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Chapter 2

The Metaphorical Structure of Mathematics: Sketching Out Cognitive Foundations for a Mind-Based Mathematics

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WARNING!

This is an essay within a new field of study—the cognitive science of mathematics. The contribution we seek ultimately to make is a new one: to characterize precisely what *mathematical ideas* are. You might think that this enterprise would leave mathematics as it exists alone and simply add to it an account of the conceptual nature of mathematical understanding. You could not be more wrong.

Studying the nature of mathematical ideas changes what we understand mathematics to be and it even changes the understanding of particular mathematical results. The reason is that a significant amount of 20th-century mathematics rests on the assumption that mathematics is not about minds and ideas, but rather about symbols and their model-theoretical interpretations. We call this 20th-century view *mind-free mathematics,* where the substance of mathematics is assumed to be independent of any human minds. Our enterprise is to bring embodied human minds, as they have come to be understood recently in cognitive science, back into mathematics, and to construct a precise *mind-based mathematics.* Mind-based mathematics is not just mind-free mathematics with some cognitive analysis added. Rather, the introduction of mind changes mathematics itself, not just mathematics education or the study of mathematical cognition.

Mind-free mathematics obscures the full beauty of mathematics. This essay is a first attempt to reveal that beauty through the characterization of a mind-based mathematics, and the changes made are in the service of revealing that beauty—and if possible, adding to it.

REASON AND MATHEMATICS: WHY COGNITIVE SCIENCE LEADS TO NEW FOUNDATIONS

Our understanding of mathematics is intimately bound up with our view of reason. The dominant tradition in Western philosophy has been to see reason as purely abstract, transcendental, culture-free, unemotional, universal, decontextualized, disembodied, and hence formal—a matter of pure form. Mathematics was seen in this tradition as the best example of reason, and hence it too was seen as having these properties. The attempt to give purely formal foundations for mathematics was a natural product of this philosophical tradition.

The tradition of the disembodied mind culminated in the enterprise of developing a disembodied, abstract, asocial, and decontextualized "Artificial Intelligence" as the purest manifestation of this belief: Reason as abstract and disembodied, and mathematics as the highest and purest form of reason. The result was the attempt, beginning in the 1950s, to ground cognitive science—the empirical study of the mind—in the field of artificial intelligence.

That movement died for two reasons. First, it failed to achieve the enthusiastic predictions of the 1950s. Nothing close to an empirically adequate view of mind was ever developed. Indeed, the more it was tried, the more the prevailing conceptions of the mind shifted towards views that consider the mind as situated in context, grounded in experience, shaped by culture, and dependent upon the peculiarities of human embodiment.

But the second and principal reason for the change came from new discoveries about the mind made in the 1970s, in the fields of neuroscience, psychology, anthropology, and linguistics. Such discoveries about the way the mind arises from the body as it functions in its everyday world have changed our understanding of the nature of reason, and have brought with them both a need for a reconceptualization of mathematics and the possibility of a breakthrough in the understanding of what mathematical ideas are.

Reason has turned out to be a product of our bodies and brains and not part of the objective nature of the universe. Human concepts are not passive reflections of some external objective system of categories of the world. Instead they arise through interactions with the world and are crucially shaped by our bodies, brains, and modes of social interaction. What is humanly universal about reason is a product of the commonalities of human bodies, human brains, physical environments and social interactions.

With those discoveries has come the discovery of new mechanisms of human reason previously undreamed of, which turn out to be very unlike the kind of symbol manipulation that formal logicians, creators of computer languages, inventors of formal mathematical proofs, as well as many philosophers of mind and orthodox cognitive scientists, had envisioned. The fundamental mechanisms of human reason include image-schemas, basic-level concepts, idealized cognitive models, prototypes of various kinds, radial categories, conceptual metaphors and metonymies, mental spaces, and conceptual blends. For an introduction to this literature, see the References, Sections A and B.

ESSENCES: THE RATIONALE FOR THE OLD FOUNDATIONS

None of these fundamental mechanisms of human reason had yet been discovered when the great 19th and 20th century inventors of formalist mathematical "foundations" did their work. When Frege, Russell, Hilbert, Weierstrass, Cantor, Gödel, Tarski and others were actively constructing 20th century mathematical philosophy and asserting that it constituted the basis for mathematics, it seemed reasonable to think of proofs as just strings of symbols to be satisfied in a formal model, rather than as mathematical narratives expressing ideas.

It seemed natural in those days to think of reason—and its best example, mathematics—as an objective and abstract feature of the universe, structuring not just this physical universe, but all possible physical universes. The utility of mathematics for describing the physical universe led to the idea that somehow the mathematics was just out there in the world—as a timeless and immutable objective fact—structuring the physical universe.

There are many variants of objectivist views of mathematics. One well known view is Platonism, that view that a unique "correct" mathematics exists independent of any minds in some "Platonic realm." As Lakoff (1987, chap. 20) has shown, such a view is incompatible with the independence results in mathematics. These results do not merely show that no formal axiom system is sufficient for mathematics. Additionally, they show that there are substantively different but equally valid forms of number theory, algebra, and topology—depending on which mathematically valid concept of "set" one uses. Since each such concept of set is equally valid mathematically, and since they give rise to substantially different but equally valid forms of mathematics, there is no unique "correct" mathematics. Thus the Platonic view of a unique, correct mathematics is mathematically incorrect.

Nonetheless, the idea that a unique mathematics is out there in the world, independently of any mind, is a powerful idea. It makes mathematicians into the ultimate scientists, discovering truths not only of this universe, but of any universe. It puts mathematicians at the top of the scientific totem pole. Results in the physical and biological sciences may be superseded. But mathematical proofs, once proved, stay proved. They are the ultimate in "truth."

The view of mathematics as being out there in the world is a natural consequence of an everyday folk theory and a collection of conceptual metaphors that has played a major role throughout Western philosophy: The Folk Theory of Kinds and The Metaphors of Essence. The Folk Theory of Kinds may be stated as follows:

- Every specific thing is a kind of thing.
- Kinds are categories, which exist as entities in the world.
- Everything has an essence, which makes it the kind of thing it is.
- Essences are causal; essences—and only essences—determine the natural behavior of things.
- The essence of a thing is part of that thing.

There are three basic metaphors for characterizing what an essence is. They are:

- Essences are substances.
- Essences are forms.
- Essences are patterns of change.

The folk theory of essences is part of what constitutes our everyday "common sense"—that is, it is part of the unconscious conceptual system that governs our everyday reasoning. Take a particular tree, for example. We understand that tree to be an instance of the general kind, Tree. The general kind is seen as having an existence of its own. When we say that there are trees in the world, we don't just mean the particular trees that happen to exist now. What is it that makes a tree a tree?

Substance: It is made of wood. If it were made of plastic, it wouldn't be a real tree. Thus substance counts as part of its essence.

Form: It has a form: trunk, bark on the trunk, roots, branches, leaves (or needles) on the branches, the roots underground, the trunk oriented relatively perpendicular to the ground, the branches extending out from the trunk. Without such a form, it wouldn't be a tree.

Change: It has a pattern of change: it grows out of a seed, matures, dies.

We apply this everyday folk theory of kinds, which works very well for things like trees, to abstract cases as well. We conceptualize people as having essences—their personality and character—that make them the kinds of people they are. A person may be friendly or mean, have a heart of gold or be rotten to the core. Depending on such judgments about their essence, we generate expectations about how people will behave.

The folk theory of kinds is also at the heart of science. Science seeks the essential properties of things, the properties that make a thing the kind of thing it is and allow us to predict its behavior. Mathematics is central to this endeavor. Mathematics is used in science to characterize the forms of things and the patterns of change a thing undergoes.

MATHEMATICS AS ESSENCE

Consider, for example, the mathematical characterization of laws of motion, which govern how things in motion behave. We understand a set of equations as constituting the essence of motion in the physical world. Since essences are metaphorically "part of" or "in" the things they are essences of, just as forms or shapes are "part of" or "in" physical objects, so the mathematical laws of motion are seen as being "in" the physical world, structuring how our world behaves. Thus, The Folk Theory of Kinds and Metaphors of Essence lead to the view that mathematics is "in the world"—an objectively real aspect of the universe, independent of any minds. According to such an objectivist view of mathematics, mathematics cannot consist in ideas, since ideas are products of minds, and the world can exist without minds.

The Folk Theory of Kinds and the Metaphors of Essence are hardly an accident. They arise because we are neural beings. Neural systems must form categories. Given our perceptual systems, we categorize neurally what we perceive. Our brains have disparate and separated systems for characterizing color, shape, motion, etc. and have other neural mechanisms for putting these together to form mental "kinds"—that is, neural categories. Colors, as we now know, do not exist in the objective world; though they are a product partly of wave-length reflectances or real world objects, color categories are just as much a product of the color cones in our retinas and the neural circuitry of our brains. What we call "colors" might better be called "chromatic experiences." Yet we perceive colors as being "in the world"—as being external to us, not just as being chromatic experiences. What our brains characterize neurally we attribute to the objective world, as if there were essences existing independently in the world. In many cases, as with trees, no practical problems arise. After all, we have the folk theories we have because they have worked well in everyday cases for generations. It is only when scientific discoveries intervene, as in the case of color, that we realize that they are only folk theories, not aspects of reality.

We claim that mathematics is no more part of the external, mind-free world than color is. Both are aspects—very different aspects—of the embodied mind. Because of the commonalities in our bodies and brains, we uniformly believe that grass *is* green, blood *is* red, and the sky *is* blue for every human being with a normally functioning color system (not counting those who are color-blind). Similarly, the reason that mathematics is common throughout the world and is stable is that the relevant parts of our bodies and brains and everyday experience is also common and stable.

When scientists model the world using humanly created mathematics, they are making a lens through which they see the world and a mechanism with which they can make predictions. But the lens is not part of, not *in*, what they see.

Mathematics, of course, differs from color in an important way. As we shall see, it arises from a wide range of common bodily experiences, makes use of the full range of the imaginative apparatus of the mind, has been actively constructed to serve important human purposes, and has come into being socially, over time, in the crucible of active debate.

What we attribute to the world by way of essences is a product of our brains—one of the most important and most functional products of our imagination. That includes mathematics. As with other abstract essences, we construct mathematical essences—essences of form and patterns of change—and we attribute them to the world, just as naturally as we attribute color to things in the world.

In short, mathematics is about essence. That is what makes mathematics "abstract." It abstracts away from all that is contingent; it doesn't care about what happens to be true about something, but only about what is necessarily true.

The ontology of mathematics has all the problems of the ontology of essences. Just as our common sense perception leads us to see color as being in the external, mind-free world, so our common sense leads us to understand mathematics as being a part of the external mind-free world. The usual reasoning is right out of the Folk Theory of Kinds and the Metaphors of Essence: If mathematics "works"—if it correctly characterizes the behavior of some aspect of the real world, and if only an essence can allow us to correctly predict behavior, then the mathematics must be real, and it must be constitutive of that essence. QED (in the folk theory of essence).

The usual reasoning is, of course false. Essences no more exist objectively in the world than colors do. Our *perceptual* systems naturally lead us to conceptualize color as being part of the objective world, when it isn't. Similarly, our *conceptual* systems naturally lead us to conceptualize the world as if essences too were in the world. Again, they aren't.

In the case of color, we have no choice but to perceive and to think in a way that, upon reflection, we know is scientifically incorrect. The same is true of essences. We may be made aware by cognitive science that essences are created by our conceptual systems. But as we go about our lives every day, we have no choice but to think in terms of essences and conceptualize them as part of the world—even though we know, upon reflection, that it is not true.

As with color, the idea that essences are a part of the world in itself is a human fabrication, reproduced naturally and spontaneously in all of us. Since we understand mathematics as characterizing essences, it seems "just common sense" to see mathematics as part of the objective world—even though contemporary cognitive science reveals that essences are no more objectively in the world than colors.

IS TWO AND TWO REALLY FOUR?

What does it mean to say that mathematics is *not* a part of the objective external world? Are we denying that two and two is four? Of course not. But now suppose that there are two chairs in this room and we bring in two more. Would we deny that there are now four chairs in the room? Of course not. But doesn't that mean that $2 + 2 = 4$ is in the world?

Of course not. The reason should be clear. The category *chair* is a human category. Suppose the four chairs in the room are different kinds of chairs—a desk chair, an armchair, a patio chair, and a recliner. Those are four very different objects. It is our human conceptual system that categorizes them all as chairs. Counting presupposes the grouping of things to be counted. But it is beings with minds that group and count things. Everything in nature is different in some way from everything else in nature. Things in nature are not grouped in and of themselves. Hence, there are no groups with numbers assigned to them in and of themselves in nature. Living beings create groupings and numberings. But we don't notice ourselves as doing so; we attribute groupings and numberings to the external world.

ESSENCE AND FOUNDATIONS

Everything that we understand as being an essence is a product of our minds, and that includes mathematics. We have learned this from recent cognitive science. But it was not always known.

At the beginning of this century the mathematical community dedicated an impressive amount of work to study the ultimate foundations of mathematics: A serious and systematic effort to find the most fundamental cornerstones on which the whole edifice of mathematics could stand. Of course, this cornerstone was conceived of as "purely mathematical," independent of ideas, minds, and people. The foundations rested on two things. First, "logic"—seen not in terms of the reasoning of human beings, but in terms of the mechanical manipulation of abstract, uninterpreted symbols. Second, the theory of sets.

There were philosophical reasons for these choices. Mathematical proof was seen as the best example of human reason. Proofs were taken technically not as psychological entities—ideas—within some theory of mind, but simply as sequences of strings of symbols. If this were so, then reason too should be characterizable in terms of sequences of strings of uninterpreted symbols.

Furthermore, there was long tradition in metaphysics in which the world was seen as being made up of discrete objects, with discrete properties and relations holding between them at any time. In this metaphysical tradition, the world also included kinds and essences (see Lakoff, 1987). Set theory was seen as a way to model such a world. The discrete entities were taken to be the entities sets were made up of. The properties were to be modeled by sets of objects, and relations by *n*-tuples of objects. The essences were to be modeled by sets of properties. And kinds were to be modeled by sets.

Gottlob Frege was the father of this tradition. He knew perfectly well that the symbols used in proofs were not meaningless—they were given interpretations. But if mathematics was to be free of the vagaries of human minds and bodies, he had to make the meanings of mathematical proofs free of human minds and bodies. So he claimed that meaning could be reduced to truth and reference, and that truth and reference could be modeled using abstract symbols and sets.

The foundations failed. First, there was Russell's paradox: Since any well-formed description should define a set of objects meeting that description, there should have been a set of the form $\{x: x \text{ is not a member of } x\}$, the set of all sets not containing themselves. But that led to a contradiction: x is in the set if and only if it isn't. In addition, it should have been possible on this approach to axiomatize all of mathematics. The most famous and devastating indication of the failure of this idea was the Gödel incompleteness proof. Gödel showed that for any finite collection of axioms for arithmetic that could either be listed or specified by rule, there will be an infinity of truths of arithmetic that cannot be proved.

By the 1940s some accomplished mathematicians started to manifest doubts about the ultimate objective foundations of mathematics stressing the fact that mathematics may well be a creative, social and original activity of man, like language or music (Wilder, 1952, 1986). But unfortunately these ideas remained at a speculative level and did not really become interesting subject matters for the community of mathematicians. The

question of how just the right kind of mathematics could get into just the right places to structure just the right aspects of the physical universe was not seriously asked until the mathematician Saunders MacLane asked it—and concluded that the mythology of a mathematics out there in the world was an impossibility. MacLane (1981) asked why we have the branches of mathematics we have, noting there is nothing in the foundations philosophy that could possibly explain why we have branches of mathematics at all, much less specific branches like number theory, geometry, topology, probability theory, and so on. MacLane concluded, reasonably, that the branches of mathematics arose from human activities like counting, building, gambling, and so on, and that the application of mathematics to the physical universe came out of a close acquaintance with mathematics together with close observation of the physical environment.

