The Memristor Inside Out

The missing element has been found!

Earlier this year HP Lab engineers announced their physical realization of the 'missing' fourth basic circuit element in electronics: the memristor. Not often a technological discovery attracted so much attention from the media. Apart from the wildest possible speculations on future applications in new non-volatile memory devices with human brain synthesizing properties and suggestions to rewrite the existing textbooks on circuit theory, the discovery met with much scepticism as well. What exactly is this memristor? Where does it come from? What will it bring us? Why didn't we miss it before?

Author: Dimitri Jeltsema

The four element quadrangle

Since electronics was developed, engineers designed, analyzed, and synthesized circuits using combinations of three basic two-terminal elements: resistors, inductors, and capacitors. From a mathematical perspective, the behavior of each of these elements, whether linear or nonlinear, is described by relationships between two of the four electrical variables: voltage, current, charge, and flux(-linkage). A resistor is described by the relationship of current and voltage, a capacitor by that of voltage and charge, and an inductor by that of current and flux. But what about the relationship between charge and flux? As Professor Leon O. Chua (the inventor of the well-known chaotic Chua circuit) from the University of California, Berkeley, pointed out in his 1971 paper, a fourth element should be added to complete the symmetry. He coined this 'missing' element the *memristor*. More specifically, if q denotes the charge and ϕ denotes the flux, then a two-terminal *charge-controlled* memristor is defined by the constitutive relationship.

$$\phi = \hat{\phi}(q). \quad (1)$$

Since flux is the time integral of voltage u (like in Faraday's law), and charge is the time integral of current *i*, or equivalently, $u = d\phi/dt$ and $i = dq/dt$, we obtain, after differentiating (1) with respect to time, the more familiar expression

$$u = M(q)i, \quad (2)$$

where $M(q) := d\hat{\phi}(q)/dq$ is called the incremental or small-signal memristance. At first glance (2) shows that a two-terminal charge-controlled memristor behaves like a linear resistor described by Ohm's law. The difference, however, is that its resistance *M(q)* is not a constant, but va-

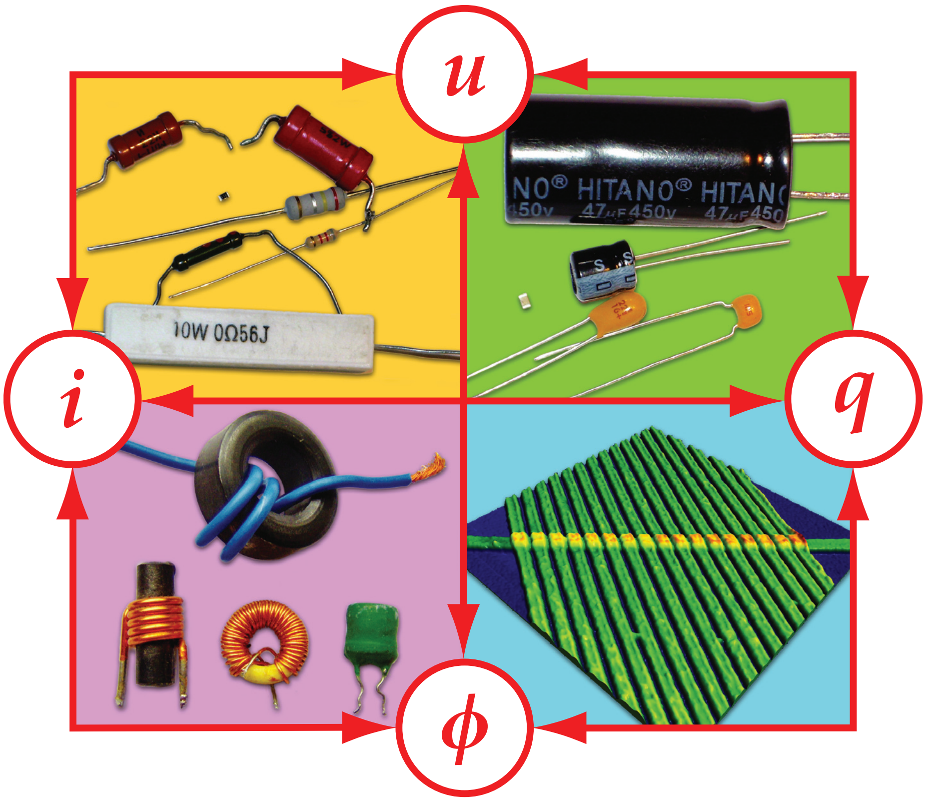

Figure 1: The four element quadrangle. An inductor corresponds to a static relationship between current i and flux ϕ, a capacitor corresponds to a static relationship between voltage u and charge q, and a resistive element corresponds to a static relationship between current and voltage. There are two dynamical relationships, one between current and charge, and the other between voltage and flux. The remaining relationship, namely between flux and charge, defines a memristor.

ries with the instantaneous value of the charge. Recalling that charge follows from the time integral of current, it thus records the past values of the current and hence motivates the name memory resistor, or memristor for short. It follows from (2) that the SI unit of memristance is the ohm [Ω], the same as that of resistance.

