal viscosity. Thus among the similarity parameters of this theory are $\mu_R\mu_V$ and $\rho\mu_S/d\mu_V$, where d is a characteristic rate of deformation or of relaxation or reciprocal of a time.

The only examples Eckart[10] gives concern infinitesimal motions, where in the usual manner all differential equations are linearized and all coefficients reduced to constants. He defines a "relaxation time" as the period of a plane wave whose phase velocity vanishes, indicates that his anelastic media have many relaxation times, and studies how these may arise in various types of waves. The value of his results is lessened by his neglect of thermal phenomena, while it was shown by Kirchhoff[11] that for a perfect gas even when relaxation is neglected the effects of viscosity and of heat conduction are of the same order of magnitude.

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[9] Thus it is quite different from the semi-empirical and purely static theory of REINER (§§45–46), in which the relaxed configuration (i.e. the final state of the body) must be known before a problem can be solved.

[10] [1948, **14**, §5] [1948, **15**].

[11] [1868, **1**].

Chapter VII. Progress and Program of the General Theories

83. Progress of the general theories. Experience, in part known for a long time and in part recently presented, exhibits a range of purely elastic or purely fluid behavior of interest both in the science of mechanics and in its practical applications, for whose description the classical theories are inadequate. We have seen that these phenomena are typical of the non-linear theories, and that a number of different theories yield qualitatively correct predictions for most or all of the indicated phenomena. The major concrete progress is the demonstration that in the common simple shearing and torsional situations *Kelvin and Poynting effects arise: the magnitudes of these effects and of the classical subjects of measurement are governed by independent moduli: the new effects are generally of the* second *order, while departure of the classical subject of measurement from its classical value is a* third *or higher order effect* (§§10, 42, 45, 71, 72).

Further, Rivlin (§§42, 72) has actually given *general solutions, both in the natural state theory of isotropically elastic incompressible bodies and in the Reiner-Rivlin theory of incompressible fluids, corresponding to the tests generally employed* for simple elastic and fluid measurements. He has conducted also a number of experiments on very large strain of rubber, obtaining in every case *full confirmation of the predictions of the theory.*

The theory of isotropic functions (Chapter II) offers simple and convenient tools for the discussion of any general theory in an important special case. The infinitesimal theory of elasticity has been generalized in three ways. Green's *method of strain energy* leads to the *classical natural state theory* (Chapter IVA), where several equivalent forms of general non-linear stress-strain relations in terms of strains measured from a certain preferred state of the body are derived. Cauchy's *method of stress-strain relations* in its broadest form leads to the more general but less convincing semi-empirical theory of Reiner (Chapter IVB), where the stress is assumed to be a function of the strain from whatever configuration the body would assume if the deforming forces were removed. If the classical linear relations be supposed to apply strictly to material rates, the simplest form of the Jaumann-Murnaghan *rate of deformation theory* (Chapter IVD) results. The classical theory of viscous compressible fluids is generalized by precise specification of the manner in which energy is dissipated, and by the dimensional moduli which occur. In the theory of the *Stokesian fluid* (Chapter VB) no modulus of the dimension of time is allowed, and the resulting higher order terms are appropriate to the description of phenonema at low pressures. The *Maxwellian fluid* (Chapter VD) is a generalization of the Stokesian fluid in which thermal gradients, pressure gradients, etc. give rise to stresses even in a fluid suffering no deformation, and thus is intended to represent phenomena occurring in rarefied gases. The *Reiner-Rivlin fluid* (Chapter VC) possesses a natural time, and its higher order terms are appropriate to phenomena in which the rate of deformation or vorticity approaches the reciprocal of this time. In In *Eckart's anelastic material* (§82) the stress is composed of two parts, one being that arising from a strain energy and the other being of a viscous type; the former is specified in terms of strain from a *varying* relaxed state, whose rate of change depends upon the present stress.

In logical development and presentation there is much difference between the various theories. The classical natural state of theory of elasticity may serve as a model of clarity and completeness, except for its defective thermodynamical foundation. Owing to a great variety of awkward notations, as well as a few misconceptions, the literature of this subject is deplorably repetitious; apparently the various authors have been unwilling to devote the pains necessary to follow the elaborate and lengthy analyses of their predecessors, preferring instead to plague posterity with yet another long list of special symbols in the derivation of results which if expressed in a general notation would immediately be seen to be equivalent to others already known (cf. §§39–41). While an author easily remembers his own notations, however ill chosen, some sort of uniformity is helpful to the reader, and here as in all the general theories *tensor notation is not merely an abbreviation but a real conceptual aid.*

84. Program of future research.[1] A. *The natural state theory of elasticity (Chapter IVA)*. The general theory is complete except for an adequate treatment of thermoelastic effects. For the technically important case of infinitesimal strain but large rotations (§49), various proposed approximations should yield equivalent results, but Seth's formulation seems most convenient for solution of special examples. Rivlin has shown what can be done when the strain is large, but his results are almost completely limited to incompressible bodies. Perhaps some further solutions, say for torsion of an elliptic cylinder, bending of a circular cylinder, and a half plane or sphere subject to concentrated force, might reveal new phenomena. Some clear examples showing the effect of compressibility should be obtained. In any case further work in elasticity should concentrate upon the fully general theory, leaving the form of the strain energy arbitrary.

