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The complex Monge-Ampère operator in pluripotential theory

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Introduction

This manuscript is a draft version of lecture notes of a course I gave at the Jagiellonian University in the academic year 1997/98. The main goal was to present the fundamental results of the pluripotential theory like Josefson's theorem on the equivalence between locally and globally pluripolar sets and Bedford-Taylor's theorem stating that negligible sets are pluripolar. The main tool is the theory of the complex Monge-Ampère operator developped by Bedford and Taylor in the 70's and 80's. Relying on their solution of the Dirichlet problem in [BT1] they wrote a breakthrough paper [BT2] which is where the most important results of these notes come from. Some of them appeared a little earlier in [Bed1] and [Bed2].

The main inspiration in writing these notes were Demailly's excellent articles [Dem2] and [Dem3]. I was fortunate to start learning the subject during my student times from a Demailly preprint survey which was later expanded into two parts [Dem2] and [Dem3]. Unfortunately, [Dem3] has never been published! One of Demailly's contributions was a major simplification of the solution of the Dirichlet problem for the homogeneous Monge-Ampère equation from [BT1].

The material presented in the first three chapters almost coincides with the core of Klimek's monograph [Kli2]. However, many proofs have been simplified. The reader is assumed to be familiar with the basic concepts of measure theory, calculus of differential forms, general topology, functional analysis and the theory of holomorphic functions of several variables. (One should mention that we make use of the solution of the Levi problem only in the proof of Theorem 1.4.8 which is later used to prove Theorem 3.2.4.)

In Chapter I we present a self-contained exposition of distributions, subharmonic and plurisubharmonic functions, regular domains in $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as the theory of nonnegative forms and currents. The exposition is by no means complete, we only concentrate on results that will be used in the next chapters. The presentation of distribution theory in Section 1.1 is mostly extracted from [Hör2]. The main result, Theorem 1.1.11, is a weak version of the Sobolev theorem and states that functions with locally bounded partial derivatives are Lipschitz continuous. This is later needed in the solution of the Dirichlet problem for the complex Monge-Ampère operator. In section 1.2 we prove basic facts concerning subharmonic functions and regular domains in $\mathbb{R}^n$. The most important one for us is due to Bouligand and asserts that a boundary point admitting a weak subharmonic barrier is regular (Theorem 1.2.8). This is one of rather few results from potential theory in $\mathbb{R}^m$, $m \geq 3$, that we will use in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. One of the sources when writing Section 1.2 was Wermer's exposition [Wer]. In Section 1.3 we collect results on nonnegative currents, the results coming mostly from Lelong's exposition [Lel] and [HK]. In Section 1.4 we prove the basic properties of plurisubharmonic functions. Then we exploit the concept of the Perron-Bremermann envelopes and a notion of a maximal plurisubharmonic function. We characterize bounded domains in $\mathbb{C}^n$ admitting strong and weak plurisubharmonic barriers (resp. B-regular and hyperconvex domains).

In Section 2.1 we define the complex Monge-Ampère operator for locally bounded plurisubharmonic functions and show the basic estimates as well as the continuity of the operator with respect to decreasing sequences. The principal result of Section 2.2 is the quasi-continuity of plurisubharmonic functions with respect to the relative capacity. Several applications are then derived, including the continuity of the complex Monge-Ampère operator with respect to increasing sequences, and the domination principle. The Dirichlet problem is solved in Section 2.3.

In Chapter III we prove Josefson's theorem and Bedford-Taylor's theorem on negligible sets and then use them to present the theory of three kinds of extremal plurisubharmonic functions: relative, global and the pluricomplex Green function. It should be pointed out that results like Propositions 3.1.3, 3.2.1, 3.3.1, 3.3.2, 3.3.3, Theorems 3.2.4, 3.2.5, 3.3.4 and parts of Propositions 3.1.9, 3.2.6, Theorems 3.2.3, 3.2.9 are proven in an elementary way, that is without using Chapter II.

We have included a number of exercises. If they are double-boxed, then it means that the result will be used later on.

In the appendix we collect some elementary facts. We have also included a list of open problems. Most of them arose while preparing the course but they are certainly not meant to be the most important questions or to represent the current development of the theory.

One more chapter on applications of pluripotential theory in complex and non-complex analysis is planned in some future...

The course which these notes are based on was given at the Jagiellonian University while I had a special research position at the Mathematical Institute of the Polish Academy of Sciences and thus no other teaching duties. This introduction was written during my stay at the Mid Sweden University in Sundsvall. I would also like to thank U. Cegrell, A. Edigarian, S. Kołodziej, P.Pflug, E. Poletsky, J. Siciak and W. Zwonek for consultations on some problems related to the present subject.

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I. Preliminaries

1.1. Distributions and the Laplacian

Let Ω be an open set in $\mathbb{R}^n$. On a vector space $C_0^k(\Omega)$, $k = 0, 1, 2, \ldots, \infty$, we introduce a topology as follows: a sequence $\{\varphi_j\}$ is convergent to φ iff

i) there exists $K \Subset \Omega$ such that $\operatorname{supp}\varphi_j \subset K$ for all j;

ii) for all multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$ we have $D^\alpha\varphi_j \longrightarrow D^\alpha\varphi$ uniformly (if $k = \infty$ then for all α).

If $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ with this topology then by $\mathcal{D}'(\Omega)$ we denote the set of all (complex) continuous linear functionals on $\mathcal{D}(\Omega)$ and call them *distributions* on Ω. We say that $u \in \mathcal{D}'(\Omega)$ is a *distribution of order* k if it can be continuously extended to $C_0^k(\Omega)$.

Theorem 1.1.1. *Let u be a linear functional on $C_0^\infty(\Omega)$. Then u is a distribution iff for all $K \Subset \Omega$ there is k and a positive constant C such that*

$$|u(\varphi)| \le C \sum_{|\alpha|\le k} \|D^\alpha\varphi\|_K\,, \quad \varphi \in C_0^\infty(\Omega),\ \operatorname{supp}\varphi \subset K. \tag{1.1.1}$$

u is a distribution of order k iff (1.1.1) holds with this k for all $K \Subset \Omega$.

Proof. That (1.1.1) implies that u is a distribution is obvious. Assume that $u \in \mathcal{D}'(\Omega)$ and that (1.1.1) does not hold for some $K \Subset \Omega$. Then for every natural j there exists $\varphi_j \in C_0^\infty(K)$ with

$$|u(\varphi_j)| > j \sum_{|\alpha|\le j} \|D^\alpha\varphi_j\|_K$$

and $u(\varphi_j) = 1$. Then for any α and $j \ge |\alpha|$ we have $|D^\alpha\varphi_j| < 1/j$. Thus $\varphi \longrightarrow 0$ in $C_0^\infty(\Omega)$ which is a contradiction. Similarly we prove the second part of the theorem. ∎

If Ω' is an open subset of Ω and $u \in \mathcal{D}'(\Omega)$ then the restriction $u|_{\Omega'}$ is well defined by the natural inclusion $C_0^\infty(\Omega') \subset C_0^\infty(\Omega)$. Being a distribution is a local property as the following result shows:

Theorem 1.1.2. *Let $\{\Omega_j\}$ be a an open covering of Ω. If $u, v \in \mathcal{D}'(\Omega)$ are such that $u = v$ on Ω_j for every j, then $u = v$ in Ω. On the other hand, if $u_j \in \mathcal{D}'(\Omega_j)$ are such that $u_j = u_k$ on $\Omega_j \cap \Omega_k$ then there exists a unique $u \in \mathcal{D}'(\Omega)$ such that $u = u_j$ on Ω_j.*

Proof. Partition of unity gives $\varphi_j \in C_0^\infty(\Omega)$ such that $\operatorname{supp}\varphi_j \Subset \Omega_j$, the family $\{\operatorname{supp}\varphi_j\}$ is locally finite and $\sum_j \varphi_j = 1$ in Ω. For $u \in \mathcal{D}'(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ we have

$$u(\varphi) = u\Big(\sum_j \varphi_j \varphi\Big) = \sum_j u\left(\varphi_j \varphi\right)$$

since the sum is in fact finite. This proves the first part. If $u_j \in \mathcal{D}'(\Omega_j)$ and $\varphi \in C_0^\infty(\Omega)$ then we set

$$u(\varphi) = \sum_j u_j\left(\varphi_j \varphi\right).$$

One can easily show that $u \in \mathcal{D}'(\Omega)$. Then for $\varphi \in C_0^\infty(\Omega_k)$ we have $\varphi_j\varphi \in C_0^\infty(\Omega_j \cap \Omega_k)$ and

$$u(\varphi) = \sum_j u_j\left(\varphi_j \varphi\right) = \sum_j u_k\left(\varphi_j \varphi\right) = u_k\Big(\sum_j \varphi_j \varphi\Big) = u_k(\varphi).$$

The uniqueness follows from the first part. ∎

By the Riesz representation theorem (see [Rud, Theorem 6.19]), distributions of order 0 can be identified with regular complex measures by the following

$$u(\varphi) = \int_\Omega \varphi d\mu, \quad \varphi \in C_0(\Omega).$$

Similarly, nonnegative distributions (that is $u(\varphi) \geq 0$ whenever $\varphi \geq 0$) are in fact nonnegative Radon measures (see [Rud, Theorem 2.14]). (Here it is even enough to assume that u is just a nonnegative linear functional on $C_0^\infty(\Omega)$.)

A function $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution of order 0:

$$u_f(\varphi) := \int_\Omega f\varphi d\lambda, \quad \varphi \in C_0(\Omega).$$

Here λ denotes the Lebesgue measure in $\mathbb{R}^n$.

Partial derivatives of a distribution are defined as follows:

$$(D_j u)(\varphi) := -u(D_j\varphi), \quad j = 1, \dots, n, \quad \varphi \in \mathcal{D}(\Omega).$$

Then $D_j u \in \mathcal{D}'(\Omega)$. Integration by parts gives $D_j u_f = u_{D_j f}$ for $f \in C^1(\Omega)$ and thus the differentiation of a distribution is a generalization of a classical differentiation.

If $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$ then we define the product

$$(fu)(\varphi) := u(f\varphi), \quad \varphi \in C_0^\infty(\Omega).$$

Then $fu \in \mathcal{D}'(\Omega)$. The same definition applies if u is a distribution of order k and $f \in C^k(\Omega)$ - then fu is a distribution of order k. One can easily show that

$$D_j(fu) = fD_ju + D_jf\,u.$$

Let $\Delta = \sum_j D_j^2$ be the Laplace operator.

Proposition 1.1.3. *If Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary and $u, v \in C^2(\overline{\Omega})$ then*

$$\int_\Omega (u\Delta v - v\Delta u)\, d\lambda = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma.$$

Proof. The Stokes theorem gives

$$\int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma = \int_{\partial\Omega} \langle u\nabla v - v\nabla u, n\rangle\, d\sigma$$

$$= \int_\Omega \operatorname{div}(u\nabla v - v\nabla u)\, d\lambda = \int_\Omega (u\Delta v - v\Delta u)\ d\lambda. \ \blacksquare$$

For a function $u(x) = f(|x|)$, where f is smooth, we have $D_ju(x) = x_jf'(|x|)/|x|$ and $\Delta u(x) = f''(|x|) + (n-1)f'(|x|)/|x|$. The solutions of the equation $y' + (n-1)y/t = 0$ are of the form $y(t) = Ct^{1-n}$, where C is a constant. For $t \geq 0$ define

$$\widehat{E}(t) := \begin{cases} \frac{1}{2\pi}\log t, & \text{if } n = 2 \\ -\frac{1}{(n-2)c_n}t^{2-n}, & \text{if } n \neq 2, \end{cases}$$

where c_n is the area of the unit sphere in $\mathbb{R}^n$. Note that for $t > 0$ $\widehat{E}'(t) = 1/\sigma(\partial B_t)$, where $B_t = B(0,t)$.

Theorem 1.1.4. *Set $E(x) := \widehat{E}(|x|)$. Then $E \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $D_jE = x_j|x|^{-n}/c_n \in L^1_{\text{loc}}(\mathbb{R}^n)$ as a distribution, $j = 1, \dots, n$. E is the fundamental solution for the Laplacian, that is $\Delta E = \delta_0$, where*

$$\delta_0(\varphi) = \varphi(0), \quad \varphi \in C_0(\mathbb{R}^n)$$

is called the Dirac delta.

Proof. On $\mathbb{R}^n \setminus \{0\}$ E is smooth and we have $D_jE = x_j|x|^{-n}/c_n$ and $\Delta E = 0$ there. For $\varphi \in C_0^\infty(B_R)$ integration by parts gives

$$D_jE(\varphi) = -E(D_j\varphi) = -\lim_{\varepsilon\to 0}\int_{B_R\setminus B_\varepsilon} E\, D_j\varphi d\lambda$$

$$= \lim_{\varepsilon\to 0}\left(\int_{B_R\setminus B_\varepsilon} \varphi(x)x_j|x|^{-n}/c_n d\lambda(x) + \int_{\partial B_\varepsilon} E\,\varphi\, n_j d\sigma\right)$$

$$= \int_{B_R} \varphi(x)x_j|x|^{-n}/c_n d\lambda(x),$$

since $\lim_{\varepsilon\to 0}\sigma(\partial B_\varepsilon)\widehat{E}(\varepsilon) = 0$. Therefore $D_j E = x_j|x|^{-n}/c_n$. Similarly, by Proposition 1.1.3

$$\begin{aligned}\Delta E(\varphi) = \lim_{\varepsilon\to 0}\int_{B_R\setminus B_\varepsilon} E\Delta\varphi &= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon}\left(\varphi\frac{\partial E}{\partial n} - E\frac{\partial\varphi}{\partial n}\right)d\sigma \\ &= \lim_{\varepsilon\to 0}\frac{1}{c_n\varepsilon^{n-1}}\int_{\partial B_\varepsilon}\varphi = \varphi(0). \blacksquare\end{aligned}$$

Exercise i) Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary. Show that $D_j\chi_\Omega = -n_j d\sigma$, where χ_Ω is the characteristic function of Ω, n_j is the j-th coordinate of the outer normal of $\partial\Omega$ and $d\sigma$ is the surface measure of $\partial\Omega$.

ii) Show that

$$\frac{\partial}{\partial\overline{z}}\left(\frac{1}{z}\right) = \pi\delta_0.$$

If $u \in L^1_{\text{loc}}(\Omega)$ and $v \in L_0^\infty(\mathbb{R}^n)$ then

$$(u * v)(x) = \int u(y)v(x-y)d\lambda(y) = \int u(x-y)v(y)d\lambda(y)$$

The integration is in fact over the set $\{x\} - \operatorname{supp} v$ and we only take those x for which $\{x\} - \operatorname{supp} v \subset \Omega$.

For a test function φ we then have

$$\begin{aligned}(u * v)(\varphi) &= \int\int u(y)v(x-y)d\lambda(y)\varphi(x)d\lambda(x) \\ &= \int\int v(x)\varphi(x+y)d\lambda(x)u(y)d\lambda(y) \\ &= u\left(\widetilde{(v * \widetilde{\varphi})}\right),\end{aligned}$$

where $\widetilde{\varphi}(x) = \varphi(-x)$.

We want to define the convolution $u * v$ when u and v are distributions. First, if $u \in \mathcal{D}'(\Omega)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ we set

$$(u * \varphi)(x) := u(\varphi(x - \cdot))$$

for x from the set

$$\Omega' := \{x \in \mathbb{R}^n : \{x\} - \operatorname{supp}\varphi \subset \Omega\}. \tag{1.1.2}$$

One can see that $u * \varphi$ is determined in a neighborhood of x by u restricted to a neighborhood of $\{x\} - \operatorname{supp}\varphi$.

Theorem 1.1.5. *For $u, v \in \mathcal{D}'(\Omega)$ and $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ we have*

i) $u * \varphi \in C^\infty(\Omega')$;

ii) $D^\alpha(u * \varphi) = u * D^\alpha\varphi = D^\alpha u * \varphi$;

iii) $\operatorname{supp}(u * \varphi) \subset \operatorname{supp} u + \operatorname{supp}\varphi$;

iv) $(u * \varphi) * \psi = u * (\varphi * \psi)$ *on the set*

$$\Omega'' = \{x \in \mathbb{R}^n : \{x\} - (\operatorname{supp}\varphi + \operatorname{supp}\psi) \subset \Omega\};$$

v) $(u * \varphi)(\psi) = u(\widetilde{\varphi} * \psi) = u\left(\widetilde{(\varphi * \widetilde{\psi})}\right)$ *if* $\operatorname{supp}\psi \subset \Omega'$;

vi) If $u * \varphi = v * \varphi$ for φ with support in an arbitrary small neighborhood of 0, then $u = v$.

Proof. Write

$$\varphi(x + h) = \varphi(x) + \sum_j h_j D_j\varphi(x) + R_h(x).$$

Then

$$(u * \varphi)(x + h) = u(\varphi(x + h - \cdot)) = u(\varphi(x - \cdot)) + \sum_j h_j\, u\,(D_j\varphi(x - \cdot)) + u(R_h(x - \cdot))$$

and $u\,(D_j\varphi(x - \cdot)) = D_j u\,(\varphi(x - \cdot))$. By Theorem 1.1.1 and Taylor's formula we have for some finite k

$$|u(R_h(x - \cdot))| \le C \sum_{|\alpha| \le k} \|D^\alpha R_h(x - \cdot)\| = o(|h|)$$

uniformly in h. Hence

$$(u * \varphi)(x + h) = (u * \varphi)(x) + \sum_j h_j\,(u * D_j\varphi)(x) + o(|h|).$$

Therefore $u * \varphi$ is differentiable (in particular continuous) and

$$D_j(u * \varphi) = u * D_j\varphi = D_j u * \varphi,$$

thus $u * \varphi \in C^1$. Iteration of this gives i) and ii).

iii) If $x \notin \operatorname{supp} u + \operatorname{supp}\varphi$ then $\operatorname{supp} u \cap (\{x\} - \operatorname{supp}\varphi) = \emptyset$ and $(u * \varphi)(x) = 0$.

iv) One can easily show that

$$\varphi * \psi = \lim_{\varepsilon \to 0} \sum_{y \in \mathbb{Z}^n} \varepsilon^n \varphi(\cdot - \varepsilon y)\psi(\varepsilon y)$$

in $C_0^\infty(\Omega'')$. Thus

$$\begin{aligned}(u * (\varphi * \psi))(x) &= u\Big(\lim_{\varepsilon \to 0} \sum_{y \in \mathbb{Z}^n} \varepsilon^n \varphi(x - \cdot - \varepsilon y)\psi(\varepsilon y)\Big) \\ &= \lim_{\varepsilon \to 0} \sum_{y \in \mathbb{Z}^n} \varepsilon^n (u * \varphi)(x - \varepsilon y)\psi(\varepsilon y) \\ &= ((u * \varphi) * \psi)(x).\end{aligned}$$

v) We have

$$\widetilde{(\varphi * \widetilde{\psi})} = \widetilde{\varphi} * \psi$$

and, similarly as above,

$$\begin{aligned}(u * \varphi)(\psi) &= \lim_{\varepsilon \to 0} \sum_{x \in \mathbb{Z}^n} \varepsilon^n u\left(\varphi(\varepsilon x - \cdot)\right) \psi(\varepsilon x) \\ &= u\Big(\lim_{\varepsilon \to 0} \sum_{x \in \mathbb{Z}^n} \varepsilon^n \varphi(\varepsilon x - \cdot) \psi(\varepsilon x)\Big) \\ &= u\left(\int \varphi(x - \cdot) \psi(x) d\lambda(x)\right) \\ &= u\left(\widetilde{\varphi} * \psi\right).\end{aligned}$$

vi) Let $x \in \Omega'$ and take $\psi \in C_0^\infty(\Omega')$ with support in a small neighborhood of x. If $\varphi := \psi(x - \cdot)$ then $\psi = \varphi(x - \cdot)$ and $u(\psi) = (u * \varphi)(x) = (v * \varphi)(x) = v(\psi)$. The conclusion follows from Theorem 1.1.2. ■

Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be such that $\rho \geq 0$, $\operatorname{supp} \rho = \overline{B}(0,1)$, $\int \rho d\lambda = 1$ and ρ depends only on $|x|$. Set $\rho_\varepsilon(x) := \varepsilon^{-n} \rho(x/\varepsilon)$. Then $\operatorname{supp} \rho_\varepsilon = \overline{B}(0, \varepsilon)$ but $\int \rho_\varepsilon d\lambda = 1$. Theorem 1.1.5 gives the following result:

Theorem 1.1.6. *If $u \in \mathcal{D}'(\Omega)$, then $u_\varepsilon := u * \rho_\varepsilon \in C^\infty(\Omega_\varepsilon)$, where*

$$\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\},$$

and $u_\varepsilon \longrightarrow u$ weakly as $\varepsilon \longrightarrow 0$ (that is $u_\varepsilon(\varphi) \longrightarrow u(\varphi)$ for every $\varphi \in \mathcal{D}(\Omega)$).

Proof. By Theorem 1.1.5.v it is enough to observe that $\widetilde{\rho}_\varepsilon * \varphi \longrightarrow \varphi$ in $C_0^\infty(\Omega)$. ■

Theorem 1.1.7. *If $u \in \mathcal{D}'(\Omega)$ then there exists a sequence $u_j \in C_0^\infty(\Omega)$ such that $u_j \longrightarrow u$ weakly.*

Proof. Take $\chi_j \in C_0^\infty(\Omega)$ such that $\operatorname{supp} \chi_j \subset \Omega_{1/j}$ and $\{\chi_j = 1\} \uparrow \Omega$ as $j \uparrow \infty$. Set

$$u_j := (\chi_j u) * \rho_{1/j}.$$

Then by Theorem 1.1.5 $u_j \in C_0^\infty(\Omega)$ and for $\varphi \in C_0^\infty(\Omega)$ and j big enough

$$u_j(\varphi) = u(\chi_j(\widetilde{\rho}_{1/j} * \varphi)) = u(\rho_{1/j} * \varphi). \ ■$$

Suppose $u \in \mathcal{D}'(\Omega)$ and let $v \in \mathcal{D}'(\mathbb{R}^n)$ have a compact support. We define

$$(u * v)(\varphi) := u\left(\widetilde{(v * \widetilde{\varphi})}\right), \quad \varphi \in C_0^\infty(\Omega'),$$

where Ω' is given by (1.1.2). By Theorem 1.1.5.v this definition is consistent with the previous one if v is smooth.

