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THE MERSENNE MEET MATRICES ON POSETS

Şerife BÜYÜKKÖSE

The University of Gazi, Turkey

Abstract

We consider Mersenne meet matrices on posets as an abstract generalization of Mersenne greatest common divisor (Mersenne GCD) matrices. Some of the most important properties of Mersenne GCD matrices are presented in terms of meet matrices.

1 Introductıon

Let $S = \{x_1, x_2, ..., x_n\}$ be a set of distinct positive integer,and let f be an arithmetical function. Then $n \times n$ matrix (S) whose i,j-entry is the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S.[1, 2, 4, 5]

The set S is said to be *factor-closed* if it contains every divisor of any element of S,and the set S is said to be *GCD-closed* if it contains the geratest common divisor of any two elements of S.[2, 3, 7]

In 1876, H. J. S. Smith showed that if S is factor-closed (FC set), then

$$\det(S) = \prod_{k=1}^{n} \phi(x_k),$$

where ϕ is Euler′s totient function. Beslin and Ligh showed that if S is GCD-closed, then

$$\det(S) = \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_t \\ x_t < x_k}} \phi(d).$$

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and the $n \times n$ matrix $[M] = (m_{ij})$, where $m_{ij} = 2^{(x_i, x_j)} - 1$, call it to be Mersenne-GCD matrix on S.[8, 9] If $S = \{x_1, x_2, \ldots, x_n\}.FC$ set then

$$det\,[M] = \prod_{i=1}^{n} g(x_i)$$

where $g(n)$ an arithmetical function defined as

$$g(n) = \sum_{d|n} \left(2^d - 1\right) \mu\left(\frac{n}{d}\right).$$

In this paper we consider an abstract generalization of Mersenne GCD matrices,namely Mersenne meet matrices on posets.Some of the most important properties of Mersenne GCD matrices are presented in terms of meet matrices.

2 Definition of Mersenne Meet Matrices

$(P, \leq)$ be a finite poset. We call P a meet semi-lattice if for any $x, y \in P$ there exists a unique $z \in P$ such that

1) $z \leq x$ and $z \leq y$ and

2) if $w \leq x$ and $w \leq y$ for some $w \in$ then $w \leq z$.In such a case z is a called the meet of x and y and is denoted $z = x \wedge y$.

Let S be a subset of P. We call S *lower-closed* if for every $x, y \in P$ with $x \in S$ and $y \leq x$ we have $y \in S$.We call S *meet-closed* if for every $x, y \in S$ we have $x \wedge y \in S$.In this case S itself is a meet semilattice.

It is clear that a lower-closed subset of a meet semilattice is always meet-closed, but not conversely. The concept of "lower-closed" and "meet-closed" are generalizations of "factor-closed" and "gcd-closed" respectively.[4, 6, 7]

In what follows, let P always denote a finite meet semilattice. S a poset that can be embedded in a meet semilatice and $\overline{S}$ the unique (u to isomorphism) minimal meet semilattice containing S.

Let x and y be two elements of the poset P. Then

$$\mu(x,y) = \begin{cases} 0 & ; \quad if \quad x \nleq y \\ 1 & ; \quad if \quad x = y \\ -\sum\limits_{z:z \leq y} \mu(x,z) & ; \quad otherwise. \end{cases}$$

Definition 1 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and the $n \times n$ matrix $S = (s_{ij})$ where $s_{ij} = 2^{x_i \wedge y_j} - 1$,is called the Mersenne meet matrix on S.*

3 Determinant of The Mersenne Meet Matrices

Theorem 1 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a meet-closed subset of P. Then*

$$\det(S) = g(x_1)g(x_2)...g(x_n)$$

where $g(x_i)$ defined by

$$g(x_i) = (2^{x_i} - 1) - \sum_{x_j \in S, x_j < x_i} g(x_j)$$

(Here $x_j < x_i$ means that $x_j \leq x_i$ and $x_j \neq x_i$)

Corollary 2 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a lower-closed subset of P then*

$$\det(S) = g(x_1)g(x_2)...g(x_n)$$

where

$$g(x_i) = \sum_{x_j \leq x_i} (2^{x_i} - 1)\, \mu(x_j, x_i)$$

or equality

$$f(x_i) = \sum_{x_j \leq x_i} g(x_j)$$

μ being the Möbius function of P.

