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# THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS FOR THE PERIODIC SOLUTION OF THE VAN DER POL EQUATION

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# The Method of Matched Asymptotic Expansions for the Periodic Solution of the Van der Pol Equation

by

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Summary

The purpose of this paper is to introduce the method of intermediate matching for asymptotic expansions and to apply this method for connecting the four local solutions of the Van der Pol equation, given by Dorodnicyn [3]. It turns out that for the approximation of the periodic solution a fifth local solution is needed. The present approach results in a reduction of the computational work. The amplitude of the periodic solution is determined up to a higher order accuracy in $\nu$ than has been done so far.

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## 1. Introduction

In this paper an asymptotic expansion is given for the periodic solution of the Van der Pol equation

$$\frac{d^2x}{dt^2} - \nu(1 - x^2)\frac{dx}{dt} + x = 0 \qquad (1.1)$$

for large value of the parameter $\nu$. Equation (1.1) has been studied extensively by many authors. It is a well-known fact that a periodic solution for this equation exists (cf. LaSalle [6]). In the phase-plane (x,p), where $p = \frac{dx}{dt}$, equation (1.1) transform into

$$p\frac{dp}{dx} - \nu(1 - x^2)\, p + x = 0. \qquad (1.2)$$

Van der Pol [7] has already pointed out that the periodic solution for large $\nu$ can be approximated by the solutions of the following reduced equations of (1.2)

$$p\frac{dp}{dx} - \nu(1 - x^2)\, p = 0 \qquad (1.3)$$

in a region of the phase-plane where p is large and

$$- \nu(1 - x^2)\, p + x = 0 \qquad (1.4)$$

in a region where p and $\frac{dp}{dx}$ are both small. The regions where these approximate solutions are valid do not overlap and it was not clear at all how these local solutions had to be matched. In 1947 Dorodnicyn [3] introduces two new regions in which he gives asymptotic solutions for (1.2). His four regions overlap, which makes it possible to find a complete solution for the whole limit-cycle. However, Dorodnicyn's way of matching is rather crude and his claim that the accuracy of the asymptotic solution can be carried up to an arbitrary order of $\nu$ is false (see Zonneveld [10]). Furthermore, some computational errors in Dorodnicyn's work have been noticed by Urabe [9], Zonneveld [10] and Ponzo and Wax [8].

If one tries to match the local solutions of Dorodnicyn by the method described by Van Dyke [4], one meets serious difficulties. Yet, it is possible to apply a well-founded method of matching, if a fifth region with its own local solution is added. Such a method of matching is based on the principle that the asymptotic expansions of two adjacent local solutions should be identical, if they are written in the local coordinates of the overlapping region. It turns out that it is possible to choose the constants of integration in such a way, that the terms of the two expansions are completely identical. In the sequel we will call this the method of intermediate matching.

Since equation (1.2) remains unchanged, if we substitute -x for x and -p for p, each solution will show radial symmetry with respect to the origin. Hence we will obtain two groups of five regions. The regions of the first group are specified as follows:

A: $\quad -x_s + r\nu^{-2} \le x \le -1 - r\nu^{-1/3}$ (III'),

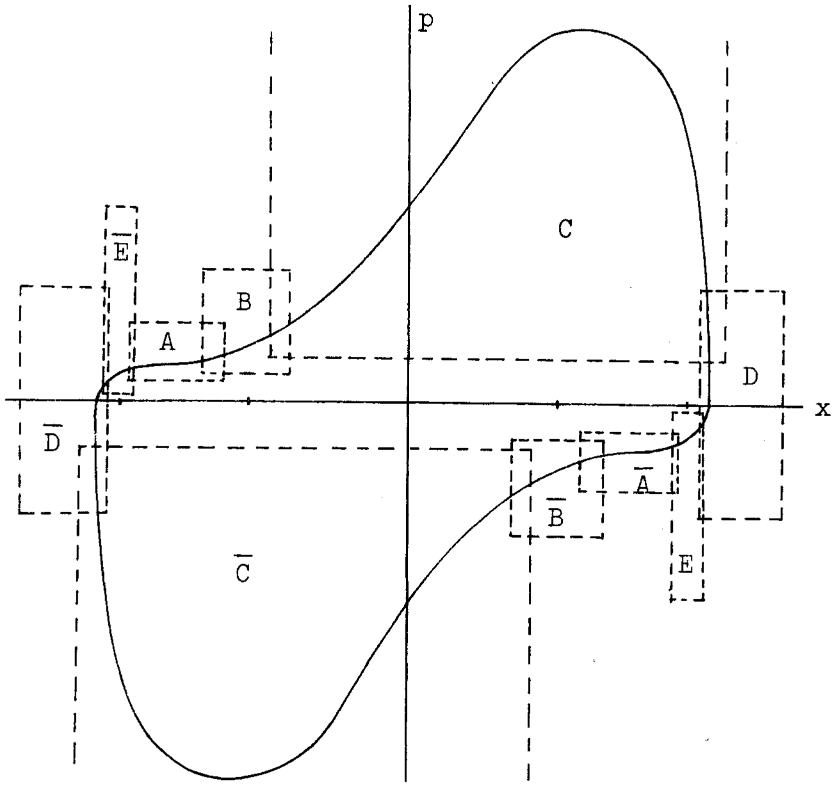
B: $\quad -1 - R\nu^{-1/3} \le x \le -1 + R\nu^{-1/3}$ (IV'),

C: $\quad -1 + r\nu^{-1/3} \le x \le 2 - r\nu^{-1}$ (I),

D: $\quad p(2 - R\nu^{-1}) \le p(x) \le p(x_s + \frac{2}{9}\nu^{-2}\log\nu - r\nu^{-2})$ (II),

E: $\quad x_s + \frac{2}{9}\nu^{-2}\log\nu - R\nu^{-2} \ge x \ge x_s - R\nu^{-2}$ (-).

The constants r and R ($0 < r << R$) are independent of $\nu$. The point $x = x_s$ ($= 2 + O(\nu^{-4/3})$) will be specified more precisely in section 10. We have given the notation of Dorodnicyn for the regions between brackets. The regions of the second group $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ and $\bar{E}$ are obtained by radial symmetry.