Theorem 1.1.8. *We have*

i) $u * v \in \mathcal{D}'(\Omega')$;

ii) $(u * v) * w = u * (v * w)$ *if* w *is a distribution with compact support;*

iii) $D^\alpha(u * v) = u * D^\alpha v = D^\alpha u * v$;

iv) $\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v$;

v) $u * v = v * u$ *if* u *has a compact support;*

vi) $u * \delta_0 = u$.

Proof. i) If $\varphi_j \longrightarrow 0$ in $C_0^\infty(\Omega')$ then $v * \varphi_j \longrightarrow 0$ in $C_0^\infty(\Omega)$.

ii) First we want to show that

$$(u * v) * \varphi = u * (v * \varphi) \tag{1.1.3}$$

if φ is a test function. We have

$$((u * v) * \varphi)(x) = (u * v)(\varphi(x - \cdot)) = u\left((v * \widetilde{\varphi(x - \cdot)})\right)$$

and

$$(v * \widetilde{\varphi(x - \cdot)})(y) = v(\varphi(x - y - \cdot)) = (v * \varphi)(x - y)$$

thus (1.1.3) follows. Now one can easily show ii) using (1.1.3) and Theorem 1.1.5.vi.

iii) It follows easily from ii), Theorems 1.1.5.ii and 1.1.5.vi.

iv) Let $\varphi \in C_0^\infty(\Omega')$ be such that $\operatorname{supp}\varphi \cap (\operatorname{supp} u + \operatorname{supp} v) = \emptyset$. By Theorem 1.1.5.iii we have

$$\operatorname{supp}(\widetilde{v * \widetilde{\varphi}}) \subset -(\operatorname{supp} v - \operatorname{supp}\varphi).$$

Since $(\operatorname{supp}\varphi - \operatorname{supp} v) \cap \operatorname{supp} u = \emptyset$, it follows that $(u * v)(\varphi) = 0$.

v) If $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ have support in a small neighborhood of 0 then by ii), Theorem 1.1.5.i and the commutativity of the convolution of functions we have

$$u * v * \varphi * \psi = u * \psi * v * \varphi = v * \varphi * u * \psi = v * u * \varphi * \psi.$$

By Theorem 1.1.5.vi used twice we conclude that $u * v = v * u$.

vi) Follows directly from the definition. ∎

If u is a distribution then by $\operatorname{sing\,supp} u$ denote the set of those x such that u is not C^∞ in a neighborhood of x.

Theorem 1.1.9. $\operatorname{sing\,supp}(u * v) \subset \operatorname{sing\,supp} u + \operatorname{sing\,supp} v$

Proof. Take $x \notin \operatorname{sing\,supp} u + \operatorname{sing\,supp} v$. We have to show that $u * v$ is C^∞ in a neighborhood of x. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ in a neighborhood of $\operatorname{sing\,supp} v$

and $x \notin \operatorname{sing\,supp} u + \operatorname{supp} \psi$. The last condition means that u is C^∞ in a neighborhood of $\{x\} - \operatorname{supp} \psi$. We have

$$u * v = u * (\psi v) + u * ((1 - \psi)v)$$

and the last term is C^∞ by Theorem 1.1.5.i, since $(1 - \psi)v \in C_0^\infty(\mathbb{R}^n)$. Let $\varphi \in C_0^\infty(\Omega)$ be such that $\varphi = 1$ in a neighborhood of $\{x\} - \operatorname{supp} \psi$ and u is C^∞ in a neighborhood of $\operatorname{supp} \varphi$. Then

$$u * (\psi v) = (\varphi u) * (\psi v) + ((1 - \varphi)u) * (\psi v).$$

The first term is C^∞ by Theorem 1.1.8.v and Theorem 1.1.5.i, since $\varphi u \in C_0^\infty(\Omega)$. By Theorem 1.1.8.iv the support of the second term is contained in the set $\operatorname{supp}(1-\varphi)+\operatorname{supp}\psi$ which does not contain x. ■

Corollary 1.1.10. *If* $u \in \mathcal{D}'(\Omega)$ *then* $\operatorname{sing\,supp} u = \operatorname{sing\,supp} \Delta u$.

Proof. Obviously $\operatorname{sing\,supp} \Delta u \subset \operatorname{sing\,supp} u$. Let $\Omega' \Subset \Omega$ and take $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in a neighborhood of $\overline{\Omega'}$. Then $\chi u = \Delta E * (u\chi) = E * \Delta(u\chi)$. Thus by Theorem 1.1.9

$$\Omega' \cap \operatorname{sing\,supp} u = \Omega' \cap \operatorname{sing\,supp} (\chi u) \subset \Omega' \cap \operatorname{sing\,supp} \Delta(\chi u) = \Omega' \cap \operatorname{sing\,supp} \Delta u$$

and the corollary follows. ■

The following result is a very weak version of the Sobolev theorem. The full version can be found in [Hör2].

Theorem 1.1.11. *Assume that Ω is a convex domain. Let $u \in \mathcal{D}'(\Omega)$ be such that $D_j u \in L^\infty(\Omega)$, $j = 1, \ldots, n$. Then u is a Lipschitz continuous function in Ω. The classical derivatives of u given by the Rademacher theorem coincide with the distributional derivatives.*

Proof. First we show that u is continuous. Take $\Omega' \Subset \Omega$ and let $\chi \in C_0^\infty(\Omega)$ be such that $\chi \geq 0$ and $\chi = 1$ in a neighborhood of $\overline{\Omega'}$. Then by Theorems 1.1.4 and 1.1.8

$$\begin{aligned} \chi u = \Delta E * (\chi u) &= \sum_j D_j E * D_j(\chi u) \\ &= \sum_j D_j E * (\chi D_j u) + \sum_j D_j E * (u D_j \chi). \end{aligned}$$

By Theorem 1.1.9 $D_j E * (u D_j \chi) \in C^\infty(\Omega')$ and $D_j E * (\chi D_j u)$ is continuous by the Lebesgue bounded convergence theorem, since $D_j E \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\chi D_j u \in L_0^\infty(\Omega)$. Hence, u is continuous.

Assume that $|D_j u| \le M$ in Ω, $j = 1, \dots, n$. If $u_\varepsilon = u * \rho_\varepsilon \in C^\infty(\Omega_\varepsilon)$ then $u_\varepsilon \longrightarrow u$ uniformly as $\varepsilon \longrightarrow 0$, $|D_j u_\varepsilon| \le M$ and by the mean value theorem

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le M|x - y|, \quad x, y \in \Omega_\varepsilon.$$

Thus u is Lipschitz continuous.

The last part of the theorem follows immediately from Proposition A1.3. ■

Theorem 1.1.12. *Let u be a distribution and $k = 1, 2, \dots$*

i) If $D^\alpha u$ is a locally bounded function for every α with $|\alpha| = k$ then $u \in C^{k-1,1}$ (that is $u \in C^{k-1}$ and partial derivatives of u of order $k - 1$ are Lipschitz continuous).

ii) If $D^\alpha u$ is a continuous function for every α with $|\alpha| = k$ then $u \in C^k$.

Proof. It is enough to prove the theorem for $k = 1$ and iterate. i) is then exactly Theorem 1.1.11 and to show ii) it suffices to observe the following elementary fact: if u is Lipschitz continuous (thus differentiable almost everywhere) and $D_j u \in L^\infty_{\text{loc}}$ can be extended to a continuous function then $D_j u$ exists in every point and is continuous. ■

Theorem 1.1.13. *Assume that $u_j \in C^{k-1,1}(\Omega)$ tend weakly to $u \in \mathcal{D}'(\Omega)$ and that $|D^\alpha u_j| \le C < \infty$ if $|\alpha| = k$. Then $u \in C^{k-1,1}(\Omega)$ and $|D^\alpha u| \le C$ if $|\alpha| = k$.*

Proof. By Theorem 1.1.12 it is enough to show that if $u_j \in L^\infty(\Omega)$, $|u_j| \le C$ and $u_j \longrightarrow u \in \mathcal{D}'(\Omega)$ weakly, then $u \in L^\infty(\Omega)$, $|u| \le C$. We have $L^\infty(\Omega) = \big(L^1(\Omega)\big)'$. By the Alouglu theorem there exists $v \in L^\infty(\Omega)$, $|v| \le C$ which is a limit of u_j in the weak* topology of $\big(L^1(\Omega)\big)'$, thus $u = v$. ■

1.2. Subharmonic functions and the Dirichlet problem

A function h is called *harmonic* if $\Delta h = 0$. By Corollary 1.1.10 every distribution with this property must be a C^∞ function. The set of all harmonic functions in Ω we denote by $H(\Omega)$.

Let $B = B(0, R)$ be a ball in $\mathbb{R}^n$. For $y \in B$ we want to find a function $u \in L^1_{\text{loc}}(B)$ such that $\Delta u = \delta_y$ and $\lim\limits_{x \to \partial B} u(x) = 0$. If we find $h \in H(B) \cap C(\overline{B})$ such that $h(x) = E(x - y)$ for $x \in \partial B$ then a function of the form $u(x) = E(x - y) - h(x)$ will be fine. For $y \neq 0$ we are thus looking for h of the form $h(x) = E(\alpha(x - \beta y))$, where $\alpha > 0$ and $|\beta| > R/|y|$. Since $E(x)$ depends only on $|x|$, it is enough to find α and β such that $|x - y| = |\alpha(x - \beta y)|$, if $|x| = R$. It is enough to have

$$R^2 - 2\langle x, y\rangle + |y|^2 = \alpha^2 R^2 - 2\alpha^2\beta\langle x, y\rangle + \alpha^2\beta^2|y|^2,$$

thus $R^2 + |y|^2 = \alpha^2 R^2 + \alpha^2\beta^2|y|^2$ and $1 = \alpha^2\beta$. Therefore it suffices to take $\alpha = |y|/R$ and $\beta = R^2/|y|^2$. We have just proved the following result:

Theorem 1.2.1. *For $x \in \overline{B}$, $y \in B$, where $B = B(0,R)$, define*

$$G_y(x) = G(x,y) = E(x-y) - E\left(\frac{|y|}{R}x - \frac{R}{|y|}y\right).$$

Then $G_y \in H(B(0,R^2/|y|) \setminus \{y\}) \cap L^1_{\mathrm{loc}}(B(0,R^2/|y|))$, $\Delta G_y = \delta_y$ and $G_y|_{\partial B(0,R)} = 0$. ■

G is called a *Green function* for B.

If $h \in H(B) \cap C^2(\overline{B})$ then smoothing G_y near y and using Proposition 1.1.3 we can show that

$$h(y) = \int_{\partial B} h(x) \frac{\partial G_y}{\partial n}(x) d\sigma(x), \quad y \in B. \tag{1.2.1}$$

We want to compute $\partial G_y / \partial n$ at $x \in \partial B$. For $t > 0$ set $\gamma(t) := \widehat{E}(\sqrt{t})$; then $\gamma'(t) = (2c_n)^{-1} t^{-n/2}$ and

$$G_y(x) = \gamma(|x|^2 - 2\langle x, y\rangle + |y|^2) - \gamma(|x|^2|y|^2/R^2 - 2\langle x,y\rangle + R^2).$$

Therefore,

$$\frac{\partial G_y}{\partial n}(x) = \frac{R^2 - |y|^2}{c_n R |x-y|^n}, \quad x \in \partial B,\ y \in B.$$

Theorem 1.2.2. *For $f \in L^\infty(\partial B)$, where $B = B(y_0, R)$, set*

$$h(y) := \int_{\partial B} f(x) \frac{R^2 - |y-y_0|^2}{c_n R |x-y|^n} d\sigma(x), \quad y \in B.$$

Then h is harmonic in B and if f is continuous at some $x_0 \in \partial B$ then

$$\lim_{y \to x_0} h(y) = f(x_0).$$

In particular, if $f \in C(\partial B)$ then $h \in C(\overline{B})$.

Proof. We may assume that $y_0 = 0$. G is symmetric and therefore

$$\Delta_y \left(\frac{\partial G_y}{\partial n}(x)\right) = \left(\frac{\partial}{\partial n}\right)_x (\Delta G_x)(y) = 0.$$

Thus h is harmonic, since we can differentiate under the sign of integration.

Take $\varepsilon, r > 0$ and set $A_r := \partial B \cap B(x_0, r)$, $a_r := \sup_{A_r} |f - f(x_0)|$ and $M := \sup_{\partial B} |f - f(x_0)|$. By (1.2.1) we have

$$\int_{\partial B} \frac{R^2 - |y|^2}{c_n R |x-y|^n} d\sigma(x) = 1, \quad y \in B.$$

If $|y-x_0|\le\varepsilon$ then $R^2-|y|^2\le 2\varepsilon R$, $|x-y|\ge r-\varepsilon$ for $x\in\partial B\setminus A_r$ and

$$\begin{aligned}|h(y)-f(x_0)| &\le \int_{\partial B}|f(x)-f(x_0)|\frac{R^2-|y|^2}{c_nR|x-y|^n}d\sigma(x)\\ &=\int_{A_r}+\int_{\partial B\setminus A_r}\\ &\le a_r+M\frac{2\varepsilon R}{(r-\varepsilon)^n}.\end{aligned}$$

If we take $r=\varepsilon+\varepsilon^{1/(n+1)}$ and let ε tend to 0 then the theorem follows. ■

A function $u:\Omega\longrightarrow[-\infty,+\infty)$ is called *subharmonic* if it is upper semicontinuous, not identically $-\infty$ on any connected component of Ω and for every ball $B_r=B(x_0,r)\Subset\Omega$

$$u(x_0)\le\frac{1}{\sigma(\partial B_r)}\int_{\partial B_r}u(x)d\sigma(x).$$

The set of all subharmonic functions in Ω we denote by $SH(\Omega)$. It follows from (1.2.1) that harmonic functions are subharmonic.

Theorem 1.2.3. *Assume that $u,v\in SH(\Omega)$ and $B_r=B(x_0,r)\Subset\Omega$. Then*

i) u satisfies the maximum principle;

ii) $u(x_0)\le\dfrac{1}{\lambda(B_r)}\displaystyle\int_{B_r}u(x)d\lambda(x)$;

iii) $u\in L^1_{\text{loc}}(\Omega)$;

iv) $\dfrac{1}{\sigma(\partial B_r)}\displaystyle\int_{\partial B_r}u(x)d\sigma(x)\downarrow u(x_0)$ as $r\downarrow 0$;

v) $u_\varepsilon=u*\rho_\varepsilon$ is subharmonic and $u_\varepsilon\downarrow u$ as $\varepsilon\downarrow 0$;

vi) A decreasing sequence of subharmonic functions on a domain is converging pointwise either to a subharmonic function or to $-\infty$;

vii) If $u\le v$ almost everywhere then $u\le v$ everywhere;

viii) If $\{u_\alpha\}$ is a family of subharmonic functions locally uniformly bounded above then u^*, where $u=\sup_\alpha u_\alpha$, is subharmonic and $u=u^*$ almost everywhere. (u^*, resp. u_*, denotes the upper, resp. lower, regularization of u.)

Proof. i) Assume that $u\le u(x_0)$ and $u(x_1)<u(x_0)$ for some x_1 with $|x_1-x_0|=r$. Then from the upper semicontinuity it follows that for some $\varepsilon>0$ we have $u\le u(x_0)-\varepsilon$ on $E\subset\partial B_r$ with $\sigma(E)>0$. Then

$$u(x_0)\le\frac{1}{\sigma(\partial B_r)}\int_{\partial B_r}u(x)d\sigma(x)\le\frac{(u(x_0)-\varepsilon)\sigma(E)+u(x_0)(\sigma(\partial B_r)-\sigma(E))}{\sigma(\partial B_r)}<u(x_0)$$

which is a contradiction.

ii) We have

$$\frac{1}{\lambda(B_r)}\int_{B_r} u(x)d\lambda(x) = \frac{1}{\lambda(B_r)}\int_0^r \int_{\partial B_t} u(x)d\sigma(x)dt$$
$$\geq \frac{1}{\lambda(B_r)}\int_0^r \sigma(\partial B_t)u(x_0)dt$$
$$= u(x_0).$$

iii) Follows easily from ii), the upper semicontinuity of u and the fact that u is not identically $-\infty$.

iv) and v) we prove simultanously. First assume that u is smooth. Let $r < R$. By Theorem 1.2.2 and (1.2.1) there is a unique $h \in H(B_R) \cap C(\overline{B}_R)$ with $h = u$ on ∂B_R. Then $h \geq u$ in B_R and

$$\frac{1}{\sigma(\partial B_r)}\int_{\partial B_r} u(x)d\sigma(x) \leq \frac{1}{\sigma(\partial B_r)}\int_{\partial B_r} h(x)d\sigma(x)$$
$$= \frac{1}{\sigma(\partial B_R)}\int_{\partial B_R} h(x)d\sigma(x) = \frac{1}{\sigma(\partial B_r)}\int_{\partial B_r} u(x)d\sigma(x).$$

Thus we have iv) for smooth functions. Now let u be arbitrary. The Fubini theorem gives

$$(u * \rho_\varepsilon)(x_0) \leq \int_{B(0,\varepsilon)} \frac{1}{\partial B_r}\int_{\partial B_r} u(x-y)\rho_\varepsilon(y)d\sigma(x)d\lambda(y)$$
$$= \frac{1}{\partial B_r}\int_{\partial B_r} (u * \rho_\varepsilon)(x)d\sigma(x)$$

and thus u_ε is subharmonic. On the other hand

$$(u * \rho_\varepsilon)(x_0) = \int_{B(0,1)} u(x_0 - \varepsilon y)\rho(y)d\lambda(y)$$
$$= \int_0^1 \frac{1}{\varepsilon^{n-1}}\int_{\partial B_{\varepsilon r}} u(x)d\sigma(x)\widetilde{\rho}(r)dr$$

and from iv) it follows that u_ε is increasing in ε. Thus we have v). Now we can approximate u and see that the mean value $\frac{1}{\sigma(\partial B_r)}\int_{\partial B_r} u(x)d\sigma(x)$ is increasing in r. The upper semicontinuity impiles that it must converge to $u(x_0)$ as $r \downarrow 0$.

vi) Follows from the upper semicontinuity and the Lebesgue monotone convergence theorem.

vii) Follows immediately from iv) and ii).

viii) The Choquet lemma (Lemma A2.3) implies that we may assume that the family $\{u_\alpha\}$ is countable and thus that u is measurable. We have

$$u(x_0) \leq \sup_\alpha \frac{1}{\sigma(\partial B_r)}\int_{\partial B_r} u_\alpha(x)d\sigma(x) \leq \frac{1}{\sigma(\partial B_r)}\int_{\partial B_r} u(x)d\sigma(x).$$

The last expression is a continuous function with respect to x_0 and thus u^* is subharmonic. In the same way as in v) we check that $u * \rho_\varepsilon$ satisfies the mean value inequality and thus is subharmonic. Since $u * \rho_\varepsilon * \rho_\delta$ is increasing in δ, it follows that $u * \rho_\varepsilon$ decreases to some subharmonic v as $\varepsilon \downarrow 0$. We have $u * \rho_\varepsilon \geq u_\alpha * \rho_\varepsilon \geq u_\alpha$, hence $u * \rho_\varepsilon \geq u^*$ and $v \geq u^*$. On the other hand $u * \rho_\varepsilon \longrightarrow u$ in L^1_{loc} and $u = v \geq u^*$ almost everywhere and viii) follows. ∎

Corollary 1.2.4. *If $u \in SH(\Omega)$ and $B \Subset \Omega$ is a ball then there exists $\widehat{u} \in SH(\Omega)$ such that $\widehat{u} \geq u$, $\widehat{u} = u$ outside B and $\widehat{u}$ is harmonic in B. If $u_j \downarrow u$ then $\widehat{u}_j \downarrow \widehat{u}$. If u is continuous then so is $\widehat{u}$.*

Proof. If u is continuous then the corollary follows from Theorem 1.2.2. If u is arbitrary then take a sequence $\{u_j\}$ of continuous subharmonic functions near $\overline{B}$ decreasing to u. Then $\widehat{u}_j$ decreases to some $\widehat{u} \in SH(\Omega)$ and one can easily show that $\widehat{u}$ has the required properties. ∎

Theorem 1.2.5. *A function u is subharmonic iff u is a distribution with $\Delta u \geq 0$.*

Proof. First assume that u is smooth. Suppose that u is subharmonic and $\Delta u < 0$ in some ball B. Let $h \in H(B) \cap C(\overline{B})$ be equal to u on ∂B and set $v := u - h$. Then $v \in SH(B) \cap C(\overline{B})$, it vanishes on ∂B and thus has a local minimum at some $x' \in B$. Then $v_{x_j x_j}(x') \geq 0$, $j = 1, \dots, n$, hence $\Delta v(x_1) \geq 0$ which is a contradiction. Now assume that $\Delta u \geq 0$. Considering $u + \varepsilon |x|^2$ instead of u we may assume that $\Delta u > 0$. We have to show that $u(x_0) \leq h(x_0)$. If there is $x'' \in B$ where $u - h$ has a local maximum then $\Delta u(x'') = \Delta(u - h)(x'') \leq 0$ which is a contradiction. Thus $u < h$ in B and u is subharmonic.