Example 1 *$S = \{1, 2, 3, 6\}$ is a lower-closed set and $(S, |)$ is a poset. Consider 4×4 Mersenne meet matrix on S.*

$$(S) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 3 \\ 1 & 1 & 7 & 7 \\ 1 & 3 & 7 & 63 \end{bmatrix}.$$

$$\det(S) = g(x_1)g(x_2)...g(x_n) = g(1)g(2)g(3)g(6).$$

By using the definition of g and $\mu(x, y)$ we obtain;

$$g(x_1) = g(1) = \sum_{x_j|1} (2^{x_i} - 1)\, \mu(x_j, 1) = \mu(1, 1) = 1$$
$$g(x_2) = g(2) = \sum_{x_j|2} (2^{x_i} - 1)\, \mu(x_j, 2) = \mu(1, 2) + 3\mu(2, 2) = 2$$
$$g(x_3) = g(3) = \sum_{x_j|3} (2^{x_i} - 1)\, \mu(x_j, 3) = \mu(1, 3) + 7\mu(3, 3) = 6$$
$g(x_4) = g(6) = \sum_{x_j|6} (2^{x_i} - 1)\, \mu(x_j, 6) = \mu(1, 6) + 3\mu(2, 6) + 7\mu(3, 6) + 63\mu(6, 6) = 54$ and where;

$$\mu(x, y) = \begin{cases} 0 & ; \quad x \nmid y \\ 1 & ; \quad x = y \\ -\sum_{z : z|y} \mu(x, z) & ; \quad \textit{otherwise.} \end{cases}$$

Thus;

$$\det(S) = g(1)g(2)g(3)g(6) = 1.2.6.54$$

Definition 2 *Let $S=\{x_1,x_2,\ldots,x_n\}$ and $T=\{y_1,y_2,\ldots,y_m\}$ be any subsets of P. Define the incidence matrix $E(S,T)$ of S and t as an $n\times m$ matrix whose i,j-entry is 1 if $y_j\leq x_i$,and 0 otherwise, namely*

$$E(S,T)=(e_{ij})_{n\times m}=\begin{cases}1 & ; \quad y_j\leq x_i\\ 0 & ; \quad otherwise.\end{cases}$$

Theorem 3 *Let $S=\{x_1,x_2,\ldots,x_n\}$ be a subset of P with $\overline{S}=\{x_1,x_2,\ldots,x_n,x_{n+1},...,x_{n+r}\}$.Let g be a function on $\overline{S}$ defined as in Theorem 1. Then*

$$(S)=E.\ diag(g(x_1),...,g(x_{n+r})).E^T$$

where $E=E(S,\overline{S})$ and E^T is the transpose of E.

Theorem 4 *Let S,$\overline{S}$, f and g be as in Theorem 2. Then*

$$\det(S)=\sum_{1\leq k_1<k_2<...<k_n\leq n+r}\det(E_{(k_1,k_2,...,k_n)})^2 g(x_{k_1})...g(x_{k_n})$$

where $E_{(k_1,k_2,...,k_n)}$ is the submatrix of $E=E(S,\overline{S})$ consisting of the $k_1th,k_2th,...,k_nth$ columns of E.Furthermore; if g is a function with positive values then

$$\det(S)\geq g(x_1)g(x_2)...g(x_n)$$

and the equality holds if and only if S is meet-closed.