Similarly, a two-terminal *flux-controlled* memristor (memductor) is defined by

$$q = \hat{q}(\phi), \quad (3)$$

Differentiation with respect to time yields

$$i = W(\phi)u, \quad (4)$$

where $W(\phi) := d\hat{q}\ q(\phi)/d\phi$ is called the incremental *memductance*. Clearly, the corresponding SI unit of memductance is the mho [℧] or Siemens [S], the same as that of conductance.

The relationships between the variables and the four basic electrical elements are summarized in the so-called four element quadrangle shown in Figure 1.

Linear versus nonlinear

In the special case that the constitutive relationship of a memristor is linear, or in other words, when the constitutive relationship defines a straight line through the origin in the flux-charge plane, a memristor becomes an ordinary linear resistor. Indeed, in such case (1) reduces to $\phi = Mq$, with constant memristance M (the slope of the line). Differentiation of both sides of the latter with respect to time yields $u = Mi$, which precisely takes the form of Ohm's law. Hence it is not possible to distinguish a two-terminal linear memristor from a two-terminal linear resistor. This perhaps explains why its existence could not be predicted from classical linear circuit theory.

A curious kind of pipe

In order to gain some intuition for what distinguishes a memristor from a resistor, as well as from an inductor or a capacitor, let us briefly consider the common analogy of an electrical resistor and a pipe that carries a fluid. The fluid can be considered analogous to charge, the pressure at the inlet of the pipe is similar to voltage, and the rate of flow of the fluid through the pipe is like current. As is the case with a resistor, the flow of fluid through the pipe is faster if the pipe is shorter or if it has a larger diameter and vice-versa.

Now, an analogy for a memristor is an peculiar kind of pipe that expands or shrinks when fluid flows through it. For example, if fluid flows through the pipe in one direction, the diameter of the pipe increases, thus enabling the fluid to flow faster. If fluid flows through the pipe in the opposite direction, the diameter of the pipe decreases, thus slowing down the flow of fluid. If the fluid pressure is turned off, the pipe retains its most recent diameter until the fluid pressure is turned back on. Unlike a bucket, which can be considered as a hydraulic capacitor, a memristive pipe does not store the fluid, but 'remembers' the amount of fluid that flowed through it. In the electrical domain this means that, like a capacitor, a memristor has a memory, but unlike a capacitor it does not store charge but just 'remembers' the last charge that passed through it. It is precisely this persisting memory feature of the memristor that could be used advantageously to create a new type of non-volatile RAM. More about that later

Quasi-static field perspective

It is well known that the circuit-theoretic definitions of resistance, inductance, and capacitance can be associated with electromagnetic systems operating in their quasi-static limit. From this point of view, a resistor or conductor corresponds to an electromagnetic system for which the first-order fields are negligible compared to its zero-order fields. Its low frequency behavior is then characterized by an instantaneous (memoryless) relationship between the zero-order electric and magnetic field intensities. Similarly, an inductor corresponds to an electromagnetic system for which both the zero-order electric field and the first-order magnetic field can be ignored. The behavior of an electromagnetic system for which both the zeroorder magnetic field and the first-order electric field can be ignored corresponds to a capacitor.

The fourth combination, in which both zero-order fields are negligible while the first-order fields are both relevant, naturally implies to correspond to a memristor type of device. Indeed, the latter situation gives rise to an instantaneous relationship between the first-order electric and magnetic field densities, which in turn correspond to charge and flux. It should be noted that this interpretation also implies that a memristor or memductor is essentially an AC device since under DC operating conditions the resistive behavior (zero-order fields) can not be ignored.

Brother or distant cousin?

So a memristor is essentially a nonlinear element described by the same fundamental set of circuit variables as the passive two-terminal resistor, inductor, and capacitor. But does that give it the right to be just as fundamental as the latter familiar three circuit elements? This, of course, depends on how we (prefer to) look at it. From a linear perspective it is senseless to complement the linear circuit elements with a linear memristor as it precisely coincides with an ordinary resistor. In the realm of impedances it is clear that linear electronics is already complete in itself; linear resistors are purely real Q impedances, linear inductors and capacitors are merely the positive and negative purely imaginary impedances »

Impedance is not passive if its real part is negative. There is simply no room to complement that.