Now, take arbitrary $u \in \mathcal{D}'(\Omega)$ with $\Delta u \geq 0$. Then $\Delta u_\varepsilon = \Delta(u * \rho_\varepsilon) \geq 0$, thus u_ε is C^∞ and subharmonic, and so is $u_{\varepsilon,\delta} = u * \rho_\varepsilon * \rho_\delta$. $u_{\varepsilon,\delta}$ is increasing in δ and $u_{\varepsilon,\delta} = u_{\delta,\varepsilon} \downarrow u_\delta$ as $\varepsilon \downarrow 0$, thus u_δ is increasing in δ. Hence, as $\delta \downarrow 0$, u_δ decreases to some $u_0 \in SH(\Omega)$ and tends weakly to u, thus $u = u_0$. (This is why u_0 cannot be identically $-\infty$.) On the other hand, if $u \in SH(\Omega)$, then $0 \leq \Delta u_\varepsilon = \Delta u * \rho_\varepsilon \longrightarrow \Delta u$ weakly, thus $\Delta u \geq 0$. ∎

Proposition 1.2.6. *If Ω is a bounded domain, $u \in SH(\Omega)$ and $h \in H(\Omega)$ then*

$$(u^* \leq h_* \text{ on } \partial\Omega) \Rightarrow ((u - h)^* \leq 0 \text{ on } \partial\Omega) \Rightarrow (u \leq h \text{ on } \Omega)\,.$$

Proof. Follows from the fact that $(u - h)^* \leq u^* - h_*$ and from the maximum principle. ∎

Throughout the rest of this section we assume that Ω is a bounded domain. The next result is the main feature of the so called Perron method.

Theorem 1.2.7. *For $f \in L^\infty(\partial\Omega)$ define*

$$h = h_{f,\Omega} := \sup\{v \in SH(\Omega) : v^*|_{\partial\Omega} \le f\}.$$

Then $h \in H(\Omega)$ (h is called the Perron envelope of f).

Proof. By $\mathcal{B}$ denote the family of all $v \in SH(\Omega)$ with $v^*|_{\partial\Omega} \le f$. Take a ball $B \Subset \Omega$, $x_0 \in B$ and let $v_j \in \mathcal{B}$ be such that $v_j(x_0) \uparrow h(x_0)$. Using Corollary 1.2.4 we may inductively define

$$u_1 := \widehat{v}_1,$$
$$u_{j+1} := (\max\widehat{\{u_j, v_{j+1}}\}).$$

Then $u_j \in \mathcal{B}$, u_j is harmonic in B, increases to some $\widetilde{h} \in H(B)$, where $\widetilde{h} \le h$. It remains to show that $\widetilde{h} = h$ in B.

Take $x_1 \in B$ and $\alpha_j \in \mathcal{B}$ such that $\alpha_j(x_1) \uparrow h(x_1)$. Define inductively

$$\beta_1 := (\max\widehat{\{u_1, \alpha_1}\}),$$
$$\beta_{j+1} := (\max\{\widehat{u_{j+1}, \alpha_{j+1}}, \beta_j\}).$$

Then $\beta_j \in \mathcal{B}$, $u_j \le \beta_j$ and β_j is increasing to some $\beta \in H(B)$. We have $\widetilde{h} \le \beta \le h$ and $\widetilde{h}(x_0) = \beta(x_0)$, thus by the maximum principle $\widetilde{h} = \beta$ in B. Now the theorem follows since $\widetilde{h}(x_1) = \beta(x_1) = h(x_1)$. ∎

A point $x_0 \in \partial\Omega$ is called *regular* if for every $f \in L^\infty(\partial\Omega)$ which is continuous at x_0 we have

$$\lim_{x \to x_0} h_{f,\Omega}(x) = f(x_0).$$

Theorem 1.2.8. *For $x_0 \in \partial\Omega$ the following are equivalent*

i) x_0 is regular;

ii) There exists a weak barrier at x_0, that is $u \in SH(\Omega)$ such that $u < 0$ and $\lim_{x\to x_0} u(x) = 0$;

iii) There exists a local weak barrier at x_0, that is a weak barrier which is defined on $\Omega \cap U$, where U is a neighborhood of x_0;

iv) There exists a strong barrier at x_0, that is a weak barrier with additional property $u^*|_{\overline{\Omega}\setminus\{x_0\}} < 0$;

v) There exists a local strong barrier at x_0.

Proof. The implications iv)⇒ii)⇒iii) and iv)⇒v) are clear. To show i)⇒iv) take $f(x) = -|x - x_0|$ and $h = h_{f,\Omega}$. Then $h \le f$ in Ω, since $f = \inf\{\widetilde{h} \in H(\mathbb{R}^n) : \widetilde{h} \ge f\}$, thus h is a strong barrier. To prove v)⇒iv) let u be a local strong barrier at x_0 defined in a neighborhood of $\Omega\cap\overline{U}$. Take $\varepsilon>0$ such that $u\le-\varepsilon$ on $\Omega\cap\partial U$. Then it is easy to show that the function

$$\begin{cases}\max\{u,-\varepsilon\} & \text{on } \Omega\cap U,\\ -\varepsilon & \text{on } \Omega\setminus U\end{cases}$$

is a global strong barrier. Thus it remains to show iii)⇒iv)⇒i).

iv)⇒i) Take $f\in L^\infty(\partial\Omega)$ such that f is continuous at x_0. We may assume that $f(x_0)=0$. If $\varepsilon>0$ and u is a strong barrier then we can find $c>0$ such that $cu^*\le f+\varepsilon$ and $cu^*\le -f+\varepsilon$ on $\partial\Omega$. The first inequality implies that $cu-\varepsilon\le h_{f,\Omega}$ on Ω. If $v\in SH(\Omega)$ is such that $v^*|_{\partial\Omega}\le f$ then $(cu+v-\varepsilon)^*\le 0$ on $\partial\Omega$ thus by the maximum principle $cu+v-\varepsilon\le 0$ on Ω. Hence

$$cu-\varepsilon\le h_{f,\Omega}\le -cu+\varepsilon$$

and $\lim_{x\to x_0} h_{f,\Omega}(x)=0$.

iii)⇒iv) Let U be a neighborhood of x_0 and $u\in SH(\Omega\cap U)$ such that $u<0$ and $\lim_{x\to x_0}u(x)=0$. Set $g(x):=|x-x_0|$ and $h:=h_{g,\Omega}$. Then $h\in H(\Omega)$ and $g\le h\le M:=\sup_\Omega g$, since g is subharmonic. It is enough to show that $h^*(x_0)=0$; then $-h$ would be a strong barrier.

Take $\varepsilon>0$ such that $B=B(x_0,\varepsilon)\Subset U$. For a compact $K\Subset\Omega\cap\partial B$ set

$$f:=\begin{cases}1 & \text{on } (\Omega\setminus K)\cap\partial B,\\ 0 & \text{elsewhere on } \partial B.\end{cases}$$

Theorem 1.2.2 gives $I\in H(B)$ such that $0\le I\le 1$,

$$\lim_{x\to(\Omega\setminus K)\cap\partial B} I(x)=1 \tag{1.2.2}$$

and

$$h(x_0)=\frac{\sigma\left((\Omega\setminus K)\cap\partial B\right)}{\sigma(\partial B)}.$$

We may choose K so that $I(x_0)\le\varepsilon$.

Now we want to find positive constants α, β and γ so that

$$h\le-\alpha u+\beta I+\gamma \quad \text{on } \Omega\cap B.$$

To have this it is enough to check that

$$v^*+\alpha u^*\le\beta I_*+\gamma \quad \text{on } \partial(\Omega\cap B) \tag{1.2.3}$$

for $v\in SH(\Omega)$ with $v^*|_{\partial\Omega}\le g$. On $\partial\Omega\cap\overline{B}$ (1.2.3) holds if $\gamma=\varepsilon$. On K $v^*\le M$ and it is enough to take

$$\alpha=\frac{M-\varepsilon}{-\max_K u},$$

whereas on $(\Omega \setminus K) \cap \partial B$, by (1.2.2), we may take $\beta = M - \varepsilon$. Thus

$$h \leq \frac{M - \varepsilon}{\max_K u} u + (M - \varepsilon) I + \varepsilon$$

and

$$h^*(x_0) \leq (M - \varepsilon)\varepsilon + \varepsilon,$$

hence $h^*(x_0) = 0$. ■

The implication iii)⇒iv) in Theorem 1.2.8 is due to Bouligand.

Theorem 1.2.9. *i) If $n = 2$ and a connected component of $\partial\Omega$ containing x_0 is not a point, then x_0 is regular.*

ii) If there exists an open cone C with a vertex at x_0 and a neighborhood U of x_0 such that $\overline{C} \cap U \cap \overline{\Omega} = \{x_0\}$ then x_0 is regular.

Proof. i) By K denote the connected component of $\partial\Omega$ containing x_0 and fix $z_1 \in K$, $z_1 \neq x_0$. Let $\widehat{\Omega}$ be a connected component of $\widehat{\mathbb{C}} \setminus K$ containing Ω (here $\widehat{\mathbb{C}}$ stands for the Riemann sphere). Then $\widehat{\Omega}$ is simply connected, thus there exist a holomorphic f in $\widehat{\Omega}$ such that $e^{f(z)} = \dfrac{z - x_0}{z - z_1}$. Set $u(z) := 1/\mathrm{Re}\, f$. For z near x_0 we have

$$u(z) = \frac{\log \dfrac{|z - x_0|}{|z - z_1|}}{|f(z)|^2} \geq \frac{1}{\log \dfrac{|z - x_0|}{|z - z_1|}},$$

hence u is negative and $\lim_{z \to x_0} u(z) = 0$.

ii) It is enough to show that for given $0 < a < 1$ the domain $\{x_1 < a|x|\}$ (which is a complement of a closed cone) is regular at the origin. Set $u(x) := |x|^\alpha g(x_1/|x|)$, where $\alpha > 0$ and g is a negative C^2 function on $[-1, a]$. One can compute that

$$\Delta u(x) = |x|^{\alpha - 2} \left((1 - t^2) g''(t) - (n - 1) t g'(t) + \alpha(\alpha + n - 2) g(t) \right),$$

where $t = x_1/|x|$. It is enough to find g with $(1 - t^2) g''(t) - (n - 1) t g'(t) > 0$ for $-1 \leq t \leq a$ and take α sufficiently small. ■

Exercise Show that i) does not hold if $n > 2$.

We say that Ω is *regular* if all its boundary points are regular.

Theorem 1.2.10. *For a bounded domain Ω the following are equivalent:*

i) Ω is regular;

ii) For every $f \in C(\partial\Omega)$ we have $h_{f,\Omega} \in H(\Omega) \cap C(\overline{\Omega})$ and $h_{f,\Omega}\Big|_{\partial\Omega} = f$;

iii) There exists a bounded subharmonic exhaustion function in Ω, that is $u \in SH(\Omega)$ such that $u < 0$ and $\lim_{x\to\partial\Omega} u(x) = 0$.

Proof. It is enough to show that i) implies iii). Take a ball $B \Subset \Omega$ and let f be equal to 0 on $\partial\Omega$ and to -1 on ∂B. Then

$$u = \begin{cases} h_{f,\Omega\setminus\overline{B}} & \text{on } \Omega \setminus \overline{B} \\ -1 & \text{on } \overline{B} \end{cases}$$

has the required properties. ■

Exercise Show that if Ω is regular then the Dirichlet problem

$$\begin{cases} u \in SH(\Omega), \\ \Delta u = \mu, \\ u_* = u^* = f \text{ on } \partial\Omega \end{cases}$$

has a unique solution provided that $f \in C(\partial\Omega)$ and μ is either a nonnegative Radon measure in Ω with compact support or $\mu \in L^p(\Omega)$ for some $p > n/2$. In the latter case the solution is continuous on $\overline{\Omega}$.

Let μ be a nonnegative Radon measure in $\mathbb{R}^n$ with compact support. We set

$$U^\mu := E * \mu.$$

U^μ is called a *potential* of the measure μ.

Theorem 1.2.11. *i) $U^\mu \in SH(\mathbb{R}^n)$;*

ii) $\Delta U^\mu = \mu$;

iii) $U^\mu(x) = \int_{\mathbb{R}^n} E(x-y)d\mu(y)$;

iv) If $\mu \in L^p_0(\mathbb{R}^n)$ for some $p > n/2$ then U^μ is continuous in $\mathbb{R}^n$;

v) If $\mu \in C^k_0(\mathbb{R}^n)$ then $U^\mu \in C^k(\mathbb{R}^n)$, $k = 0, 1, 2, \ldots, \infty$.

Proof. i) and ii) follow from Theorems 1.1.8 and 1.2.5.

iii) For $\varphi \in C^\infty_0(\Omega)$ we have

$$U^\mu(\varphi) = E\left(\widetilde{(\mu * \widetilde{\varphi})}\right) = \int\int E(x)\varphi(x+y)d\mu(y)d\lambda(x) = \int\int E(x-y)d\mu(y)\varphi(x)d\lambda(x)$$

and iii) follows.

iv) If μ is a bounded function then it follows easily from the Lebesgue bounded convergence theorem. Let μ be arbitrary and let $\mu_j := \max\{\mu, j\}$. Then $\mu_j \longrightarrow \mu$ in $L^p(\mathbb{R}^n)$ as $j \longrightarrow \infty$ and from the Hölder inequality we infer

$$|U^{\mu_j}(x) - U^{\mu}(x)| \leq ||E||_{L^q(\{x\}-\operatorname{supp}\mu)} ||\mu_j - \mu||_{L^p(\mathbb{R}^n)},$$

where $1/p + 1/q = 1$, hence $q < n/(n-2)$. Thus $U^{\mu_j} \longrightarrow U^{\mu}$ locally uniformly in $\mathbb{R}^n$.

v) Follows from iv) and Theorem 1.1.12. ■

Exercise Show that subharmonic functions are in L^p_{loc} for every $p < n/(n-2)$.

1.3. Nonnegative forms and currents

By $\mathbb{C}_{(p,q)}$, $p, q = 0, 1, \ldots, n$, we denote the set of complex forms

$$\alpha = \sum_{\substack{|J|=p\\|K|=q}}{}' \alpha_{JK} \left(\frac{i}{2}\right)^p dz_J \wedge d\overline{z}_K, \quad \alpha_{JK} \in \mathbb{C}.$$

Here $\sum'$ denotes the summation over increasing multi-indices and $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$, $d\overline{z}_K = d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}$. The volume form is given by

$$d\lambda = \frac{i}{2} dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge \frac{i}{2} dz_n \wedge d\overline{z}_n.$$

If $\alpha_1, \ldots, \alpha_p \in \mathbb{C}_{(1,0)}$ then the form

$$\frac{i}{2}\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge \frac{i}{2}\alpha_p \wedge \overline{\alpha}_p \in \mathbb{C}_{(p,p)}$$

is called an *elementary nonnegative form.*

Take $\alpha \in \mathbb{C}_{(p,p)}$. We say that α is *nonnegative* and write $\alpha \geq 0$ if $\alpha \wedge \beta \geq 0$ for all elementary nonnegative forms in $\mathbb{C}_{(n-p,n-p)}$. We say that α is *real* if $\overline{\alpha} = \alpha$ (that is $\alpha_{JK} = \overline{\alpha}_{KJ}$ for all J, K with $|J| = |K| = p$).

Proposition 1.3.1. *Write*

$$\alpha = \sum_{j,k=1}^{n} \alpha_{jk} \frac{i}{2} dz_j \wedge d\overline{z}_k \in \mathbb{C}_{(1,1)}.$$

Then α is nonnegative iff the matrix (α_{jk}) is nonnegative.

Proof. Take

$$\alpha_j = \sum_{k=1}^{n} a_{jk} dz_k \in \mathbb{C}_{(1,0)}, \quad j = 1, \ldots, n-1.$$

Then

$$\alpha_1 \wedge \cdots \wedge \alpha_{n-1} = \sum_{k=1}^{n} M_k dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \cdots \wedge dz_n,$$

where $M_k = \det\left(a_{st}\right)_{\substack{s=1,\ldots,n-1 \\ t=1,\ldots,n,\, t\neq k}}$. Therefore

$$\alpha \wedge \frac{i}{2}\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge \frac{i}{2}\alpha_{n-1} \wedge \overline{\alpha}_{n-1} = \sum_{j,k=1}^{n} \alpha_{jk} M_j \overline{M}_k d\lambda$$

and the proposition follows. ■

Theorem 1.3.2. *Let $\alpha \in \mathbb{C}_{(p,p)}$ and $\beta \in \mathbb{C}_{(1,1)}$ be nonnegative forms. Assume moreover that β is real. Then $\alpha \wedge \beta \geq 0$.*

Proof. Write

$$\beta = \sum_{j,k} a_{jk} \frac{i}{2} dz_j \wedge d\overline{z}_k$$

and $A = (a_{jk})$. Since β is real, A is a hermitian matrix. Let P be a unitary matrix (that is $P^T \overline{P} = (\delta_{jk})$) such that $B := P^{-1}AP$ is a diagonal matrix. Then $A = PB\overline{P}^T = (\sum_l p_{jl} b_l \overline{p}_{kl})$ and

$$\beta = \sum_l b_l \, \frac{i}{2} \Big(\sum_j p_{jl} dz_j\Big) \wedge \overline{\Big(\sum_j p_{jl} dz_j\Big)},$$

where $b_l \geq 0$. Therefore if γ is an elementary nonnegative form in $\mathbb{C}_{(n-p-1,n-p-1)}$ then

$$\alpha \wedge \beta \wedge \gamma = \sum_l b_l \alpha \wedge \frac{i}{2} \Big(\sum_j p_{jl} dz_j\Big) \wedge \overline{\Big(\sum_j p_{jl} dz_j\Big)} \wedge \gamma \geq 0. \;■$$

Theorem 1.3.2 implies in particular that elementary nonnegative forms are nonnegative.

Lemma 1.3.3. *The set of all elementary nonnegative forms in $\mathbb{C}_{(p,p)}$ spans $\mathbb{C}_{(p,p)}$ over $\mathbb{C}$.*

Proof. We have

$$\left(\frac{i}{2}\right)^p dz_J \wedge d\overline{z}_K = (-1)^{p(p-1)/2} \frac{i}{2} dz_{j_1} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge \frac{i}{2} dz_{j_p} \wedge d\overline{z}_{k_p}$$

and

$$dz_j \wedge d\bar{z}_k = \frac{1}{2}(dz_j + dz_k) \wedge (d\bar{z}_j + d\bar{z}_k) + \frac{i}{2}(dz_j + i dz_k) \wedge (d\bar{z}_j - i d\bar{z}_k)$$
$$- \frac{i+1}{2}(dz_j \wedge d\bar{z}_j + dz_k \wedge d\bar{z}_k). \blacksquare$$

Let Ω be an open set in $\mathbb{R}^n$. Continuous linear functionals on $\mathcal{D}_{(n-p)}(\Omega)$ are called *currents* on Ω of degree p or dimension $n-p$. The set of them will be denoted by $\mathcal{D}'_{(p)}(\Omega)$. For $T \in \mathcal{D}'_{(p)}(\Omega)$ we may write

$$T = \sum_{|I|=p}{}' T_I dx_I, \quad T_I \in \mathcal{D}'(\Omega)$$

where $T_I(\varphi) = T(\varphi\omega_I)$ and $\{\omega_I\}$ is a dual basis to dx_I (that is $dx_I \wedge \omega_{I'} = \delta_{II'} d\lambda$).

A current T is said to be *of order 0* if it can be continuously extended to $C_{0,(n-p)}(\Omega)$. This is equivalent to the fact that all coefficients T_I are distributions of order 0, that is complex measures. In this case

$$T(\Psi) = \int_\Omega T \wedge \Psi, \quad \Psi \in C_{0,(n-p)}(\Omega).$$

If T is of order 0 and E is a Borel subset of Ω then a total mass of T on E is defined by

$$||T||_E = \sum_I{}' |T_I|(E),$$

where $|\mu|$ denotes o variation of a complex measure μ.

A current T is called *closed* if $dT = 0$.

The following result is the Stokes theorem for currents.

Theorem 1.3.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary. Assume that T is a current in Ω of order $n-1$ which is C^1 on $\overline{\Omega} \setminus U$, where $U \Subset \Omega$. If moreover dT is of order 0 then*

$$\int_{\partial\Omega} T = \int_\Omega dT.$$

Proof. Take $F \in C^1_{(n-1)}(\overline{\Omega})$ such that $F = T$ in a neighborhood of $\partial\Omega$. Then

$$\int_{\partial\Omega} T = \int_{\partial\Omega} T = \int_\Omega dF$$

by the classical Stokes theorem. Thus we may assume that T has a compact support in Ω. Set

$$T_\varepsilon := T * \rho_\varepsilon = {\sum_I}' T_I * \rho_\varepsilon dx_I$$

and take $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ in a neighborhood of $\operatorname{supp} T$. Then $dT_\varepsilon \longrightarrow dT$ weakly and by the classical Stokes theorem again we have

$$\int_\Omega dT = \int_\Omega \varphi dT = \lim_{\varepsilon\to 0} \int_\Omega \varphi dT_\varepsilon = \lim_{\varepsilon\to 0} \int_\Omega dT_\varepsilon = 0. \blacksquare$$

Let now Ω be an open set in $\mathbb{C}^n$. Currents of the form

$$T = {\sum_{\substack{|J|=p \\ |K|=q}}}' T_{JK} \left(\frac{i}{2}\right)^p dz_J \wedge d\overline{z}_K, \quad T_{JK} \in \mathcal{D}'(\Omega)$$

we call *complex currents* of bidegree (p,q) or bidimension $(n-p, n-q)$. We have $\mathcal{D}'_{(p,q)}(\Omega) = \big(\mathcal{D}_{(n-p,n-q)}(\Omega)\big)'$.

Let $T \in \mathcal{D}'_{(p,p)}(\Omega)$. Then, similarly as in the case of constant forms, we say that T is *nonnegative* and write $T \geq 0$ if $T \wedge \alpha \geq 0$ for all elementary nonnegative forms α from $\mathbb{C}_{(n-p,n-p)}$. We say that T is *real* if $\overline{T} = T$ (that is $T_{JK} = \overline{T}_{KJ}$ for all J, K with $|J| = |K| = p$).