The main modifications we made in Dorodnicyn's work are the following:

a. The region A(III') is taken as the starting-point of the calculations.

b. In the formal expansions of the regions C(I) and D(II) also fractional powers of $\nu$ are included.

c. Between the regions D(II) and $\overline{A}$(III) a fifth region E(-) with its own local solution is added.

d. For the local solutions of C and D and of D and E we apply intermediate matching, since the Van Dyke matching fails here. In order to obtain a uniform representation we also apply intermediate matching in the other cases, although the Van Dyke matching is applicable there.

These modifications result in a reduction of the computational work, so that we determine the amplitude of the periodic solution up to a higher order accurracy in $\nu$.

## 2. Solutions in region A

In region A the main term of the asymptotic expansion of the solution originates from the solution of the approximate equation (1.4). It turns out that $p = O(\nu^{-1})$ and that the formal expansion of the solution of (1.2) takes the form

$$p = \sum_{n=0}^{\infty} y_n(x)\, \nu^{-2n-1} \quad . \tag{2.1}$$

Substitution of (2.1) into (1.2) and equalization of the coefficients of $\nu^{-2m}$ (m = 0, 1, 2, ...) lead to the system

$$y_0(x) = \frac{x}{1-x^2} \quad , \tag{2.2}$$

$$y_n(x) = \frac{1}{1-x^2} \sum_{k=0}^{n-1} y_k'(x)\, y_{n-k-1}(x) \quad , \tag{2.3}$$

(n = 1, 2, ...).

For n = 1 and n = 2 we find

$$y_1(x) = \frac{x(x^2+1)}{(1-x^2)^4} \quad , \tag{2.4}$$

$$y_2(x) = \frac{2x(3x^4 + 6x^2 + 1)}{(1-x^2)^7} \quad . \tag{2.5}$$

## 3. Solution in region B

When $x \uparrow -1$ the terms of the local expansion (2.1) become singular and the solution can no longer be represented by (2.1). Following Dorodnicyn we introduce the local coordinate u and the local dependent variable Q by

$$x = -1 - u\nu^{\alpha} \quad (\alpha > 0) \quad , \tag{3.1}$$

$$p = \nu^{\beta} Q(u;\, \nu) \quad . \tag{3.2}$$

Equation (1.2) will be transformed into an equation, which will have all the terms of the same order O(1), if

$$2\beta - \alpha = 1 + \alpha + \beta = 0 \quad . \tag{3.3}$$

Hence, by choosing $\alpha = -\frac{2}{3}$, $\beta = -\frac{1}{3}$ in (3.1) and (3.2), we obtain the appropriate local variables for the solution in region B. For $Q(u;\, \nu)$ we construct a formal expansion of the type

$$Q(u;\ \nu) = \sum_{n=0}^{\infty} Q_n(u)\nu^{-2n/3} \quad , \tag{3.4}$$

so that the coefficients satisfy the following recurrent system

$$Q_0 \frac{dQ_0}{du} - 2uQ_0 + 1 = 0 \quad , \tag{3.5}$$

$$Q_0^2 \frac{dQ_1}{du} - Q_1 = u^2 Q_0^2 - uQ_0 \quad , \tag{3.6}$$

$$Q_0^2 \frac{dQ_n}{du} - Q_n = u^2 Q_0 Q_{n-1} - \sum_{k=1}^{n-1} Q_0 Q_k \frac{dQ_{n-k}}{du} \quad . \tag{3.7}$$

Equation (3.5) can be reduced to the Riccati type if one puts $Q_0 = \frac{du}{dz}$ and for the general solution of (3.5) one finally finds

$$Q_0(u) = u^2 - z(u) \quad , \tag{3.8}$$

where $z = z(u)$ is the inverse functions of

$$u = \frac{-\ C_0\ Ai'(z) - \hat{C}_0\ Bi'(z)}{C_0\ Ai(z) + \hat{C}_0\ Bi(z)} \quad . \tag{3.9}$$

($Ai(z)$ and $Bi(z)$ are the so-called Airy functions, $C_0$ and $\hat{C}_0$ are constants). For the solutions of (3.6) and (3.7) one finds

$$Q_1(u) = \frac{1}{A(u)} \{C_1 + \int_0^u A(v)\ (v^2 - \frac{v}{Q_0})\ dv\} \quad , \tag{3.10}$$

where $A(u) = \exp\ (- \int_0^u \frac{dv}{Q_0^2(v)})$ ,

$$Q_n(u) = \frac{1}{A(u)} [C_n + \int_0^u A(v)\ \{\sum_{k=1}^{n-1} Q_k(v)\ Q'_{n-k}(v) - v^2\ Q_{n-1}(v)\} \frac{dv}{Q_0(v)}],$$

$$n = 2, 3, \ldots \tag{3.11}$$

It is still necessary to determine the constants occurring in (3.9) until (3.11).

## 4. Matching of the solutions for regions A and B

In order to obtain matching relations for the coefficients of the expansions (2.1) and (3.4) we introduce the intermediate coordinates by

$$x = -1 - s\nu^{-\mu} \qquad , \; 0 < \mu < \frac{2}{3}. \tag{4.1}$$

Then, after substitution of this new coordinate and after reordening of the terms, (2.1) and (3.4) have to represent identical expansions. In order to avoid unneccessarily complicated formal computations we select one value of $\mu$ and take $\mu = \frac{1}{3}$. Substitution of $x = -1 - s^{-1/3}$ into (2.1) yields the expansion

$$\begin{aligned} p = {} & \frac{1}{2} s^{-1} \nu^{-2/3} + \frac{1}{4} \nu^{-1} - \frac{1}{8} s\nu^{-4/3} + \frac{1}{16} (s^2 - 2s^{-4}) \nu^{-5/3} \\ & - \frac{1}{32} s^3 \nu^{-2} + \frac{1}{64} s\nu^{-7/3} - \frac{1}{128} (s^5 - 20s^{-7}) \nu^{-8/3} + \\ & \frac{1}{256} (s^6 + 2 - 4s^{-6}) \nu^{-3} - \frac{1}{512} (s^7 + 8s + 8s^{-5}) \nu^{-10/3} \\ & + O(\nu^{-11/3}). \end{aligned} \tag{4.2}$$