4 Inverse of Mersenne Meet Matrices

Theorem 5 *Let $S=\{x_1,x_2,\ldots,x_n\}$ be a lower-closed subset of P and let $g(x_i)=\sum\limits_{x_j\leq x_i}(2^{x_i}-1)\,\mu(x_j,x_i)\neq 0$ for all $x_j\in S$. Then (S) is invertible and $(S)^{-1}=(b_{ij})$, where*

$$b_{ij}=\sum_{\substack{x_i\leq x_k\\ x_j\leq x_k}}\frac{1}{g(x_k)}\mu(x_i,x_k)\mu(x_j,x_k).$$

Example 2 *(S) is a Mersenne meet matrix on lower-closed set $S=\{1,2,3,6\}$. By Theorem 8*

$$(S)^{-1}=B=(b_{ij})$$

where,using Example 1

$$b_{11} = \sum_{1|x_k} \frac{1}{g(x_k)}\mu(1,x_k)^2 = \frac{\mu(1,1)^2}{g(1)} + \frac{\mu(1,2)^2}{g(2)} + \frac{\mu(1,3)^2}{g(3)} + \frac{\mu(1,6)^2}{g(6)} = \frac{91}{54}$$

$$b_{12} = \sum_{\substack{1|x_k \\ 2|x_k}} \frac{1}{g(x_k)}\mu(1,x_k)\mu(2,x_k) = \frac{\mu(1,2)\mu(2,2)}{g(2)} + \frac{\mu(1,6)\mu(2,6)}{g(6)} = -\frac{28}{54}$$

$$b_{13} = \sum_{\substack{1|x_k \\ 3|x_k}} \frac{1}{g(x_k)}\mu(1,x_k)\mu(3,x_k) = \frac{\mu(1,3)\mu(3,3)}{g(3)} + \frac{\mu(1,6)\mu(3,6)}{g(6)} = -\frac{10}{54}$$

$$b_{14} = \sum_{\substack{1|x_k \\ 4|x_k}} \frac{1}{g(x_k)}\mu(1,x_k)\mu(4,x_k) = \frac{\mu(1,6)\mu(6,6)}{g(6)} = -\frac{1}{54}$$

$$b_{22} = \sum_{2|x_k} \frac{1}{g(x_k)}\mu(2,x_k)^2 = \frac{\mu(2,2)^2}{g(2)} + \frac{\mu(2,6)^2}{g(6)} = \frac{28}{54}$$

$$b_{23} = \sum_{\substack{2|x_k \\ 3|x_k}} \frac{1}{g(x_k)}\mu(2,x_k)\mu(3,x_k) = \frac{\mu(2,6)\mu(3,6)}{g(6)} = \frac{1}{54}$$

$$b_{24} = \sum_{\substack{2|x_k \\ 4|x_k}} \frac{1}{g(x_k)}\mu(2,x_k)\mu(6,x_k) = \frac{\mu(2,6)\mu(6,6)}{g(6)} = -\frac{1}{54}$$

$$b_{33} = \sum_{3|x_k} \frac{1}{g(x_k)}\mu(3,x_k)^2 = \frac{\mu(3,3)^2}{g(3)} + \frac{\mu(3,6)\mu(6,6)}{g(6)} = \frac{10}{54}$$

$$b_{34} = \sum_{\substack{3|x_k \\ 6|x_k}} \frac{1}{g(x_k)}\mu(3,x_k)\mu(6,x_k) = \frac{\mu(3,6)\mu(6,6)}{g(6)} = -\frac{1}{54}$$

$$b_{44} = \sum_{6|x_k} \frac{1}{g(x_k)}\mu(6,6)^2 = \frac{\mu(6,6)^2}{g(6)} = -\frac{1}{54}$$

Therefore since $(S)^{-1} = B$ is the symmetric we have

$$(S)^{-1} = B = \begin{bmatrix} \frac{91}{54} & -\frac{28}{54} & -\frac{10}{54} & \frac{1}{54} \\ -\frac{28}{54} & \frac{28}{54} & \frac{1}{54} & -\frac{1}{54} \\ -\frac{10}{54} & \frac{1}{54} & \frac{10}{54} & -\frac{1}{54} \\ \frac{1}{54} & -\frac{1}{54} & -\frac{1}{54} & \frac{1}{54} \end{bmatrix}.$$

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