Theorem 1.3.5. *Nonnegative currents are of order 0.*

Proof. By Lemma 1.3.3 we may find a basis $\{\alpha_j\}$ of $\mathbb{C}_{(n-p,n-p)}$ consisting of elementary nonnegative forms. Let $\{\beta_j\}$ be a basis in $\mathbb{C}_{(p,p)}$ dual to $\{\alpha_j\}$. Then

$$T = {\sum_{J,K}}' T_{JK} \left(\frac{i}{2}\right)^p dz_J \wedge d\overline{z}_K = \sum_j T_j \beta_j,$$

where $T_j d\lambda = T \wedge \alpha_j \geq 0$, that is T_j are nonnegative Radon measures. We may write

$$\beta_j = {\sum_{J,K}}' c^j_{JK} \left(\frac{i}{2}\right)^p dz_J \wedge d\overline{z}_K, \quad c^j_{JK} \in \mathbb{C}.$$

Then

$$T_{JK} = \sum_j C^j_{JK} T_j$$

and thus T_{JK} are complex measures. $\blacksquare$

From Proposition 1.3.1 and Theorem 1.3.2 the next two results easily follow by approximation.

Proposition 1.3.6. *If*

$$T = \sum_{j,k} T_{jk} \frac{i}{2} dz_j \wedge d\overline{z}_k \in \mathcal{D}'_{(1,1)}(\Omega)$$

then $T \geq 0$ *iff* $(T_{jk}) \geq 0$. ■

Theorem 1.3.7. *Let* $T \in \mathcal{D}'_{(p,p)}(\Omega)$ *be nonnegative and* $F \in C_{(1,1)}(\Omega)$ *be nonnegative and real. Then* $T \wedge F \geq 0$. ■

Note that $T \wedge F$ in Theorem 1.3.7 makes sense by Theorem 1.3.5.

A *fundamental Kähler form* is defined by

$$\omega := \sum_{j=1}^{n} \frac{i}{2} dz_j \wedge d\overline{z}_j.$$

Later on we shall use the following estimate.

Lemma 1.3.8. *For every nonnegative current*

$$T = {\sum_{J,K}}' T_{JK} \left(\frac{i}{2}\right)^p dz_J \wedge d\overline{z}_K \in \mathcal{D}'_{(p,p)}(\Omega)$$

we have

$$|T_{JK}| \leq c_n \, T \wedge \omega^{n-p}.$$

Proof. Let $\{\alpha_l\}$ be a basis of $\mathbb{C}_{(n-p,n-p)}$ consisting of elementary nonnegative forms and $\{\omega_{JK}\}$ a basis in $\mathbb{C}_{(n-p,n-p)}$ dual to the basis $\{(i/2)^p dz_j \wedge d\overline{z}_K\}$ in $\mathbb{C}_{(p,p)}$. Write

$$\omega_{JK} = \sum_l c^l_{JK} \alpha_l.$$

Then

$$|T_{JK}| = |T \wedge \omega_{JK}| = \left| \sum_l c^l_{JK} T \wedge \alpha_l \right| \leq c'_n \max_{l,J,K} |c^l_{JK}| \, T \wedge \alpha_l.$$

We may write $\alpha_l = \alpha^1_l \wedge \cdots \wedge \alpha^p_l$ where α^j_l are nonnegative real $(1,1)$ forms. By Proposition 1.3.1 and the matrix theory we have $\alpha^j_l \leq c''_n \, \omega$. By Theorem 1.3.7

$$T \wedge \alpha_l \leq c''_n \, T \wedge \alpha^1_l \wedge \cdots \wedge \alpha^{p-1}_l \wedge \omega \leq \cdots \leq (c''_n)^p \, T \wedge \omega^p$$

and the lemma follows. ■

Corollary 1.3.9. *Let T be a nonnegative current of bidegree (p,p) such that $T\wedge\omega^{n-p}=0$. Then $T=0$.* ■

1.4. Plurisubharmonic functions and regular domains in $\mathbb{C}^n$

Let Ω be an open set in $\mathbb{C}^n$. A function $u:\Omega\longrightarrow[-\infty,+\infty)$ we call *plurisubharmonic* if it is upper semicontinuous, not identically $-\infty$ on any connected component of Ω and for every $z\in\Omega$ and $w\in\mathbb{C}^n$ the function $\zeta\longmapsto u(z+\zeta w)$ is subharmonic in a neighborhood of 0 in the complex plane (in other words u is subharmonic on every complex plane cutting Ω). The set of all plurisubharmonic functions on Ω we denote by $PSH(\Omega)$.

Theorem 1.4.1. *i) $PSH(\Omega)\subset SH(\Omega)$;*

ii) If $u\in PSH(\Omega)$ then $u*\rho_\varepsilon\in PSH(\Omega_\varepsilon)$;

iii) If u_j is a decreasing sequence of plurisubharmonic functions on Ω to some u then on every connected component of Ω u is either plurisubharmonic or $-\infty$;

iv) If $\{u_j\}$ is a family of plurisubharmonic functions locally uniformly bounded above then $(\sup_j u_j)^*$ is a plurisubharmonic function;

v) Suppose $u\in\mathcal{D}'(\Omega)$. Then $u\in PSH(\Omega)$ iff the matrix $(\partial^2u/\partial z_j\partial\overline{z}_k)$ is nonnegative;

vi) If Ω_1 and Ω_2 are domains in $\mathbb{C}^n$, $T:\Omega_1\longrightarrow\Omega_2$ is a holomorphic mapping and $u\in PSH(\Omega_2)$ then $u\circ T\in PSH(\Omega_1)$;

vii) If $u\in PSH(\mathbb{C}^n)$ is bounded above then it must be constant;

viii) Assume that u is plurisubharmonic and let χ be a convex and increasing function in the range of u. Then $\chi\circ u$ is plurisubharmonic.

Proof. i) Take a ball $B\Subset\Omega$. We may assume that $B=B_n$ is the unit ball in $\mathbb{C}^n$. Use the parametrization of ∂B_n defined by $e^{it}(z',\sqrt{1-|z'|^2})$ for $t\in(0,2\pi]$ and $z'\in B_{n-1}$. Then, since u is plurisubharmonic,

$$\int_{\partial B_n}u(z)d\sigma(z)=\int_{B_{n-1}}|z'|^{2n-1}\int_0^{2\pi}u(e^{it}(z',\sqrt{1-|z'|^2}))dt\,d\lambda(z')$$
$$\geq\int_{B_{n-1}}|z'|^{2n-1}\int_0^{2\pi}u(0)dt\,d\lambda(z')=\sigma(\partial B_n)u(0).$$

ii) Follows easily from the Fubini theorem.

iii), iv) Follow directly from the definition and related properties of subharmonic functions.

v) By ii) we may assume that u is smooth. Then it is enough to compute

$$\frac{1}{4}\Delta_\zeta u(z+\zeta w)=\frac{\partial^2u(z+\zeta w)}{\partial\zeta\partial\overline{\zeta}}=\sum_{j,k=1}^n\frac{\partial^2u}{\partial z_j\partial\overline{z}_k}(z+\zeta w)w_j\overline{w}_k.$$

vi) We may assume that u is smooth and that $n = 1$. Then

$$\frac{\partial^2(u \circ T)}{\partial z \partial \overline{z}} = \frac{\partial^2 u}{\partial w \partial \overline{w}} \circ T \left| \frac{\partial T}{\partial z} \right|^2 .$$

vii) We may assume that $n = 1$ and $\sup_{\mathbb{C}} u = 0$. The function

$$v(z) = \begin{cases} u(1/z) & \text{if } z \neq 0 \\ \limsup_{\zeta \to 0} u(1/\zeta) & \text{if } z = 0 \end{cases}$$

is subharmonic and bounded in a punctured neighorhood of the origin. By the maximum principle $v(0) = 0$. Moreover, in the unit disk we have $v = (\sup_j v_j)^*$, where $v_j(z) = u(z) + 1/j \log |z|$. By iii) v is subharmonic in $\mathbb{C}$ and vii) follows from the maximum principle.

viii) Again, we may assume that both u, χ are smooth and $n = 1$. Then we compute

$$\frac{\partial^2(\chi \circ u)}{\partial z \partial \overline{z}} = f'' \left| \frac{\partial u}{\partial z} \right|^2 + f' \frac{\partial^2 u}{\partial z \partial \overline{z}}. \blacksquare$$

Note that vii) is not true for subharmonic functions in $\mathbb{R}^n$, $n \geq 3$.

Exercise Show that plurisubharmonic functions are in L^p_{loc} for every $p < \infty$.

We have the operators

$$\partial : \mathcal{D}'_{(p,q)} \longrightarrow \mathcal{D}'_{(p+1,q)}$$

and

$$\overline{\partial} : \mathcal{D}'_{(p,q)} \longrightarrow \mathcal{D}'_{(p,q+1)}$$

so that $d = \partial + \overline{\partial}$. Set

$$d^c := i(\overline{\partial} - \partial).$$

Then $dd^c = 2i\partial\overline{\partial}$. It follows that u is plurisubharmonic iff the $(1,1)$-current $dd^c u$ is nonnegative.

A function is called *pluriharmonic* in Ω if it is plurisubharmonic in Ω and harmonic on every complex plane intersecting Ω. The set of all pluriharmonic functions in Ω we denote by $PH(\Omega)$. Obviously we have $PH \subset H \subset C^\infty$.

Proposition 1.4.2. *For a real smooth function u the following are equivalent*

i) u is a pluriharmonic function;

ii) $\partial^2 u/\partial z_j \partial \overline{z}_k = 0$ for all $j, k = 1, \dots, n$;

iii) Locally we can find a holomorphic function f such that $u = \operatorname{Re} f$.

Proof. i) implies that for every $w \in \mathbb{C}^n$ we have $\sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k = 0$ and ii) follows. Assume that ii) holds. Then $dd^c u = 0$ and one can easily check that $d^c u$ is a real 1-form. By the Poincare lemma in every ball there exists a real C^1 function v such that $d^c u = dv$. It means that $\overline{\partial}(u + iv) = 0$, thus $f = u + iv$ gives iii). Obviously i) follows from iii). ■

A plurisubharmonic function u said to be *strongly plurisubharmonic* in Ω if for every open $U \Subset \Omega$ there exists $\lambda > 0$ such that the function $u(z) - \lambda|z|^2$ is plurisubharmonic in U (that is $dd^c u \geq 4\lambda\, \omega$).

Exercise Assume that u, v are plurisubharmonic and negative. Show that the function $-(uv)^{1/2}$ is plurisubharmonic and that it is strongly plurisubharmonic if so is u.

We want to use the Perron method for plurisubharmonic functions. If Ω is a bounded domain in $\mathbb{C}^n$ and $f \in L^\infty(\partial\Omega)$, the function

$$u = u_{f,\Omega} := \sup\{v \in PSH(\Omega) : v^*|_{\partial\Omega} \leq f\}. \tag{1.4.1}$$

is called a *Perron-Bremermann envelope* of f in Ω. However, contrary to the real case, $u_{f,\Omega}$ need not be even upper semicontinuous in general, as the following example shows.

Exercise Let $\Omega = \Delta^2$ be a bidisk and

$$f(z, w) := \begin{cases} -1 & \text{if } z = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $u_{f|_{\partial\Delta^2}, \Delta^2} = f$ in Δ^2.

It means that there can be no counterpart of Theorem 1.2.7 in the complex case. Instead, our main tool will be the following result due to Walsh [Wal].

Theorem 1.4.3. *Assume that Ω is a bounded domain and $f \in C(\partial\Omega)$ is such that $u^* = u_* = f$ on $\partial\Omega$, where $u = u_{f,\Omega}$. Then u is continuous in Ω.*

Proof. We have $u = u^* \in PSH(\Omega)$, hence it is enough to show that u is lower semicontinuous. Fix $z_0 \in \Omega$ and $\varepsilon > 0$. Since $\partial\Omega$ is compact, we can find $\delta > 0$ such that

$$z \in \Omega,\ w \in \partial\Omega,\ |z - w| \leq \delta \implies |u(z) - f(w)| \leq \varepsilon. \tag{1.4.2}$$

Take $\widetilde{z} \in \Omega$ with $|\widetilde{z} - z_0| \leq \delta/2$ and define

$$v(z) := \begin{cases} \max\{u(z), u(z + z_0 - \widetilde{z}) - 2\varepsilon\} & \text{if } z \in \Omega \cap \widetilde{\Omega}, \\ u(z) & \text{if } z \in \Omega \setminus \widetilde{\Omega}, \end{cases}$$

where $\widetilde{\Omega} = \Omega - (z_0 - \widetilde{z})$. By (1.4.2) $v = u$ in a neighborhood of $\Omega \cap \partial\widetilde{\Omega}$ and thus $v \in PSH(\Omega)$. Moreover, if $z \in \Omega \cap \widetilde{\Omega}$ and $w \in \partial\Omega$ are such that $|z - w| \leq \delta/2$ then $|z + z_0 - \widetilde{z} - w| \leq \delta$ and from (1.4.2) again it follows that $u(z + z_0 - \widetilde{z}) - 2\varepsilon \leq f(w) - \varepsilon \leq u(z)$. Therefore $v(z) \leq u(z)$ if $\operatorname{dist}(z, \partial\Omega) \leq \delta/2$ and thus $v \leq u$. We obtain $u(\widetilde{z}) \geq v(\widetilde{z}) \geq u(z_0) - 2\varepsilon$ and it follows that u is lower semicontinuous. ■

Proposition 1.4.4. *Assume that $\Omega \Subset \mathbb{C}^n$ is a regular domain (as a domain in $\mathbb{R}^{2n}$) and $f \in C(\partial\Omega)$. Then $u^*_{f,\Omega}|_{\partial\Omega} \leq f$ and $u_{f,\Omega}$ is plurisubharmonic in Ω. If $f_j \in C(\partial\Omega)$ are such that $f_j \downarrow f$ then $u_{f_j,\Omega} \downarrow u_{f,\Omega}$. If f satisfies*

$$\text{(1.4.3)} \qquad \exists\ v \in PSH(\Omega) \cap C(\overline{\Omega}) \text{ such that } v|_{\partial\Omega} = f$$

then $u_{f,\Omega} \in PSH(\Omega) \cap C(\overline{\Omega})$.

Proof. We can find $h \in H(\Omega) \cap C(\overline{\Omega})$ such that $h = f$ on $\partial\Omega$. It follows that $u_{f,\Omega} \leq h$ and thus $u^*_{f,\Omega}|_{\partial\Omega} \leq f$ and $u_{f,\Omega} \in PSH(\Omega)$. The sequence $u_{f_j,\Omega}$ is decreasing to some $u \in PSH(\Omega)$ such that $u \geq u_{f,\Omega}$. But $u^*|_{\partial\Omega} \leq f$ and thus $u = u_{f,\Omega}$. The last part of the proposition follows from Theorem 1.4.3. ■

If $\Omega \Subset \mathbb{C}^n$ fulfills (1.4.3) for every $f \in C(\partial\Omega)$ then it is called *B-regular*. The following characterization of B-regular domains is due to Sibony [Sib].

Theorem 1.4.5. *For a bounded domain Ω in $\mathbb{C}^n$ the following are equivalent*

i) Every boundary point of Ω admits a strong plurisubharmonic barrier;

ii) Ω is B-regular;

iii) There exists a continuous plurisubharmonic function ψ in Ω such that $\lim_{z \to \partial\Omega} \psi(z) = 0$ and the function $\psi(z) - |z|^2$ is plurisubharmonic (i.e. ψ is "uniformly" strongly plurisubharmonic in Ω).

Proof. Observe that every condition implies that Ω is regular. Of course i) follows from ii). To prove i)⇒ii) assume that Ω is B-regular and let $f \in C(\partial\Omega)$. Set $u := u_{f,\Omega}$. By Theorem 1.4.3 it is enough to show that $u^* = u_* = f$ on $\partial\Omega$. By Proposition 1.4.4 $u^*|_{\partial\Omega} \leq f$. Fix $z_0 \in \partial\Omega$ and $\varepsilon > 0$. By i) there is $v \in PSH(\Omega)$ such that $v^*|_{\overline{\Omega}\setminus\{z_0\}} < 0$ and $\lim_{z \to z_0} v(z) = 0$. Then $f(z_0) + Av^* - \varepsilon \leq f$ on $\partial\Omega$ for A big enough, thus $f(z_0) + Av - \varepsilon \leq u$ in Ω. In particular $u_*(z_0) \geq f(z_0) - \varepsilon$ and ii) follows.

If Ω is B-regular then we can find $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u(z) = -|z|^2$ for $z \in \partial\Omega$. Then the function $\psi(z) = u(z) + |z|^2$ gives iii). Assume therefore that iii) holds and it remains to show that Ω is B-regular. Take $f \in C(\partial\Omega)$ and set $u := u_{f,\Omega}$. By Theorem 1.4.3 and Proposition 1.4.4 it is enough to show that $u_*|_{\partial\Omega} \geq f$. For every $\varepsilon > 0$ we can find a smooth g in a neighborhood of $\overline{\Omega}$ such that $f \leq g \leq f + \varepsilon$ on $\partial\Omega$. For A big enough $g + A\psi \in PSH(\Omega)$ and thus $g + A\psi - \varepsilon \leq u$ which implies that $u_*|_{\partial\Omega} \geq g - \varepsilon \geq f - \varepsilon$. ■

A domain Ω in $\mathbb{C}^n$ is called *hyperconvex* if there exists a negative plurisubharmonic exhaustion function in Ω, that is $u \in PSH(\Omega)$, $u < 0$ in Ω such that for every $c < 0$ we have $\{u < c\} \Subset \Omega$. If Ω is bounded then hyperconvexity can be characterized as follows.

Theorem 1.4.6. *For a bounded domain Ω in $\mathbb{C}^n$ the following are equivalent*
i) Every boundary point of Ω admits a (global) weak plurisubharmonic barrier.
ii) There exists a continuous strongly plurisubharmonic exhaustion function in Ω.

Proof. Obviously ii) implies i). Assume therefore that i) holds and let $K \Subset \Omega$ be a closed euclidean ball (or any other compact subset of Ω such that $\Omega \setminus K$ is regular in the real sense). Set

$$u := \sup\{v \in PSH(\Omega) : v \le 0, v|_K \le -1\}.$$

We have $\lim_{z\to\partial\Omega} u(z) = 0$ and from Theorem 1.4.3 applied to the domain $\Omega \setminus K$ it follows that u is continuous. Put $\psi(z) := -((|z|^2 - M)u(z))^{1/2}$, where $M > 0$ is such that $|z|^2 - M < 0$ for $z \in \Omega$. Then ψ is strongly plurisubharmonic in Ω. ■

Exercise Polydisks in $\mathbb{C}^n$, $n \ge 2$, are hyperconvex but not B-regular.

So, contrary to the real case (Theorem 1.2.8), there is no equivalence between the existence of weak and strong plurisubharmonic barriers.

Ω is called *pseudoconvex* if there exists $\psi \in PSH(\Omega)$ such that $\lim\limits_{z\to\partial\Omega} \psi(z) = \infty$. It can be shown that Ω is pseudoconvex iff the function $-\log \operatorname{dist}(z, \partial\Omega)$ is plurisubharmonic in Ω and the famous result obtained independently by Oka, Bremermann and Norguet states that this is equivalent to the fact that Ω is a domain of holomorphy (see e.g. [Hör1]). If $n = 1$ then all domains are pseudoconvex.

Exercise $\Omega = \{(z,w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\}$ is called a *Hartogs triangle*. Show that Ω is a regular pseudoconvex but not hyperconvex domain.

The next result says that hyperconvexity is a local property of a boundary. It is due to Kerzman and Rosay [KR] and the proof we present is taken from [Dem1]. An analoguous result for B-regular domains is obvious (since local strong barriers immediately give global strong barriers) and for pseudoconvex domains it can be found for example in [Hör1].

Theorem 1.4.7. *Suppose that Ω is bounded domain in $\mathbb{C}^n$ such that for every $z_0 \in \partial\Omega$ there exists a neighborhood U of z_0 such that $\Omega \cap U$ is hyperconvex. Then Ω is hyperconvex.*

Proof. There are domains $U_1, \dots, U_p$ such that $\partial\Omega \subset \bigcup_j U_j$ and $\Omega \cap U_j$ are hyperconvex. Let u_j be negative plurisubharmonic continuous functions in $\Omega \cap U_j$ and such that $\lim_{z\to\partial\Omega} u_j(z) = 0$. Choose domains $U_j' \Subset U_j$ such that $\partial\Omega \subset \bigcup_j U_j'$. By Lemma A2.4 there is a convex, increasing function $\chi : (-\infty, 0) \longrightarrow (0, +\infty)$ such that $\lim_{t\to 0^-} \chi(t) = +\infty$ and $|\chi \circ u_j - \chi \circ u_k| \leq 1$ on $U'_j \cap U'_k \cap \Omega$. (To use Lemma A2.4 we set

$$f(t) := \max\{u_j(z) : z \in \overline{U'_j} \cap \Omega,\ j = 1, \dots, p,\ \mathrm{dist}(z, \partial\Omega) \geq -\varepsilon\},$$
$$g(t) := \min\{u_j(z) : z \in \overline{U'_j} \cap \Omega,\ j = 1, \dots, p,\ \mathrm{dist}(z, \partial\Omega) \leq -\varepsilon\}.)$$

From the convexity of χ it follows that

$$|\chi(u_j(z) - \varepsilon) - \chi(u_k(z) - \varepsilon)| \leq 1, \quad \varepsilon > 0,\ z \in U'_j \cap U'_k \cap \Omega.$$

Let $U''_j \Subset U'_j$ be such that $\overline{\Omega} \setminus V \subset \bigcup_j U''_j$ for some $V \Subset \Omega$ and take smooth φ_j with $\operatorname{supp} \varphi_j \subset U'_j$, $0 \leq \varphi_j \leq 1$ and $\varphi_j = 1$ in a neighborhood of $\overline{U''_j}$. Moreover there are constants M, λ such that $|z|^2 - M \leq 0$ in Ω and $\varphi_j + \lambda(|z|^2 - M)$ is plurisubharmonic for every j. Set

$$v_{j,\varepsilon}(z) := \chi(u_j(z) - \varepsilon) + \varphi_j(z) - 1 + \lambda(|z|^2 - M).$$

We have $v_{j,\varepsilon} \leq v_{k,\varepsilon}$ in a neighborhood of $\partial U'_j \cap \overline{U''_k} \cap \Omega$, thus $v_\varepsilon(z) := \max\{v_{j,\varepsilon}(z), \chi(a) - 1 + \lambda(|z|^2 - M)\}$ is plurisubharmonic in Ω, where a is such that $\sup_{V \cap U_j} u_j < a < 0$ and ε is small enough. Then $w_\varepsilon := v_\varepsilon / \chi(-\varepsilon) - 1$ is ≤ 0, $\geq -\lambda M / \chi(-\varepsilon)$ on $\partial\Omega$ and $\leq (\chi(a) - 1)/\chi(-\varepsilon) - 1$ on $V \setminus \bigcup_j U'_j$. It follows that the function u defined in the proof of Theorem 1.4.8 satisfies $\lim_{z \to \partial\Omega} u(z) = 0$. ∎

A domain Ω in $\mathbb{C}^n$ is called *balanced* if $z \in \Omega$, $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ implies $\lambda z \in \Omega$. The function

$$f_\Omega(z) := \inf\{t > 0 : t^{-1} z \in \Omega\}$$

is called a *Minkowski functional* of Ω. Since Ω is open, one can show that f_Ω is upper semicontinuous in $\mathbb{C}^n$ and $\Omega = \{f_\Omega < 1\}$. The following result is due to Siciak [Sic4].