From a contemporary perspective on cognitive science and neuroscience, it now seems bizarre that 20th century mathematical foundations ever evolved at all. They are, after all, mathematical "foundations" without an account of mathematical ideas. They have no account of ideas at all. "Proofs" are just set-theoretical structures—sequences of meaningless symbol strings—and "interpretations" are just mappings of those strings onto other set-theoretical structures, and the mappings are also set-theoretical structures. According to these foundations, mathematics is nothing but set-theoretical structures. It is as though mathematics had nothing to do with the mind. In an age when so little was known about the mind, that might have seemed reasonable. But given what we are learning about the mind nowadays, this view does not make sense any longer. In order to understand mathematics, minds, and reason, we simply need new views. New accounts of the mind and of reason are being developed within cognitive science. We want to extend that development to mathematics.

GROUNDING IN EMBODIMENT

Let us stop for a moment, step back and think: Mathematics, one of the most imaginative activities in the history of humankind, has been supplied with foundations that seek not to explicate mathematical imagination precisely, but rather to ban from the characterization of mathematics all forms of imagination—images, metaphorical thought, signs, pictures, narrative forms. Of course, practicing mathematicians go on using their minds with all of their unconscious imaginative mechanisms working silently, unobserved, and undescribed in the highest of high gears. The shame has been that mathematics has not been properly understood as a product of inspired human imagination, and has been taught—generation after generation—as if it were something else, a mere grasping of transcendental truths, something lesser and less interesting.

Of course, there is a historical reason why this is so. Imagination used to be thought of as a matter of personal inspiration, differing wildly from person to person, and showing up primarily in art and poetry. That view has been shown to be inadequate. Imaginative mechanisms, such as metaphorical thought, schematic mental imagery, and forms of narrative are part of the general mental endowment of all normal human beings, and they arise in a regular way from commonalities of our bodies, brains, and experiences interacting in the physical and social world. A great deal of metaphorical thought is universal, and the image-schemas that characterize spatial reasoning (like container-schemas, source–path–goal schemas, and part–whole schemas) are conceptual universals realized in common neural structures (Regier, 1996). Much of what is "abstract" in mathematics, as we shall begin to see in what follows, concerns coordination of meanings and sensemaking based on common image-schemas and forms of metaphorical thought. Abstract reasoning and cognition are thus genuine embodied processes (see references, Section C).

One of the properties of commonplace conceptual metaphors is that they preserve forms of inference by preserving image-schema structure. As we shall see, conceptual metaphors are central to mathematical ideas, and one of the reasons why the inference structures of mathematical proofs is stable is that the inference structures of commonplace metaphors is stable. That feature of metaphors is one of the reasons why theorems, once proved, stay proved.

The stability of basic mathematical ideas over hundreds, sometimes thousands of years, requires that the neural structures used must be commonplace and readily configurable. It requires that mathematical ideas make use of the most commonplace of everyday experiences and ideas—concepts like motion, spatial relations, object manipulation, space, time, and so on. The study of the conceptual structure of mathematics must show how mathematics is built up imaginatively out of such commonplace experiences and ideas. Mathematics thus, is not the purely "abstract" discipline that it has been made out to be by formal foundationalists, orthodox philosophers of mind, and proponents of classical artificial intelligence. Our mathematical conceptual system, like the rest of our conceptual systems, is grounded instead in our sensorimotor functioning in the world, in our very bodily experiences.

OUR GOALS

The enterprise of studying the mathematical conceptual system can be understood in at least three ways.

- As the empirical study of the unconscious conceptual system that constitutes mathematical thought.
- As the task of identifying and clearly describing the collection of ideas that constitutes what mathematics *is*.
- As a helpful task for mathematics education. If we are going to teach mathematical ideas, it is useful to know what ideas are to be taught and what the human conceptual system for those ideas is.

Much of the interest in mathematical conceptual systems has come from the third concern on the list—education—by means of studies of didactic processes, curricula, and pedagogical devices, but also by painstaking explorations of classroom realities, everyday mathematical thinking, informal and noninstitutional teaching, and so on. We loudly applaud those educators who have been working to figure out the nature of mathematical understanding for the sake of teaching mathematics. Most of the contributions of this volume, are concerned with the third issue, and are good examples of this endeavor.

But the educational process as such is only a peripheral concern of ours here. We are mainly concerned with the first and second enterprises. We want to study the details of the conceptual system from which mathematics arises, and by which it is conceived and understood. Through that, we hope to describe what that mathematics is as a conceptual system. This entails a *new philosophy of mathematics, in which mathematics is a product of the embodied human mind*—especially its imaginative capacities such as image-schemas and conceptual metaphors. Because we begin with results in cognitive science, and understand mathematics as product of embodied minds functioning in the world, we are neither platonists nor constructivists, neither objectivists nor subjectivists (see Lakoff, 1987; Núñez, 1995). Those a priori philosophical perspectives do not describe what mathematical ideas are and they simply cannot explain why mathematics has the structure it has nor how it can be learned anew by human bodies/brains generation after generation with an amazingly stable content.

We are aware that analyzing the conceptual structure of mathematics is a major enterprise, and we are at the very beginning of that enterprise. In the following pages we give a general idea about the richness, complexity, and beauty of the imaginative and embodied processes that make mathematics possible. In doing this, we sketch some elements we consider crucial in the study of the cognitive foundations of mathematics. We analyze a few interesting and important examples that show how mathematics is permanently created and sustained by imaginative resources such as conceptual metaphors, metonymies and blends, shared by those who practice, teach, and learn mathematics.

FUNDAMENTAL TYPES OF CONCEPTUAL METAPHORS IN MATHEMATICS

There is a branch of cognitive linguistics concerned with metaphor theory and we use the results of that discipline throughout. The major results are these:

- There is an extensive conventional system of conceptual metaphors in every human conceptual system.
- Metaphors are cross-domain conceptual mappings. That is, they *project* the structure of a source domain onto a target domain. Such projections or mappings can be stated precisely. (These terms are not the same as the terms *mapping* and *projection* in the formal mathematics. They have a different precise meaning in cognitive semantics.)
- Metaphorical mappings are not arbitrary, but are motivated by our everyday experience—especially bodily experience.
- Metaphorical mappings are not isolated, but occur in complex systems and combine in complex ways.
- As with the rest of our conceptual systems, our system of conventional conceptual metaphors is effortless and below the level of conscious awareness.
- Metaphor does not reside in words; it is a matter of thought. Metaphorical linguistic expressions are surface manifestations of metaphorical thought.
- Unlike mathematical mappings, metaphorical mappings may add structure to a target domain.
- The inferential structure of the source domain is preserved in each mapping onto a target domain, except for those cases where target domain structure overrides the mappings (see Lakoff, 1993, for details).
- The image-schema structure of the source domain is preserved in the mappings.
- The evidence on which these general claims are based comes from eight sources: generalizations over polysemy, generalizations over inference patterns, extensions to novel cases, historical semantic change, psycholinguistic experiments, language acquisition, spontaneous gestures, and American sign language.
- Novel metaphors that we consciously concoct use the mechanisms of our everyday unconscious conventional metaphor system.

Other than conceptual metaphors we will also have need to refer to conceptual metonymies, or "metonymyc mappings." Such mappings link two elements in a single conceptual schema, in such a way that the first *stands for* the second. An everyday example would be a sentence such as "Table six left without paying" said by a waitress to another. Here "table six" stands for "the customer at table six." Metonymy occurs in mathematical discourse in cases like "the function approaches zero," where "the function" stands for "the value of the function."

For basic references, see Lakoff and Johnson, 1980; and Lakoff, 1993.

MATHEMATICAL AGENTS

We use metaphor to conceptualize mathematics, and in the process we create what we call "mathematical agents." A mathematical agent is a metaphorical idealized actor, that is, an idealized actor in the source domain of a metaphor characterizing some aspect of mathematics. For example, when addition is conceptualized as putting objects in a collection, the mathematical agent is the one who does the collecting. In this case, the agent does nothing but collect objects; we call such an agent a *Collector.* Similarly, when addition is conceptualized as taking steps of a certain length in a certain direction, the one who does the moving is a metaphorical mathematical agent, and correspondingly, we call him a *Traveler.*

As we saw earlier, mathematics is about essence. The universe of mathematics is thus a universe of abstract objects and actions. Mathematical agents, when they appear, have only the minimal essential features needed to perform the kind of action performed. Indeed, the properties of the agent are often so minimal that it is difficult or impossible to distinguish the agent from the action. For example, take a source domain when there is an agent that moves. All that can be mapped by a mathematical metaphor onto the target domain is that the agent moves. No particular qualities of an agent, like hair color or gender, can be mapped, because hair color and gender play no roles in the metaphors that ground our mathematical understanding.

THE LITERAL BASIS OF COGNITIVE ARITHMETIC

It is well known that human beings, as well as many other higher animals, have the capacity to instantaneously perceive quantities of objects. Human beings can accurately perceive and distinguish up to six objects at a glance. Some monkeys can do better—eight or more.

This capacity is technically called "subitizing" (from the Latin *subitare,* meaning to "arrive suddenly"). It is, of course, combined with other basic cognitive capacities such as the ability to form mental images, remember, form groupings, superimpose images, and so on. Such capacities have allowed human beings to create a literal, but primitive cognitive arithmetic. But it is metaphor that has allowed us to move beyond such a relatively primitive mathematical capacity to form an abstract mathematics of dizzying complexity.

GROUNDING METAPHORS AND LINKING METAPHORS

There are two fundamental types of metaphors used in forming mathematical ideas: *grounding metaphors* and *linking metaphors.* Grounding metaphors ground mathematical ideas in everyday experience. For example, they allow us to conceptualize arithmetic operations in terms of forming collections, constructing objects, or moving through space. Since metaphors preserve inference structure, such metaphors allow us to project inferences about collecting, constructing, and moving onto the abstract domain of arithmetic. Metaphorical projections preserve the structure of image-schemas—cognitive schemas for such things as containers (bounded regions in space), paths, entities, links, and so on. Consequently, grounding metaphors allow us to project precise yet abstract image-schema structure from everyday domains that we know and understand intimately to the domain of mathematics. And correspondingly, grounding metaphors project inferences about our everyday world that we implicitly understand as well as we understand anything onto the domain of mathematics. In short, our understanding of arithmetic rests on our intimate and precise understandings of domains like collecting, constructing objects, and moving.

While grounding metaphors allow us to ground our understanding of mathematics in familiar domains of experience, linking metaphors allow us to link one branch of mathematics to another. For example, when we metaphorically understand numbers as points on a line, we are linking arithmetic and geometry. Such metaphors allow us to project one field of mathematical knowledge onto another. In this case, we project our knowledge of geometry onto arithmetic in a precise way via metaphor. It is the conceptual metaphor that tells us exactly how our knowledge of geometry is to be projected onto arithmetic.

As we shall see shortly, both grounding and linking metaphors can be presupposed within definitions. We will call such cases *metaphorical definitions.* When metaphors are given as definitions, we shall call such cases *definitional metaphors.*

Let us now turn to the details. Here are some of our most basic grounding metaphors. The name of the metaphor is given at the top of the list, and each bullet demarcates a submapping. An expression of the form "*B* Is *A*" is to be understood as the asymmetric source-to-target metaphorical mapping: "$A \rightarrow B$."

Arithmetic Is Object Collection

- Numbers Are Collections of Physical Objects of uniform size.
- The Mathematical Agent Is a Collector of Objects.
- Arithmetic Operations Are Acts of Forming a collection of objects.
- The Result of an Arithmetic Operation Is A Collection of Objects.
- The Unit (One) Is The Smallest collection.
- The Size of the Number Is The Physical Size (volume) of the collection.
- The Quantity Measured by a number Is the Weight of the Collection.
- Equations Are Scales Weighing collections that balance.
- Addition Is Putting Collections Together with other collections to form larger collections.
- Subtraction Is Taking smaller collections from larger collections to form other collections.
- The Number of Times an action is performed Is The Collection Formed by adding a unit for each performance of the action.
- Multiplication Is The Repeated Addition of collections of the same size a given number of times.
- Division Is The Repeated Dividing up of a given collection into as many smaller collections of a given size as possible.
- Zero Is An Empty Collection.

Arithmetic Is Object Construction

- Numbers Are Physical Objects.
- The Mathematical Agent Is a Constructor of Objects.
- Arithmetic Operations Are Acts of object construction.
- The Result of an Arithmetic Operation Is A Constructed Object.
- The Unit (One) Is the Smallest whole object.
- The Size of the Number Is the Size of the Object.
- The Measure of the size of a number Is The Collection of smallest whole objects needed to construct the object.
- The Quantity Measured by a number Is the Weight of the Object.
- Equations Are Scales Weighing objects that balance.
- Addition Is Putting Objects Together with other objects to form larger objects.

- Subtraction Is Taking smaller objects from larger objects to form other objects.
- The Number of Times an action is performed Is The Object Formed by adding a unit for each performance of the action.
- Multiplication Is The Repeated Addition of objects of the same size a given number of times.
- Division Is The Repeated Segmentation of a given object into as many objects of a given smaller size as possible.
- Zero Is The Absence of Any Object.

These conceptual metaphors are not only used for conceptualizing arithmetic, but they also form the basis of the language we use for talking about arithmetic. Here are some linguistic examples of these conceptual metaphors. First, there is a group of cases which instantiate both the Collection and Object Construction metaphors. The reason that the examples fit both is that both metaphors are instances of a more general metaphor: Arithmetic Is Object Manipulation, in which Numbers Are Physical Objects and Adding Is Putting Objects Together.

A trillion is a *big* number.
How many 5's are there *in* 20?
There are 4 5's *in* 23, and 3 *left over.*
Five from 12 *leaves* 7.
12 less 5 *leaves* 7.
How many times does 2 *go into* 10?
7 is too *big* to *go into* 10 more than once?
If 10 is on one side of the equation and 7 is on the other, what do you have to add to 7 to *balance* the equation?

There are of course linguistic examples that distinguish the Object Construction metaphor from the Object Collection metaphor.

Object Construction:

If you put 2 and 2 together, it *makes* 4.
What is the *product* of 5 and 7?
2 is a *small fraction* of 248.

Object Collection:

How many *more* than 5 is 8?
8 is 3 *more* than 5.

Now let us turn to the Motion metaphor.

Arithmetic Is Motion

- Numbers Are Locations on a Path.
- The Mathematical Agent Is A Traveler along that path.
- Arithmetic Operations Are Acts of Moving along the path.
- The Result of an arithmetic operation Is A Location on the path.
- Zero Is the Origin (starting point).
- The Smallest Whole Number (One) Is A Step Forward from the origin.
- The Size of the Number Is The Length of the trajectory from the origin to the location.
- The Quantity Measured by a Number Is the Distance From the Origin to the location.
- Equations Are Routes to the same location.
- Addition of a Given Quantity Is Taking Steps a given distance to the right (or forward).
- Subtraction of a Given Quantity Is Taking Steps a given distance to the left (or backward).
- The Number of Times an Action is performed Is The Location reached by starting at the origin and taking one step for each performance of the action.
- Multiplication Is The Repeated Addition of quantities of the same size a given number of times.
- Division Is The Repeated Segmentation of a path of a given length into as many smaller paths of a given length as possible.

As in the ealier cases, this conceptual metaphor also provided language for talking about arithmetic.

How *close* are these two numbers?
37 is *far away from* 189,712.
4.9 is *almost* 5.
The result is *around* 40.
Count up to 20, without *skipping* any numbers.
Count *backwards* from 20.
Count *to* 100, *starting at* 20.
Name all the numbers *from* 2 *to* 10.

The linguistic examples are important here in a number of respects. First, they illustrate how the language of object manipulation and motion can be recruited in a systematic way to talk about arithmetic. The conceptual mappings characterize what is systematic about this use of language. Second, these usages of language provide evidence for the existence of the conceptual mappings—evidence that comes not only from the words, but also from what the words mean. The metaphors can be seen as stating generalizations not only over the use of the words, but also over the inference patterns that these words supply from the source domains of object collection, object construction and motion, which are then used in reasoning about arithmetic.

EDUCATIONAL EXTENSIONS OF NATURAL GROUNDING METAPHORS

In the Object Collection and Object Construction metaphors, zero is not the same kind of thing as a number. It represents the absence of attributes—the absence of a collection or constructed object. It is only in the Motion metaphor that zero is the same kind of thing as a number—it is a location in space. Because the Collection and Construction metaphors we use are so basic to the conception of number, we can see why it took so long for zero to be included and why there was so much resistance to calling zero a number.