One the other hand, apart from the fact that linear elements can be considered as a special case (small-signal or local approximation) of nonlinear elements, a few arguments in favor of the memristor as the fourth fundamental passive circuit element can be given as follows. A fundamental property of a resistor, inductor, and capacitor, whether linear or nonlinear, is that the values of their associated incremental or small-signal resistance, inductance, and capacitance, respectively, do not change with the frequency of an infinitesimally small sinusoidal variation about any fixed point of operation. The same property holds true for a memristor. Furthermore, there does not exist a combination of two-terminal passive resistors, inductors, and/or capacitors that duplicates the properties of a memristor (although including active elements like op-amps can do so). These features make the memristor just as fundamental as the existing three elements.

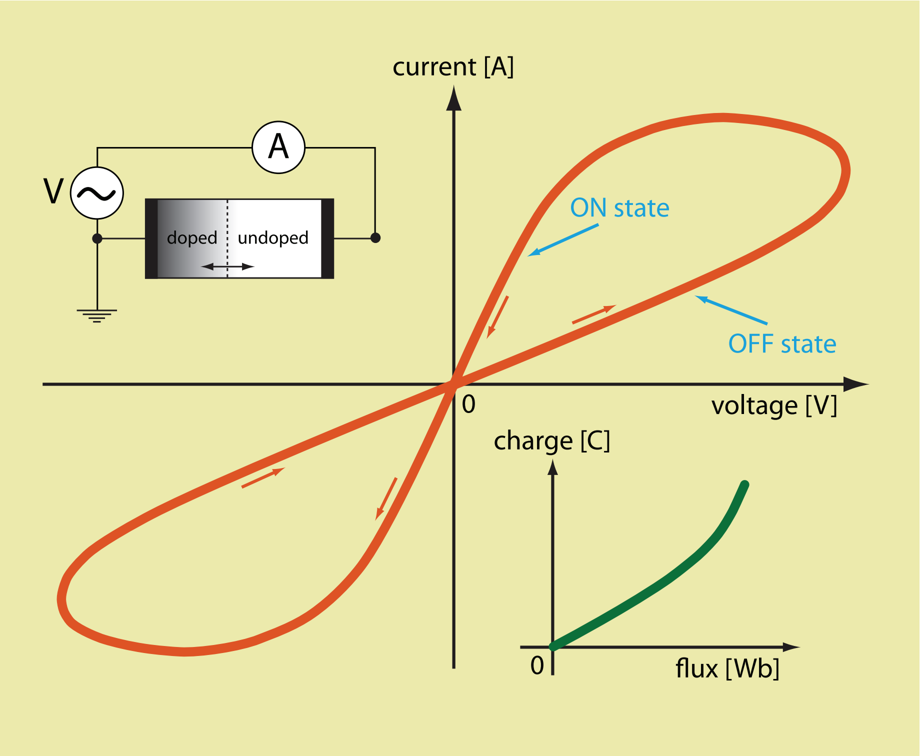

Figure 2: Current-voltage plot demonstrating hysteretic phenomena of HP Lab's memristor. The application of a sinusoidal voltage (V) across the device will move the boundary between the doped and undoped regions causing the charged dopants to drift. The distribution of the dopants, and thus the resistance of the device, is proportional to the charge that passes through. Note the corresponding charge-flux plot shows a much simpler non-hysteretic relationships.

About HP Lab's device

Now that we know some ins and outs about the theoretical background of the memristor, let us briefly look at what the engineers at HP Lab have actually created. HP Lab's memristor is a two-terminal, two-layer semiconductor constructed from layers of titanium oxide (a substance we also find in toothpaste and suncream) sandwiched between two metal electrodes in a crossbar architecture. One layer of titanium oxide is doped with oxygen vacancies and the adjacent layer is undoped, leaving it in its natural state as an insulator. Under the influence of a bias voltage, oxygen vacancies move from the doped layer of titanium dioxide to the undoped layer. A high concentration of dopants results in a relatively low resistance. Likewise, if the polarity of the voltage is reversed, oxygen vacancies migrate back into the doped layer, thus turning to the region with relatively high resistance. The most typical feature of HP's device is that, after reversing the polarity of bias voltage, the current does not take the same reverse path, an effect we know as hystereses.