B. *Dynamic elasticity*. Results predicted from the natural state theory of elasticity for bodies in motion are not realistic when internal damping is neglected. While numerous more or less plausible theories of solid bodies endowed with viscosity exist (§81), none simple enough to yield solutions gives results in agreement with experiment. A wholly new theory, physically adequate and yet mathematically tractable, is most instantly needed.

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[1] Fuller proposals for both the experiment and theory are given in [1952, **2**].

C. *The Jaumann-Murnaghan rate of deformation theory* (*Chapter IVC*). This theory offers a natural approach to problems of large initial stress. The energetics should be worked out in detail. Some special problems should be solved in as exact a fashion as possible and compared with the results of experiment—preferably not by curve-fitting, but with formulae using the classical coefficients of elasticity.

D. *Theories of fluid dynamics* (*Chapter V*). Here there is one great hindrance to future progress: *lack of proper boundary conditions* (§79). In a fluid it is natural to expect the boundary conditions to be at least partially kinematical. Thus for example Rivlin in giving a general solution for flow in a pipe (§72B) tacitly assumes that the fluid adheres to the walls without slipping. Since the differential equations for $\dot{x}^i$ increase in degree with the degree of approximation adopted for the stress formulae, we should expect Rivlin's solution to be only one of several satisfying the same boundary condition. In the case of the Maxwellian fluid the situation is still more complicated because not only the degree but also the order of the differential equations may be arbitrarily high.

E. *Eckart's theory of anelasticity* (§82). Eckart's notion of specifying the relaxation of the material by specifying the functional dependence of the reference rate tensor **r** is promising. While the energetics of the theory are fairly clear, no illustration of its dynamical aspects has been given, except for some dubious linearizations. Here as in the rate of deformation theory an exact treatment of simple extension, simple shear, simple torsion, etc., perhaps in the special cases of fluids with slight elasticity of shape or solids with slight viscosity, should be the next objective.

F. *Connection with plasticity*. Some of the general theories presented here represent some aspects of plasticity. It should be possible to study plastic materials with corresponding generality, and corresponding emphasis upon essential non-linear phenomena.

G. *Reacting media*. The framework of §§25, 32 may serve as a foundation for the theory of deformation of media in which chemical reactions take place. I have not been able to examine Jaumann and Lohr's treatments of the subject with the attention they deserve.

H. *Use of structure theories in conjunction with continuum methods*. An example of the use of a structure theory to suggest a special form of a continuum theory may be found in the recent investigations of rubber (§54), leading to Rivlin's neo-Hookean material. In this case the only purpose actually served by the structure theory was to call attention to the simplest possible exact continuum theory—a theory so simple that apparently no one had thought of investigating it, and one which turned out moreover to be insufficient. In general, the extreme idealization of the molecular theories severely limits their value for the quantitative prediction of gross phenomena (cf. e.g. §76). Their chief advantage lies in their yielding specific values for the various moduli, which in a continuum theory must be experimentally determined—but to obtain these values still further idealizations are necessary, so that the results are often at variance with experiment except for the simplest substances, and should be used only in case no experimental value is available. In the study of gross phenomena the structure theories might possibly be useful as priming to suggest the variables suitable for a proper continuum analysis.

I. *Methods of solution in general.* All solutions of special problems in the general theories given so far have been obtained by purely inverse methods. As in other branches of the mechanics of continua, it is likely that many valuable and informative results can be obtained by semi-inverse methods, in which some boundary conditions are prescribed, while others are replaced by some geometrical or kinematical property[2].

J. *Comparison of results with experiment.* The non-linear theories offer the temptation of a large number of empirical moduli, by which the results of almost any single experiment can be fitted. This temptation should be resisted, and a new term should be retained in the equations only when it is positively demonstrated to yield a new effect not present in the classical theories but significant in the experiment. The introduction of "apparent" moduli, depending on the motion as well as the material, while common in engineering practice, is confusing and useless; moduli should be absolute constants, or functions of the thermodynamic state only.

K. *Decisive experimental tests.* Rigorous solutions of the equations of all the proposed theories for simple cases, particularly in shearing, torsion, combined torsion and extension, etc., should be calculated and put side by side[3], and then carefully compared with experiments on various materials, particular attention being paid to the Poynting effect. Response of the material in these cases is a more significant test for the non-linear theories than for the classical theories because even though one-dimensional in one of the tensors these situations are two-dimensional or three-dimensional in another (§10).

[2] A stimulating survey is given by NEMÉNYI [1951, **5**]. Note added at press time: semi-inverse methods have just been successfully applied to the problem of moderate twist of an incompressible cylinder by GREEN & SHIELD [1951, **15**].

[3] Such a comparison for some of the approximate theories of elasticity is unsuccessfully attempted by SETH [1950, **18**, §§5–13]; cf. [1951, **14**].

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