It should be clear that the Collection and Construction metaphors also work just for the natural numbers. Multiplication by zero, for example, is not defined. Nor are negative numbers, rational numbers, and the reals. Many Mathematics teachers attempt to use these metaphors to teach arithmetic—as they must, if students are to grasp the subject at all. But, unfortunately, such teachers often fail to grasp the limited role played by grounding metaphors, which arise quite naturally. Grounding metaphors are partial, limited only to the natural numbers and basic operations. They are the metaphors from which the most basic ideas of arithmetic arise. However, the arithmetic characterized by these metaphors has been greatly extended over the centuries through linking metaphors and metonymies, and there is no way to extend these metaphors in a consistent natural manner to cover all those extensions. But many teachers nonetheless concoct nonnatural novel and ad hoc extensions in attempt to take those metaphors beyond their natural domain.

Consider the example of negative numbers. The Object Construction and Object Collection metaphors do not give rise to simple natural extensions of the negative numbers. So some teachers extend the natural grounding metaphors in unnatural ways. For example, suppose we want to teach the equation using the object collection metaphor $(-1) + (-2) = (-3)$. A teacher might introduce the ad hoc extension Negative Numbers Are Helium Balloons, and use it together with Quantity Is Weight and Equations Are Scales. Here helium balloons are seen as having negative weight, offsetting positive weight for the purpose of measuring weight on a scale. This ad hoc extension will work for this case, but not for multiplying by negative numbers. In addition, it must be used with care, because it has a very different cognitive status than the largely unconscious natural grounding metaphor. It cannot be added and held constant as one moves to multiplication by negative numbers.

Or take another ad hoc extension to the Object Collection metaphor: Negative Numbers are Objects Made of Anti-Matter. This is sometimes used to teach why $3 + (-3) = 0$: 3 and -3 annihilate one another! Again, this can't be extended to teach multiplication by negative numbers. Moreover, the concept of anti-matter may be harder to teach than the concept of negative number by other means.

The easiest natural extension of one of the grounding metaphors to negative numbers is the Motion Metaphor. If numbers are locations reached by moving in uniform steps in a given (positive) direction from a source, and if zero is the origin, then negative numbers are also locations, but locations reached by moving in the opposite direction. Thus, if positive numbers are to the right of the origin, negative numbers are locations to the left of the origin. Addition and subtraction of negative numbers can then be given by a relatively easy extension of the metaphor: when you encounter a negative number, turn around in place. Multiplication by a negative number is turning around and multiplying by a positive number.

But however straightforward these extensions are, they are still concocted novel extensions of the natural grounding metaphor, and so will seem a bit artificial because it is a bit artificial. Such metaphors are neither natural grounding metaphors, nor are they linking metaphors. They belong neither to the realm of the natural grounding of arithmetic, not to the linking of one branch to another, but rather to the domain of teaching by making up extensions of the natural grounding metaphors. As such they are neither part of the natural grounding of mathematical ideas nor the linking of one branch of mathematics to another. Hence, they stand outside of mathematics proper and are part of imaginative, and sometimes forced, methods of mathematical education. Since such purely educational metaphors, however useful for certain aspects of teaching, are not the subject matter of this paper, we will not discuss them further. (For an explicit statement of such educational metaphors, see Chiu, 1996.) Instead we will proceed with our discussion of the natural grounding metaphors for basic mathematical ideas and the linking metaphors used to map one branch of mathematics onto another.

METAPHORS FOR SET THEORY

Set theory is grounded in two kinds of related experiences:

1. Grouping objects into conceptual containers
2. Comparing the number of objects in two groupings

The source domain of the metaphor uses a container-schema, which specifies a bounded region of space, with an interior, a boundary, and an exterior. Container schemas may have a physical realization (a jar, a plot of ground marked off by a line, etc.), but, as used here, the boundary of the container is purely imaginative and need not be physically realized.

An image-schematic container consists of the boundary of the container schema plus the interior of that container schema. Objects within the boundary are *In* the container. When we conceptually group objects together we are superimposing a single container-schema on the objects so that the objects are conceptualized as being inside the boundary and overlapping with the interior of the schema.

Sets in mathematics have traditionally been conceptualized as container-schemas, and the members of the sets as objects inside the container-schema.

The Sets-As-Container-Schemas Metaphor

- A Set Is A Container-Schema.
- A Member of a Set Is An Object in a Container-Schema.
- A Subset of a Set Is A Container-Schema Within a Container-Schema.

One of the properties of conceptual groupings is that the grouping is determined by the choice of the objects. A choice of different objects is a different grouping. Two choices of the same objects do not constitute different conceptual groupings. This property of objects conceptually grouped together by superimposition of a container-schema is an important property. This property is mapped by the Sets-As-Container-Schemas Metaphor into The Axiom of Extensionality in set theory, the axiom that states that a set is uniquely characterized by its members. In this case, an entailment of a grounding metaphor is an axiom—a self-evident truth that follows from the grounding metaphor. Not all axioms arise in this way. Some require additional metaphors.

Incidentally, this metaphor can be extended in a fairly obvious way to metaphorically define unions, intersections, and complements. But there are many aspects of naive set theory in mathematics that are not consequences of this metaphor and cannot be defined using it. Container-schemas are just cognitive mechanisms that impose conceptual groups. Though you can have sets of objects further grouped into subsets by additional container schemas inside an outer one, those internal container schemas are not themselves made objects by the Sets-As-Container-Schemas Metaphor, and since they are not objects, they cannot be members of the set.

However there is a metaphor that can turn such container schemas into objects, which would make it possible for them to be set members. This is not an ordinary grounding metaphor that naturally characterizes our nonmathematical understanding of a set. Instead it is a special metaphor for grounding the technical discipline of set theory.

The Sets Are Objects Metaphor

- Sets Are Objects.

This extremely powerful metaphor, when combined with the Sets Are Container-Schemas Metaphor allows not just for subsets, but for sets to be *members* of other sets, since members are conceptualized as objects. But even though this metaphor allows sets to be members of other sets, combining this metaphor with the Sets Are Container-Schemas Metaphor will guarantee that it will still make no sense to conceptualize a set as being a member of itself. The reason is that container-schemas cannot be inside themselves by their very nature. Boundaries of containers cannot be parts of their own interiors. It therefore follows that, when The Sets Are Container-Schemas Metaphor is being used to conceptualize sets, expressions like "sets that are members of themselves" and "sets that are not members of themselves" are nonsense and therefore cannot designate anything at all. Relative to this metaphor, Russell's classical set-theoretical paradox concerning the set of all sets which are not members of themselves cannot arise. It arises only in a mind-free mathematics where set theory is developed axiomatically without an explicit characterization of ideas. To get around the constraint that sets cannot be members of themselves, a theory of hypersets has been developed and we will discuss it shortly.

Consequences of Sets Are Objects

Once sets are conceptualized metaphorically as objects and not just cognitive container-schemas that accomplish grouping, then the submetaphor that Members of a Set Are Objects In a Container-Schema can apply and allow subsets of a set to be members of that set. And once a set is conceptualized as a container-object, it possible to conceptualize the empty set—the container-object containing no objects. Putting together the Sets Are Container-Schemas Metaphor and the Sets Are Objects Metaphor, we get new entailments. For example, it is entailed that every set now has a power set—the set of all its subsets. This is another fundamental axiom of set theory.

The Ordered Pair Metaphor

Once we conceptualize sets as objects and subsets as members, we can construct a metaphorical definition for ordered pairs. Intuitively, an ordered pair can be conceptualized nonmetaphorically as a subitized pair of elements (call it a Pair-Schema) structured by a Path-Schema, where the source of the path is seen as the first member of the pair and the goal of the path is seen as the second member.

With the addition of the Sets Are Objects Metaphor, we can conceptualize subsets of sets as members of sets. We can now use the idea of a subset that is also a member of set to conceptualize the concept of the ordered pair metaphorically in terms of sets.

An Ordered Pair (a,b) Is The Set $\{a, a,b\}\}$

Using this metaphorical concept of an ordered pair, one can go on to metaphorically define relations, functions, and so on in terms of sets. One of the most interesting things that one can do with this metaphorical ordered pair definition is to metaphorically conceptualize the natural numbers in terms of sets that have other sets as members, as von Neumann did.

The Natural Numbers Are Sets Metaphor

- Zero Is The Empty Set, $\varnothing$
- A Natural Number Is The Set of its predecessors.

Entailments:

- One Is The Set containing the empty set, $\{\varnothing\}$
- Two Is $\{ \varnothing, \{\varnothing\} \}$ (that is, $\{0,1\}$)
- Three is $\{ \varnothing , \{\varnothing\} , \{ \varnothing , \{\varnothing\} \} \}$ (that is, $\{0,1,2\}$)

Here, zero has no members, one has one member, two has two members, and so on. By virtue of this metaphor, every set containing three members is in 1–1 correspondence with the number three. Using these metaphors, one can metaphorically construct the natural numbers out of nothing but sets.

This is the basic metaphor linking set theory to arithmetic. It allows one to conceptualize one branch of mathematics, arithmetic, in terms of another branch, set theory. This metaphor projects truths of set theory onto arithmetic.

SAME SIZE AND MORE THAN: THE EVERYDAY COMMON SENSE CRITERIA

Our ordinary everyday conceptual system includes the concepts Same Number and More Than. They are based, of course, on our experience with finite, not infinite, collections. Among the criteria characterizing our ordinary everyday versions of these concepts are:

> Same Number: Group *A* has the same number of elements as group *B* if, for every member of *A*, you can take away a member of *B* and at the end of the process there are no objects left in *B*.
>
> More Than: Group *B* has more objects than group *A* if, for every member of *A*, you can take away a member of *B* and at the end of the process there will still be objects left in *B*.

It is important to contrast our everyday concept of Same Number with Georg Cantor's concept of Equipollence, that is, being able to be put in a 1–1 correspondence. For finite collections of objects, having the Same Number entails Equipollence. Two finite sets with the same number of objects can be put in a 1–1 correspondence. This does not mean that Same Number and Equipollence are the same idea. The ideas are different in a significant way, but they happen to have the same truth conditions for finite sets.

Compare the set of natural numbers and the set of even integers. As Cantor observed, they are equipollent. That is, they can be put into a 1–1 correspondence. Just multiply the natural numbers by two, and you will set up the 1–1 correspondence. Of course, these two sets do not have the same number of elements according to our everyday criterion. If you take the even numbers away from the natural numbers, there are still all the odd numbers left over. Therefore, according to our everyday concept of "more than" there are, of course, more natural numbers than even numbers. The concepts "same number" and "equipollence" are different concepts.

Our everyday concepts of "same number" and "more than" are, of course, linked to other everyday quantitative concepts, like "how many" or "as many as," "size" as well as to the concept "number" itself—the basic concept in arithmetic. In his investigations into the properties of infinite sets, Cantor and other mathematicians have used the concept of equipollence in place of our everyday concept of same number. In doing so, Cantor established a metaphor, which we can state as:

Cantor's Metaphor

- Equinumerosity Is Equipollence.

That is, our ordinary concept of having the same number of elements (equinumerosity) is metaphorically conceptualized, especially for infinite sets, in terms of the very different concept of being able to be put in a 1–1 correspondence.

But this has never been stated explicitly as a metaphor. As a result, it has produced confusion for generations of students. Consider a statement of the sort made by many mathematics teachers: Cantor proved that there are just as many even numbers as natural numbers. Given our ordinary concept of "just as many as," Cantor proved no such thing. He only proved that the sets were equipollent. But if you use Cantor's metaphor, then he did prove that, metaphorically, there are just as many even numbers as natural numbers.

The same comment holds for other proofs of Cantor's. Literally, there are more rational numbers than natural numbers, since if you take the natural numbers away from the rational numbers, there will be lots left over. But Cantor did prove that two sets are equipollent, and hence they metaphorically can be said (via Cantor's metaphor) to have the same number of elements.

The point of all this is that the conceptualization of arithmetic in terms of set theory is not literal truth. Numbers are not literally sets. They don't have to be understood in terms of sets. But you can use the set-theoretical metaphor system for arithmetic if you want to study the consequences of those metaphors. It is literally false that there are the same number of natural numbers and rational numbers. But it is metaphorically true if you choose to use Cantor's metaphor and follow out its consequences.

The fact that the set-theoretical conceptualization of arithmetic is metaphorical does not mean that there is anything wrong with it. It just means that it does not provide literal, objective foundations for arithmetic.

HYPERSETS

The normal intuitive concept of a set, as we have just seen, is grounded in metaphor that Sets Are Container-Schemas, with set members insider the container-schemas. Since container-schemas cannot be inside themselves, sets understood in this way cannot be members of themselves. This has led to Russell's paradox and to the frustration of model-theorists with this constraint.

Model-theorists do not depend upon our ordinary everyday concept of a set. They use sets, as formally characterized, as models for axiom systems and for other symbolic expressions within formal languages. Certain model-theorists have found that our ordinary grounding metaphor that Sets Are Container-Schemas gets in the way of certain kinds of phenomena they want to model, especially recursive phenomena. For example, take expressions like

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}$$

which is equivalent to

$$x = 1 + \frac{1}{x}$$

Such recursive expressions are common in mathematics and computer science, and the requirement that sets, which cannot contain themselves, be used in modeling limits the possibilities for modeling. Set-theorists have realized that a new metaphor is needed, and have explicitly constructed one (see Barwise & Moss, 1991). The idea is to use graph theory, first to model set theory, and then to go beyond what sets can do.

The implicit metaphor is:

The Set Theory Is Graph Theory Metaphor

- The Membership Relation Is An Arrow (called an "Edge") Relating Two Nodes In A Directed Graph.
- Sets Are Nodes That Are Tails of Arrows.
- Members Are Nodes that are Heads of Arrows.
- The Mapping From Tail Nodes To Sets Is A Labeling (A "Decoration") Of the Nodes By The Sets.

The effect of this metaphor is to eliminate the notion of containment from the elements used in model-theory. The graphs have no notion of containment built into them at all. And containment is not modeled by the graphs.

The set-theoretical Axiom of Foundation, which imposes some of the properties of containment can now be replaced by an Anti-Foundation Axiom modeled by graphs. Set-theory becomes "The Theory of Hypersets," where hypersets are not sets at all, since they have no notion of containment, but only graph-theoretical structure. For this reason, the term "hyperset" is a bit misleading, since hypersets are not kinds of traditional sets, but rather a generalization from them. Nevertheless, they play the same role as sets technically. The old axioms of "set theory" can be satisfied model-theoretically by such graphs. The metaphor that Numbers Are Sets can now be replaced by a new metaphor, Numbers Are Graphs. Adjusting the von Neumann definition of numbers in terms of sets to the new graph-theoretical models, zero (the empty set) becomes a node with no arrows leading from it. One becomes the graph with two nodes and one arrow leading from one node to the other. Two and three are represented by the graphs in Figs. 2.1a and 2.1b.

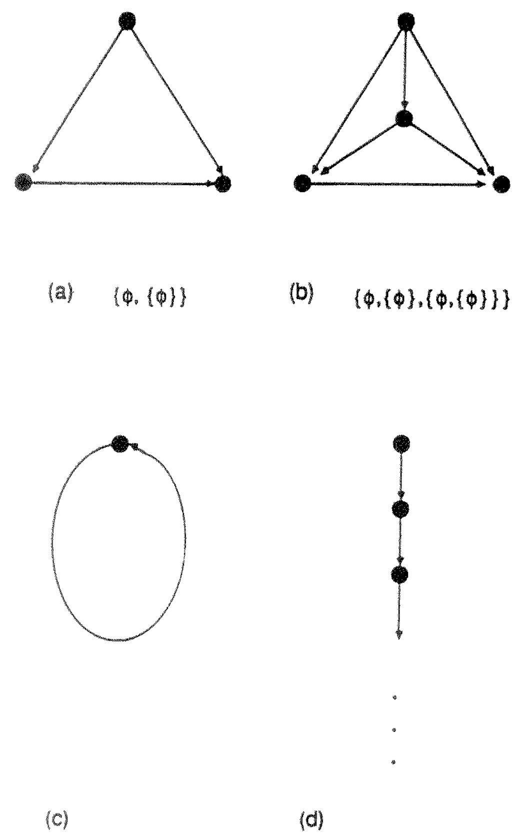

FIG. 2.1. Hypersets: Graphs for modeling sets.