Theorem 1.4.8. *For a balanced domain Ω in $\mathbb{C}^n$ the following are equivalent*

i) Ω is pseudoconvex;

ii) $\log f_\Omega \in PSH(\mathbb{C}^n)$;

iii) Ω is convex with respect to homogeneous polynomials, that is, if $K \subset \Omega$ is compact then the compact set

$$\widehat{K}^H = \{z \in \mathbb{C}^n : |Q(z)| \leq ||Q||_K, \ \text{ for all homogeneous polynomials } Q \text{ in } \mathbb{C}^n\}$$

is contained in Ω.

Proof. If iii) holds then Ω is in particular holomorphically convex, thus it is a domain of holomorphy and pseudoconvex. Obviously ii) implies that $\Omega = \{\log f_\Omega < 0\}$ is pseudoconvex. Assume therefore that Ω is a domain of holomorphy and it is enough to show that ii) and iii) hold. There exists a holomorphic function F in Ω which cannot be continued holomorphically beyond Ω. Since Ω is balanced, expanding F in the Taylor series about the origin gives $F=\sum_{j=0}^{\infty} Q_j$, where Q_j are homogeneous polynomials of degree j and the series $\sum Q_j$ converges pointwise in Ω. Set $\psi := \limsup_{j\to\infty} |Q_j|^{1/j}$. By the Cauchy criterion we have $\Omega \subset \{\psi \le 1\}$.

We claim that the sequence $|Q_j|^{1/j}$ is locally uniformly bounded above. Indeed, for $m\ge 1$ the sets $E_m := \bigcap_j \{|Q_j|\le m\}$ are closed, increasing and $\Omega\subset\bigcup_m E_m$. By the Baire theorem for some m the set E_m has nonempty interior. Thus $|Q_j|\le m$ in $B(z_0,r)$. If $z\in B(0,r)$ then for $\lambda\in\mathbb{C}$, $|\lambda|=1$ we have

$$|Q_j(z_0+\overline{\lambda}z)| = |Q_j(\lambda z_0 + z)| \le m,$$

since Q_j are homogeneous. From the maximum principle for holomorphic functions of one variable we deduce that the last inequality is valid also for $\lambda=0$ and therefore $|Q_j|\le m$ in $B(0,r)$. It follows that $|Q_j(z)|^{1/j}\le m^{1/j}|z|/r$ for every $z\in\mathbb{C}^n$, which proves the claim.

It is easy to show that the series $\sum Q_j$ is locally uniformly convergent, and thus is a holomorphic function in $\{\psi^* < 1\}$. Since F is not extendable beyond Ω, it follows that $\{\psi^*<1\}\subset\Omega\subset\{\psi\le 1\}$. From the fact that ψ is homogeneous of degree 1 one can deduce that $\mathrm{int}\{\psi\le 1\}=\{\psi^*<1\}$, hence $\Omega=\{\psi^*<1\}$ and $\psi^*=f_\Omega$. The functions $u_k=(\sup_{j\ge k}\frac{1}{j}\log|Q_j|)^*$ are plurisubharmonic in $\mathbb{C}^n$ and $u_k\downarrow \log f_\Omega$ as $k\uparrow\infty$, which gives ii).

To show iii) let $K\subset\Omega$ be compact. There is a such that $\max_K \psi^* < a < 1$. We have $\max_K e^{u_k}\downarrow \max_K\psi^*$ as $k\uparrow\infty$, thus there is j_0 such that for every $j\ge j_0$ we have $|Q_j|^{1/j}\le a$ on K. Then

$$\widehat{K}^H \subset \bigcap_{j\ge j_0}\{|Q_j|\le ||Q_j||_K\} \subset \bigcap_{j\ge j_0}\{|Q_j|^{1/j}\le a\}\subset\{\psi\le a\}\subset\Omega$$

and iii) follows. ∎

A function $u\in PSH(\Omega)$ is called *maximal* if for every $v\in PSH(\Omega)$ such that $v\le u$ outside a compact subset of Ω we have $v\le u$ in Ω. If $n=1$ then maximal means precisely harmonic. If $n\ge 2$ then it is easy to show that for example plurisubharmonic functions independent of one variable are maximal.

Exercise i) The function $\log|z|$ is maximal in $\mathbb{C}^n\setminus\{0\}$ but not in $\mathbb{C}^n$.

ii) Let F be a holomorphic function on an open subset of $\mathbb{C}^n$, $n\ge 2$. Then for $\alpha\ge 0$ the functions $|F|^\alpha$ and $\log|F|$ are maximal. This is false if $n=1$.

Proposition 1.4.9. *i) A decreasing sequence of maximal plurisubharmonic functions converges either to a maximal function or to $-\infty$;*

ii) If u is maximal in an open Ω then for every $G\Subset\Omega$ there is a sequence of continuous maximal plurisubharmonic functions in G decreasing to u.

Proof. i) Follows directly from the definition.

ii) We may assume that G is smooth, in particular regular. Let v_j be a sequence of continuous plurisubharmonic functions in a neighborhood of $\overline{G}$ decreasing to u. Set $u_j := u_{v_j|_{\partial G}, G}$. Then u_j is a decreasing sequence of continuous functions on $\overline{G}$, maximal in G. From i) it easily follows that $u_j \downarrow u$. ∎

The following counterpart of Corollary 1.2.4 can be easily obtained from Proposition 1.4.4.

Proposition 1.4.10. *Assume that u is plurisubharmonic in Ω and let $G \Subset \Omega$ be a regular domain. Then there is $\widehat{u} \in PSH(\Omega)$ such that $\widehat{u} \geq u$, $\widehat{u} = u$ on $\Omega \setminus G$ and $\widehat{u}$ is maximal in G. If $u_j \downarrow u$ then $\widehat{u}_j \downarrow \widehat{u}$. If u is continuous then so is $\widehat{u}$.* ∎

II. The complex Monge-Ampère operator

2.1. The definition and basic properties

We start with a formula which is useful when one integrates by parts.

Proposition 2.1.1. *If* $\Psi \in C^\infty_{(p,p)}$ *and* $T \in \mathcal{D}'_{(q,q)}$ *with* $p+q=n-1$ *then*

$$\Psi \wedge dd^cT - dd^c\Psi \wedge T = d\left(\Psi \wedge d^cT - d^c\Psi \wedge T\right).$$

Proof. We have

$$d\left(\Psi \wedge d^cT - d^c\Psi \wedge T\right) = d\Psi \wedge d^cT + \Psi \wedge dd^cT - dd^c\Psi \wedge T + d^c\Psi \wedge dT$$

and, since $p+q+1=n$,

$$d\Psi \wedge d^cT = i\left(\partial\Psi \wedge \overline{\partial}T - \overline{\partial}\Psi \wedge \partial T\right) = -d^c\Psi \wedge dT. \blacksquare$$

By Proposition 2.1.1 for every current $T \in \mathcal{D}'_{(q,q)}(\Omega)$, Ω open in $\mathbb{C}^n$, we have

$$dd^cT(\Psi) = T(dd^c\Psi), \quad \Psi \in C^\infty_{0,(n-q-1,n-q-1)}(\Omega). \tag{2.1.1}$$

Let T be a nonnegative closed current of bidegree (q,q) and u a locally bounded plurisubharmonic function on Ω. By Theorem 1.3.5 the coefficients of T are complex measures and thus uT is a well defined current. We define

$$dd^cu \wedge T := dd^c(uT).$$

By (2.1.1)

$$\int_\Omega dd^cu \wedge T \wedge \Psi = \int_\Omega u\,T \wedge dd^c\Psi, \quad \Psi \in C^\infty_{0,(n-q-1,n-q-1)}(\Omega). \tag{2.1.2}$$

Proposition 2.1.2. $dd^cu \wedge T$ *is a nonnegative closed current of bidegree* $(q+1,q+1)$.

Proof. It is enough to show the nonnegativity. If $|u| \leq M$ then $|u_\varepsilon| \leq M$, where $u_\varepsilon = u * \rho_\varepsilon$. By the Lebesgue bounded convergence theorem $u_\varepsilon T \longrightarrow uT$ weakly, hence $dd^c(u_\varepsilon T) \longrightarrow dd^c(uT)$ weakly. Since u_ε is smooth, we have $dd^c(u_\varepsilon T) = dd^c u_\varepsilon \wedge T$ in the usual sense and $dd^c u_\varepsilon \wedge T \geq 0$ by Theorem 1.3.6. ∎

Therefore we may define inductively a nonnegative closed current

$$dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T \in \mathcal{D}'_{(p+q,p+q)}$$

for $u_1, \ldots, u_p \in PSH \cap L^\infty_{\text{loc}}$ and $T \in \mathcal{D}'_{(q,q)}$ with $T \geq 0$, $dT = 0$, $p + q \leq n$. In particular, we may take $T = dd^c v$, where v is an arbitrary plurisubharmonic function.

Proposition 2.1.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ and let $u, v \in PSH \cap L^\infty_{\text{loc}}(\Omega)$ be such that $u, v \leq 0$, $\lim_{z \to \partial\Omega} u(z) = 0$ and $\int_\Omega dd^c v \wedge T < \infty$. Assume that $T \in \mathcal{D}'_{(n-1,n-1)}(\Omega)$ is nonnegative and closed. Then*

$$\int_\Omega v\, dd^c u \wedge T \leq \int_\Omega u\, dd^c v \wedge T.$$

In particular, if in addition $\lim_{z \to \partial\Omega} v(z) = 0$ and $\int_\Omega dd^c u \wedge T < \infty$,

$$\int_\Omega v\, dd^c u \wedge T = \int_\Omega u\, dd^c v \wedge T.$$

Proof. For $\varepsilon > 0$ set $u_\varepsilon := \max\{u, -\varepsilon\}$. Then by the Lebesgue monotone convergence theorem

$$\int_\Omega u\, dd^c v \wedge T = \lim_{\varepsilon \to 0} \int_\Omega (u - u_\varepsilon) dd^c v \wedge T$$

and

$$\int_\Omega (u - u_\varepsilon) dd^c v \wedge T = \lim_{j \to \infty} \int_\Omega (u - u_\varepsilon) * \rho_{1/j}\, dd^c v \wedge T.$$

Let $\Omega' \Subset \Omega$ be such that $\{u - u_\varepsilon \neq 0\} \Subset \Omega' \Subset \Omega$. From (2.1.2) for j big enough we infer

$$\int_\Omega (u - u_\varepsilon) * \rho_{1/j}\, dd^c v \wedge T = \int_\Omega v\, dd^c \left((u - u_\varepsilon) * \rho_{1/j} \right) \wedge T \geq \int_{\Omega'} v\, dd^c (u * \rho_{1/j}) \wedge T$$

and the proposition follows from Lemma A2.1. ∎

The next estimate is called the Chern-Levine-Nirenberg inequality [CLN].

Theorem 2.1.4. *If $K \Subset \Omega$ then*

$$||dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T||_K \leq C_{K,\Omega} ||u_1||_{L^\infty(\Omega)} \ldots ||u_p||_{L^\infty(\Omega)} ||T||_\Omega$$

for $u_1,\dots,u_p\in PSH\cap L^\infty(\Omega)$ *and* $T\in\mathcal{D}'_{(q,q)}(\Omega)$ *with* $T\geq 0$, $dT=0$, $p+q\leq n$.

Proof. We may assume that $p=1$. Take $\varphi\in C_0^\infty(\Omega)$ such that $\varphi\geq 0$ and $\varphi=1$ on K. If $\beta\in\mathbb{C}_{(n-q-1,n-q-1)}$ is as in Lemma 1.3.8 then

$$||dd^cu\wedge T||_K\leq C'\int_K dd^cu\wedge T\wedge\beta\leq C'\int_\Omega \varphi\, dd^cu\wedge T\wedge\beta.$$

By (2.1.2) and since $dd^c(\varphi\beta)=dd^c\varphi\wedge\beta$,

$$\int_\Omega\varphi\, dd^cu\wedge T\wedge\beta=\int_\Omega u\,T\wedge dd^c\varphi\wedge\beta\leq C''||u||_{L^\infty(\Omega)}||T||_\Omega. \blacksquare$$

Exercise Assume that $v\in PSH(\Omega)$ and $K\Subset\Omega$. Show that

$$||dd^cv||_K\leq C_{K,\Omega}||v||_{L^1(\Omega)}.$$

The following approximation theorem is due to Bedford and Taylor [BT2].

Theorem 2.1.5. *Let* $u_0^j,u_1^j,\dots,u_p^j\in PSH\cap L^\infty_{\mathrm{loc}}$, $0\leq p\leq n$, $j=1,2,\dots,$ *be sequences decreasing to* $u_0,\dots,u_p\in PSH\cap L^\infty_{\mathrm{loc}}$ *respectively. Let* T *be a closed nonnegative current of bidegree* (q,q), $p+q\leq n$. *Then*

$$u_0^jdd^cu_1^j\wedge\dots\wedge dd^cu_p^j\wedge T\longrightarrow u_0dd^cu_1\wedge\dots\wedge dd^cu_p\wedge T$$

weakly.

Proof. Suppose that u_k^j and T are defined in a neighborhood of $\overline{B}$, where $B=B(z_0,r)$. We may assume that for some positive constant M we have $-M\leq u_k^j\leq -1$ in a neighborhood of $\overline{B}$. If we take $B'\Subset B$ and $\psi(z):=|z-z_0|^2-r^2$ then for A big enough $\max\{u_k^j,A\psi\}=u_k^j$ on B' and $\max\{u_k^j,A\psi\}=A\psi$ in a constant neighborhood of ∂B. We may therefore assume that $u_k^j=u_k=A\psi$ in a neighborhood of ∂B.

The further proof is by induction with respect to p. The theorem is obviously true if $p=0$. Let $p\geq 1$ and assume the theorem holds for $p-1$. It follows that

$$S^j:=dd^cu_1^j\wedge\dots\wedge dd^cu_p^j\wedge T\longrightarrow dd^cu_1\wedge\dots\wedge dd^cu_p\wedge T=:S$$

weakly. By the Chern-Levine-Nirenberg inequality (Theorem 2.1.4) the sequence S^j is relatively compact in the weak* topology. It therefore remains to show that if $u_0^jS^j\longrightarrow\Theta$ weakly then $\Theta=u_0S$.

By Lemma A2.2 we have $\Theta \wedge \alpha \le u_0 S \wedge \alpha$ for every elementary nonnegative α of bidegree $(n-p-q, n-p-q)$, hence $u_0 S - \Theta \ge 0$. By Corollary 1.3.9 it is enough to show that $\int_B (u_0 S - \Theta) \wedge \omega^{n-p-q} \le 0$. By Proposition 2.1.3

$$\begin{aligned}\int_B u_0 dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T \wedge \omega^{n-p-q} &\le \int_B u_0^j dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T \wedge \omega^{n-p-q} \\ &= \int_B u_1 dd^c u_0^j \wedge dd^c u_3 \wedge \dots \wedge dd^c u_p \wedge T \wedge \omega^{n-p-q} \\ &\le \dots \le \int_B u_0^j dd^c u_1^j \wedge \dots \wedge dd^c u_p^j \wedge T \wedge \omega^{n-p-q}\end{aligned}$$

and the theorem follows from Lemma A2.1. ■

Remark. The above theorem is much easier to prove if all considered functions are continuous. For then the convergence $u_k^j \longrightarrow u_k$, $k = 0, \dots, p$, is uniform and we may write

$$u_0^j S^j - u_0 S = (u_0^j - u_0) S^j + u_0 (S^j - S).$$

It is easy to show that both terms tend weakly to 0.

From Theorem 2.1.5 it follows in particular that for every nonnegative closed current $T \in \mathcal{D}'_{(q,q)}(\Omega)$ the mapping

$$(PSH \cap L^\infty_{\text{loc}}(\Omega))^p \ni (u_1, \dots, u_p) \longmapsto dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T \in \mathcal{D}'_{(p+q,p+q)}(\Omega)$$

is symmetric.

One can easily compute that

$$(dd^c u)^n = n! 4^n \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) d\lambda \tag{2.1.3}$$

if $u \in C^2$. We have defined the left hand-side of (2.1.3) if $u \in PSH \cap L^\infty_{\text{loc}}$. The right hand-side of (2.1.3) is a nonnegative Radon measure if $u \in PSH \cap W^{2,n}$ (that is $D^\alpha u \in L^n_{\text{loc}}$ if $|\alpha| = 2$).

Proposition 2.1.6. *(2.1.3) holds if u is a $W^{2,n}$ locally bounded plurisubharmonic function.*

Proof. By Theorem 2.1.5 it is enough to show that if $u_\varepsilon = u * \rho_\varepsilon$ then

$$\det \left(\frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} \right) \longrightarrow \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right)$$

weakly. In fact, it is easy to see that we even have convergence in L^1_{loc} using the following fact which is a consequence of the Hölder inequality: if $f^k_\varepsilon \longrightarrow f^k$ in L^n_{loc}, $k = 1, \dots, n$, then $f^1_\varepsilon \dots f^n_\varepsilon \longrightarrow f^1 \dots f^n$ in L^1_{loc}. ■

$(dd^c)^n$ is called the *complex Monge-Ampère operator* and we have defined it for locally bounded plurisubharmonic functions. The following exercise shows that a good definition of $(dd^c u)^n$ as a nonnegative Borel measure for an arbitrary plurisubharmonic function u is not possible.

Exercise For $a \in (0,1)$ set

$$u(z) := (-\log|z_1|)^a\,(|z_2|^2 + \cdots + |z_n|^2 - 1).$$

Compute the following

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\right) = \frac{a(-\log|z_1|)^{na-2}}{4|z_1|^2}(1 - a - |z_2|^2 - \cdots - |z_n|^2)$$

if $z_1 \neq 0$. Conclude that u is plurisubharmonic on the set

$$\{|z_1| < 1, |z_2|^2 + \cdots + |z_n|^2 < 1 - a\}.$$

Show that if $a \geq 1/n$ then

$$\int\limits_{B(0,\varepsilon)\setminus\{z_1=0\}} \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) d\lambda = \infty.$$

The above example is due to Kiselman [Kis]. The first example of this kind has been constructed by Shiffman and Taylor (see [Siu]).

Exercise Show that $(dd^c \log^+|z|)^n = (2\pi)^n d\sigma/\sigma(\partial B)$, where $d\sigma$ is the surface measure of the unit sphere.

The following estimate is essentially due to Cegrell [Ceg, Proposition 6.2] (see also [Dem2, Theorem 1.8]).

Theorem 2.1.7. *If $K \Subset \Omega$ then for every $v \in PSH(\Omega)$ and $u_1, \ldots, u_p \in PSH \cap L^\infty(\Omega)$, $p \leq n$ we have*

$$||v\, dd^c u_1 \wedge \cdots \wedge dd^c u_p||_K \leq C_{K,\Omega} ||v||_{L^1(\Omega)} ||u_1||_{L^\infty(\Omega)} \ldots ||u_p||_{L^\infty(\Omega)}.$$

Proof. Similarly as before we may reduce the problem to the following situation: $\Omega = B$ is the unit ball, $v \le 0$, $\psi \le u_k \le 0$ on B and $u_k = \psi$ in a neighborhood of ∂B, $k = 1, \dots, p$. If $T \in \mathcal{D}'_{(n-1,n-1)}(\Omega)$ is nonnegative and closed then

$$\begin{aligned}
-\int_B v\, dd^c u_k \wedge T &= -\int_B v\, dd^c\psi \wedge T - \int_B v\, dd^c(u_k - \psi) \wedge T \\
&= -\int_B v\, dd^c\psi \wedge T - \int_B (u_k - \psi) dd^c u_k \wedge T \\
&\le -\int_B v\, dd^c\psi \wedge T.
\end{aligned}$$

This implies that

$$-\int_K v\, dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge \omega^{n-p} \le -\int_B v(dd^c\psi)^p \wedge \omega^{n-p} = -C\int_B v\, d\lambda$$

and the theorem follows from Lemma 1.3.8. ■

Exercise Show that if f is C^1 then $df \wedge d^c f \ge 0$ but $(df \wedge d^c f)^2 = 0$.