After substitution of $u = \nu^{1/3}s$ the leading term of expansion (3.4) has to be $O(\nu^{-1/3})$. It follows from (3.8), that this is only possible if $u \to \infty$ as $z \to \infty$. The asymptotic expansions of the Airy functions (see [1], ch. 10) show that $\hat{C}_0$ in (3.9) has to be chosen 0. In fact, we have

$$u = -\frac{Ai'(z)}{Ai(z)} = z^{1/2} \{1 + \frac{1}{4} z^{-3/2} + O(z^{-3})\} \qquad (z \to \infty) \quad .$$

Thus, by (3.8) we may conclude that

$$\begin{aligned} \nu^{-1/3} Q_0(\nu^{1/3}s) = {} & \frac{1}{2} s^{-1} \nu^{-2/3} - \frac{1}{8} s^{-4} \nu^{-5/3} + \frac{5}{32} s^{-7} \nu^{-8/3} \\ & - \frac{11}{32} s^{-10} \nu^{-11/3} + O(\nu^{-14/3}). \end{aligned} \tag{4.3}$$

Estimation of the order of magnitude of the solutions (3.10) and (3.11) for $u = \nu^{1/3}s$ and comparison with (4.2) lead to the following choice for $Q_n(u)$:

$$Q_1(u) = \frac{1}{A(u)} \int_u^\infty A(v) \left\{\frac{v}{Q_0(v)} - v^2\right\} dv \quad ,$$

$$Q_n(u) = \frac{1}{A(u)} \int_u^\infty A(v) \left\{\sum_{k=1}^{n-1} Q_k(v)\, Q'_{n-k}(v) - v^2\, Q_{n-1}(v)\right\} \frac{dv}{Q_0(v)} \quad ,$$

$n = 2, 3, \ldots$ .

By taking the asymptotic expansions of $Q_n(u)$ for large values of u, one obtains

$$\nu^{-1}\, Q_1(\nu^{1/3}s) = \frac{1}{4}\nu^{-1} - \frac{1}{64}\, s^{-6}\, \nu^{-3} + \frac{57}{128}\, s^{-9}\, \nu^{-4} + O(\nu^{-5}) \ , \quad (4.4)$$

$$\nu^{-5/3}\, Q_2(\nu^{1/3}s) = -\frac{1}{8}\, s\nu^{-4/3} - \frac{1}{64}\, s^{-5}\, \nu^{-10/3} + O(\nu^{-13/3}) \ , \quad (4.5)$$

$$\nu^{-7/3}\, Q_3(\nu^{1/3}s) = \frac{1}{16}\, s^2\, \nu^{-5/3} + O(\nu^{-11/3}) \ , \quad (4.6)$$

$$\nu^{-3}\, Q_4(\nu^{1/3}s) = -\frac{1}{32}\, s^3\, \nu^{-2} + \frac{1}{128}\, \nu^{-3} + O(\nu^{-4}) \ , \quad (4.7)$$

$$\nu^{-11/3}\, Q_5(\nu^{1/3}s) = \frac{1}{64}\, s^4\, \nu^{-7/3} - \frac{1}{64}\, s\nu^{-10/3} + O(\nu^{-13/3}) \ , \quad (4.8)$$

$$\nu^{-\frac{2n+1}{3}}\, Q_n(\nu^{1/3}s) = (-1)^{n-1}\, \frac{s^{n-1}}{2^{n+1}}\, \nu^{-\frac{n+2}{3}} + O(\nu^{-\frac{n+3}{3}}) \ , \ n = 6, 7, \ldots \quad (4.9)$$

It is also possible to derive these asymptotic expansions directly from the differential equations (3.5), (3.6) and (3.7).

Comparing expansions (4.2) with the expansion given by (4.3) until (4.9), one concludes that the expansions are identical indeed.

## 5. Solution for region C

For $u \to -\infty$ the first two terms of (3.4) behave as

$$Q_0(u) = u^2 + \alpha + O(u^{-1}) \quad , \quad Q_1(u) = \frac{1}{3}\, u^3 + O(1) \quad (5.1)$$

The constant $\alpha$ denotes the first zero of $Ai(z)$ : $Ai(-\alpha) = 0$, $\alpha = 2.338107$. From equation (1.2) we learn that in the case that p is large ($p = O(\nu)$), the first two terms of (1.2) are dominant. Therefore we suppose that the solution in region C can be expanded as follows

$$p = f_0(x,\nu)\ \nu + \sum_{n=1}^{\infty} f_n(x,\nu)\ \nu^{(1-2n)/3} \quad . \qquad (5.2)$$

It is assumed that for any constant $\varepsilon > 0$ the relations $\lim_{\nu\to\infty} |f_n(x,\nu)|\ \nu^{\varepsilon} = \infty$ and $\lim_{\nu\to\infty} f_n(x,\nu)\ \nu^{-\varepsilon} = 0$ are valid for $n = 0, 1, 2, \ldots$ . For the function $f_n$ we have the recurrent system

$$f_0' = 1 - x^2 \quad , \quad f_1' = 0 \quad , \quad f_0\ f_2' = -x \quad ,$$

$$f_3' = 0 \quad , \quad f_0^2\ f_4' = xf_1 \quad , \quad f_0^2\ f_5' = xf_2 \quad , \qquad (5.3)$$

$$f_0^3\ f_6' + xf_1^2 - xf_0\ f_3 = 0 \quad , \quad f_0^3\ f_7' + 2xf_1\ f_2 + xf_0\ f_4 = 0$$

etc.