Set-theoretical models can now be replaced by graph-theoretical models. Such models allow for recursion, that is, they allow a node to be both head and tail of an arrow. That is, an arrow can bend back on itself. If the old terminology of sets if misleadingly retained, then such "sets" of "hypersets" (really nodes in graphs) can "contain themselves" (that is, they can be both heads and tails of arrows in graphs). Actually, there is no containment at all.

When such a case arises, where an arrow bends back on itself as in Fig. 2.1c, we get recursion—the equivalent of an infinitely long chain of nodes with arrows not bending back on themselves, as in Fig. 2.1d.

Here we see the power of metaphor in the foundations of mathematics. Sets do not have the right properties to model everything needed. So we can now metaphorically model sets themselves (minus containment) using certain kinds of graphs, generalize sets to those graphs, and replace set

theory by a special case of graph theory. The only confusing thing is that this special case of graph theory is still called "set theory" for historical reasons.

Because of this misleading terminology, it is sometimes said that the theory of hypersets is "a set theory in which sets can contain themselves." This is not strictly true because it is not a theory of sets as we ordinarily understand them in terms of containment.

GROUNDING METAPHORS FOR FUNCTIONS

Let us look next at the metaphors that ground our understanding of functions.

A Function Is A Machine

- The Domain of the function Is A Collection of acceptable input objects.
- The Range of the function Is A Collection of output objects.
- The Operation of the function Is The Making of a unique output object from each collection of input objects.

The notion of an algorithm is, of course, based on the machine metaphor. An algorithm, as a metaphorical machine *performs operations* sequentially on input objects to yield output objects. Since the machine is metaphorical and not real, the operations and objects are conceptual in nature, and they always apply perfectly in exactly the same way, since imperfections of physical objects are not mapped onto conceptual objects by the metaphor.

For arithmetic functions, this metaphor extends the grounding metaphor of Arithmetic As Object Construction. Here, the construction of output objects from input objects is done by a machine.

Here are some examples of expressions that make use of this metaphor.

Linguistic Examples:

Nonprime numbers are *made* up of primes.

Multiplication *makes* nonprime numbers out of collections of prime numbers.

The function $f(x) = x^2 + 5$ *takes* a number, *first squares* it *and then adds* 5, *to yield* a *new* number.

The function $f(x) = e^x$ *starts producing* huger and huger numbers as you put *in* moderately larger values of x.

A Function Is a Collection of Objects With Directional Links

- The Domain of the function Is A Collection of objects.
- The Range of the function Is A Collection of objects.
- The Function Is A Collection of Unique Paths from objects in the domain to objects in the range.
- The operation of the function Is the transportation of the domain object to the range object by an agent.

Metonymy: The function stands for the agent performing the function.

There are also common expressions in the English of mathematicians that use this metaphor. All of these cases use the metonymy in which the function stands for the agent performing the function. The function itself is seen as doing the carrying, sending, and projecting.

Linguistic Examples:

$f(x) = x^2$ *takes/carries* 3 into 9

This function *projects* the integers onto the even numbers.

This function *sends* numbers into their inverses.

Given the metaphor that Ordered Pairs Are Sets, this grounding metaphor can be reformulated as a linking metaphor, characterizing functions metaphorically in terms of sets:

- A Function Is a Set of Ordered Pairs, in which the first member of the pair uniquely determines the second.

At this point, we are able to discuss one of the central ideas in classical mathematics—the Cartesian Plane.

THE METAPHORICAL STRUCTURE OF THE CARTESIAN PLANE

Linking metaphors are commonly versions of grounding metaphors. For example, the fundamental metaphor linking set theory to arithmetic, Numbers Are Sets, is a technical version of the natural grounding metaphor Arithmetic is Object Collection. We can now turn to a second case in which there is a linking metaphor that is a technical version of a grounding metaphor. Corresponding to Arithmetic is Motion, with Numbers As Locations, there is a metaphor linking Arithmetic to Geometry, in which Numbers Are Points on a line.

The Arithmetic Is Geometry Metaphor

(Assume a Euclidean plane, with the truths of Euclidean Geometry.)

- Numbers Are Points on a line.
- Zero Is the Origin.
- Quantities Are Distances (from the origin to a point).
- Greater than Is Above (for vertically oriented lines).
- Greater than Is To the right of (for horizontally oriented lines).

This metaphor maps truths of Euclidean Geometry onto Arithmetic! That is what makes this a metaphor that links the domains of Geometry and Arithmetic.

What is particularly interesting about this metaphor is that it is used to form a metaphoric blend, a composite of the source and target domains of the metaphor (Turner, 1996; Turner & Fauconnier, 1995), that is, a composite of numbers and points on a line known as the *number line*.

The Number Line Blend

A number line is a conceptual blend formed from the superimposition of the source domain (Geometry) of the Arithmetic As Geometry metaphor onto the target domain (Arithmetic). The entities in this blend are number-points—numbers that are metaphorically points.

The blend combines truths of geometry with truths of arithmetic, to yield new inferences about the target domain, arithmetic. Using the metaphorical concept of the number line, we can construct a much more complex metaphorical concept, the Cartesian Plane.

The Cartesian Plane

A metaphorical definition is a definition that presupposes one or more metaphors. All the definitions given below assume the Arithmetic Is Geometry metaphor. We use the double-bullet to mark metaphorical definitions.

- • The x-axis Is a Number line.
- • The y-axis Is a Number Line perpendicular to the x-axis.
- • The Intersection of the x- and y-axes Is An Origin for both axes.
- • A Coordinate Point Is a Point on an axis.

- • The x-coordinate line Is the Line parallel to the y-axis that crosses the x-axis at coordinate point x.
- • The y-coordinate line Is the Line parallel to the x-axis that crosses the y-axis at coordinate point y.
- • A Cartesian Coordinate system Is the Collection of x- and y-coordinate lines.

The Cartesian Point Metaphor

Like any other plane in Euclidean geometry, the Cartesian Plane is a collection of points. But in the Cartesian Plane, a point in the plane is conceptualized in one of two ways, either (a) as an intersection of an x- and a y- coordinate, or (b) as an ordered pair of numbers on the x- and y-axes, respectively. Each of these conceptions is metaphorical. Points don't have to be conceptualized as intersections of line, nor as ordered pairs of other points. But these alternate conceptualizations are inherent to an understanding of what the Cartesian Plane is.

The Points Are Intersections Metaphor

- A Point in the Cartesian Plane Is The Intersection of an x-coordinate line and a y-coordinate line.

The Point-for-Coordinate-Line Metonymy

- A Coordinate Point P can Stand For the Coordinate Line L that intercepts an axis at point P.

By forming the composition of the Points Are Intersections Metaphor and the Point-For-Coordinate-Line Metonymy, we arrive at:

The Cartesian Point Metaphor

- Each Point in the Cartesian Plane Is an Ordered Pair of points (x,y) on the x- and y-axes.

The Cartesian Point Metaphor allows us to define a Cartesian Graph.

- • A Cartesian Graph Is A Collection of points in the Cartesian plane.

This metaphorical definition allows us to conceptualize equations with two variables. (It presupposes the use of a very important metaphor that we are not going to discuss here: Variables Are Numbers. This metaphor lies at the base of the conceptualization of elementary algebra. The combination of this metaphor with the Arithmetic Is Geometry metaphor lies at the base of Analytic Geometry.)

- • An Equation with two variables Is A Cartesian Graph.

Thus, we have the familiar examples:

$y = ax + b$ Is a Line.

$x^2 + y^2 = r^2$ Is a Circle of radius r with a center at the origin.

It is important to realize that these are metaphorical conceptualizations of equations, not the equations themselves.

Cartesian Functions

The Cartesian Plane is, as we have seen, a blend of arithmetic and geometry in which arithmetic is conceptualized in terms of geometry through the concept of the number line. The Cartesian Point metaphor conceptualized points in the plane as ordered pairs of number-points on the x- and y-number lines. This allows us to define a Cartesian Function using the metaphor discussed above, that a function is a set of ordered pairs. In this case, because we have a blend of numbers and points, a Cartesian Function maps two things at once. It maps points onto points and numbers onto numbers. The point-to-point mappings are constrained by geometry, while the number-to-number mappings are constrained by arithmetic.

A Cartesian Function

- • A Cartesian Function Is A Collection of points in the Cartesian Plane, where each two points in the collection have different x-coordinates.

Since each point in the Cartesian Plane is metaphorically an ordered pair of points on the x- and y-axes respectively, Cartesian Functions can be conceptualized in terms of set theory as sets of ordered pairs, which are in turn metaphorically conceptualized as sets with a certain structure (as described earlier).

The Arithmetic of Functions Metaphor

Literally, functions are not numbers. Addition and multiplication tables do not include functions. But when we conceptualize functions as ordered pairs of points in the Cartesian Plane, we can create an extremely useful metaphor. The operations of arithmetic can be metaphorically extended from numbers to functions, so that functions can be metaphorically added, subtracted, multiplied and divided in a way that is consistent with arithmetic. Here is the metaphor:

The Arithmetic of Functions Metaphor

- An Arithmetic Operation on functions Is That Operation on the y-values at each point x of the functions.

This is what we have called a *definitional metaphor*, in that it extends the definition of arithmetic operations to functions via metaphor. To see how this works, consider addition.

Let $f = \{(x,y)\}$, that is, a set of pairs of values taken from the x- and y-axes respectively.
Let $g = \{(x,y')\}$, where x and y' are also values taken from the x- and y-axes respectively.

Consider only x-values in the intersection of the domains of the functions.

By the expression, "$f + g$" we mean "$\{(x,y)\} + \{(x,y')\}$". To understand what *that* means, we need a metaphor—a special case of the Arithmetic of Functions metaphor, restricted to addition. The special case of that metaphor is:

- $\{(x,y)\} + \{(x,y')\}$ Is $\{(x, y + y')\}$

Or, if you prefer English:

- The Sum of two functions Is The Sum of the y-values at each point x of the functions.

This is usually written as: $(f + g)\ (x) = f(x) + g(x)$. But notice that the "+" on the left is a metaphorical + and that the "=" specifies a metaphorical definition.

The same applies for the rest of the arithmetic functions:

- $\{(x,y)\} - \{(x,y')\}$ Is $\{(x, y - y')\}$
- $\{(x,y)\} \cdot \{(x,y')\}$ Is $\{(x, y \cdot y')\}$
- $\{(x,y)\} \div \{(x,y')\}$ Is $\{(x, y \div y')\}$ (where $y' \neq 0$)

The following are therefore all metaphoric:

- $(f + g)(x)$ Is $f(x) + g(x)$
- $(f - g)(x)$ Is $f(x) - g(x)$
- $(f \cdot g)(x)$ Is $f(x) \cdot g(x)$
- $(f \div g)(x)$ Is $f(x) \div g(x)$, where $g(x) \neq 0$

In the Cartesian Plane, simple arithmetic functions of one variable can be conceptualized metaphorically as curves. The Arithmetization of Functions metaphor allows us to conceptualize what it means to add, subtract, multiply, and divide such curves. This is a central idea in classical mathematics.

EULER'S CONCEPTION OF CONTINUOUS FUNCTIONS

Euler assumed the Cartesian Plane and characterized a continuous function as "a curve described by freely leading the hand" across the Cartesian Plane. This intuitive characterization of a function has a number of important properties.

First, the continuous function is formed by motion, which takes place over time.
Second, there is a directionality in the function.
Third, the continuity arises from the motion.
Fourth, since there is motion, there is some entity moving, in this case, the hand.
Fifth, the motion results in a static line with no "jumps."
Sixth, the line has no directionality.

THE FICTIVE MOTION METAPHOR

Within the Cartesian plane, we commonly conceptualize variables as if they were point-sized travelers in motion, moving either along one of the axes or across the plane. An independent variable is conceptualized as a traveler moving along the x-axis, and the dependent variable as a traveler moving along the y-axis. Time, is of course, implicit here. At each time, each of the traveler-variables is at some unique point on one of the axes. That point is the value of the variable at that time. Here are the metaphorical mappings characterizing these notions of what a variable is:

- An Independent Variable Is A Traveler, Traveler X, who moves independently over the x-axis.

- A Dependent Variable Is A Traveler, Traveler Y, whose movements are determined by the movements of Traveler X plus the function in the following way: when Traveler X is at location x on the x-axis, Traveler Y is at Location $y = f(x)$, on the y-axis.

Correspondingly, the function relating the variables can itself be conceptualized as a traveler moving across the plane. Its motion is determined by the other two variable-travelers. The mapping characterizing this conceptualization of a dynamic function is as follows:

- A Dynamic Function Is the Path traced by a Traveler F, whose motion is determined by the motion of Travelers X and Y in the following way: when Traveler X is at point x on the x-axis and Traveler y is a point y on the y-axis, then Traveler F is at the point (x,y) in the Cartesian plane.
- An Instance of a Function Is A Time at which Traveler X is at location x, Traveler Y is at location $f(x)$, and Traveler F is at (x,y).

These are, of course, grounding metaphors. Their role is to ground our mathematical understanding of variables in terms of nonmathematical understanding—here what is used as the source domain of the metaphor is our common understanding of motion. For example, we are conceptualizing the *varying* of the variable in terms of the *motion* of a traveler. Since the metaphor maps our common everyday inferences about motion onto mathematics, our mathematical understanding depends on everyday commonplace understanding of motion.

In the Cartesian Plane, the x- and y-axes are lines and, following Euler, the continuous functions are lines as well. Moreover, it is commonplace to conceptualize continuity of a function as Euler did, in terms of nonskipping motion along the line characterizing the function. We normally conceptualize this in terms of a commonplace metaphor, the Fictive Motion metaphor (see Talmy, 1988).

The Fictive Motion Metaphor

- A Line Is The Motion of a Traveler tracing that line.

Examples:

Highway 80 *goes to* San Francisco.
Just before Highway 24 *reaches* Walnut Creek, it *goes through* the Caldecott Tunnel.
So far, the new freeway has gotten only *half way to* Los Angeles.

Here a highway, which is a static linear object, is conceptualized in terms of a traveler moving along the route of the highway. Accordingly, via this metaphor, we can speak of a function as *moving, growing, oscillating,* and *reaching* limits.

What we have just done is provide a characterization in cognitive terms of the intuitive dynamic and static conceptualizations of a continuous function. As in the case of Euler's characterization, the continuity is characterized by motion in the Fictive Motion version. This is a cognitive account of Euler's intuitive notion of continuity for a function in terms of elements of ordinary human cognition. It shows how mathematical ideas are constituted out of ordinary ideas.

SCANNING AND THE CONCEPT OF CONTINUITY FOR A LINE

Scanning is a form of motion, the motion of our gaze. The concept of a gaze uses the common metaphor

- Seeing Is Touching

in which seeing is conceptualized in terms of a limblike "gaze" that reaches out from our eyes and "touches" what we are seeing. It is this metaphor that characterizes what we mean by "scanning along" a line.

Our capacity to scan mentally is central to our everyday concept of continuity. We understand a line to be continuous if and only if we can scan along it with no jumps, that is, without any place at which our gaze fails to be in contact with the line. For this reason, the concept of contact also enters crucially into our everyday concept of continuity.

From the perspective of cognitive science all this is natural. Scanning is a fundamental cognitive capacity and contact is a fundamental image-schematic concept. These, of course, need to be further characterized at the level of the brain. For a discussion of how this might work for the concept of contact, see Regier (1996). We mention this because, when it comes to characterizing a concept, there is a considerable difference between what is sufficient for a cognitive scientist and what is sufficient for a mathematician. When we define an interval in cognitive terms, we will be using the everyday concept of a line, which in turn uses the everyday notion of continuity.