Theorem 2.1.8. *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that $u_1, \dots, u_n, v, w \in PSH \cap L^\infty(\Omega)$ are such that $u_1, \dots, u_n \le 0$, $v \le w$ and $\lim_{z \to \partial\Omega}(w(z) - v(z)) = 0$. Then*

$$\begin{aligned}
&\int_\Omega (w - v)^n dd^c u_1 \wedge \dots \wedge dd^c u_n \\
&\qquad \le n! ||u_1||_{L^\infty(\Omega)} \dots ||u_{n-1}||_{L^\infty(\Omega)} \int_\Omega |u_n| (dd^c v)^n
\end{aligned} \tag{2.1.4}$$

and, for every $p > n$,

$$\begin{aligned}
&\int_\Omega (w - v)^p dd^c u_1 \wedge \dots \wedge dd^c u_n \\
&\qquad \le p(p-1)\dots(p-n+1) ||u_1||_{L^\infty(\Omega)} \dots ||u_n||_{L^\infty(\Omega)} \int_\Omega (w - v)^{p-n} (dd^c v)^n.
\end{aligned} \tag{2.1.5}$$

Proof. For $\varepsilon > 0$ set $w_\varepsilon = \max\{v, w - \varepsilon\}$, then $w_\varepsilon \uparrow w$ as $\varepsilon \downarrow 0$ and $w_\varepsilon = v$ in a neighborhood of $\partial\Omega$. By the Lebesgue monotone convergence theorem we may therefore assume that $w = v$ in a neighborhood of $\partial\Omega$. Set $v_j := v * \rho_{1/j}$ and $w_j := w * \rho_{1/j}$. By Theorem 2.1.5

$$|u_n|(dd^c v_j)^n \longrightarrow |u_n|(dd^c v)^n$$

weakly, and

$$(w_j - v_j)^{p-n}(dd^c v_j)^n \longrightarrow (w - v)^{p-n}(dd^c v)^n, \tag{2.1.6}$$

provided that $p = n+1$. Right now, when proving (2.1.5), we shall restrict ourselves only to this case and postpone the general one to Section 2.2, after we have shown that plurisubharmonic functions are quasi-continuous. In the proof of quasi-continuity we will use (2.1.5) only for $p = n+1$.

We may therefore reduce the proof to the case when w, v are smooth and equal near $\partial\Omega$. For a nonnegative closed $T \in \mathcal{D}'_{(n-1,n-1)}(\Omega)$ we then have

$$\int_\Omega (w-v)^p dd^c u_1 \wedge T = \int_\Omega u_1 dd^c (w-v)^p \wedge T.$$

Since

$$\begin{aligned} -dd^c(w-v)^p &= -p(p-1)(w-v)^{p-2} d(w-v) \wedge d^c(w-v) - p(w-v)^{p-1} dd^c(w-v) \\ &\le p(w-v)^{p-1} dd^c v, \end{aligned}$$

we obtain

$$\int_\Omega (w-v)^p dd^c u_1 \wedge T \le p \int_\Omega |u_1| (w-v)^{p-1} dd^c v \wedge T \le p ||u_1||_{L^\infty(\Omega)} \int_\Omega (w-v)^{p-1} dd^c v \wedge T.$$

Iteration of this easily gives (2.1.4) and (2.1.5) (provided that (2.1.6) holds). ∎

The first part of Theorem 2.1.8 was proved in [Bło1].

Let u, v be plurisubharmonic and locally bounded. For a nonnegative closed current T of bidegree $(n-1, n-1)$ we want to define $du \wedge d^c v \wedge T$. If u, v are smooth then $du \wedge d^c v \wedge T = dv \wedge d^c u \wedge T$ (because they are of full degree), hence by polarization we may assume that $u = v$ and $u \ge 0$. Then we set

$$du \wedge d^c u \wedge T := \frac{1}{2} dd^c u^2 \wedge T - u \, dd^c u \wedge T$$

so that it agrees with the smooth case. In particular, $du \wedge d^c v \wedge T$ is a complex measure. Directly from this definition and Theorem 2.1.5 we obtain the following approximation result:

Theorem 2.1.9. *Let u_j, v_j and w_k^j, $k = 1, \dots, p$, be sequences of locally bounded plurisubharmonic functions decreasing to $u, v, w_k \in PSH \cap L^\infty_{\text{loc}}$ respectively. Then, if T is a nonnegative closed current of bidegree $(n-p-1, n-p-1)$, we have the weak convergence of measures*

$$du_j \wedge d^c v_j \wedge dd^c w_1^j \wedge \dots \wedge dd^c w_p^j \longrightarrow du \wedge d^c v \wedge dd^c w_1 \wedge \dots \wedge dd^c w_p. \ \blacksquare$$

The next two theorems were proved in [Bło2]:

Theorem 2.1.10. *Assume that* $u, v \in PSH \cap L^\infty_{\mathrm{loc}}$ *and* $2 \le p \le n$. *Then*

$$(dd^c \max\{u,v\})^p = dd^c \max\{u,v\} \wedge \sum_{k=0}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-1-k} - \sum_{k=1}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-k}.$$

Proof. We may assume that u, v are smooth. A simple inductive argument reduces the proof to the case $p = 2$. Set $w := \max\{u, v\}$ and, for $\varepsilon > 0$, $w_\varepsilon := \max\{u + \varepsilon, v\}$. In an open set $\{u + \varepsilon > v\}$ we have $w_\varepsilon - u = \varepsilon$, whereas $w - v = 0$ in $\{u < v\}$. Therefore $dd^c(w_\varepsilon - u) \wedge dd^c(w - v) = 0$ for every $\varepsilon > 0$ and taking the limit we conclude that $dd^c(w - u) \wedge dd^c(w - v) = 0$. ∎

Theorem 2.1.11. *Assume that* u_k, $k = 1, 2$, *is a nonnegative plurisubharmonic function in a domain* $\Omega_k \subset \mathbb{C}^{n_k}$, *such that*

$$\int_{\{u_k > 0\}} (dd^c u_k)^{n_k} = 0, \quad k = 1, 2.$$

Then, treating u_1, u_2 *as functions on* $\Omega_1 \times \Omega_2$, *we have*

$$(dd^c \max\{u_1, u_2\})^{n_1+n_2} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.$$

Proof. Set $w := \max\{u_1, u_2\}$ and $\alpha := u_1 - u_2$. Since $(dd^c u_k)^{n_k+1} = 0$, $k = 1, 2$, and by Theorem 2.1.10 we have

$$(2.1.7) \qquad \begin{aligned}(dd^c w)^{n_1+n_2} = dd^c w \wedge &\left[(dd^c u_1)^{n_1-1} \wedge (dd^c u_2)^{n_2} + (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2-1}\right] \\ &- (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.\end{aligned}$$

Let $\chi : \mathbb{R} \to [0, +\infty)$ be smooth and such that $\chi(x) = 0$ if $x \le -1$, $\chi(x) = x$ if $x \ge 1$ and $0 \le \chi' \le 1$, $\chi'' \ge 0$ everywhere. Define

$$\psi_j := u_2 + \frac{1}{j}\chi(j\alpha).$$

We can easily check that $\psi_j \downarrow w$ as $j \uparrow \infty$. An easy computation gives

$$dd^c(\chi(j\alpha)/j) = \chi'(j\alpha) dd^c\alpha + j\chi''(j\alpha) d\alpha \wedge d^c\alpha.$$

Therefore

$$dd^c\psi_j = \chi'(j\alpha) dd^c u_1 + (1 - \chi'(j\alpha)) dd^c u_2 + j\chi''(j\alpha) d\alpha \wedge d^c\alpha$$

and, in particular, ψ_j is plurisubharmonic.

Using the hypothesis on u_1, u_2 we may compute

$$\begin{aligned} dd^c\psi_j \wedge (dd^c u_1)^{n_1-1} \wedge (dd^c u_2)^{n_2} \\ &= \left[\chi'(0)(dd^c u_1)^{n_1} + j\chi''(ju_1) du_1 \wedge d^c u_1 \wedge (dd^c u_1)^{n_1-1}\right] \wedge (dd^c u_2)^{n_2} \\ &= dd^c(\chi(ju_1)/j) \wedge (dd^c u_1)^{n_1-1} \wedge (dd^c u_2)^{n_2}. \end{aligned}$$

Since $\chi(ju_1)/j \downarrow u_1$ as $j \uparrow \infty$, it follows that

$$dd^c w \wedge (dd^c u_1)^{n_1-1} \wedge (dd^c u_2)^{n_2} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$$

and, similarly,

$$dd^c w \wedge (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2-1} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.$$

This, together with (2.1.7), finishes the proof. ∎

2.2. Quasi-continuity of plurisubharmonic functions and applications

If Ω is open in $\mathbb{C}^n$ and E is a Borel subset of Ω, we define

$$c(E) = c(E,\Omega) := \sup\left\{\int_E (dd^c u)^n : u \in PSH(\Omega), -1 \le u \le 0\right\}.$$

c is called a *relative Monge-Ampère capacity*. As well as almost all results of this section it comes from [BT2]. By the Chern-Levine-Nirenberg inequality $c(E,\Omega)$ is finite if $E \Subset \Omega$.

Proposition 2.2.1. *i) If $\Omega \subset B(z_0, R)$ then $c(E,\Omega) \ge 4^n n! R^{-2n} \lambda(E)$;*
ii) If $E_1 \subset E_2 \subset \Omega_1 \subset \Omega_2$ then $c(E_1,\Omega_2) \le c(E_2,\Omega_1)$;
iii) $c(\bigcup_{j=1}^\infty E_j) \le \sum_{j=1}^\infty c(E_j)$;
iv) If $E \subset \omega \Subset \Omega_1 \subset \Omega_2 \Subset \mathbb{C}^n$ then $c(E,\Omega_1) \le C_{\omega,\Omega_1,\Omega_2} c(E,\Omega_2)$;
v) If $E_j \uparrow E$ then $\lim_{j\to\infty} c(E_j) = c(E)$.

Proof. i) It is enough to take $u(z) = |z - z_0|^2/R^2 - 1$.
ii) is clear.
iii) We may write

$$c(\bigcup E_j) = \sup_u \sum_j \int_{E_j} (dd^c u)^n \le \sum_j \sup_u \int_{E_j} (dd^c u)^n = \sum_j c(E_j).$$

iv) If we cover ω by finite number of balls contained in Ω_1 then using ii) and iii) we may reduce the problem to the case when $\omega = B(z_0, r)$ and $\Omega_1 = B(z_0, R_1)$ are concentric balls. Take $u \in PSH(\Omega_1)$ such that $-1 \le u \le 0$. If $\psi(z) = (R_1^2 - r^2)^{-1}(|z - z_0|^2 - R_1^2)$ then $\psi = 0$ on $\partial\Omega_1$ and $\psi \le -1$ on ω. Set

$$\widetilde{u} = \begin{cases} \max\{u, \psi\} & \text{on } \Omega_1, \\ \psi & \text{on } \Omega_2 \setminus \Omega_1 \end{cases}$$

and $v = (a+1)^{-1}(\widetilde{u} - a)$, where $a = ||\psi||_{\Omega_2}$. Then $v \in PSH(\Omega_2)$, $-1 \le v \le 0$ and $(dd^c v)^n = (a+1)^{-n}(dd^c u)^n$ on ω, hence $c(E, \Omega_1) \le (a+1)^{-n} c(E_2, \Omega_2)$.

v) We may write

$$\lim_{j\to\infty} c(E_j) = \sup_{j,u} \int_{E_j} (dd^c u)^n = c(E). \blacksquare$$

The following result shows that plurisubharmonic functions are quasi-continuous with respect to c.

Theorem 2.2.2. *Let v be a plurisubharmonic function on an open subset Ω of $\mathbb{C}^n$. Then for every $\varepsilon > 0$ there exists an open subset G of Ω such that $c(G, \Omega) < \varepsilon$ and v is continuous on $\Omega \setminus G$.*

For the proof of Theorem 2.2.2 we need two propositions.

Proposition 2.2.3. *If $v \in PSH(\Omega)$ and $K \Subset \Omega$ then*

$$\lim_{j\to\infty} c(K \cap \{v < -j\}, \Omega) = 0.$$

Proof. Take $u \in PSH(\Omega)$ with $-1 \le u \le 0$ and an open ω such that $K \Subset \omega \Subset \Omega$. By Theorem 2.1.7

$$\int\limits_{K\cap\{v<-j\}} (dd^c u)^n \le \frac{1}{j} \int\limits_K |v|(dd^c u)^n \le \frac{C_{K,\omega}}{j} ||v||_{L^1(\omega)}. \blacksquare$$

Proposition 2.2.4. *Let $v_j \in PSH \cap L^\infty_{\text{loc}}(\Omega)$ be a sequence decreasing to $v \in PSH \cap L^\infty_{\text{loc}}(\Omega)$. Then for every $K \Subset \Omega$ and $\delta > 0$*

$$\lim_{j\to\infty} c(K \cap \{v_j > v + \delta\}, \Omega) = 0.$$

Proof. We may easily reduce it to the case when Ω is a ball. Using Proposition 2.2.1.iv and similarly as in the proof of Theorem 2.1.5 we may also assume that $v_j = v = A\psi$ in a neighborhood of ∂B, where $B = B(z_0, R) = \Omega$ and $\psi(z) = |z - z_0|^2 - R^2$. Let $u \in PSH(B)$ be such that $-1 \leq u \leq 0$. Then by Theorem 2.1.8 (with $p = n + 1$)

$$\int\limits_{K\cap\{v_j>v+\delta\}} (dd^c u)^n \leq \frac{1}{\delta^{n+1}} \int\limits_B (v_j - v)^{n+1} (dd^c u)^n \leq \frac{1}{\delta^{n+1}} \int\limits_B (v_j - v)(dd^c v)^n$$

and the last term tends to 0 by the Lebesgue monotone convergence theorem. ∎

Proof of Theorem 2.2.2. Take $\omega \Subset \Omega$. We claim that it is enough to show that there exists an open $G \subset \omega$ such that $c(G, \Omega) < \varepsilon$ and v is continuous on $\omega \setminus G$. Indeed, we may then take $\omega_j \Subset \Omega$ with $\omega_j \uparrow \Omega$ and open $G_j \subset \omega_j$ such that $c(G_j, \Omega) < 2^{-j}\varepsilon$ and v is continuous on $\omega_j \setminus G_j$. Setting $G = \bigcup G_j$ we obtain $c(G) < \varepsilon$ and for every open $U \Subset \Omega$ we have $U \setminus G \subset \omega_j \setminus G_j$ for some j, thus v is continuous on $\Omega \setminus G$.

Let $G_1 = \omega \cap \{v < -j\}$, where j is such that $c(G_1, \Omega) < \varepsilon/2$ (by Proposition 2.2.3). Set $\widetilde{v} = \max\{v, -j\}$ and let v_k be a sequence of continuous plurisubharmonic functions defined in a neighborhood of $\overline{\omega}$ decreasing to $\widetilde{v}$. By Proposition 2.2.4 for every $j = 2, 3, \ldots$ we can find $k(j)$ such that for $G_j = \omega \cap \{v_{k(j)} > v + \delta\}$ we have $c(G_j, \Omega) < 2^{-j}\varepsilon$. If $G = \bigcup G_j$ then $c(G, \Omega) < \varepsilon$ and on $\omega \setminus G$ we have uniform convergence $v_{k(j)} \longrightarrow \widetilde{v} = v$. ∎

In what follows we shall derive several useful applications of the quasi-continuity of plurisubharmonic functions. The first one is a generalization of Theorem 2.1.5 to sequences of locally bounded plurisubharmonic functions increasing to a plurisubharmonic function almost everywhere (with respect to the Lebesgue measure). As well as almost all results of this chapter it is due to Bedford and Taylor [BT2]. It was Cegrell [Ceg] who observed that it can be proved without using the solution of the Dirichlet problem and that a complicated inductive procedure on the dimension from [BT2] can be avoided.

Theorem 2.2.5. *Let $u_0^j, u_1^j, \ldots, u_p^j \in PSH \cap L^\infty_{\text{loc}}$, $0 \leq p \leq n$, $j = 1, 2, \ldots,$ be monotone sequences (either decreasing or increasing) converging almost everywhere to $u_0, \ldots, u_p \in PSH \cap L^\infty_{\text{loc}}$ respectively. Then*

$$u_0^j dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \longrightarrow u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

weakly.

Proof. We will modify the proof of Theorem 2.1.5. In the same way as there we may reduce the problem to the situation where all considered functions are defined in a ball $B = B(z_0, r)$, $\geq A\psi$ in B and equal to $A\psi$ in a neighborhood of ∂B, where $A > 0$ and $\psi(z) = |z - z_0|^2 - r^2$. The proof is by induction in p. Of course the theorem holds if $p = 0$. Let therefore $p \geq 1$ and assume the theorem is true for $p - 1$. It means in particular that

$$S^j := dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j \longrightarrow dd^c u_1 \wedge \cdots \wedge dd^c u_p =: S$$

weakly. By the Chern-Levine-Nirenberg inequality it is enough to show that if $u_0^j S^j \longrightarrow \Theta$ weakly then $\Theta = u_0 S$. If u_0^j is decreasing then Lemma A2.2 implies that $\Theta \le u_0 S$. If u_0^j is increasing then $u_0^j S^j \le u_0 S_j$ and again by Lemma A2.2 every weak limit of $u_0 S^j$ is $\le u_0 S$. Thus also in this case we have $\Theta \le u_0 S$. By Corollary 1.3.9 it therefore remains to show that $\int_B (\Theta - u_0 S) \wedge \omega^{n-p} \ge 0$, that is that

$$\lim_{j\to\infty} \int_B u_0^j S^j \wedge \omega^{n-p} \ge \int_B u_0 S \wedge \omega^{n-p}. \tag{2.2.1}$$

Using quasi-continuity of plurisubharmonic functions we will show that for every bounded plurisubharmonic function u in B which is equal to $A\psi$ in a neighborhood of ∂B we have

$$\lim_{j\to\infty} \int_B u\, S^j \wedge \omega^{n-p} = \int_B u\, S \wedge \omega^{n-p}. \tag{2.2.2}$$

First we show how (2.2.2) implies (2.2.1). If u_0^j is decreasing then $u_0^j \ge u_0$ and (2.2.1) follows directly from (2.2.2) applied to u_0. Assume that u_0^j is increasing. Then for every k by (2.2.2) and Proposition 2.1.3 we have

$$\begin{aligned}
\lim_{j\to\infty} \int_B u_0^j S^j \wedge \omega^{n-p} &\ge \lim_{j\to\infty} \int_B u_0^k S^j \wedge \omega^{n-p} \\
&= \int_B u_0^k S \wedge \omega^{n-p} \\
&= \int_B u_1 dd^c u_0^k \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-p}.
\end{aligned}$$

If we now let k tend to ∞ and use (2.2.2) again, we get (2.2.1).

Hence, it remains to prove (2.2.2). By Theorem 2.2.2 for every $\varepsilon > 0$ we can find an open $G \Subset B$ such that u is continuous on $F := \overline{B} \setminus G$ and $c(G, B) < \varepsilon$. For simplicity we denote $\mu_j = S^j \wedge \omega^{n-p}$ and $\mu = S \wedge \omega^{n-p}$. Write

$$\int_B (u\, d\mu_j - u\, d\mu) = \int_G + \int_F .$$

We have $\mu_j \le (dd^c(u_1^j + \cdots + u_p^j + (n-p)\psi))^n$ and

$$\left| \int_G (u\, d\mu_j - u\, d\mu) \right| \le C_1 \varepsilon$$

where C_1 is a constant independent of j and ε. Let φ be a continuous function in $\overline{B}$ such that $\varphi = u$ on F and $-A \le \varphi \le 0$ in B. Then, since $u = \varphi = 0$ on ∂B,

$$-\int_F u\, d\mu = -\int_F \varphi\, d\mu \ge \overline{\lim_{j\to\infty}} \left(-\int_F \varphi\, d\mu_j \right) = -\underline{\lim_{j\to\infty}} \int_F u\, d\mu_j .$$

On the other hand

$$-\int_F u\,d\mu \le -\int_B \varphi\,d\mu = -\lim_{j\to\infty}\int_B \varphi\,d\mu_j$$

and

$$-\int_B \varphi\,d\mu_j = -\int_F - \int_G \le -\int_F u\,d\mu_j + C_2\varepsilon.$$

We infer

$$\overline{\lim_{j\to\infty}}\left|\int_B (u\,d\mu_j - u\,d\mu)\right| \le (C_1+C_2)\varepsilon$$

and (2.2.2) follows since ε was arbitrary. ∎

We can also finish the proof of Theorem 2.1.8:

End of proof of Theorem 2.1.8. We have to show that (2.1.6) holds for arbitrary $p > n$. It is no loss of generality to assume that $-1 \le v, w \le 0$. Let Ω' be open and such that $\{v < w\} \subset \Omega' \Subset \Omega$ and let $\varepsilon > 0$. By Theorem 2.2.2 we can find an open $G \subset \Omega$ with $c(G,\Omega) < \varepsilon$ such that v, w are continuous on $F = \Omega \setminus G$. Set $f_j := (w_j - v_j)^{p-n}$, $f := (w-v)^{p-n}$, $\mu_j := (dd^c v_j)^n$, $\mu := (dd^c v)^n$, $G' := G \cap \Omega'$ and $F' := F \cap \Omega'$. We have

$$\int_{\Omega'} (f_j d\mu_j - f\,d\mu) = \int_{G'} (f_j d\mu_j - f\,d\mu) + \int_{F'} (f_j - f) d\mu_j + \int_{F'} f(d\mu_j - d\mu).$$

Since f_j and f have compact supports in Ω', in the same way as in the proof of Theorem 2.2.5 we can show that the first and the third terms tend to 0, whereas on F' $f_j \longrightarrow f$ uniformly and thus so does the second term. ∎

The next application is the domination principle.

Theorem 2.2.6. *Assume that Ω is a bounded domain in $\mathbb{C}^n$. Let $u, v \in PSH \cap L^\infty(\Omega)$ be such that $(u-v)_* \ge 0$ on $\partial\Omega$. Then*

$$\int_{\{u<v\}} (dd^c v)^n \le \int_{\{u<v\}} (dd^c u)^n.$$

Proof. If instead of u we consider $u+\delta$, $\delta > 0$, then $\{u+\delta < v\} \uparrow \{u < v\}$ as $\delta \downarrow 0$ and it follows that we may assume that $(u-v)_* \ge \delta > 0$ on $\partial\Omega$. Then $\{u < v\} \Subset \Omega$.