An example of a typical current-voltage characteristic observed by the HP engineers is shown by the so-called Lissajous plot of Figure 2. In relation to the features highlighted above, the two approximately straight lines segments within the curve correspond to the two distinct resistance states. The connecting end parts are the transition regions between these two states. Obviously, a memristor can be used as a switching device, where the low resistance or conduction state can be associated with its 'ON' state and the high resistance state as the 'OFF' state. As already emphasized before, the main advantage of the memristor is that its resistance changes are non-volatile, and remain until a reversed bias voltage is applied.

Criticism

The main criticism received by HP Lab's discovery is that memristors, or the memristance phenomenon in particular, already existed. Indeed, a variety of physical devices, including thermistors, discharge tubes, Josephson junctions, and even ionic systems like the Hodgkin-Huxley model of a neuron, were shown to exhibit memristive effects. Apart from the fact that these devices belong to a broader class of systems that generalize the memristor, called memristive systems, there remains a lack between the mathematics and the physical properties. Furthermore,

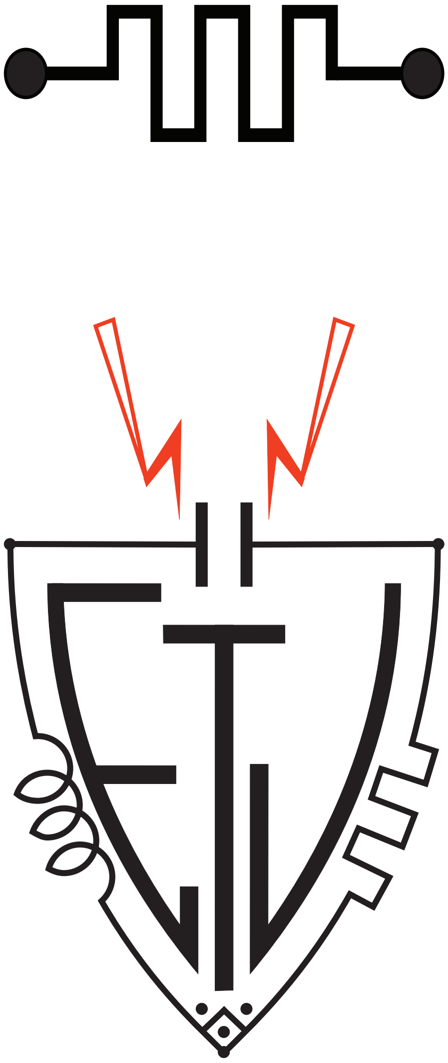

Figure 3: Memristor symbol (upper) and the logo of the ETV (lower).

it is also known that there have been many researchers before who observed similar peculiar hysteretic current-voltage characteristics in various materials. However, most of these observations were reported as anomalous or interpreted as difficult time-varying conductances, often leading to paradoxes and confusion.

Admittedly, the actual order of events at HP Lab was alike. The HP Lab engineers were also puzzled by their creation and it took them years to realize that their device satisfied the equations of Chua's memristor. For that reason, the main contribution of HP Lab is that they provide a physical passive two-terminal model that allows a better understanding of the mechanism behind memristance and the hysteretic current-voltage characteristics observed in many nanoscale electronic devices. This understanding might gain the possibility to create new and useful devices.

Concluding remarks

The question why we did not really miss the memristor before can most likely find its answer in the fact that so far the majority of practical devices are still reasonably well modeled by some (though often artificial) combination of standard circuit building blocks such as the resistor, inductor, capacitor, and their nonlinear and multiport versions. As nanoscale electronic devices become more and more important and complex, it might be beneficial, and on the longer term maybe even necessary, to enlarge our repertoire of modeling building blocks that establishes a closer connection between the mathematics and the observed physics. This of course does not mean that we have to abandon our classical and familiar tools, or that we have to rewrite the existing textbooks on linear circuit theory. We should, however, avoid to confine ourselves too much to the things we ought to think of as safe, sound and complete.

In conclusion, it would be interesting to see what future applications arise from the concept of memristance. As remarked by Stan Williams from HP: "the most valuable applications of memristors will most likely come from some young student who learns about these devices and has an inspiration for something totally new." There seems to be a fairly big chance that this will be somebody from our faculty as the existence of the memristor was already apparent from the logo (see Figure 3) of the Electrotechnische Vereeniging (ETV), even 65 years before it was postulated as the fourth element.

Selected references

This text is largely based on a variety of resources which are too many to cite here. Some key references that are recommended to the interested reader are listed below. An article nuancing some of the overblown media statements can be found in the 10th issue of Bits & Chips magazine of 2008 (in Dutch).

- L.O. Chua. Memristor—the missing circuit element. IEEE Trans. on Circ. Theory, CT-18(2):507–519, September 1971.
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