THE INFINITY AS A POINT METAPHOR

As we scan along the indefinitely long line in our imagination, fixed distances between two points on a line appear closer the further away they are. As we scan toward the horizon our gaze reaches a single point at which the further infinite extension of the line is contained visually within that point. This is the basis of a metaphor by which we conceptualize infinity—the infinite extension of that line—as a point. We will refer to that metaphor as The Infinity As A Point Metaphor.

- The Infinite Extension of a Line Is a Location (a Point) at the End of that Line. This point is referred to as "∞".

This metaphor, together with the metaphors just discussed, jointly entail the following metaphorical conclusion, which is obviously literally false:

- The further a traveler moves along a line of infinite extent, the closer he comes to the point ∞.

Though this is literally false, it is metaphorically true; that is, it is entailed by metaphors that we use for the purpose of conceptualizing infinity. It is by means of the Infinity-as-a-Point metaphor that we understand expressions like "as *x* approaches infinity" or "the value of the function at infinity".

THE METAPHOR FOR APPROACHING A LIMIT

Using these metaphors, we get the following metaphorical definition of "approaching a limit" for a dynamic function in the Cartesian Plane. Suppose that, as the variable *x* gets closer and closer to the point *a* from either side, $f(x)$ gets correspondingly closer and closer to a unique value *L*. We define *L* to be "the Limit of $f(x)$ as *x* approaches *a*."

The metaphorical character of this definition should be clear. It assumes that we know from the source domain what "approach" and "closer and closer" mean. These are concepts from our everyday understanding of motion through space. Given all the metaphors for dynamic functions that we have just characterized, the metaphor for approaching a limit is straightforward:

- The Existence of A Limit *L* (for a function *f*) At Point *a* Is The Existence Of A Unique Location *L* Such That Traveler *Y* Approaches *L* Whenever Traveler *X* Approaches *a*.

The notion of a limit becomes interesting when *x* approaches *a* but does not reach *a*, and when $f(x)$ approaches *L* but does not reach *L*. Indeed, *a* may not even be in the domain of *f*. In such a case, we may want to extend the function in a consistent way to give it a value at *a*. Since the function does not literally take *a* as a value, conceptual metaphor is used to accomplish this.

Intuitively, this is done in terms of the notions of open and closed intervals. From a cognitive perspective, such intervals are conceptualized in terms of container-schemas. This is the image-schema used in mind-based set theory, which we discussed earlier. Each container-schema has an interior, a boundary, and an exterior. We form an interval by imposing a container-schema on a line, so that the container-boundary on each side of the interval coincides with a single point on the line. The interior and exterior on the interval have extent. The boundary on each side is a point, minimally differentiating the interior from the exterior.

Given the metaphor that numbers are points on a line, and the characterization of real numbers as being the totality of those points, open and closed intervals are conceptualized metaphorically as sets of real numbers, which are simultaneously, via the Number Line Blend, points on a line.

OPEN AND CLOSED INTERVALS

Here are metaphorical definitions of open and closed intervals.

- • An Open Interval Is The Intersection of the real number line with the interior of a container schema.
- • A Closed Interval Is The Intersection of the real number line with the Interior and Boundary of a container schema.

Suppose we take the metaphorical conceptualization of an independent variable of a function as a traveler moving along the *X*-axis in a Cartesian Plane. Suppose the function is defined only over an open interval. In that case, the variable can keep moving closer and closer to the boundary of that interval and never reach it, even if it keeps moving for an infinitely long stretch of time. For any given observer, the position of the traveler and the boundary can get so close as to be indistinguishable for that observer, though still not be literally identical.

At this point two commonplace everyday metaphors in our conceptual system become relevant:

- Existence Is Visibility
- Nonexistence Is Invisibility

- Similarity Is Closeness
- Difference Is Distance
- Identity Is Identity of location.

Thus, when we say the difference between the location of the traveler and boundary becomes "vanishingly small" and "disappears," we conceptualize that *distance* and, hence, the metaphorical *difference* to be metaphorically nonexistent. This occurs when the function $f(x)$ approaches a limit L as x approaches a. At some point, the difference between $f(x)$ and L will become "vanishingly small" and will "disappear" as x approaches a.

The interesting cases are those where the function f is not defined at a, and yet the difference between $f(x)$ and L becomes vanishingly small as x approaches a. In many cases over the centuries, mathematicians have metaphorically extended the function to the value a, saying metaphorically that $f(a) = L$. In other words, they were using what we shall call The General Limit Metaphor. Suppose L is the limit of $f(x)$ as x approaches a, when f is not defined at a.

The General Limit Metaphor

- $f(a)$ Is L.

Instances of The General Limit Metaphor are derivatives and integrals, convergent sequences, power series, Taylor series, Fourier series, and the Hilbert Space-Filling Curve (to be discussed later). Even Euclid's conception of idealized points, lines, and planes can be seen as instances of this metaphor. For example, a point can be conceptualized in terms of a sequence of circles whose radii get progressively smaller, with zero as the limit. The circle at the limit is a point.

The General Limit Metaphor is especially interesting when it is combined with the metaphor that Infinity Is A Point, that is, when $a = \infty$ or $L = \infty$. Consider the case where $f(x) = 1/x$. As x gets larger and larger, $1/x$ gets closer and closer to zero, but does not literally reach it. That is, there is no literal number in the number line for which $1/x$ is zero. However, mathematics texts commonly write:

$$\lim_{x\to\infty}(1/x) = 0$$

Literally, this makes no sense, since infinity is not a number and $1/x$ does not equal zero for any number. But this makes perfect sense given the metaphors that Infinity Is A Point, that Numbers Are Points, and The Limit Metaphor that $f(a)$ Is L. In this case, metaphorically $a = \infty$, $f(\infty) = 1/\infty$, and $L = 0$.

Notice that this extends the function f to what is technically a new function, partly defined by metaphor. Since functions are metaphorically conceptualized as numbers that can be added, subtracted, multiplied and divided, this new function with its metaphorical value is subject to such operations. In other words, limits of functions are subject to the same arithmetic operations as functions themselves.

Now consider $f(x) = 1/x$ once more. As x gets very small, $1/x$ grows without bound. The function $1/x$ is not literally defined for $x = 0$. And as x approaches zero, there is no number L that $1/x$ approaches as a limit. Yet mathematics texts often write:

$$\lim_{x\to 0}(1/x) = \infty$$

Again, this makes perfect sense given the metaphors that Infinity Is A Point, Numbers Are Points, and The Limit Metaphor. In this case, metaphorically $a = 0$, $f(0) = 1/0$, and $L = \infty$. But because classical arithmetic operations are not defined for the metaphorical number ∞, this limit cannot be seen as extending the function $1/x$ for the purpose of performing arithmetical operations on the function.

RIGOR AND FORMALISM

In contemporary textbooks, the intuitive concept of continuity, as used by Euler and as described precisely in cognitive terms, is seen as something to be apologized for, overcome and replaced by more rigorous tools. Here is a typical quotation from a well-known text:

> In everyday speech, a "continuous" process is one that proceeds without gaps or interruptions or sudden changes. Roughly speaking, a function $y = f(x)$ is continuous if it displays similar behavior, that is, if a small change in x produces a small change in the corresponding value $f(x)$ . . . Up to this stage, our remarks about continuity have been rather loose and intuitive, and intended more to explain than to define. (Simmons, 1985)

As we shall see, however, the "more rigorous" version is just as metaphorical if not moreso.

CONCEPTUALIZING A LINE: THE LINE AS SET OF POINTS METAPHOR

We have two importantly different ways of conceptualizing a line (either a curved or straight line). The first is nondiscrete, not made up of discrete elements: a line is absolutely continuous and points are locations (infinitely precise locations) *on* a line. Similarly, a plane is absolutely continuous and points are infinitely precise locations on that plane. In this sense, a line is an entity distinct from the points, that is, locations on that line, just as a road is a distinct entity from the locations on that road. We will refer to such a nondiscrete notion of a line as a *natural continuum*. From the perspective of our everyday geometric intuition, lines are natural continua in this sense.

The second way to conceptualize a line is metaphorical: A Line Is A Set Of Points. According to this metaphor, the points are not locations *on* the line; they are entities *constituting* the line. Similarly, a plane, or any other *n*-dimensional space, can also be conceptualized metaphorically as being a set of points. The distinction between these two ways of conceptualizing lines, planes, and *n*-dimensional spaces has been crucial throughout the history of mathematics, and the failure to distinguish between them has led to considerable confusion. Both conceptualizations are used. Neither is "right" or "wrong." But they have very different properties.

Both conceptions are natural, in that both arise from our everyday conceptual system. A basic division in out conceptual system is the mass-count division. We normally conceptualize the substance water as a mass; it is undifferentiated, continuous and uncountable. The expression "five water" is ill-formed. A chair, on the other hand, is discrete, differentiated from other chairs, and countable. The expression "five chairs" is well-formed.

We can conceptualize a multiplicity of countable entities metaphorically as a mass, via the metaphor Mass Is Multiplicity. The metaphor is grounded in a common perceptual experience. Suppose that you are close to a flock of sheep and can pick out the individual sheep, which are clustered together. As you step further and further back, you reach a point at which the individual sheep are no longer distinguishable as individuals, but rather blend into an indistinguishable mass. Yet you *know* that what you see as a mass is a collection (or multiplexity) of individuals. The same is true of salt, which comes in individual crystals, but which looks from a distance like a fluid mass. Similarly, we see a line on a tv screen from a distance as continuous and unbroken, even though we know it to be made up of dots that we can see up close. In all these cases, we are able to conceptualize the mass that we see as the multiplicity we know it to be. This experience of conceptualizing what we see in terms of what we know is the basis of the Mass As Multiplicity metaphor, in which a continuous mass is conceptualized as a multiplicity of individuals. This metaphor is grammaticized in English via the derivation suffix -(e)ry, as in the words *shrubbery* and *weaponry*. These words each designate a mass which is conceptualized in terms of a multiplicity of discrete individuals, shrubs, and weapons. The word *artillery* also designates a mass conceptualized in terms of multiplicity of discrete individuals, even though there is no corresponding word for the individuals, no *artil. The Line As Set Of Points metaphor is a mathematical version of this Mass As Multiplicity metaphor.

Interestingly, we also have in our everyday conceptual system the converse metaphor, in which a multiplicity is conceptualized as a mass. This Multiplicity As Mass metaphor can be seen in an ordinary sentence like "The cook spread the salt thickly over the surface of the chicken so as to keep the juices in." Here the salt, which comes in individual crystals, is conceptualized as a mass that can be "spread thickly." Another example is "The flies covered the wall" in which the multiplicity of separate flies is conceptualized as a mass capable of "covering." In conceptualizing mathematics, we often use a version of this metaphor: An Ordered Set of Points (with additional properties) Is A Line. For example, when we conceptualize the real numbers as such a set of points, we often further conceptualize the set of points metaphorically as being a continuous line with no jumps or gaps or discrete parts.

To complicate the picture a bit further, we also have a metaphor in our everyday conceptual systems in which A Line Is A Trajectory Traced By Motion, which we referred to earlier as The Fictive Motion Metaphor. We see this metaphor at work in everyday sentences like "The lines of steeple *come together* at the pinnacle" in which the shape of the steeple is conceptualized by points tracing out the lines and "coming together" or "meeting" at the pinnacle. Altogether, this gives us three ways of conceptualizing a continuous line: (a) A static, continuous, undifferentiated, masslike literal notion of a line, which is not conceptualized in terms of points or in terms of motion; (b) a metaphorical notion in which the line *is* conceptualized in terms of points; (c) and another metaphorical view in which the line is traced by motion, creating a undifferentiated continuous trajectory.

The fictive-motion concept of a line as traced by a moving, pointlike object has a long history within mathematics. For example, as we saw, Euler characterized a continuous function as a curve in the Cartesian plane "described by freely leading the hand." As late as 1899, James Pierpont, Professor of Mathematics at Yale, felt compelled to address the American Mathematics Society arguing against the fictive-motion concept of a curve, which was widely taken for granted at that time. Pierpont's address (Pierpont, 1899) is revealing in that it occurs at a point in history where our ordinary everyday intuitive literal notion of the continuous, undifferentiated, masslike line was being challenged by the Line As Set Of Points metaphor, which Pierpont was defending.

PIERPONT'S ADDRESS

Pierpont's address concerned perhaps the three most important intellectual movements within mathematics at the end of the 19th century: (a) the arithmetization of calculus, following Weierstrass; (b) the set-theoretical foundations movement, following Cantor; and (c) the philosophy of formalism, following Frege. These movements were separate in their goals, but linked in detail—and all of them required conceptualizing lines, planes, and *n*-dimensional spaces as sets of points. From the perspective of our everyday, intuitive conceptual system for geometry, this meant using what we have been calling the Line As Set Of Points metaphor.

It is important to distinguish these three movements and the ways in which each of them depended on the Line As Set Of Points Metaphor. Cantor's set theory required that lines, planes, and spaces all be conceptualized as sets of points; otherwise, there could be no set-theoretical foundation for geometry. The philosophy of formalism claimed that mathematical axioms, theorems, and proofs consisted only of meaningless symbols, which were to be interpreted in terms of set theoretical structures. If this formalist view of axioms as no more than strings of meaningless symbols was to be applied to geometry, the axioms had to be interpreted model-theoretically in terms of set-theoretical structures containing sets of points. Finally, Weierstrass' arithmetization of calculus reconceptualized the geometric ideas of Leibniz and Newton, such as fluxions, fluents, and tangents. Central to this enterprise, was conceptualizing of a line as a set of points. Cauchy and Dedekind had extended the Numbers As Sets metaphor to characterize the real numbers as infinite sets of rational numbers. Given this set-theoretical construction of the real numbers, the Numbers As Points metaphor could be extended to conceptualize the real numbers as the points constituting the real line. Weierstrass put together the late 19th century versions of all these metaphors—The Line As A Set Of Points, Numbers As Sets, and Numbers as Points—in his classical arithmetization of the concepts of limit and continuity.

As we can see, the issue as to whether to accept the Line As Set Of Points Metaphor was central to all of these major intellectual projects at the turn of the century, and its widespread acceptance has played a central role in 20th century mathematics.

Pierpont's address is noteworthy mainly because it contains an especially clear discussion of the issues at stake in the choice between the Line As Set Of Points Metaphor versus the literal concept of the line as static, undifferentiated, absolutely continuous, and masslike and the Fictive Motion metaphor version in which an undifferentiated, absolutely continuous, masslike line is traced by the motion of a pointlike object.

We should say at the outset that the static version of the undifferentiated line has no concept of direction, while the metaphorical, fictive-motion version does have direction. Pierpont discusses the fictive-motion version for the following reason: the conceptualization of the line as traced by a moving point had been widely used in characterizing the notion of "approaching a limit," the central idea in calculus.

Pierpont correctly and insightfully lists what a cognitive scientist would now call eight "prototypical" properties of what he calls a "curve", that is, a line in 3-dimensional space that is either straight or curved. Those prototypical properties of a "curve" are:

1. It can be generated by the motion of a point.
2. It is continuous.
3. It has a tangent.
4. It has a length.
5. When closed it forms the complete boundary of a region.
6. This region has an area.
7. A curve is not a surface.
8. It is formed by the intersection of two surfaces.

He presents these as "intuitive" and "more or less undisputed."

A contemporary cognitive semanticist would immediately recognize that these are properties that define a central prototype of the category. Indeed, these are necessary and sufficient conditions for the central prototype of a "curve." However, as cognitive science has long recognized, most human categories are not defined by necessary and sufficient conditions. That does not necessarily mean that the categories are vague. They may be "radial," as it appears the category of a curve is. A radial category can have a clearly defined central prototype, with clearly defined variations (Lakoff, 1987). The variations may be cases where one or more specifiable properties of the prototype does not hold. Other kinds of variations may be metaphorical, where the conceptual metaphor is precisely specifiable.

Thus, there may be no necessary and sufficient conditions characterizing *all* the members of a radial category, but the internal structure of the category may be given by (1) a clearly defined prototype, (2) clearly defined principles extending the prototype to other subcategories, and (3) clearly defined non-central subcategories. As a result, a radial category can be described in clearly defined ways, if we take care to note its precise internal structure. It does not have to be vague.