Hence,

$$f_0 = a_0(\nu) + x - \frac{1}{3}x^3 \quad , \quad f_1 = a_1(\nu) \quad . \qquad (5.4)$$

In order to avoid unnecessary difficulties, we determine the constants $a_0(\nu)$ and $a_1(\nu)$ immediately by matching with (5.1). We obtain

$$p = \frac{1}{3}(x+1)^2(2-x)\ \nu + \alpha\nu^{-1/3} + O(\nu^{-1}) \quad , \qquad (5.5)$$

so that $a_0(\nu) = \frac{2}{3}$ and $a_1(\nu) = \alpha$. The higher order terms become

$$f_2 = -\frac{1}{x+1} + \frac{2}{3}(\log|2-x| - \log|x+1|) + a_2(\nu) \quad , \qquad (5.6)$$

$$f_3 = a_3(\nu) \quad , \qquad (5.7)$$

$$f_4 = [\frac{5}{27}(\log|x+1| - \log|2-x|) + \frac{2}{9}\frac{1}{2-x} - \frac{1}{3}\frac{1}{x+1} - \frac{1}{6}\frac{1}{(x+1)^2} + \frac{1}{3}\frac{1}{(x+1)^3}]\alpha + a_4(\nu), \qquad (5.8)$$

$$f_5 = \frac{2}{27}\frac{1}{2-x}(2\log|2-x| + 1 - 2\log|x+1| + 3a_2(\nu)) + O(\log^2|2-x|) \quad , \tag{5.9}$$

$$f_6 = -\frac{1}{27}\frac{\alpha^2}{(2-x)^2} + O(2-x)^{-1}) \quad , \tag{5.10}$$

$$f_7 = -\frac{4}{81}\alpha\frac{\log|2-x|}{(2-x)^2} + \frac{\alpha}{81}\frac{2+4\log 3-6a_2(\nu)}{(2-x)^2} + O(\frac{\log|2-x|}{2-x}), \tag{5.11}$$

$$f_8 = \frac{2}{243}\alpha^3\frac{1}{(2-x)^3} + O(\frac{\log|2-x|}{(2-x)^2}) \quad , \tag{5.12}$$

$$f_{10} = -\frac{1}{2.3^6}\frac{\alpha^4}{(2-x)^4} + O(\frac{\log|2-x|}{(2-x)^3}) \quad . \tag{5.13}$$

## 6. Matching of the solutions for regions B and C

The matching relations are derived by substitution of

$$x = -1 + t\nu^{-1/3} \tag{6.1}$$

into the expansion (5.2) and of $u = -t\nu^{1/3}$ into (3.4). The choice of the intermediate coordinate $t$ in (6.1) is again made in order to avoid difficult formal computations. Any transformation of the type $x = -1 + t\nu^{-\mu}$ with $0 < \mu < 2/3$ would work.

We first investigate the functions $Q_k(u)$ for large negative values of $u$. The function $u = -\frac{Ai'(z)}{Ai(z)}$ has a simple pole at $z = -\alpha$ with residue $-1$. Thus, in the neighbourhood of $z = -\alpha$ we may write

$$z = -\alpha - \frac{1}{u} + \ldots \quad .$$

Since $u \to -\infty$ if $z \downarrow -\alpha$ we finally obtain

$$\nu^{-1/3}Q_0(-\nu^{1/3}t) = t^2\nu^{1/3} + \alpha\nu^{-1/3} - t^{-1}\nu^{-2/3} + \frac{1}{3}\alpha t^{-3}\nu^{-4/3} - \frac{1}{4}t^{-4}\nu^{-5/3} - \frac{1}{5}\alpha^2 t^{-5}\nu^{-2} + \frac{7}{18}\alpha t^{-6}\nu^{-7/3} + \frac{1}{7}(\alpha^3 - \frac{5}{4})t^{-7}\nu^{-8/3} + O(\nu^{-3}) \quad , \tag{6.2}$$

$$\nu^{-1}\, Q_1(-\nu^{1/3}t) = -\frac{1}{3}t^3 + \left(b_1 - \frac{2}{9}\log\nu - \frac{2}{3}\log|t|\right)\nu^{-1} - \frac{1}{6}\alpha t^{-2}\,\nu^{-5/3}$$

$$+ \left(\frac{1}{27} + \frac{1}{3}b_1 - \frac{2}{27}\log\nu - \frac{2}{9}\log|t|\right)t^{-3}\,\nu^{-2}$$

$$-\left(\frac{1}{450} + \frac{2}{5}b_1 - \frac{4}{45}\log\nu - \frac{4}{15}\log|t|\right)\alpha t^{-5}\,\nu^{-8/3}$$

$$+ o(\nu^{-8/3}) \quad , \qquad (6.3)$$

$$\nu^{-5/3}\, Q_2(-\nu^{1/3}t) = -\frac{2}{9}t\nu^{-4/3} + b_2\,\nu^{-5/3} - \frac{1}{3}\alpha t^{-1}\,\nu^{-2}$$

$$+ \left(\frac{1}{27}\log\nu + \frac{1}{9}\log|t| + \frac{1}{9} - \frac{1}{6}b_1\right)t^{-2}\,\nu^{-7/3}$$

$$+ \frac{1}{9}\alpha^2\,t^{-3}\,\nu^{-8/3} + o(\nu^{-8/3}) \qquad (6.4)$$

$$\nu^{-7/3}\, Q_3(-\nu^{1/3}t) = -\frac{1}{27}t^2\,\nu^{-5/3} +$$

$$+ \left(\frac{5}{27}\alpha\log|t| + \frac{5}{81}\alpha\log\nu + b_3\right)\nu^{-7/3}$$

$$+ o(\nu^{-7/3}) \quad , \qquad (6.5)$$

$$\nu^{-3}\, Q_4(-\nu^{1/3}t) = -\frac{2}{243}t^3\,\nu^{-2} + o(\nu^{-7/3}) \quad , \qquad (6.6)$$

$$\nu^{-11/3}\, Q_5(-\nu^{1/3}t) = -\frac{1}{486}t^4\,\nu^{-7/3} + o(\nu^{-7/3}). \qquad (6.7)$$

These asymptotic expressions may also be obtained directly from the differential equations for $Q_n(-\nu^{1/3}t)$. The constants are given by

$$\alpha = 2.33810741 \quad ,$$

$$b_1 = \frac{1}{A(-\infty)}\int_{-\infty}^{\infty} A(v)\left\{\frac{v}{Q_0(v)} - \frac{v^3}{3Q_0^2(v)} - \frac{2}{3v} + \frac{2}{3}\frac{\log|v|}{Q_0^2(v)}\right\}dv \ , \qquad (6.8)$$

$$b_2 = \frac{1}{A(-\infty)}\int_{-\infty}^{\infty} A(v)\left\{\frac{Q_1^2(v)}{Q_0^3(v)} - \frac{vQ_1(v)}{Q_0^2(v)} + \frac{2}{9} - \frac{2}{9}\frac{v}{Q_0^2(v)}\right\}dv \ , \qquad (6.9)$$

$$b_3 = \frac{1}{A(-\infty)} \int_{-\infty}^{\infty} A(v) \{ \frac{2Q_1 Q_2}{Q_0^3} - \frac{vQ_2}{Q_0^2} - \frac{Q_1^3}{Q_0^4} + \frac{vQ_1^2}{Q_0^3} + \frac{1}{27} \frac{v^2}{Q_0^2} - \frac{2}{27} v$$

$$+ \frac{5}{27} \frac{\alpha}{v} - \frac{5}{27} \frac{\alpha \log|v|}{Q_0^2} \} dv \quad . \tag{6.10}$$