First we assume that u and v are continuous. Then $\Omega' := \{u < v\}$ is open, u, v are continuous on $\overline{\Omega'}$ and $u = v$ on $\partial\Omega'$. For $\varepsilon > 0$ set $u_\varepsilon := \max\{u+\varepsilon, v\}$. Then $u_\varepsilon \downarrow v$ on Ω' as $\varepsilon \downarrow 0$ and $u_\varepsilon = u + \varepsilon$ in a neighborhood of $\partial\Omega'$. By the Stokes theorem

$$\int_{\{u<v\}} (dd^c u_\varepsilon)^n = \int_{\{u<v\}} (dd^c u)^n$$

and by Theorem 2.1.5

$$\int_{\{u<v\}} (dd^c v)^n \le \underline{\lim}_{\varepsilon\to 0} \int_{\{u<v\}} (dd^c u_\varepsilon)^n,$$

hence the theorem follows if u and v are continuous.

Let now u and v be arbitrary and let ω be a domain such that $\{u \le v+\delta/2\} \Subset \omega \Subset \Omega$. There are sequences u_j and v_k of smooth plurisubharmonic functions in a neighborhood of $\overline{\omega}$ decreasing to u and v respectively and such that $u_j \ge v_k$ on $\partial\omega$ for every j, k. We may assume that $-1 \le u_j, v_k \le 0$. Take $\varepsilon > 0$ and let G be open in Ω such that $c(G, \Omega) < \varepsilon$ and u, v are continuous on $F = \Omega \setminus G$. There is a continuous φ on Ω such that $v = \varphi$ on F. We have

$$\int_{\{u<v\}} (dd^c v)^n = \lim_{j\to\infty} \int_{\{u_j<v\}} (dd^c v)^n.$$

Since $\{u_j < v\} \subset \{u_j < \varphi\} \cup G$ and since $\{u_j < \varphi\}$ is open

$$\int_{\{u_j<v\}} (dd^c v)^n \le \int_{\{u_j<\varphi\}} + \int_G \le \underline{\lim}_{k\to\infty} \int_{\{u_j<\varphi\}} (dd^c v_k)^n + \varepsilon.$$

From $\{u_j < \varphi\} \subset \{u_j < v\} \cup G$ and $\{u_j < v\} \subset \{u_j < v_k\}$ it follows that

$$\int_{\{u_j<\varphi\}} (dd^c v_k)^n \le \int_{\{u_j<v\}} + \int_G \le \int_{\{u_j<v_k\}} (dd^c v_k)^n + \varepsilon.$$

By the first part of the proof

$$\int_{\{u_j<v_k\}} (dd^c v_k)^n \le \int_{\{u_j<v_k\}} (dd^c u_j)^n,$$

thus

$$\int_{\{u<v\}} (dd^c v)^n \le \underline{\lim}_{j\to\infty} \underline{\lim}_{k\to\infty} \int_{\{u_j<v_k\}} (dd^c u_j)^n + 2\varepsilon \le \underline{\lim}_{j\to\infty} \int_{\{u_j\le v\}} (dd^c u_j)^n + 2\varepsilon.$$

Further,

$$\int_{\{u_j\le v\}} (dd^c u_j)^n \le \int_{\{u_j\le v\}\cap F} (dd^c u_j)^n + \varepsilon$$

and, since the set $\{u \le v\} \cap F$ is compact and $\{u_j \le v\} \subset \{u \le v\}$,

$$\overline{\lim}_{j\to\infty} \int_{\{u_j\le v\}\cap F} (dd^c u_j)^n \le \int_{\{u\le v\}\cap F} (dd^c u)^n \le \int_{\{u\le v\}} (dd^c u)^n.$$

Because $\varepsilon > 0$ was arbitrary, we obtain

$$\int\limits_{\{u<v\}} (dd^c v)^n \leq \int\limits_{\{u\leq v\}} (dd^c u)^n.$$

This implies that for every $\eta > 0$

$$\int\limits_{\{u+\eta<v\}} (dd^c v)^n \leq \int\limits_{\{u+\eta\leq v\}} (dd^c (u+\eta))^n = \int\limits_{\{u+\eta\leq v\}} (dd^c u)^n.$$

The theorem follows since $\{u+\eta < v\} \uparrow \{u < v\}$ and
$\{u+\eta \leq v\} \uparrow \{u < v\}$ as $\eta \downarrow 0$. ■

Corollary 2.2.7. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $u, v \in PSH \cap L^\infty(\Omega)$ be such that $u \leq v$ and $\lim_{z\to\partial\Omega} u(z) = \lim_{z\to\partial\Omega} v(z) = 0$. Then*

$$\int_\Omega (dd^c v)^n \leq \int_\Omega (dd^c u)^n.$$

Proof. For $\lambda > 1$ we have $\lambda u < v$ in Ω, thus the corollary is a direct consequence of Theorem 2.2.6. ■

Exercise Show that $(dd^c \log^+(|z|/R))^n = (2\pi)^n d\sigma$ where $d\sigma$ is the unitary surface measure of $\partial B(0,R)$.

The domination principle also easily implies the comparison principle.

Corollary 2.2.8. *Let Ω, u and v be as in Theorem 2.2.6. Assume moreover that $(dd^c u)^n \leq (dd^c v)^n$. Then $v \leq u$.*

Proof. Set $\psi(z) = |z|^2 - M$, where M is so big that $\psi < 0$ in Ω. Suppose that the set $\{u < v\}$ is nonempty. Then for some $\varepsilon > 0$ $\{u < v + \varepsilon\psi\}$ is nonempty and thus of positive Lebesgue measure. By Theorem 2.2.6

$$\begin{aligned}\int\limits_{\{u<v+\varepsilon\psi\}} (dd^c u)^n &\geq \int\limits_{\{u<v+\varepsilon\psi\}} (dd^c (v+\varepsilon\psi))^n \\ &\geq \int\limits_{\{u<v+\varepsilon\psi\}} (dd^c v)^n + 4^n n! \varepsilon^n \lambda\left(\{u < v+\varepsilon\psi\}\right) \\ &> \int\limits_{\{u<v+\varepsilon\psi\}} (dd^c v)^n\end{aligned}$$

which is a contradiction. ■

Corollary 2.2.9. *If Ω is a bounded domain in $\mathbb{C}^n$ and $u, v \in PSH \cap L^\infty(\Omega)$ are such that $(dd^c u)^n = (dd^c v)^n$ and $\lim_{z \to \partial\Omega}(u(z) - v(z)) = 0$ then $u = v$.* ■

The next result is due to Demailly [Dem3].

Theorem 2.2.10. *For $u, v \in PSH \cap L^\infty_{\text{loc}}$*

$$(dd^c \max\{u, v\})^n \geq \chi_{\{u \geq v\}}(dd^c u)^n + \chi_{\{u<v\}}(dd^c v)^n.$$

Proof. It is enough to show this inequality of measures on the set $\{u \geq v\}$, since then we can exchange u and v to get it on the complement. Let $K \subset \{u \geq v\}$ be compact. We may assume that u, v are defined in a neighborhood of $\overline{\Omega}$ and $-1 \leq u, v \leq 0$. Let $u_j = u * \rho_{1/j}$, $v_j = v * \rho_{1/j}$ be the regularizations of u, v, then $-1 \leq u_j, v_j \leq 0$ in Ω. By Theorem 2.2.2 there is an open $G \subset \Omega$ such that $c(G, \Omega) \leq \varepsilon$ and u, v are continuous on $\Omega \setminus G$. The convergences $u_j \longrightarrow u$ and $v_j \longrightarrow v$ are uniform on compact subsets of $\Omega \setminus G$, thus for every $\delta > 0$ there is an open neighborhood U of K such that $u_j + \delta \geq v_j$ on $U \setminus G$ for j big enough. Hence

$$\int_K (dd^c u)^n \leq \varliminf_{j\to\infty} \int_U (dd^c u_j)^n \leq \varepsilon + \varliminf_{j\to\infty} \int_{U\setminus G} (dd^c u_j)^n$$
$$= \varepsilon + \varliminf_{j\to\infty} \int_{U\setminus G} (dd^c \max\{u_j + \delta, v_j\})^n.$$

If we let $\varepsilon \longrightarrow 0$ and $j \longrightarrow \infty$, we get

$$\int_K (dd^c u)^n \leq \int_{\overline{U}} (dd^c \max\{u + \delta, v\})^n$$

and, if $U \downarrow K$,

$$\int_K (dd^c u)^n \leq \int_K (dd^c \max\{u + \delta, v\})^n.$$

The desired estimate now follows if we let $\delta \longrightarrow 0$. ■

Note that if u, v are continuous then Theorem 2.2.10 is much easier to prove. For then it is enough to show the inequality on the set $\{u = v\}$ and for compact $K \subset \{u = v\}$ we have

$$\int_K (dd^c \max\{u, v\})^n \geq \varlimsup_{\varepsilon\downarrow 0} \int_K (dd^c \max\{u + \varepsilon, v\})^n = \int_K (dd^c u)^n.$$

2.3. The Dirichlet problem

The main goal of this section is to prove the following theorem due essentially to Bedford and Taylor [BT1], [BT2].

Theorem 2.3.1. *Let u be a locally bounded plurisubharmonic function in an open subset Ω of $\mathbb{C}^n$. Then u is maximal in Ω iff $(dd^c u)^n = 0$. In particular, being a locally bounded maximal plurisubharmonic function is a local property.*

The main tool in proving Theorem 2.3.1 will be the following regularity result.

Theorem 2.3.2. *Let P be a polydisk and assume that $f \in C^{1,1}(\partial P)$ (that is f is $C^{1,1}$ in a neighborhood of ∂P) is such that (2.3.2) holds on P. Then $u_{f,P} \in C^{1,1}(P)$.*

Proof. By Proposition 1.4.4 $u := u_{f,P} \in PSH(P) \cap C(\overline{P})$. We may assume that $P = (\Delta(0,1))^n$ is the unit polydisk. Take $r < 1$ and let $P_r = (\Delta(0,r))^n$. For $z \in \overline{P}$, $a \in P$ and h small enough define

$$T_{a,h}(z) = T(a,h,z) = \left(\frac{h_1 + (1 - |a_1|^2 - \overline{a}_1 h_1) z_1}{1 - |a_1|^2 - a_1 \overline{h}_1 + \overline{h}_1 z_1}, \dots, \frac{h_n + (1 - |a_n|^2 - \overline{a}_n h_1) z_n}{1 - |a_n|^2 - a_n \overline{h}_n + \overline{h}_n z_n} \right).$$

Then T is C^∞ smooth in a neighborhood of the set $\overline{P}_r \times \overline{P}_{(1-r)/2} \times \overline{P}$ and $T_{a,h}$ is a holomorphic automorphism of P such that $T_{a,h}(a) = a + h$ and $T_{a,0}(z) = z$. Set

$$V(a,h,z) := u(T_{a,h}(z))$$

and

$$v_{a,h}(z) := \frac{1}{2}(V(a,h,z) + V(a,-h,z)).$$

$v_{a,h} \in PSH(P) \cap C(\overline{P})$ and we claim that for K big enough we have $v_{a,h} - K|h|^2 \le u$ for every $a \in P_r$ and $h \in P_{(1-r)/2}$. It is enough to show that $v_{a,h} - K|h|^2 \le f$ on ∂P and since the both functions are continuous it is enough to prove this inequality on

$$R := \bigcup_{j=1}^{n} \Delta^{j-1} \times \partial\Delta \times \Delta^{n-j}.$$

But this follows from Proposition A1.5, since V is $C^{1,1}$ on $P_r \times P_{(1-r)/2} \times R$ and $|D^2 V| \le K$ there, where K depends only on n, r and $\sup |D^2 f|$. Therefore, for $z = a$ we obtain

$$u(a+h) + u(a-h) - 2u(a) \le K|h|^2$$

for $a \in P_r$ and $h \in P_{(1-r)/2}$. To finish the proof of Theorem 2.3.2 it suffices to use the following fact.

Proposition 2.3.3. *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that u is a plurisubharmonic function in a neighborhood of $\overline{\Omega}$ such that for a positive constant K and h sufficiently small it satisfies the estimate*

$$u(z+h) + u(z-h) - 2u(z) \leq K|h|^2, \quad z \in \Omega.$$

Then u is $C^{1,1}$ in Ω and $|D^2 u| \leq K$ there.

Proof. Let $u_\varepsilon = u * \rho_\varepsilon$ denote the standard regularizations of u. Then for $z \in \Omega_\varepsilon := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \varepsilon\}$ and h sufficiently small we have

$$u_\varepsilon(z+h) + u_\varepsilon(z-h) - 2u_\varepsilon(z) \leq K|h|^2.$$

This implies that

$$D^2 u_\varepsilon . h^2 \leq K|h|^2. \tag{2.3.1}$$

We have

$$D^2 u_\varepsilon . h^2 = \sum_{j,k=1}^{n} \left(\frac{\partial^2 u_\varepsilon}{\partial z_j \partial z_k} h_j h_k + 2 \frac{\partial^2 u_\varepsilon}{\partial z_j \partial \overline{z}_k} h_j \overline{h}_k + \frac{\partial^2 u_\varepsilon}{\partial \overline{z}_j \partial \overline{z}_k} \overline{h}_j \overline{h}_k \right)$$

and, since u_ε is plurisubharmonic,

$$D^2 u_\varepsilon . h^2 + D^2 u_\varepsilon . (ih)^2 = 4 \sum_{j,k=1}^{n} \frac{\partial^2 u_\varepsilon}{\partial z_j \partial \overline{z}_k} h_j \overline{h}_k \geq 0.$$

Therefore by (2.3.1)

$$D^2 u_\varepsilon . h^2 \geq -D^2 u_\varepsilon . (ih)^2 \geq -K|h|^2.$$

This implies that $|D^2 u_\varepsilon| \leq K$ on Ω_ε and the proposition follows from Theorem 1.1.13. ∎

Exercise Let $f \in C(\partial P)$, where P is a polydisk. Show that there exists $v \in PSH(P) \cap C(\overline{P})$ with $v|_{\partial P} = f$ iff f is subharmonic on every analytic disk embedded in ∂P.

One can modify the proof of Theorem 2.3.2 using the holomorphic automorphisms of the the unit ball to get an analogous regularity in euclidean balls **Exercise** (this is an original result from [BT1]). The following example of Gamelin and Sibony shows that it is not possible to get a better regularity.

Exercise Let B be the unit ball in $\mathbb{C}^2$. For $(z, w) \in \partial B$, set

$$f(z,w) := \left(|z|^2 - 1/2\right)^2 = \left(|w|^2 - 1/2\right)^2.$$

In particular, $f \in C^\infty(\partial B)$. Show that

$$u(z,w) = u_{f,B}(z,w) = \big(\max\{0, |z|^2 - 1/2, |w|^2 - 1/2\}\big)^2,$$

so that u is $C^{1,1}$ but not C^2.

Proof of Theorem 2.3.1. If $u \in PSH \cap L^\infty_{\text{loc}}(\Omega)$ is such that $(dd^c u)^n = 0$ then it follows immediately from the comparison principle that u is maximal in Ω. Assume therefore that u is maximal in a neighborhood of a polydisk P. Let $f_j \in C^{1,1}(\partial P)$ be a sequence deacresing to u on ∂P, then $u_{f_j,P} \downarrow u$ on $\overline{P}$ by Proposition 1.4.9. Hence, by Theorem 2.3.2 we may assume that u is $C^{1,1}$ in P and we have to show that $(dd^c u)^n = 0$ there. By the Rademcher theorem u is twice differentiable almost everywhere in the classical sense and, by Proposition 2.1.6, (2.1.3) holds. Let $z_0 \in P$ be such that $D^2 u(z_0)$ exists and assume that $\det(\partial^2 u/\partial z_j \partial \overline{z}_k) > 0$. The Taylor expansion gives

$$u(z_0 + h) = \text{Re } P(h) + \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z_0) h_j \overline{h}_k + o(|h|^2) \geq \text{Re } P(h) + c|h|^2 + o(|h|^2),$$

where

$$P(h) = u(z_0) + 2\sum_{j=1}^n \frac{\partial u}{\partial z_j}(z_0) h_j + \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial z_k}(z_0) h_j h_k,$$

and $c > 0$. We can find $r > 0$ such that $u(z_0+h) > \text{Re } P(h)$ if $|h| = r$ but $u(z_0) = \text{Re } P(0)$ which contradicts the maximality of u. ∎

Theorem 2.3.1 allows to use the methods from the theory of the complex Monge-Ampère operator in order to show certain elementary properties of maximal plurisubharmonic functions. For example, Theorem 2.2.5 immediately gives the following:

Theorem 2.3.4. *Let u_j be a sequence of maximal plurisubharmonic functions increasing to a plurisubharmonic function u almost everywhere. Then u is maximal.* ∎

As a direct application of Theorem 2.1.10 we can get a result from [Zer]:

Theorem 2.3.5. *Assume that u_j, $j = 1, 2$, is a maximal plurisubharmonic function in domain $\Omega_j \subset \mathbb{C}^{n_j}$. Then $\max\{u_1, u_2\}$ is maximal in $\Omega_1 \times \Omega_2$.* ∎

Theorem 2.3.1 also makes it easy to solve the homogeneous Dirichlet problem for the complex Monge-Ampère operator

$$\begin{cases} u \in PSH \cap L^\infty(\Omega) \\ (dd^c u)^n = 0 \\ u^* = u_* = f \text{ on } \partial\Omega. \end{cases} \tag{2.3.2}$$

Here Ω is a bounded domain in $\mathbb{C}^n$ and $f \in C(\partial\Omega)$. By Corollary 2.2.9 the solution, if exists, must be unique and by Corollary 2.2.8 it must be equal to $u_{f,\Omega}$ (as defined by (1.4.1)). By Theorem 1.4.3 the solution has to be continuous on $\overline{\Omega}$. Theorem 2.3.1 coupled with the results of section 1.4 immediately gives the following.

Theorem 2.3.6. *Assume that $\Omega \Subset \mathbb{C}^n$ is a regular domain. Let $f \in C(\partial\Omega)$ be extendable to a plurisubharmonic, continuous function on $\overline{\Omega}$ (that is (1.4.3) holds). Then there exists a unique continuous solution of (2.3.2).* ∎

In particular, the problem (2.3.2) has a continuous solution for every $f \in C(\partial\Omega)$ iff Ω is B-regular.

III. Pluripolar sets and extremal plurisubharmonic functions

3.1. Pluripolar sets and the relative extremal function

A set $P \subset \mathbb{C}^n$ is called *pluripolar* if for every $z_0 \in P$ there exists an open neighborhood U of z_0 and $u \in PSH(U)$ such that $P \cap U \subset \{v = -\infty\}$. Let Ω be an open subset of $\mathbb{C}^n$. Subsets of sets of the form $\{u < u^*\}$, where $u = \sup_\alpha u_\alpha$ and $\{u_\alpha\}$ is a family of plurisubharmonic functions in Ω locally uniformly bounded above, are called *negligible in* Ω.

Observe that negligible sets are of Lebesgue measure 0 (by Theorem 1.2.3.viii). Also, if $E \subset \{u = -\infty\}$ for some $u \in PSH(\Omega)$ with $u \leq 0$ then E is negligible in Ω. Indeed, $\{u = -\infty\} = \{v < v^*\}$, where $v = \sup\limits_{\alpha \in (0,1)} \alpha u$.

Also note that by Proposition 2.2.3, if $u \in PSH \cap L^\infty_{\text{loc}}$, then $(dd^c u)^n$ takes no mass at pluripolar sets.

The main goal of this section is to prove the following two theorems:

Theorem 3.1.1. *If $P \subset \mathbb{C}^n$ is pluripolar then there exists $u \in PSH(\mathbb{C}^n)$ such that $P \subset \{u = -\infty\}$. Moreover, u can be chosen to have a logarithmic growth, that is there is $C > 0$ such that*

$$u(z) \leq \log^+ |z| + C, \quad z \in \mathbb{C}^n.$$

Theorem 3.1.2. *Negligible sets are pluripolar.*

The first part of Theorem 3.1.1 is due to Josefson [Jos] and the logarithmic estimate was obtained by Siciak [Sic2]. Theorem 3.1.2 as well as the proof of Theorem 3.1.1 below is due to Bedford and Taylor [BT2].

The main tool in proving Theorems 3.1.1 and 3.1.2 will be a *relative extremal function* which we already encountered in the proof of Theorem 1.4.6: if E is a subset of a domain Ω in $\mathbb{C}^n$ then we set

$$u_E = u_{E,\Omega} = \sup\{v \in PSH(\Omega) : v \leq 0,\ v|_E \leq -1\}.$$

Here are the basic properties.

Proposition 3.1.3. *i) If $E_1 \subset \Omega_1$ and $E_2 \subset \Omega_2$ are such that $E_1 \subset E_2$ and $\Omega_1 \subset \Omega_2$ then $u_{E_1,\Omega_1} \geq u_{E_2,\Omega_2}$ on Ω_1;*

ii) u_E^* is maximal in $\Omega \setminus \overline{E}$;

iii) $u_P^* \equiv 0$ iff there is $u \in PSH(\Omega)$ such that $u \leq 0$ and $P \subset \{u = -\infty\}$;

iv) $u_{E\cup P}^* = u_E^*$ if P is as in iii);

v) If K_j is a sequence of compact subsets of Ω decreasing to K then $u_{K_j} \uparrow u_K$;

vi) If $K \subset \Omega$ is compact and Ω is bounded and hyperconvex then the supremum in the definition of $u_{K,\Omega}$ can be taken only over continuous functions. In particular, in such a case $u_{K,\Omega}$ is lower semicontinuous;

vii) If K, Ω are as in vi) and K is such that $\Omega \setminus K$ is a regular domain (in the real sense), then $u_{K,\Omega}$ is continuous on $\overline{\Omega}$ (with $u_{K,\Omega} = 0$ on $\partial\Omega$).