But Pierpont, of course, knew nothing of this possibility from contemporary cognitive science. All he knew was that a category was supposed to be defined by necessary and sufficient conditions; if an intuitive idea could not be characterized in this way, he saw it as "hazy and vague." Most mathematicians nowadays, unaware of the idea of a radial category, still think the same way.

There was an important period in mathematics in which all mathematical functions were assumed to be characterizable in terms of curves in the Cartesian plane that had all the intuitive geometric properties of the prototype just described. However, toward the end of the 19th century, "pathological," "monster" functions, failing to have one or more of these properties, were discovered. These were, from our perspective, just noncentral members of the category, differing from the prototype by failing to have one or more specifiable properties—just as a whale is a nonprototypical mammal or an ostrich is a nonprototypical bird. Biologists can distinguish whales from other mammals without saying that biology is beset with pathological cases. But those mathematicians who expected formalizations to either accord with intuitive mathematical ideas or not, found cases that did not do so to be pathological. Not having radial categories at their disposal, their only way of eliminating the pathology was to eliminate the intuitions, which were taken as sources of vagueness.

Pierpont, following the lead of important mathematicians like Hilbert, Klein, Peano and others, argued that such "monsters" arise from a lack of "rigor" and an overdependence on geometrical intuition as characterized by the above eight properties. ". . . The notions arising from our intuitions are vague and incomplete . . . The practice of intuitionists of supplementing their analytical reasoning at any moment by arguments drawn from intuition cannot therefore be justified" (Pierpont, 1899, p. 405).

The antidote, Pierpont argues, is to adopt the arithmetization project, and with it, implicitly, the set-theoretical foundations and the philosophy of formalism. From our perspective, this means at the very least, taking the metaphors of Lines As Sets Of Points, Points on Lines As Real Numbers, and Numbers As Sets to be literally true and superior to our everyday geometric concepts. Though this was a minority view at the turn of the century, it has come to be the standard view today. That does not mean is it is "right"; nor does it mean that it is "wrong." Pierpont felt he had to choose. From the perspective of a cognitive scientist looking at this situation, there was, and still is, no objective reason to choose.

What has come to be the common wisdom is that Weierstrass' arithmetization of calculus was "more rigorous," and Pierpont echoes this view as well. Following upon the heels of the formalization of the real numbers by Cauchy and Dedekind, Pierpont (1899) assumes that,

> There are, however, a few standards which we shall all gladly recognize when it becomes desirable to place a great theory on the securest foundations possible . . . What can be proved should be proved. In attempting to carry out conscientiously this program, analysts have been forced to arithmetize their science. (p. 395)
>
> . . . The quantities we deal with are numbers; their existence and laws rest on an arithmetic and not on an intuitional basis . . . and therefore, if we are endeavoring to secure the most perfect form of demonstrations, it must be wholly arithmetical. (p. 397)

This attitude is the norm a century later.

What was interesting about Pierpont is that he knew better. He knew that ideas are necessary in mathematics and that one cannot, within mathematics, rigorously put ideas into symbols. The reason is that ideas are in our minds; even mathematical ideas are not entities within formal mathematics and there is no branch of mathematics that concerns ideas. The link between mathematical formalisms using symbols and the ideas they are to represent is part of the study of the mind, of cognitive science, not part of mathematics. Formalisms using symbols have to be understood, and that understanding is not a rigorous part of mathematics. This is not only true of mathematical formulas, but is also true of their model-theoretic interpretations. The formal models and the formalism used in the modeling also have to be understood in terms of ideas. That understanding is not part of mathematics proper, and therefore it cannot be made mathematically rigorous. Pierpont (1899) understood this:

> From our intuition we have the notions of curves, surfaces, continuity, etc. . . . No one can show that the arithmetic formulations are *coextensive* with their corresponding intuitional concepts. (pp. 400–401)

As a result he felt tension between this wisdom and the appeal of the three linked intellectual programs: the arithmetization of calculus, the set-theoretical foundations, and the philosophy of formalism (which denied any role at all for mathematical ideas).

Pierpont was torn. He understood that mathematics was irrevocably about ideas, but he could not resist the vision of total rigor offered by the three intellectual programs:

> The mathematician of today, trained in the school of Weierstrass, is fond of speaking of his science as "die absolut klare Wissenschaft" [the absolutely clear science]. Any attempts to drag in metaphysical speculations are resented with indignant energy. With almost painful emotions, he looks back at the sorry mixture of metaphysics and mathematics which was so common in the last century and at the beginning of this. The analysis of today is indeed a transparent science built up on the simple notion of number, its truth are the most solidly established in the whole range of human knowledge. It is, however, not to be overlooked that the price paid for this clearness is appalling, it is total separation from the world of our senses. (Pierpont, 1899, p. 406)

This is a remarkable passage. Pierpont knows what is going to happen when mathematics comes to be conceived of primarily, mainly, or only as being about rigorous formalism. Mathematical ideas—he uses the unfortunate term "intuition," which misleadingly suggests vagueness and lack of rigor—will not only be downplayed, but will be seen as the enemy, a form of mathematical evil to be fought and overcome. He can't help himself. He has been converted and comes down on the side of "rigor," but he sees the cost and it is "appalling."

We believe that, because of advances in cognitive sciences, the time has come to better understand the claims of "rigor" for certain modes of doing mathematics. This is necessary concomitant of carefully studying mathematical ideas. As we have seen, anyone who says that a mathematical formalism in the form of symbols expresses certain ideas with mathematical rigor is simply mistaken in the most obvious of ways, since ideas are not part of formal mathematics.

If no formalism could possibly express an idea with mathematical rigor, then what are we to make of the claims of the use of formalism as being rigorous? Is the use of formalism always an advance in mathematics? If so, of what kind? Has the myth that formalism can express ideas with mathematical rigor been damaging? Has the passion for "rigor" brought with it something "appalling," as Pierpont feared?

Such questions cannot be asked in a vacuum. To raise them, we have to look at details. As a case study, we will take one of the most celebrated instances where one form of mathematics was seen as "more rigorous" than previous approaches, the one that Pierpont discusses: Weierstrass' arithmetization of calculus, especially his central ideas of limits and continuity. These are taught, in virtually every modern text, as a great advance made by expressing ideas—limits and continuity—with formal rigor. Weierstrass is portrayed as the monster tamer and the curer of pathological functions. To understand what he did and did not do, we will have to first characterize the pre-Weierstrass notions of limits and continuity, second, examine the Weierstrass versions, and finally look at some of the more celebrated "monsters" and "pathologies." This will, in turn, require us to introduce some new terminology.

NATURAL CONTINUITY VERSUS GAPLESSNESS

According to our everyday intuition, a line is what we called earlier a *natural continuum*. It is not conceptualized as made up of points; rather points are conceptualized as *locations* on the line. The line itself is an entity distinct from the point-locations on it. We understand lines, that is, natural continua, without any jumps or gaps, as being continuous. We will use the term *naturally continuous* to refer to our everyday notion of continuity as it applies to our everyday notion of a line, that is, a natural continuum.

The term "continuity," as used in discussion of mathematics, can mean three distinct ideas. One of them is natural continuity, and the other two do not have their own distinct names. For the sake of clarity, we will give those ideas names here: *Gaplessness* (for lines as sets of points) and *Preservation of Closeness* (for functions).

Let us start with the naturally continuous line. Points are locations on that line, but distinct from the line itself. As we move along a line, we go through point-locations. Suppose we move continuously along the line from location *A* to location *B*. We will then go through all point-locations on the line between *A* and *B*, without skipping over any, that is, without leaving any gaps between the point-locations. In this case we will say that the collection of point-locations between *A* and *B* is *gapless* when the line segment *A B* is naturally continuous.

Now let us apply the metaphor that A Line Is A Set of Points. This metaphor identifies the point-locations on a line, that is, a natural continuum, as constituting the line itself. Such a metaphorical "line" *is not* a natural continuum, but only a set of points. Given a naturally continuous line segment *A B*, the point-locations on that line will be *gapless*. Similarly, when a naturally continuous line segment is conceptualized as a set of points, that set of points will be *gapless*. Thus, the metaphor A Line Is A Set Of Points entails a submetaphor:

- Natural Continuity Is Gaplessness.

Therefore, a line conceptualized as a set of points cannot be *naturally continuous* but only *gapless*. This terminology distinguishes two distinct ideas that have previously both been called "continuity."

Before we discuss the third idea that has been called "continuity"—preservation of closeness—we need to discuss the Weierstrass arithmetization of calculus.

WEIERSTRASS' ARITHMETIZATION OF CALCULUS

The work of Weierstrass on limits and continuity is the crystallization of a philosophical and theoretical enterprise—"the arithmetization of analysis," that is, the reduction of calculus to the ontology of arithmetic, namely, numbers and arithmetic relationships. This entailed an attempt to completely eliminate from calculus all concepts that were seen as standing outside a literal characterization of arithmetic. Since numbers and arithmetic relationships were seen as timeless and static, all reference to time and motion had to be eliminated. This meant eliminating Euler's intuitive characterization of continuous functions in terms of motion and intuitive attempts by mathematicians like Newton and Leibniz down to Cauchy to define a limit in terms of approaching a location in space.

The critique was that attempts like those of Newton and Leibniz were "vague." But "vagueness" covered two sins: lack of precision and the failure to adhere to a literal ontology taken from arithmetic. It is important to see that these were different sins. To see this, let us begin with the Weierstrass definitions of limits and continuity.

The Weierstrass program of arithmetizing calculus depended crucially on accepting two other major mathematical programs: set-theoretical foundations and formalism. It was therefore necessary for Weierstrass to reject the literal notion of a line as a natural continuum and instead to use the conception of a line as a set of points, which is metaphorical from the cognitive perspective. Accordingly, Weierstrass also assumed that surfaces and n-dimensional spaces were also to be conceptualized as sets of points. Weierstrass therefore assumed implicitly the metaphor that Numbers Are Points, and Dedekind's metaphor that Real Numbers Are Sets Of Rational Numbers. Given the metaphor of A Line As A Set of Points, the metaphor that Points On The Real Line Are Real Numbers (Infinite Decimals) gives rise to the conception of the real line as a set of real numbers. As Dedekind argued, the set of real numbers, and hence the real line, is gapless. Correspondingly, Weierstrass implicitly assumed the metaphor that we have called Natural Continuity Is Gaplessness. When Weierstrass speaks of an open interval on the real line, he is therefore taking it to be a gapless set of real numbers.

In order to arithmetize calculus, Weierstrass had to define limits, continuity, and differentiability in arithmetic terms, using only numbers. Since calculus was defined in terms of the Cartesian plane, Weierstrass had to reconceptualize the Cartesian plane in terms of arithmetic alone, eliminating all geometric concepts. The x- and y-axes, originally conceptualized as natural continua—lines with points conceptualized as locations on them—were replaced via the metaphors of A Line Is A Set Of Points, Points On An Axis Are Real Numbers, and Real Numbers Are Sets. The points in the Cartesian plane were thereby reconceptualized as pairs of real numbers. Thus, the project of making calculus rigorous required new mathematical ideas, especially this collections metaphors. These metaphors were a substantial and absolutely necessary part of what counted as "rigor."

Ontologically, all geometry was eliminated. Where before there was a Cartesian plane, a surface with points and perpendicular lines as axes, now, via these metaphors, there were only sets of numbers, with numbers themselves reducible to sets alone! All definitions, axioms, and proofs could now be formalized within the framework of set theory and formal logic.

Within this framework, all intuitive lines, that is, all natural continua, are eliminated, as is the concept of natural continuity. The axes in the Cartesian plane, instead of being naturally continuous lines are replaced by gapless sets of real numbers. Confusing the picture, the word "continuity" is used for gaplessness in the set of real numbers constituting an axis in the Cartesian "plane," which is no longer a surface.

Weierstrass must now give a definition of a limit consistent with these metaphors, a definition without the idea of variable values of functions as travelers moving and without the idea of "approaching" a limit, which is

the way we ordinarily understand what a limit is. His definition must be purely static and nongeometric, using only ideas from arithmetic as formalized set-theoretically using formal logic. The only way he can do this is by giving a metaphorical definition of a limit that fits all his metaphors. He reconceptualize the notion of approaching a limit in purely static, discrete terms using another concept—the concept of preservation of closeness in the neighborhood of a real number. Here is his definition:

WEIERSTRASS' DEFINITION OF "LIMIT"
(REALLY PRESERVATION OF CLOSENESS
NEAR A REAL NUMBER)

Let a function f be defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement

$$\lim_{x\to a} f(x) = L$$

means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\text{if } 0 < | x - a | < \delta, \text{ then } | f(x) - L | < \varepsilon.$$

Here there is no motion, no time, and no "approach." Instead, there are static conditions. The "open interval" is a gapless set of real numbers; there are no lines and no points and no surfaces in this metaphorical ontology for the Cartesian plane. The plane itself is a made up of a set of pairs of real numbers. The gaplessness of the set of real numbers in the open interval in the definition is Weierstrass' metaphorical version replacing the natural continuity of the intuitive line in Newton's geometric idea of a limit.

The idea of the function f approaching a limit L as x approaches a is replaced by a different idea, that is, preservation of closeness near a real number: $f(x)$ is arbitrarily close to L when x is sufficiently close to a. The epsilon-delta condition expresses this precisely in formal logic. What Weierstrass has done is give a new metaphor:

- Approaching A Limit Is Preservation Of Closeness Near A Point.

Now, when Weierstrass "defines continuity" for a function, he does not and cannot mean the natural continuity assumed by Newton for ordinary lines, that is, natural continua. Again, he must use metaphors that allow him to reconceptualize geometry using arithmetic. Just as he needed a new metaphor for approaching a limit, he needed a new metaphor for continuity of a function. He characterizes this new metaphor in two steps: first at a single arbitrary real number and then throughout an interval.

His new metaphor for continuity uses the same basic idea as his metaphor for a limit: preservation of closeness. Continuity at a real number is conceptualized as preservation of closeness not just near a real number but also *at* it. Continuity of a function throughout an interval is preservation of closeness near and at every real number in the interval.

WEIERSTRASS' DEFINITION OF CONTINUITY OF A FUNCTION AT A REAL NUMBER (ACTUALLY PRESERVATION OF CLOSENESS AT A REAL NUMBER)

A function f preserves closeness at a number a if the following three conditions are satisfied:

1. f is defined on an open interval containing a,
2. $\lim_{x\to a} f(x)$ exists, and
3. $\lim_{x\to a} f(x) = f(a)$.

WEIERSTRASS' DEFINITION OF CONTINUITY OF A FUNCTION THROUGHOUT AN INTERVAL (ACTUALLY PRESERVATION OF CLOSENESS THROUGHOUT AN INTERVAL)

A function preserves closeness in an open interval if it preserves closeness at every real number in that interval.

For the sake of brevity, we will use "preservation of closeness" to designate "preservation of closeness throughout an interval."

Here we have the third of the three concepts of "continuity" that we mentioned above: first there is the literal notion of natural continuity; then there is gaplessness for a set of real numbers; finally there is preservation of closeness for functions. In the course of the following sections, we will see that these are very distinct ideas and must not be confused.

These concepts are commonly confused, and there is a reason. Consider the collection of prototypical curves characterized earlier by Pierpont. Those curves are all instances of intuitively goemetric lines—that is, they are natural continua and therefore they are naturally continuous. All the continuous functions discussed by Newton, Leibniz, Euler and so on were like this. The Weierstrass "definition of continuity" was seen as fitting all these cases. But since that definition applies to sets of real numbers not natural geometric continua, what exactly can this mean?

As natural continua, these functions conceptualized as straight or curved lines were not just sets of points. Now suppose we apply the Lines As Sets of Points metaphor to them, and then further conceptualize the points constituting the lines as ordered pairs of real numbers. What we get are Weierstrass' arithmetized versions of these natural continua. According to the Weierstrass definition of preservation of closeness, all the functions that were naturally continuous when conceptualized as lines will preserve closeness when conceptualized as sets of ordered pairs of real numbers. In other words, Weierstrass' so-called "definition of continuity" as preservation of closeness will work for every classical case of natural continuity when arithmetized via Weierstrass' metaphors. Thus, the Weierstrass definition of preservation of closeness is seen as "fitting our intuitions" about natural continuity. It only "fits our intuitions" after Weierstrass' metaphors have been applied. It does not "fit our intuitions" literally.