Substitution of (6.1) into expansions (5.2) yields

$$\nu f_0(x,\nu) = t^2 \nu^{1/3} - \frac{1}{3} t^3 \quad , \tag{6.11}$$

$$\nu^{-1/3} f_1(x,\nu) = \alpha \nu^{-1/3} \quad , \tag{6.12}$$

$$\nu^{-1} f_2(x,\nu) = - t^{-1} \nu^{-2/3} + (-\frac{2}{3} \log|t| + \frac{2}{9} \log \nu + \frac{2}{3} \log 3 + a_2(\nu)) \nu^{-1}$$

$$- \frac{2}{9} t\nu^{-4/3} - \frac{1}{27} t^2 \nu^{-5/3} - \frac{2}{243} t^3 \nu^{-2} - \frac{1}{486} t^4 \nu^{-7/3}$$

$$+ o(\nu^{-7/3}) \quad , \tag{6.13}$$

$$\nu^{-5/3} f_3(x,\nu) = a_3(\nu) \nu^{-5/3} \quad , \tag{6.14}$$

$$\nu^{-7/3} f_4(x,\nu) = \frac{\alpha}{3} t^{-3} \nu^{-4/3} - \frac{\alpha}{6} t^{-2} \nu^{-5/3} - \frac{\alpha}{3} t^{-1} \nu^{-2}$$

$$+ (\frac{5}{27} \alpha \log|t| - \frac{5}{81} \alpha \log \nu - \frac{5}{27} \alpha \log 3 + \frac{2}{27} \alpha$$

$$+ a_4(\nu)) \nu^{-7/3} + o(\nu^{-7/3}) \quad , \tag{6.15}$$

$$\nu^{-3} f_5(x,\nu) = - \frac{1}{4} t^{-4} \nu^{-5/3} + (\frac{2}{9} \log 3 + \frac{1}{3} a_2(\nu) + \frac{1}{27} - \frac{2}{9} \log|t|$$

$$+ \frac{2}{27} \log \nu) t^{-3} \nu^{-2} + o(\nu^{-2}) \quad , \tag{6.16}$$

$$\nu^{-11/3} f_6(x,\nu) = \frac{1}{5} t^{-5} \nu^{-2} + o(\nu^{-2}) \quad . \tag{6.17}$$

We note that all terms depending on t are identical in both expansions. By equating the constant terms we obtain the matching relations

$$a_2(\nu) = b_1 - \frac{4}{9} \log \nu - \frac{2}{3} \log 3 \quad , \tag{6.18}$$

$$a_3(\nu) = b_2 \quad ,$$

$$a_4(\nu) = b_3 + \frac{10}{81} \alpha \log \nu + \frac{5}{27} \alpha \log 3 - \frac{2}{27} \alpha \quad . \tag{6.19}$$

## 7. Solution for region D

The expansion (5.2) is no longer valid if x is in the neighbourhood of 2. Therefore, we consider region D, where $p = 0(\nu^{-1})$ and, because of (5.5), $x = 2 + \frac{1}{3} \alpha\nu^{-4/3} + 0(\nu^{-2})$. Since p(x) is a double-valued function in this region, we consider the inverse function x(p). Furthermore, the local variable q is defined by

$$p = q\nu^{-1} - \frac{2}{3} \nu^{-1} + \frac{5}{27} \alpha\nu^{-7/3} \quad , \tag{7.1}$$

so that equation (1.2) transforms into

$$\{(q - \frac{2}{3} + \frac{5}{27} \alpha\nu^{-4/3}) (x^2 - 1) + x\} \frac{dx}{dq} = \nu^{-2}q - \frac{2}{3} \nu^{-2} + \frac{5}{27} \alpha\nu^{-10/3} \quad . \tag{7.2}$$

We suppose that there exists an expansion of the form

$$x = 2 + \frac{1}{3} \alpha\nu^{-4/3} + \sum_{n=2}^{\infty} X_n (q,\nu) \nu^{-2(n+1)/3} \quad . \tag{7.3}$$

It is assumed that for any $\varepsilon > 0$ the functions $X_n(q,\nu)$ , n = 2, 3, ... satisfy the relations $\lim_{\nu\to\infty} |X_n(q,\nu)| \nu^{\varepsilon} = \infty$ and $\lim_{\nu\to\infty} X_n(q,\nu) \nu^{-\varepsilon} = 0$. For $X_n(q,\nu)$ we find the following equations

$$- 3q \frac{dX_2}{dq} = q - \frac{3}{2} \quad , \tag{7.4}$$

$$\frac{dX_3}{dq} = 0 \quad , \tag{7.5}$$

$$- 3 q \frac{dX_4}{dq} - \frac{4}{3} \alpha q \frac{dX_2}{dq} = \frac{5}{27} \alpha \quad , \tag{7.6}$$

$$- 3 q \frac{dX_5}{dq} - (4 q - \frac{5}{3}) X_2 \frac{dX_2}{dq} = 0 \quad . \tag{7.7}$$

etc.