WHAT IS PRECISE IN THE WEIERSTRASS DEFINITIONS?

Many students of mathematics are falsely led to believe that it is the epsilon-delta portion of these definitions that constitutes the rigor of this arithmetization of calculus. The epsilon-delta portion actually plays a far more limited role. What the epsilon-delta portion accomplishes is a precise characterization of the notion "correspondingly" that occurs in the dynamic definition of a limit where the values of $f(x)$ get "correspondingly" closer to L as x gets closer to a. That is the only vagueness that is made precise by the epsilon-delta definition. Indeed, we can show this by returning to the dynamic notion of approaching a limit and adding the epsilon-delta portion of Weierstrass' definition to make the notion of a limit precise for a function defined in terms of motion. Here is our revised notion of such a limit.

THE DYNAMIC EPSILON-DELTA LIMIT

Assume the dynamic definition of a function given earlier, where variation in values was conceptualized metaphorically as motion by a traveler. Let a function f be defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement

$\lim_{x\to a} f(x) = L$
means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that
as x moves toward a and gets and stays within the distance δ of a,
$f(x)$ moves toward L and gets and stays within the distance ε of L.

This eliminates the vagueness of the term "correspondingly" in the old dynamic definition of a limit. But this would hardly satisfy Weierstrass, since it retains the notions of motion and time. It remains geometric and not arithmetized.

The point is that it is not the imprecision of the notion "correspondingly" that is at issue. Nor is it inability to comprehend the metaphorical idea of a value of a function "approaching" something. The metaphor we employed characterizes "approaching" using precisely formulated mappings, based on the clear, commonplace idea of a traveler approaching a physical location. There is nothing incomprehensible or vague about this. The metaphor of dependent variable approaching the limit L as the independent variable gets correspondingly closer to the value a is made precise by the metaphor plus the epsilon-delta definition.

The issue is not imprecision. It is the fact that this formulation is not arithmetized and does make exclusive use of set theoretical foundations and formal logic.

And this is not the only issue. There is the issue of the "monsters"—the "pathological" cases that do not fit the properties defining the prototypical central case of a "curve." The Weierstrass arithmetization is not justified merely on the grounds of rigorously meeting the standards of set-theoretical foundations and formal logic. As we saw in the case of Pierpont, it was justified on the grounds of being able to tame the monsters and cure the pathologies. Let us now look at the monstrous pathological cases and see if this traditional account makes any sense.

THE MONSTERS

What makes a function a monster? Suppose you believed with Descartes, Newton, and Euler that a function could be characterized in terms of (one or more) natural geometric curves—natural continua—in a classical Cartesian Plane. Your understanding of a function would then be characterized by your understanding of curves. As Pierpont pointed out, the prototypical curve has the following properties:

1. It can be generated by the motion of a point.
2. It is continuous.
3. It has a tangent.
4. It has a length.
5. When closed it forms the complete boundary of a region.
6. This region has an area.
7. A curve is not a surface.
8. It is formed by the intersection of two surfaces.

What are called "monsters" are functions that fail to have all these properties.

Consider the following two monsters:

$$\text{Monster 1: } f(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$$\text{Monster 2: } f(x) = \begin{cases} x\sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

These are represented in Figs. 2.2 and 2.3.

As x gets close to zero, $1/x$ grows to infinity (when x is positive) and minus infinity (when x is negative). For every amount of 2π in the growth of $1/x$, $f(x)$ oscillates once. As x approaches zero and $1/x$ approaches positive or negative infinity, both functions oscillate with indefinitely increasing frequency.

Monster 1 oscillates between -1 and 1 all the way up to zero (see Fig. 2.2). Monster 2 is more controlled. It oscillates between two straight lines each at a 45 degree angle form the x-axis and intersecting at the origin. But since x gets progressively smaller as it approaches zero, the function goes through progressively smaller and smaller oscillations (see Fig. 2.3).

Do monsters (1) and (2) have all the properties 1–8 of prototypical curves? Suppose we ask if they can be generated by the motion of a point? The answer is no. Such a point would have to be moving in a direction

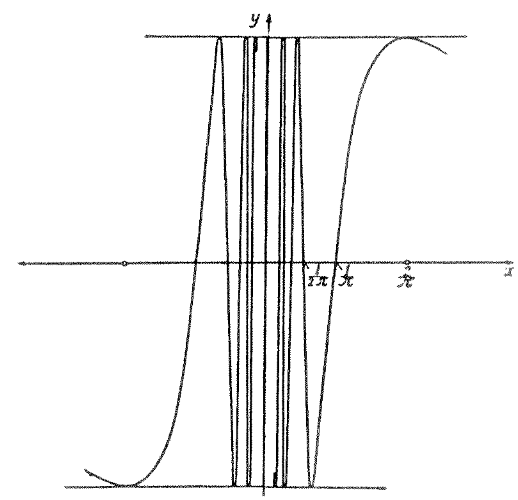

FIG. 2.2. The graph of the function $f(x) = \sin(1/x)$ (Monster 1). Its oscillations between 1 and -1 become more and more frequent as x approaches zero. The frequency of the oscillations becomes infinite *at* $x = 0$.

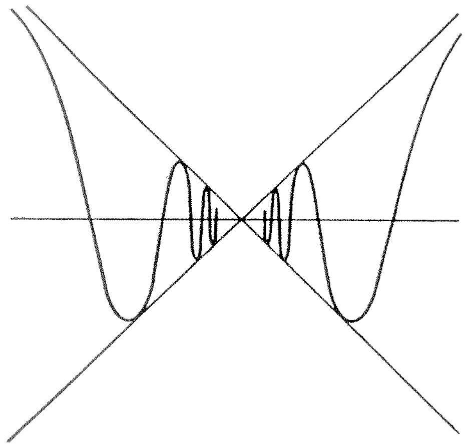

FIG. 2.3. The graph of the function $f(x) = x \sin(1/x)$ (Monster 2). As its oscillations become infinitely frequent, the amplitude of the oscillations becomes infinitely small.

at every point, including zero. But as each function approaches zero it oscillates—that is, changes direction—more and more often. As it passes through the origin, the oscillation, or change of direction, approaches infinity. What direction is the curve going in as it passes through the origin? There is no well-defined direction. Hence, the function cannot be a curve traced by the motion of a point. Since no direction can be assigned to the function as it passes through zero, the function cannot be said to have any fixed tangent at zero. Additionally, consider an arc of each function in a region including zero, for example, an arc between $x = -0.1$ and $x = 0.1$. What is the length of such an arc? Because the function oscillates indefinitely, if does not have a fixed length. In short, monsters (1) and (2) fail to have properties 1, 3, and 4. What about property 2: Are they "continuous"? Continuity for prototypical curves—that is, for natural continua—means "natural continuity."

If natural continuity is characterized, as Euler proposed, by the motion of a point, then the answer is no. Since neither can be so characterized, neither monster function is naturally continuous. Thus, they also fail to have property 2. Since they lack half of the properties of prototypical curves, and since properties 5 and 6 don't apply to them, they lack four out of six of the relevant properties of curves. That is what makes them monsters from the classical perspective.

From Weierstrass' perspective, the definitions of "continuity" (that is, preservation of closeness) and differentiability apply to such cases exactly as they would to any other function. Neither curve is differentiable at the origin; since they have no tangent there, they could not be differentiable

on Weierstrass' account. But Weierstrass' account gives different answers so far as preservation of closeness is concerned. Monster 1 does not preserve closeness at the origin. If you pick an epsilon less than, say, 1/2, there will be no delta near zero that will keep $f(x)$ within 1/2. The reason is that, as x approaches zero, $f(x)$ oscillates between 1 and -1 with indefinitely increasing frequency—and so cannot be held to within the value of 1/2 when x is anywhere near zero. Since preservation of closeness is what Weierstrass means by "continuity," Monster 1 is not Weierstrass-continuous. In this case, preservation of closeness matches natural continuity: both are violated by monster 1.

Monster 2 is very different for Weierstrass. Because $f(x)$ gets progressively smaller as x approaches zero, it does preserve closeness. If you pick some number epsilon much less than 1, then for every delta less than epsilon, the value of $f(x)$ for monster 2 will stay within epsilon. Given that preservation of closeness is Weierstrass' metaphor for continuity, Weierstrass' "definition of continuity" designates monster 2 as "continuous" by virtue of preserving closeness.

This does not mean that Monster 2 is naturally continuous, while Monster 1 is not naturally continuous. Neither is naturally continuous. It only means that Monster 2 preserves closeness while Monster 1 does not. Weierstrass-continuity means nothing more than preservation of closeness.

"SPACE-FILLING CURVES"

Monster 3 and 4 are examples of so-called "space-filling" curves. We will consider two, one proposed by Cantor that we will call the odd-even function, and the celebrated Hilbert "Space-filling Curve." Both of these are functions from the interval [0, 1] on the x-axis to points (y,z) in the y–z plane, where the values of y and z each lie in the [0, 1] interval. That is, both are functions from a line segment of length 1 to points in a square of area 1.

Monster 3: The Cantor Odd-Even Function. Consider the infinite decimal representations of the real numbers between 0 and 1 on the x-axis. For each such infinite decimal x, form two infinite decimals y and z, such that y consists of the sequence of digits in the odd-numbered positions in the infinite decimal representation of x, while z consists of the sequence of digits in the even-numbered positions in the infinite decimal representation of x. The function maps each infinite decimal representing x in the [0, 1] interval on the x-axis to the pair (y,z) in the y–z plane.

For example, suppose $x = 0.35872961\ldots$; then $y = 0.3826\ldots$ and $z = 0.5791\ldots$. Thus, for every point (y,z) in the square, there will be a point x on the interval [0, 1] on the x-axis whose infinite decimal expansion is mapped onto (y,z) by the function f. Thus yields a 1–1 correspondence between the points on the line and the points in the square.

Monster 4: The Hilbert Space-Filling Curve. This is the limit at infinity of a sequence of functions from [0, 1] on the x-axis to the sequence of curves in Fig. 2.4.

Consider, for example, the simple curve in the leftmost box—the first step in the construction of the function. The mapping at this step has the following parts:

i. When x varies from 0 to 1/3 in the interval [0, 1/3], $f(x)$ varies continuously along the lower horizontal line going from left to right.
ii. When x varies from 1/3 to 2/3 in the interval (1/3, 2/3], $f(x)$ varies continuously along the vertical line from bottom to top.
iii. When x varies between 2/3 and 1 in the interval (2/3, 1], $f(x)$ varies continuously along the upper horizontal line from right to left.

At each stage the mapping preserves closeness; that is, it is Weierstrass-continuous. At each successive stage, the curve gets longer and longer, "filling up more of the square," until at the limit at infinity, the curve "goes through" every point in the square, thus "filling the square." Technically, the curve doesn't "go" anywhere, much less "go through" any points. Each "curve" in the sequence is a set of points, the square is a set of points, and at the limit, the set of points constituting the "curve" maps onto the set of points constituting the square. Indeed, in certain cases, more than one point on the line will map onto the same point on the square.

Both of these monsters violate properties 1–4, and 7–8 of the properties of a prototypical curve. And in both cases, 5 and 6 are inapplicable. Neither case can be generated by a moving point. The reason is that such a point must move in a direction at each point of the function. But in neither function is there such a direction. This is clear in the Odd–Even function. In the Hilbert Curve, which is defined only at infinity, the curve changes direction at each point. Therefore, it can have no single direction of motion at any point.

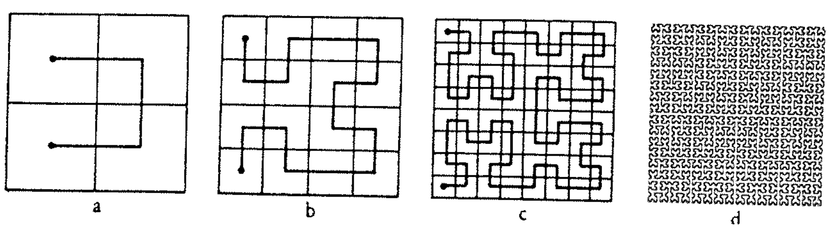

FIG. 2.4. The first few steps in the construction of the Hilbert Curve (Monster 4).

Since neither function can be generated by a moving point, neither function is naturally continuous in Euler's sense. Moreover, since both functions have no direction at any point, they have no tangent at any point. Additionally, neither function has a specific length for any arc. In the odd–even function, it is hard to know what an "arc" could mean. In the Hilbert curve, every arc is of indefinitely long length. Thus both monsters fail properties 1–4.

Since both functions map onto all the points in the square, they each map onto the points on a surface, in violation of 7, and they cannot be formed by an intersection of surfaces, in violation of 8. These are thus even more monstrous than monsters 1 and 2. You can see why anyone who insisted on defining a function as an instance of a prototypical curve in the Cartesian plane would find all these cases "pathological."

Now let us consider what the Weierstrass definitions say about monsters 3 and 4. Both monsters 3 and 4 preserve closeness, according to the Weierstrass definition. That is, points that are close on the x-axis map onto points that are close on the y–z plane, where corresponding closeness is defined by the epsilon-delta condition. Since preservation of closeness is what Weierstrass called "continuity," these functions are Weierstrass-continuous even though they are not naturally continuous.

So far as differentiability is concerned, neither function is differentiable at any point on the Weierstrass criterion, just as neither function has a well-defined tangent at any point.

DO "SPACE-FILLING CURVES" ACTUALLY FILL SPACE?

There are two different interpretations on the Hilbert Curve and the Cantor Odd-Even Function: The first is based on natural continua, in which points are locations on lines and surfaces and in n-dimensional spaces, but do not constitute lines, surfaces, and n-dimensional spaces. The second is based on the metaphorical idea that lines, surfaces, and n-dimensional spaces in general are constituted by sets of points. It is only on the second interpretation and not on the first that the Odd-Even Function and the Hilbert Curve "fill a surface." In other words, the idea that they "fill a surface" is a metaphorical idea, based on the metaphor that lines and surfaces are sets of points.

Consider the first interpretation. Lines, surfaces, and n-dimensional spaces are naturally continuous entities that exist independently of the points that define locations on them. What the Odd–Even Function and the Hilbert Curve do on this interpretation is to map point-location on a line segment onto all the point-locations on the squares. Since these functions are defined only for point-locations, not for naturally continuous surfaces, they do not "fill" those naturally continuous surfaces. Recall that points are zero-dimensional. On a line, a point extends over *no* distance. On a plane, a point covers *no* area. These are *sizeless* entities. They cannot "fill" anything. You can "fit" as many as you want anywhere. It should not be surprising that as many zero-dimensional sizeless elements can "fit" onto a 1 × 1 square as onto the length of line 1. It should be recalled that on this interpretation, the naturally continuous line is *not* being mapped onto the naturally continuous square. Only the sizeless points *on* the line are being mapped onto the sizeless points *on* the square.

Moreover, recall that the Limit Metaphor is indeed a metaphor. Both the Hilbert Curve and the Cantor Odd-Even Function use these metaphor. The Hilbert Curve is defined as the limit of a sequence of functions. And the Cantor Odd-Even function makes use of infinite decimal representations, which are also defined by the Limit Metaphor. It takes still one additional metaphor—the Spaces Are Sets Of Points Metaphor—to reach the conclusion that curves can "fill spaces."

What is the significance of this? These two functions have been taken as defying our ordinary intuitions about dimensionality, showing that these intuitions cannot be trusted. They are taken as showing that it is not the case that all lines are one-dimensional and all surfaces are two-dimensional. Moreover, they have historically been the catalyst for the extension of dimensionality to fractional dimensions (fractals), which also defy our ordinary spatial intuitions.

But these functions, by themselves, do not show any of these things. To reach these conclusions, one must make use of a considerable number of metaphors: Cantor's Metaphor, The Limit Metaphor, The Space As Sets Of Points Metaphor, and the metaphors that Continuity (For Functions) Is Points Metaphor, and the metaphors that Continuity (For Functions) Is Preservation of Closeness and that The Points On The Real Line Are Real Numbers (Infinite Decimals). What defies our ordinary spatial intuitions is the joint operation of all these metaphors. It is these metaphors, taken together, that are at odds with our ordinary intuitions about dimensionality.

These metaphors define important mathematical ideas, like the idea of a fractional dimension. There is nothing wrong with using them. But one should be aware when they are being used. And one should not make the claim that their joint entailments are literal.

HAS WEIERSTRASS TAMED THE MONSTERS?

It depends on what you mean by "tamed." The Weierstrass definitions categorize these monsters the same as they would any other function. Does Weierstrass-continuity match our idea of natural continuity, the intuitive basis of our intuition of continuity? The answer is no. Monsters 2, 3, and 4 are Weierstrass-continuous but not naturally continuous. If the match between Weierstrass-continuity and natural continuity is a criterion, then Weierstrass continuity does not tame the monsters. If that is not a criterion, then it does.

But how wild are the monsters after all? Suppose we take properties 1-8 as defining the central prototype of a radial category (Lakoff, 1987). Then monsters 1 and 2, which share properties 7 and 8 with the prototypical curves, will form a clearly defined noncentral subcategory of the category, which bears clearly defined family resemblances to the central members. monsters 3 and 4 will simply not be in the category. On this account, not all functions are represented equally well by curves. The prototypical curves are fully members of the category of curves, and the monsters 1 and 2 are members of a particular well-defined subcategory, while monsters 3 and 4 are out. From a cognitive point of view, there is nothing imprecise or vague or nonrigorous about such a radial categorization. The radial category categorizes these monster functions the same way it would categorize any function. But the categories are different than Weierstrass', since his metaphors are not used.

If rigor is *defined* not cognitively, but via the philosophy of formalism and the program of set-theoretical foundations, then such a precise cognitive characterization would not be "rigorous." The issue of rigor here is not a matter of the arithmetization of calculus, but a matter of your philosophy of mathematics: Do you or don't adhere to the philosophy of formalism and the program given by the set-theoretical foundations? Do you accept cognitive science as characterizing mathematical ideas, and do you consider mathematical ideas as central to mathematics? It is the answer to these questions that determines what "rigor" is to mean.

But even if you adhere to the philosophy of requiring formal set-theoretical foundations, you cannot characterize rigor in the case of Weierstrass' definitions without extensive use of conceptual metaphor in your mathematical ideas: The metaphors that A Line Is A Set of Points, The Points Constituting the Real Line Are The Real Numbers, Continuity (For Axes) Is Gaplessness, and Continuity (For Functions) Is Preservation Of Closeness.

GAPLESSNESS VERSUS PRESERVATION OF CLOSENESS

We claimed above that the Weierstrass "definition of continuity" for functions really defines the idea of preservation of closeness, which is a second metaphorical concept for continuity, in addition to gaplessness. We now need to demonstrate that such a third notion of continuity is real and is required in addition to gaplessness.

We will demonstrate this in the following way. We will modify the domain requirement of Cantor's "definition of continuity" so as to allow massive gaps in the domain of a function. Then we will show that the remainder of Weierstrass' definition holds for the function, but that the function is neither naturally continuous nor gapless. Instead, the function preserves closeness.

MORE MONSTERS

Here is a well-known function that we will call Monster 5:

$$h(x) = \begin{cases} 1 \text{ if } x \text{ is irrational} \\ 0 \text{ if } x \text{ is rational} \end{cases}$$

So far as natural continuity is concerned, Monster 5 is nowhere naturally continuous. At every point p, there is a point infinitely close to p where the function "jumps." It also does not preserve closeness. That is, it fails to meet Weierstrass' "definition of continuity."

Suppose now that we redefine the function, calling it $h^*(x)$, to apply over the irrationals taken separately and to be undefined over the rationals. Suppose additionally we change the Weierstrass definition of preservation of closeness so that the "open interval" need not be gapless.

It should be clear that the function redefined in this way preserves closeness, which has been redefined to allow gaps in the domain of the function. $h^*(x)$ is now a constant, since it equals one for all irrationals, and since it is defined only for the irrationals. Its domain has gaps everywhere. The graph of the function in the Cartesian plane also has gaps everywhere. But since $h^*(x)$ is a constant, the values of $h^*(x)$ will be within distance epsilon (indeed, they will be the same) for any choice of delta.

$h^*(x)$ is neither naturally continuous nor gapless, but it does preserve closeness, which is exactly what the Weierstrass definition defines.

Monster 6: The Cantor Set Example. The same point can be made with an even more interesting example: Let us replace the gapless open interval in the Weierstrass definition this time with one that is "almost" gapless everywhere, and yet still has an uncountable infinity of gaps. What we have in mind is the complement of the Cantor Set.

Let us first define the Cantor Set. Start with the open interval (0, 1). Remove the middle third, that is, the closed interval [1/3, 2/3]. That leaves two segments, (0, 1/3) and (2/3, 1). Now continue removing the middle closed interval of each, ad infinitum. For example, the next closed intervals that drop out are [1/9, 2/9] and [7/9, 8/9]. What you have is an infinite set of subsets of the interval (0, 1), getting sparser and sparser as more segments are removed. Now take the infinite union of the members of that infinite set. This is the Cantor Set.

Now consider the complement of the Cantor Set. Think of constructing it out of the complement of each constituent set. The first complement

contains the closed interval [1/3, 2/3]. The second complement is the set of closed intervals {[1/9, 2/9], [1/3, 2/3], [7/9, 8/9]}. And so on, ad infinitum. The infinite union of these sets is the complement of the Cantor Set.

The complement sets in this construction keep accumulating more and more pieces of gapless closed intervals, ad infinitum. Yet at each step there are gaps. Indeed, at the nth step, there are 2^n gaps. As more and more gapless pieces are added at each step, the number of gaps increases exponentially with the steps. The number of steps in the construction is denumerable, that is, $\aleph_0$. The number of gaps is therefore $2^{\aleph_0}$. Since $2^{\aleph_0} \geq \aleph_1$, the number of gaps in the complement of the Cantor Set is nondenumerable.

This complement set is a subset of the interval (0, 1) and has measure 1. In addition, it is gapless "almost everywhere," that is, it will have infinitely many gapless subsets. Moreover, any neighborhood around any point in the set will either be gapless or contain infinitely many gapless subsets. Despite all this, it nonetheless has massively many gaps.

Suppose now that we loosen the "open interval" condition on the Weierstrass definition of preservation of closeness, as follows: allow the complement of the Cantor set in addition to the usual open intervals. Now take the continuous function $f(x) = x^2$ and restrict its domain to the complement of the Cantor Set. Call this $g(x)$. Because its domain has massively many gaps, $g(x)$ is not gapless. Indeed, it has a nondenumerable number of gaps. Does this function preserve closeness, as redefined?

The answer is yes. The reason is this: Every point in the domain will either be in a gapless interval or not. If so, the function will preserve closeness at that point. If not, that point will be arbitrarily close to a gapless interval where the function does preserve closeness. But being arbitrarily close is good enough for the Weierstrass definition of preservation of closeness.

The point is that the Weierstrass "definition of continuity" does not define either natural continuity or gaplessness; it defines preservation of closeness, which will happen to coincide with natural continuity and gaplessness in the case of functions that can be characterized as prototypical curves.

GAPLESSNESS IN, GAPLESSNESS OUT

Weierstrass formulated his "definition of continuity" with the explicit conditions that the function is defined over an open interval. He assumed this open interval to be gapless. Since gaplessness was his way of metaphorically conceptualizing continuity on the real line, he was assuming a "continuous," that is, gapless input. What he showed really was that (a) when his metaphors hold, especially when lines are metaphorically conceptualized as sets of real numbers, and (b) when the input of the function is gapless, and (c) when the function preserves closeness, then (d) the output is also gapless.

In short, there is a reason why it has been widely assumed that Weierstrass' definition of preservation of closeness was instead a "definition of continuity." First, it has been assumed, falsely, that Weierstrass' metaphors match our ordinary intuitions. Given the metaphor that a line is a set of real numbers, then natural continuity can only be conceptualized metaphorically as gaplessness. Since Weierstrass' open interval condition guaranteed that the inputs to his function were always gapless, it is no surprise that preservation of closeness for a function with a gapless input guarantees a gapless output. If his input is metaphorically continuous (that is, gapless), then the output is going to be metaphorically continuous (gapless). Since the metaphors were not noticed as being metaphorical or controversial in any way, and since the open interval condition hid the continuity (gaplessness) required in the input, Weierstrass' definition appeared even to Weierstrass to be a definition of continuity, when in fact, all it did was guarantee that a gapless input for a function gives a gapless output. Gaplessness in, gaplessness out.

THE "GREAT TASK"

Herman Weyl, one of the greatest mathematicians of the 20th century, notes in his classic work, *The Continuum* (Weyl, 1987):

> We must point out that, in spite of Dedekind, Cantor, and Weierstrass, the great task which has been facing us since the Pythagorean discovery of the irrationals remains today as unfinished as ever; that is, the continuity given to us immediately by intuition (in the flow of time and in motion) has yet to be grasped mathematically as a totality of discrete "stages" in accordance with that part of its content which can be conceptualized in an "exact" way. (p. 24)

Why should it, as Weyl says, be a "task of mathematics" to "grasp" the continuum as "a totality of discrete stages"? Why does mathematics have to understand the continuous in terms of the discrete? Each attempt to understand the continuous in terms of the discrete is necessarily metaphorical—an attempt to understand one kind of thing in terms of another kind of thing. Indeed, it is an attempt to understand one kind of thing—the continuum—in terms of its very opposite—the discrete. It is strange at the very least why it should be seen as a central task of mathematics to provide a metaphorical characterization of the continuum in terms of its opposite. Any such metaphor is bound to miss aspects of what the continuum is.

If the "great task" is to provide absolute, literal foundations for mathematics, then the attempt to conceptualize the continuous in terms of the discrete is self-defeating. First, such foundations cannot be literal; they can only be metaphorical. Second, as Weyl himself says, only "part of its content" can be conceptualized discretely. The rest must be left out. If Weyl is right, the task cannot be done. Any discrete "foundations" will not be adequate to characterize the totality of the continuum, with nothing left out.

We believe there is a greater task: understanding mathematical ideas.

MORALS

It is time to derive whatever lessons we can from all this discussion. The morals we can draw concern our deepest understanding of what mathematics is and isn't. They concern the nature of mathematics, the nature of mathematical ideas, the status of mathematical formalism, and fundamental ideas such as number, point, line, surface, space, set, size, infinity, function, limit, continuity, and so on. By looking at mathematics from a cognitive perspective, we can see these ideas anew, and when we do, we are led to challenge many received dogmas. Mathematics from a cognitive perspective—a mind-based perspective—is not like mathematics from a mind-free perspective.

Moral 1: Mathematical ideas cannot be expressed in formalisms with mathematical rigor.

The reason is that ideas, including mathematical ideas, are not part of the subject matter of formal mathematics. They are part of the subject matter of cognitive science, especially cognitive semantics. What mathematical ideas are and how they are or are not expressed in mathematical formalism is not, and cannot be, part of the subject matter of formal mathematics.

Moral 2: Mind-free mathematics is a myth.

Mathematics is about mathematical ideas—about alternative ways of conceptualizing central concepts like number, space, size, set, and so on. Formal notations and formal proofs are interesting only insofar as they express mathematical ideas.

Moral 3: There can be no ultimate foundations for mathematics within mathematics itself.

Mathematical ideas are central to mathematics, not extraneous. Mathematics is not mathematics without mathematical ideas. Since mathematical ideas are not part of the subject matter of mathematics, mathematics in itself cannot characterize what mathematics is.

Moral 4: The formal foundations program is constituted by a set of ideas.

The formal foundations program, one of the great monuments of twentieth-century intellectual life, is constituted by a set of ideas, only one of which concerns formal notations, formal proofs, and the use of formal logic. Most of those ideas are conceptual metaphors, which include: Lines (And Other *n*-dimensional Spaces) Are Sets of Points, The Points On A Line Are Real Numbers, Real Numbers Are Sets of Rational Numbers, Numbers Are Sets, Sameness of Set Size Is One-to-One Correspondence, Continuity Is Preservation of Closeness, and Infinity Is A Point. Much of the mathematical content of the formal foundations program is a consequence of these metaphors, which have unwittingly taken as literal and which have been given hegemonic status. These metaphors are no more right or wrong, beneficial or harmful than other metaphors. But what is harmful is their hegemonic status and the failure to note their metaphorical character.

Since these are ideas and since ideas are technically not part of any formal notations nor part of the subject matter of mathematics proper, the formal foundations program cannot be characterized within formal mathematics itself. Nor can it be characterized within metamathematics, which is another branch of formal mathematics.

Moral 5: The study of ideas need not be vague or hazy.

Ideas can be studied within cognitive science, especially cognitive semantics, in very great detail. Cognitive science is developing methods of modeling ideas with precision. The study of mathematical ideas can therefore be as precise as the cognitive science of a given era can make it.

Moral 6: Mathematics is ultimately grounded in the human body, the human brain, and in everyday human experiences.

Our everyday conceptual system and capacity for reason is grounded in the human body, brain, and everyday experience. Basic mathematical concepts are expert versions of everyday human concepts. As we saw, our basic mathematical ideas are metaphorically grounded in everyday experiences and make use of our commonplace conceptual system.

Moral 7: Most interesting mathematics arises through the use of linking metaphors.

That is, through metaphors by which we conceptualize one mathematical domain or idea in terms of others.

Moral 8: Because metaphors preserve inference, proofs using metaphorical ideas stay proved.

A Platonic worldview is not necessary to explain why proofs stay proved.

Moral 9: Not all ='s mean the same thing, and most ='s are metaphorical.

We have seen that in the grounding metaphors there are many different uses of '=' in numerical equations; e.g., balance, different routes to the same place, and so on. Moreover, every linking metaphor introduces metaphorical uses of '=' via which expressions for source domain ideas can be substituted in proofs for expressions of target domain ideas.

Moral 10: The continuous cannot be literally defined in terms of its opposite, the discrete.

Any attempt to characterize the natural continuum in terms of discrete elements, like points or numbers, can only be done using metaphor, because continuity and discreteness are opposite ideas. This is an example of why one cannot expect absolute, literal, ultimate foundations for mathematics.

Moral 11: One should neither expect nor seek absolute, literal, definitive foundations for mathematics.

The reason comes from Moral 6: Traditional set-theoretical foundations were metaphorical, not literal. Those metaphors led to interesting results. There will always be more interesting mathematics to be done by metaphorically conceptualizing basic concepts in new ways. Recent examples include the metaphors conceptualizing sets in terms of hypersets (that is, graphs) and Conway numbers, in which numbers are metaphorically conceptualized in terms of advantages in combinatorial games. The very act of seeking an absolute foundation is an attempt to make own mathematical metaphors hegemonic, and to rule out the possibility of alternative interesting forms of mathematics.

Moral 12: A cognitively based philosophy of mathematics is needed.

The existing philosophical approaches to mathematics can make no sense of what we have discussed in this paper. Neither formalism, nor constructivism, nor Platonism has any room for an account of mathematical ideas. What we are suggesting instead is a cognitively based philosophy of mathematics. Results from cognitive science have made obsolete the foundations for mathematics using formal logic and set theory.

Such a philosophy of mathematics would not impose any hegemonic foundations. Instead, it would be dedicated to keeping mathematics open and dynamic, and to studying mathematical ideas and making them explicit.

Moral 13: Mathematics should be taught in terms of mathematical ideas.

Mathematics is a supreme intellectual endeavor. It is all about ideas, and it should be taught as being about ideas. To overstress either techniques of formal proof or techniques of calculation is to shortchange students. Students have a right to understand mathematics in terms of its ideas—and especially when the ideas are controversial and conflict with one another. Since a great many mathematical ideas are metaphorical, teaching mathematics necessarily requires teaching the metaphorical structure of mathematics. This should have the beneficial effect of dispelling the myth that mathematics is literal, is inherent in the structure of the universe, and exists independent of human minds.

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