The solutions are

$$X_2(q,\nu) = -\frac{1}{3}q + \frac{2}{q}\log|q| + d_2(\nu) \quad , \tag{7.8}$$

$$X_3(q,\nu) = d_3(\nu) \quad , \tag{7.9}$$

$$X_4(q,\nu) = \frac{4}{27}\alpha q - \frac{13}{81}\alpha\log|q| + d_4(\nu) \quad , \tag{7.10}$$

$$\begin{aligned} X_5(q,\nu) = &-\frac{2}{27}q^2 + \left(\frac{5}{81} + \frac{8}{81}\log|q| + \frac{4}{9}d_2(\nu)\right)q + \\ &\left(-\frac{13}{27}d_2(\nu)\log|q| - \frac{10}{243}\log|q| - \frac{13}{243}\log^2|q| + d_5(\nu)\right) + \\ &\left(-\frac{20}{729}\log|q| - \frac{20}{729} - \frac{10}{81}d_2(\nu)\right)\frac{1}{q} \quad , \end{aligned} \tag{7.11}$$

Moreover, we have

$$X_6(q,\nu) = -\frac{7}{27}\alpha^2 q + \frac{4}{27}b_2 q + \dots \quad , \tag{7.12}$$

$$X_7(q,\nu) = \frac{7}{81}\alpha q^2 + \dots \quad , \tag{7.13}$$

$$X_8(q,\nu) = -\frac{7}{243}q^3 + \dots \quad . \tag{7.14}$$

## 8. Matching of the solutions for regions C and D

In order to derive matching relations we substitute in (5.2) an intermediate coordinate $X$ given by

$$x = 2 - X\nu^{-1}. \tag{8.1}$$

Reordening the terms we obtain

$$\begin{aligned} p = \; &3X + \alpha\nu^{-1/3} + \left(-2X^2 - \frac{1}{3} + \frac{2}{3}\log|X| - \frac{10}{9}\log\nu - \frac{4}{3}\log 3 + b_1\right)\nu^{-1} \\ &+ \frac{2}{9}\alpha X^{-1}\nu^{-4/3} + \left(b_2 - \frac{1}{27}\alpha^2 X^{-2}\right)\nu^{-5/3} + \left(\frac{1}{3}X^3 + \frac{1}{9}X + \frac{4}{27}X^{-1}\log|X| \right. \\ &\left. - \frac{20}{81}X^{-1}\log\nu - \frac{8}{27}X^{-1}\log 3 + \frac{2}{9}b_1 X^{-1} + \frac{2}{27}X^{-1} + \frac{2}{243}\frac{\alpha^3}{X^3}\right)\nu^{-2} \\ &+ \left(b_3 + \frac{10}{27}\alpha\log 3 + \frac{25}{81}\alpha\log\nu - \frac{31}{162}\alpha - \frac{4}{81}\alpha\frac{\log|X|}{X^2} + \frac{20}{243}\frac{\alpha\log\nu}{X^2} \right. \\ &\left. - \frac{2}{27}\frac{\alpha b_1}{X^2} + \frac{8}{81}\frac{\alpha\log 3}{X^2} + \frac{2\alpha}{81X^2} - \frac{1}{2.3^6}\frac{\alpha^4}{X^4}\right)\nu^{-7/3} + o(\nu^{7/3}) \; . \end{aligned} \tag{8.2}$$

Substitution of $q = p\nu + \frac{2}{3} - \frac{5}{27}\alpha\nu^{-4/3}$ into (7.13) yields

$$\chi = \frac{1}{3}p - \frac{1}{3}\alpha\nu^{-1/3} + \{\frac{2}{9} - \frac{2}{9}\log|p| - \frac{2}{9}\log\nu + \frac{2}{27}p^2 - d_2(\nu)\}\,\nu^{-1}$$
$$- \frac{4}{27}\alpha p\nu^{-4/3} - d_3(\nu)\,\nu^{-5/3} + \{-\frac{4}{27p} + \frac{1}{27}p - \frac{8}{81}p\log|p|$$
$$- \frac{8}{81}p\log\nu - \frac{4}{9}p\,d_2(\nu) + \frac{7}{243}p^3\}\,\nu^{-2} + \{-\frac{5}{81}\alpha - \frac{8}{81}\alpha$$
$$+ \frac{13}{81}\alpha\log|p| + \frac{13}{81}\alpha\log\nu - d_4(\nu) - \frac{7}{81}\alpha p^2\}\,\nu^{-7/3} + o(\nu^{-7/3}). \tag{8.3}$$

The expansions (8.2) and (8.3) have to represent the same intermediate solution, so that substitution of (8.2) into (8.3) must lead to an identity. Working out this identity we obtain the following matching relations

$$d_2(\nu) = \frac{1}{9} - \frac{16}{27}\log\nu - \frac{2}{3}\log 3 + \frac{b_1}{3}\quad, \tag{8.4}$$

$$d_3(\nu) = \frac{b_2}{3} - \frac{2}{27}\alpha^2\quad, \tag{8.5}$$

$$d_4(\nu) = \frac{b_3}{3} + \frac{13}{27}\alpha\log 3 + \frac{104}{243}\alpha\log\nu - \frac{139}{486}\alpha - \frac{4}{27}\alpha b_1\quad. \tag{8.6}$$

## 9. Solution for region E

Expansion (7.3) is singular in $q = 0$. In order to reveal the behaviour of the solution near this singularity we introduce the local coordinate $\xi$ by

$$x = x_s + \xi\nu^{-2}\quad. \tag{9.1}$$

The coordinate $x = x_s$ is defined in such a way that the exact solution there takes the value

$$p(x_s) = -\frac{2}{3}\nu^{-1} + \frac{5}{27}\alpha\nu^{-7/3}\ .$$

If we suppose that the following expansions exist

$$p = -\frac{2}{3}\nu^{-1} + \frac{5}{27}\alpha\nu^{-7/3} + \sum_{n=2}^{\infty}\eta_n(\xi,\nu)\,\nu^{-(5+2n)/3}\quad, \tag{9.2}$$

$$x_s = 2 + \frac{\alpha}{3}\nu^{-4/3} + \sum_{n=2}^{\infty}x_n(\nu)\,\nu^{-2(n+1)/3}\quad, \tag{9.3}$$

then by substituting (9.1) and (9.2) into (1.2) we obtain the recurrent system

$$\frac{d\eta_2}{d\xi} - \frac{9}{2}\eta_2 + \frac{5}{2}(x_2(\nu) + \xi) = 0 \quad , \tag{9.4}$$

$$\frac{d\eta_3}{d\xi} - \frac{9}{2}\eta_3 + \frac{5}{2}\ x_3(\nu) - \frac{7}{27}\alpha^2 = 0 \quad , \tag{9.5}$$

$$\frac{d\eta_4}{d\xi} - \frac{9}{2}\eta_4 + \frac{5}{2}x_4(\nu) - \frac{4}{9}\alpha(x_2(\nu) + \xi) - \frac{5}{18}\alpha\frac{d\eta_2}{d\xi} - 2\ \alpha\eta_2 = 0 \ , \tag{9.6}$$

$$\frac{d\eta_5}{d\xi} - \frac{9}{2}\eta_5 + \frac{5}{2}\ x_5(\nu) - \frac{4}{9}x_3(\nu) - \frac{5}{18}\alpha\frac{d\eta_3}{d\xi} - 2\ \alpha\eta_3 - \frac{3}{2}\eta_2\frac{d\eta_2}{d\xi}$$
$$- 6\eta_2(x_2(\nu) + \xi) + (x_2(\nu) + \xi)^2 - \frac{5}{162}\alpha^3 = 0 \quad , \tag{9.7}$$

with the condition $\eta_k(0,\nu) = 0$ , $k = 2, 3, \ldots$, since $x = x_s$ was chosen such that $p = -\frac{2}{3}\nu^{-1} + \frac{5}{27}\alpha\nu^{-7/3}$ exactly. The solutions are

$$\eta_2 = \frac{5}{9}\xi + (\frac{10}{81} + \frac{5}{9}x_2(\nu))\ (1 - e^{9\xi/2}) \quad , \tag{9.8}$$

$$\eta_3 = (\frac{5}{9}x_3(\nu) - \frac{14}{243}\alpha^2)\ (1 - e^{9\xi/2}) \quad , \tag{9.9}$$

$$\eta_4 = (\frac{5}{9}x_4(\nu) - \frac{28}{81}\ \alpha x_2(\nu) - \frac{121}{729}\alpha)\ (1 - e^{9\xi/2}) - \frac{28}{81}\alpha\xi$$
$$- (\frac{65}{162}\alpha + \frac{65}{36}\alpha\ x_2(\nu))\ \xi e^{9\xi/2} \quad , \tag{9.10}$$

$$\eta_5 = \frac{3}{2}(\frac{10}{81} + \frac{5}{9}x_2(\nu))^2\ e^{9\xi} + \ldots \quad . \tag{9.11}$$

## 10. Matching of the solutions for regions D and E

In region D we have $q = O(1)$, whereas in region E $q = O(\nu^{-2})$. Therefore, we introduce an intermediate coordinate $\hat{q} = q\nu$ in order to get the matching relations. Substitution in expansion (7.3) yields

$$x = 2 + \frac{1}{3}\alpha\nu^{-4/3} + (\frac{2}{9}\log|\hat{q}| - \frac{2}{9}\log\nu + d_2(\nu))\nu^{-2} + d_3(\nu)\ \nu^{-8/3}$$
$$+ \{-\frac{1}{3}\hat{q} + (-\frac{20}{729}\log|\hat{q}| + \frac{20}{729}\log\nu - \frac{20}{729} - \frac{10}{81}d_2(\nu))\frac{1}{\hat{q}}\}\ \nu^{-3}$$
$$+ \{-\frac{13}{81}\alpha\log|\hat{q}| + \frac{13}{81}\alpha\log\nu + d_4(\nu)\}\ \nu^{-10/3} + o(\nu^{-10/3}) \quad . \tag{10.1}$$

In order to obtain an intermediate coordinate $\hat{\xi}$ such that the term $\eta_2(\xi,\nu)$ in expansion (9.2) becomes of order $O(\nu)$, which is necessary for having $p + \frac{2}{3}\nu^{-1} = O(\nu^{-2})$, we use the following transformation

$$\xi = \hat{\xi} + \frac{2}{9}\log\nu \quad . \tag{10.2}$$

In the intermediate region expansion (9.2) transforms into

$$\begin{aligned}
\hat{q} &= \eta_2(\xi,\nu)\,\nu^{-1} + \eta_3(\xi,\nu)\,\nu^{-5/3} + \eta_4(\xi,\nu)\,\nu^{-7/3} + \eta_5(\xi,\nu)\,\nu^{-3} + \ldots \\
&= \left(-\frac{10}{81} - \frac{5}{9}x_2(\nu)\right) e^{9\hat{\xi}/2} + \left(-\frac{5}{9}x_3(\nu) + \frac{14}{243}\alpha^2\right) e^{9\hat{\xi}/2}\,\nu^{-2/3} \\
&\quad + \left\{\frac{5}{9}\,\hat{\xi} + \frac{10}{81}\log\nu + \frac{10}{81} + \frac{5}{9}x_2(\nu) + \frac{3}{2}\left(\frac{10}{81} + \frac{5}{9}x_2(\nu)\right)^2 e^{9\hat{\xi}}\right\}\nu^{-1} \\
&\quad + \left\{-\frac{5}{9}x_4(\nu) + \frac{28}{81}\alpha x_2(\nu) + \frac{121}{729}\alpha - \frac{65}{729}\alpha\log\nu - \frac{65}{162}\alpha x_2(\nu)\log\nu \right. \\
&\quad \left. - \frac{65}{162}\alpha\hat{\xi} - \frac{65}{36}\alpha x_2(\nu)\,\hat{\xi}\right\} e^{9\hat{\xi}/2}\,\nu^{-4/3} + \left(\frac{5}{9}x_3(\nu) - \frac{14}{243}\alpha^2\right)\nu^{-5/3} \\
&\quad + o(\nu^{-5/3}) \,.
\end{aligned} \tag{10.3}$$

Inserting (10.3) into (10.1) we obtain an identity, when (9.1), (9.3) and (10.2) are used. Consequently we obtain the matching relations

$$x_2(\nu) = -\frac{28}{27}\log\nu + \frac{1}{9} - \frac{2}{3}\log 3 + \frac{1}{3}b_1 + \frac{2}{9}\log\left|\frac{10}{81} + \frac{5}{9}x_2(\nu)\right| \,, \tag{10.4}$$

$$x_3(\nu) = \frac{b_2}{3} - \frac{2}{27}\alpha^2 + \frac{2}{135}\,\frac{135\,x_3(\nu) - 14\alpha^2}{2 + 9\,x_2(\nu)} \quad , \tag{10.5}$$

$$\begin{aligned}
x_4(\nu) &= -\frac{13}{81}\alpha\log\left|\frac{10}{81} + \frac{5}{9}x_2(\nu)\right| - \frac{1}{9}\left(\frac{135\,x_3(\nu) - 14\,\alpha^2}{15(2 + 9x_2(\nu))}\right)^2 + \\
&\quad \frac{2(405\,x_4(\nu) - 252\,\alpha x_2(\nu) - 121)}{81(10 + 45\,x_2(\nu))} + \frac{b_3}{3} + \frac{13}{27}\alpha\log 3 - \frac{139}{486}\alpha \\
&\quad - \frac{4}{27}\alpha b_1 + \frac{108}{243}\alpha\log\nu \,.
\end{aligned} \tag{10.6}$$

From these relations $x_2(\nu)$ , $x_3(\nu)$ and $x_4(\nu)$ may be computed.

## 11. Matching of the solutions for regions E and $\bar{A}$

In expansion (2.1) we substitute

$$x = 2 + \frac{1}{3}\alpha\nu^{-4/3} + (x_2(\nu) + \xi)\,\nu^{-2} + x_3(\nu)\,\nu^{-8/3} + x_4(\nu)\,\nu^{-10/3} + \dots \tag{11.1}$$

and we obtain

$$p = -\frac{2}{3}\nu^{-1} + \frac{5}{27}\alpha\nu^{-4/3} + \{\frac{5}{9}(x_2(\nu) + \xi) + \frac{10}{81}\}\,\nu^{-3} + \\ + \{\frac{5}{9}x_3(\nu) - \frac{14}{243}\alpha^2\}\,\nu^{-11/3} + \{\frac{5}{9}x_4(\nu) - \frac{28}{81}\alpha(x_2(\nu) + \xi) - \\ - \frac{121}{729}\alpha\}\,\nu^{-13/3} + \dots \quad , \tag{11.2}$$

which agrees with (9.2) for $\xi << -1$. In this case there are no matching relations: the solutions fit exactly.

## 12. The amplitude

In order to determine the amplitude $a_\nu$ of the periodic solution we have to insert

$$q = \frac{2}{3} - \frac{5}{27}\alpha\nu^{-4/3} \tag{12.1}$$

into expansion (7.3). The constants $d_2(\nu)$, $d_3(\nu)$ and $d_4(\nu)$ which occur in this expansion are computed in (8.4), (8.5) and (8.6). We obtain

$$a_\nu = 2 + \frac{1}{3}\alpha\nu^{-4/3} + (\frac{1}{3}b_1 - \frac{16}{27}\log\nu - \frac{1}{9} + \frac{2}{9}\log 2 - \frac{8}{9}\log 3)\,\nu^{-2} \\ + (\frac{1}{3}b_2 - \frac{2}{27}\alpha^2)\,\nu^{-8/3} + (\frac{1}{3}b_3 + \frac{104}{243}\alpha\log\nu - \frac{4}{27}\alpha b_1 - \frac{91}{486}\alpha + \\ + \frac{52}{81}\alpha\log 3 - \frac{13}{81}\alpha\log 2)\,\nu^{-10/3} + o(\nu^{-10/3}) \quad . \tag{12.2}$$

In formula (12.2) $\alpha = 2.33810741$ and the constants $b_1$, $b_2$ and $b_3$ are given by (6.8), (6.9) and (6.10). Numerical values for $b_1$, $b_2$ and $b_3$ will be given in a subsequent paper.

## 13. The period

The computation of the asymptotic expression for the period cannot be reduced. Therefore, we refer to Dorodnicyn's paper [3] or Bavinck and Grasman [2] , where all the details have been worked out. Integration of

$$T = 2 \int_{-a_\nu}^{a_\nu} \frac{dx}{p(x)} \tag{13.1}$$

over the five regions yields

$$\begin{aligned} T = {} & (3 - 2 \log 2)\, \nu + 3\, \alpha \nu^{-1/3} - 2\, \nu^{-1} \log \nu + \\ & + (\log 2 - \log 3 + 3\, b_1 - 1 - \log \pi - 2 \log Ai'(-\alpha))\, \nu^{-1} \\ & + o(\nu^{-1}) \quad . \end{aligned} \tag{13.2}$$

It should be noticed that the period computed in [3] and [9] contains computational errors. We remark that in Dorodnicyn's paper the integration (13.1) has been carried out over intervals in different regions, which are separated by concrete points. It can be shown that one may take arbitrary points in the regions of overlapping for the points of separation of the different intervals of integration. In fact, it turns out that in the final summing up of the contributions from the five regions the coordinates of these points cancel. For the Volterra-Lotka type of relaxation oscillations, where we have the same situation, this can be verified rather directly (see [5] ).

References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, Dover (1965).

[2] H. Bavinck and J. Grasman, Relaxatietrillingen, MC Syllabus 8 Mathematisch Centrum, Amsterdam (1969)

[3] A.A. Dorodnicyn, Asymptotic Solution of the Van der Pol Equation, Prikl. Mat. Mekh. vol. 11 (1947), 313-328 (russian), Am. Math. Soc. Transl. ser. 1 vol. 4 (1962), 1-23.

[4] M. Van Dyke, Perturbation Methods in Fluid Mechanics Acad. Press (1964).

[5] J. Grasman and E. Veling, An Asymptotic Formula for the Period of a Volterra - Lotka System, Mathematisch Centrum TN 70/72.

[6] J. LaSalle, Relaxation Oscillations, Quart. J. Appl. Math. vol.7 (1949), 1-19.

[7] B. van der Pol, On Relaxation Oscillations, Phil. Mag. vol. 2 (1926), 978-992.

[8] P.J. Ponzo and N. Wax, On Certain Relaxation Oscillations : Asymptotic Solutions, J. Soc. Indust. Appl. Math. vol. 13 (1965), 740-766.

[9] M. Urabe, Remarks on Periodic Solutions of Van der Pol's Equation, J. Sci. Hiroshima Univ. vol. A24 (1960), 197-199.

[10] J.A. Zonneveld, Periodic Solutions of the Van der Pol Equation, Nederl. Akad. Wetensch. Proc. Ser. A69 = Indag. Math. 28 (1969), 620-622.