Proof. i) is obvious.

ii) By the Choquet lemma (Lemma A2.3) there is a sequence $u_j \in PSH(\Omega)$ such that $u_j \leq 0$, $u_j|_K \leq -1$ and $u_E^* = \sup_j u_j$. Considering the functions $\max\{u_1, \dots, u_j\}$ instead of u_j we may assume that u_j is increasing to u_E^* almost everywhere. If $B \Subset \Omega \setminus \overline{E}$ is a ball then by Proposition 1.4.10 we may assume that u_j are maximal in B. Theorem 2.3.4 gives ii).

iii) If $u = -\infty$ on P then $\varepsilon u|_P \leq -1$ for every $\varepsilon > 0$, hence $u_P = 0$ on $\{u > -\infty\}$ and consequently $u_P^* \equiv 0$. Conversely, if $u_P^* \equiv 0$ then by the Choquet lemma there exists a sequence $v_j \in PSH(\Omega)$ increasing to 0 almost everywhere and such that $v_j \leq 0$ and $v_j|_P \leq -1$. Choosing a subsequence if necessary, we may assume that $\int_B |v_j| d\lambda \leq 2^{-j}$ for a fixed $B \Subset \Omega$. Therefore the function $u := \sum_j v_j$ is plurisubharmonic and $u = -\infty$ on P.

iv) We have to show that $v \leq u_{E\cup P}^*$ for every $v \in PSH(\Omega)$ with $v \leq 0$ and $v|_E \leq -1$. If u is as in iii), then $v + \varepsilon u \leq -1$ on $E \cup P$ for every $\varepsilon > 0$. Thus $v + \varepsilon u \leq u_{E\cup P}$ and $v \leq u_{E\cup P}$ almost everwhere, hence $v \leq u_{E\cup P}^*$ everywhere.

v) We have $u_{K_j} \uparrow w \leq u_K$. It remains to show that $v \leq w$ for every $v \in PSH(\Omega)$ with $v \leq 0$ and $v|_K \leq -1$. For every $\varepsilon > 0$ the set $\{v < -1 + \varepsilon\}$ is an open neighborhood of K, so there is j such that $v|_{K_j} \leq -1 + \varepsilon$, hence $v - \varepsilon \leq u_{K_j} \leq w$.

vi) Let $\psi \in PSH(\Omega) \cap C(\overline{\Omega})$ be such that $\psi|_{\partial\Omega} = 0$ and $\psi|_K \leq -1$. Take $v \in PSH(\Omega)$ with $\psi \leq v \leq 0$, $v|_K = -1$ and fix $\delta > 0$. If $\varepsilon > 0$ is small enough, the regularization $v_\varepsilon = v * \rho_\varepsilon$ is defined in a neighborhood of $\{\psi \leq -\delta\}$. From an elementary property of upper semicontinuous functions it follows that $m(\varepsilon) := \max_K v_\varepsilon \downarrow -1$ as $\varepsilon \downarrow 0$. Set

$$w := \begin{cases} \max\{\psi, v_\varepsilon - (m(\varepsilon) + 1) - \delta\}, & \text{if } \psi \leq -\delta, \\ \psi & \text{otherwise.} \end{cases}$$

Then w is continuous, plurisubharmonic in Ω, $w \leq 0$ and $w|_K \leq -1$. Thus, if ε is sufficiently small, then on $\{\psi \leq -\delta\}$ we have $v - 2\delta \leq v_\varepsilon - 2\delta \leq w \leq u_{K,\Omega}$.

vii) The same as the proof of Theorem 1.4.6. ∎

Exercise Prove that if $G \subset \Omega$ is open then $u_G = u_G^*$.

Exercise Let $E \subset \Omega$ be such that $E \cap \partial E$ is compact. Show that there exists a sequence of open set $G_j \subset \Omega$ decreasing to E such that $u_{G_j} \uparrow u_E$.

Exercise Let K and Ω be as in vi). Show that the $PSH(\Omega)$-hull of K is given by $\widehat{K}_\Omega = \{u_{K,\Omega} = -1\}$ and $u_{\widehat{K}_\Omega,\Omega} = u_{K,\Omega}$.

If c is a set function defined on Borel subsets of Ω then for arbitrary $E \subset \Omega$ we set

$$c^*(E) := \inf_{E \subset G,\ G \text{ open}} c(G)$$
$$c_*(E) := \sup_{K \subset E,\ K \text{ compact}} c(K).$$

Theorem 3.1.4. *Suppose E is an arbitrary relatively compact subset of a bounded hyperconvex domain Ω. Then*

$$c^*(E, \Omega) = \int_\Omega (dd^c u_E^*)^n.$$

If $K \subset \Omega$ is compact then $c(K, \Omega) = c^*(K, \Omega)$.

Proof. First we want to show that

$$c(K, \Omega) = \int_K (dd^c u_K^*)^n, \quad K \Subset \Omega,\ K \text{ compact}. \tag{3.1.1}$$

The inequality "$\geq$" follows directly from the definition of $c(K, \Omega)$. To show the converse, take $u \in PSH(\Omega)$ with $-1 \leq u \leq 0$ and $\varepsilon > 0$. Since Ω is hyperconvex, there is $\psi \in PSH(\Omega) \cap C(\overline{\Omega})$ with $\psi|_{\partial\Omega} = 0$ and $\psi|_K \leq -1$. By the Choquet lemma we can find a sequence $v_j \in PSH(\Omega)$ with $\psi \leq v_j \leq 0$, $v_j|_K \leq -1$ and $v_j \uparrow v$, $v^* = u_K^*$. Set

$$u_j := \max\{v_j, (1 - 2\varepsilon)u - \varepsilon\}.$$

Then $-1 + \varepsilon \leq u_j \leq -\varepsilon$, $u_j = (1 - 2\varepsilon)u - \varepsilon$ in a neighborhood of K and $u_j = v_j$ on $\{\psi \geq -\varepsilon\}$. Therefore for ε small enough

$$(1 - 2\varepsilon)^n \int_K (dd^c u)^n = \int_K (dd^c u_j)^n \leq \int_{\{\psi \leq -\varepsilon/2\}} (dd^c u_j)^n = \int_{\{\psi \leq -\varepsilon/2\}} (dd^c v_j)^n.$$

By Theorem 2.2.5 $(dd^c v_j)^n \longrightarrow (dd^c u_K^*)^n$ weakly. Thus

$$(1 - 2\varepsilon)^n \int_K (dd^c u)^n \leq \overline{\lim_{j \to \infty}} \int_{\{\psi \leq -\varepsilon/2\}} (dd^c v_j)^n \leq \int_{\{\psi \leq -\varepsilon/2\}} (dd^c u_K^*)^n = \int_K (dd^c u_K^*)^n$$

by Proposition 3.1.3.ii and (3.1.1) follows.

Next, observe that we also have

$$c(G,\Omega) = \int_\Omega (dd^c u_G^*)^n, \quad G \Subset \Omega,\ G \text{ open.} \tag{3.1.2}$$

Indeed, let K_j is a sequence of compact sets increasing to G. Then $u_{K_j}^*$ is decreasing to some plurisubharmonic v and $v \geq u_G^* = u_G$. If $B \Subset G$ then $u_{K_j}^* = -1$ on B for j big enough, thus $v = -1$ on G and $u_{K_j}^* \downarrow u_G^*$. Now (3.1.1) and Proposition 2.2.1.v implies (3.1.2).

Let now $E \Subset \Omega$ be arbitrary and let $G \Subset \Omega$ be an open neighborhood of E. We may assume that $\psi \leq -1$ on G. Then $\psi \leq u_G^* \leq u_E^* \leq 0$ and by Corollary 2.2.7 and (3.1.2)

$$\int_\Omega (dd^c u_E^*)^n \leq \int_\Omega (dd^c u_G^*)^n = c(G,\Omega).$$

Hence $\int_\Omega (dd^c u_E^*)^n \leq c^*(E,\Omega)$. Let v_j be a sequence as above obtained from the Choquet lemma with $v_j \uparrow u_E^*$ almost everywhere. If $\lambda_j \uparrow 1$ and $G_j = \{v_j < -\lambda_j\}$ then G_j are open, decreasing and $\lambda_j^{-1} v_j \leq u_{G_j}$. Therefore $u_{G_j}^* \uparrow u_E^*$ almost everywhere and Theorem 2.2.5 implies that

$$\lim_{j\to\infty} \int_\Omega (dd^c u_{G_j}^*)^n = \int_\Omega (dd^c u_E^*)^n. \ \blacksquare$$

Exercise Let $B_r = B(0,r)$ and $r < R$. Show that

$$u_{\overline{B}_r, B_R}(z) = \max\left\{ \frac{\log|z| - \log R}{\log R - \log r}, -1 \right\}$$

and

$$c(\overline{B}_r, B_R) = \left(\frac{2\pi}{\log R - \log r} \right)^n.$$

Corollary 3.1.5. *Assume that $P \Subset \Omega \Subset \mathbb{C}^n$ and that Ω is hyperconvex. Then $c^*(P,\Omega) = 0$ iff there exists $u \in PSH(\Omega)$ with $u \leq 0$ and $u = -\infty$ on P.*

Proof. It follows immediately from Theorem 3.1.4 and Proposition 3.1.3.iii. $\blacksquare$

Proof of Theorem 3.1.1. We can find bounded hyperconvex domains Ω_j such that $\bigcup_j \Omega_j = \mathbb{C}^n$ and $u_j \in PSH(\Omega_j)$ with $P_j := P \cap \Omega_j \subset \{u_j = -\infty\}$. By Corollary 3.1.5 $c^*(P_j, \Omega_j) = 0$. Set $B_k := B(0, e^{2^k})$. Let $j(k)$ be a sequence of positive integers such that each of them is repeated infinitely many times and $\Omega_{j(k)} \Subset B_k$. By Proposition 2.2.1.iv

$c^*(P_{j(k)}, B_{k+2}) = 0$, hence $u^*_{P_{j(k)},B_{k+2}} = 0$ and there is $v_k \in PSH(B_{k+2})$ with $-1 \le v_k \le 0$, $v_k = -1$ on $P_{j(k)}$ and $\int_{B_1} |v_k| d\lambda \le 2^{-k}$. Set

$$\widetilde{v}_k(z) := \begin{cases} \max\{v_k(z), 2^{-k}(\log|z| - 2^{k+1})\} & \text{if } z \in B_{k+2}, \\ 2^{-k}(\log|z| - 2^{k+1}) & \text{if } z \in \mathbb{C}^n \setminus B_{k+2}. \end{cases}$$

The expression $2^{-k}(\log|z| - 2^{k+1})$ is ≥ 0 on $\mathbb{C}^n \setminus B_{k+1}$, ≤ 0 on B_{k+1} and ≤ -1 on B_k. Therefore $\widetilde{v}_k \in PSH(\mathbb{C}^n)$, $\widetilde{v}_k \le 0$ on B_{k+1} and $\widetilde{v}_k = v_k$ on B_k. Thus $u := \sum_{k=1}^{\infty} \widetilde{v}_k \in PSH(\mathbb{C}^n)$, since $\int_{B_1} |u| d\lambda \le 1$. Moreover, for z big enough we infer

$$u(z) \le \sum_{2^{k+1} \le \log|z|} \widetilde{v}_k(z) \le \sum_{2^{k+1} \le \log|z|} 2^{-k}(\log|z| - 2^{k+1}) \le \log|z|.$$

We have $\widetilde{v}_k = -1$ on $P_{j(k)}$, thus $u = -\infty$ on $P = \bigcup_j P_j$, since each P_j is repeated infinitely many times. ∎

Corollary 3.1.6. *A countable union of pluripolar sets is pluripolar.*

Proof. Let P_j, $j = 1, 2, \ldots$, be pluripolar. Fix a ball B in $\mathbb{C}^n$. By Theorem 3.1.1 we can find $u_j \in PSH(B)$ with $u_j < 0$, $P_j \cap B \subset \{u_j = \infty\}$ and $\int_B |u_j| d\lambda \le 2^{-j}$. Then $u := \sum u_j \in PSH(\Omega)$ and $\bigcup P_j \cap B \subset \{u = -\infty\}$. ∎

Proof of Theorem 3.1.2. By Corollary 3.1.6 and the Choquet lemma we may assume that $N = \{u < u^*\} \cap \widetilde{K}$, where $u = \sup_j u_j$, $u_j \in PSH(\Omega)$, $u_j \le 0$, Ω is a bounded hyperconvex domain and $\widetilde{K}$ is a compact subset of Ω. By Theorem 2.2.2 for every $\varepsilon > 0$ there is an open $G \subset \Omega$ with $c(G, \Omega) \le \varepsilon$ and such that u^*, u_j are continuous on $\Omega \setminus G$. Therefore u is lower semicontinuous on $\Omega \setminus G$ and for every α and β with $\alpha < \beta \le 0$ the set

$$K = K_{\alpha\beta} = \{z \in \widetilde{K} \setminus G : u(z) \le \alpha < \beta \le u^*(z)\}$$

is compact. We claim that $c^*(K, \Omega) = 0$. To prove this we may assume that $\alpha = -1$. We have $u_j \le -1$ on K, thus $u_j \le u_K$ and $u^* \le u_K^*$, so $u_K^* \ge \beta > -1$ on K. From Theorems 2.3.4 and 2.2.10 we infer

$$c^*(K, \Omega) = c(K, \Omega) = \int_K (dd^c u_K^*)^n \le \int_K (dd^c \max\{u_K^*, \beta\})^n \le |\beta|^n c(K, \Omega),$$

thus $c^*(K_{\alpha\beta}, \Omega) = 0$. Moreover, $N \subset G \cup \bigcup_{\alpha,\beta \in \mathbb{Q}} K_{\alpha\beta}$ and we can easily construct an open $\widetilde{G}$ with $N \subset \widetilde{G} \subset \Omega$ and $c(\widetilde{G}, \Omega) \le 2\varepsilon$, so that $c^*(N, \Omega) = 0$ and N is pluripolar by Corollary 3.1.5. ∎

We can now prove a further property of the relative extremal function:

Theorem 3.1.7. *Assume that* $E_j \subset \Omega_j$, $j = 1, 2, \ldots$, *are such that* $E_j \uparrow E$, $\Omega_j \uparrow \Omega$ *and* Ω *is bounded. Then* $u^*_{E_j,\Omega_j} \downarrow u^*_{E,\Omega}$.

Proof. We have $u^*_{E_j,\Omega_j} \downarrow u \geq u^*_{E,\Omega}$ and $u \in PSH(\Omega)$. The set $P = \bigcup_j \{u_{E_j,\Omega_j} < u^*_{E_j,\Omega_j}\}$ is pluripolar, thus there exists $v \in PSH(\Omega)$ such that $v = -\infty$ on P and $v \leq 0$ (because Ω is bounded). Since $u^*_{E_j,\Omega_j} = -1$ on $E_j \setminus P$, it follows that $u = -1$ on $E \setminus P$. Therefore by Proposition 3.1.3.iv, $u \leq u^*_{E\setminus P,\Omega} = u^*_{E,\Omega}$. ■

Theorem 3.1.8. *If* Ω *is a bounded hyperconvex domain then* $c^*(\cdot, \Omega)$ *is a generalized capacity on* Ω *(see A3 for the definition). For every Borel set* $E \subset \Omega$ *we have* $c_*(E) = c(E) = c^*(E)$.

Proof. Obviously $E_1 \subset E_2$ implies $c^*(E_1) \leq c^*(E_2)$. If $K_j \downarrow K$ and K_j are compact then by Proposition 3.1.3.v $u^*_{K_j} \uparrow u^*_K$ almost everywhere and $c^*(K_j) \downarrow c^*(K)$ by Theorem 3.1.4. In the same way, using Theorem 3.1.7, we can prove that if $E_j \uparrow E$ then $c^*(E_j) \uparrow c^*(E)$ provided that E is relatively compact. If E is arbitrary, it is no loss of generality to assume that $E_j \Subset \Omega$ for every j. Fix $\varepsilon > 0$. Since $c^*(\{u_{E_j} < u^*_{E_j}\}) = 0$, there is an open G with $\bigcup_j \{u_{E_j} < u^*_{E_j}\} \subset G$ and $c(G) \leq \varepsilon$. Fix $\alpha > -1$ and set $U_j := \{u^*_{E_j} < \alpha\}$. Then $u^*_{E_j}/|\alpha| \leq u_{U_j} = u^*_{U_j}$ and by Corollary 2.2.7

$$c(U_j) = \int_\Omega \left(dd^c u^*_{U_j}\right)^n \leq |\alpha|^{-n} \int_\Omega \left(dd^c u^*_{E_j}\right)^n = |\alpha|^{-n} c^*(E_j).$$

The set $V = G \cup \bigcup_j U_j$ is an open neighborhood of E and

$$c(V) \leq \varepsilon + \lim_{j\to\infty} c(U_j) \leq \varepsilon + |\alpha|^{-n} \lim_{j\to\infty} c^*(E_j),$$

which implies that $c^*(E) \leq \lim_{j\to\infty} c^*(E_j)$. This shows that c^* is a generalized capacity.

Now by Theorem A3.1 applied to c^* and the second part of Theorem 3.1.4 for every Borel set $E \subset \Omega$ we have

$$c^*(E) = \inf_{K\subset E,\ K \text{ compact}} c^*(K) = c_*(E). \blacksquare$$

The first part of the following proposition is due to Sadullaev [Sad] and the second one to Bedford and Taylor [BT2].

Proposition 3.1.9. *Let* Ω *be a bounded domain in* $\mathbb{C}^n$ *and* E *an arbitrary subset of* Ω. *Then there is a decreasing sequence of open sets* G_j *with* $E \subset G_j \subset \Omega$ *such that* $u^*_{G_j} = u_{G_j} \uparrow u^*_E$ *almost everywhere, as* $j \uparrow \infty$. *If* $E \subset \Omega$ *is Borel then there is an increasing sequence of compact sets* $K_j \subset E$ *such that* $u^*_{K_j} \downarrow u^*_E$.

Proof. By the Choquet lemma there is a sequence $u_j \in PSH(\Omega)$ with $u_j \le 0$ and $u_j|_E \le -1$ such that $u_j \uparrow u_E^*$ almost everywhere. The sets $G_j = \{u_j < -1 + 1/j\}$ are open, decreasing and contain E. Since $u_j - 1/j \le u_{G_j} \le u_E$, it follows that $u_{G_j} \uparrow u_E^*$ almost everywhere.

To show the second part define

$$\gamma(A) := \int_\Omega |u_A^*| d\lambda, \quad A \subset \Omega.$$

From Proposition 3.1.3.v and Theorem 3.1.7 it follows that γ is a generalized capacity. If E is Borel then by Theorem A3.1 we have

$$\gamma(E) = \sup_{K \subset E,\ K \text{ compact}} \gamma(K),$$

thus there is an increasing sequence of compact sets $K_j \subset E$ with $\gamma(K_j) \uparrow \gamma(E)$. Then, if $F = \bigcup K_j$, we have $u_{K_j}^* \downarrow u_F^* \ge u_E^*$ and $u_F^* = u_E^*$ almost everywhere, hence everywhere. ■

Proposition 3.1.10. *Assume that E is a subset of a bounded domain Ω in $\mathbb{C}^n$. Then*

$$\int_{\{u_E^* > -1\}} (dd^c u_E^*)^n = 0.$$

Proof. Note that

$$\int_{\{u_E^* > -1\}} (dd^c u_E^*)^n = 0 \iff \int_\Omega (dd^c u_E^*)^n = \int_\Omega -u_E^* (dd^c u_E^*)^n$$

and that $(dd^c u_E^*)^n = 0$ in $\Omega \setminus \overline{E}$. If E is compact then it is enough to observe that the set $\overline{E} \cap \{u_E^* > -1\}$ is contained in $\{u_E < u_E^*\}$ and thus pluripolar. Next, if E is an F_σ set and $K_j \uparrow E$ are compact then $u_{K_j}^* \downarrow u_E^*$ and by Theorem 2.1.5

$$\int_\Omega -u_E^* (dd^c u_E^*)^n = \lim_{j\to\infty} \int_\Omega -u_{K_j}^* (dd^c u_{K_j}^*)^n = \lim_{j\to\infty} \int_\Omega (dd^c u_{K_j}^*)^n = \int_\Omega (dd^c u_E^*)^n.$$

Finally, let E be arbitrary. Then sets G_j given by Proposition 3.1.9 are open, in particular F_σ. Now using Theorem 2.2.5 in the same way as in the case of F_σ sets we conclude that

$$\int_\Omega (dd^c u_E^*)^n = \int_\Omega -u_E^* (dd^c u_E^*)^n. \ ■$$

We finish this section with a product property for the relative extremal function (see [NS] and [Bł2]):

Theorem 3.1.11. *Let Ω_j be bounded pseudoconvex domains in $\mathbb{C}^{n_j}$, $j = 1, 2$. If $K_j \subset \Omega_j$ are compact then*

(3.1.3) $$u_{K_1 \times K_2, \Omega_1 \times \Omega_2} = \max\{u_{K_1,\Omega_1}, u_{K_2,\Omega_2}\}.$$

If $E_j \subset \Omega_j$ are arbitrary then

$$u^*_{E_1\times E_2,\Omega_1\times\Omega_2} = \max\{u^*_{E_1,\Omega_1}, u^*_{E_2,\Omega_2}\}, \tag{3.1.4}$$

$$\left(dd^c u^*_{E_1\times E_2,\Omega_1\times\Omega_2}\right)^{n_1+n_2} = \left(dd^c u^*_{E_1,\Omega_1}\right)^{n_1} \wedge \left(dd^c u^*_{E_2,\Omega_2}\right)^{n_2}, \tag{3.1.5}$$

and, if E_j are relatively compact in Ω_j and Ω_j are hyperconvex,

$$c^*(E_1\times E_2, \Omega_1\times\Omega_2) = c^*(E_1,\Omega_1)c^*(E_2,\Omega_2). \tag{3.1.6}$$

Proof. By Theorems 3.1.7 and 3.1.4 we may assume that Ω_j are hyperconvex. First note that for arbitrary $E_j \subset \Omega_j$ we have the following inequalities

$$\max\{u_{E_1,\Omega_1}, u_{E_2,\Omega_2}\} \le u_{E_1\times E_2,\Omega_1\times\Omega_2} \le -u_{E_1,\Omega_1}u_{E_2,\Omega_2}. \tag{3.1.5}$$

The first inequality follows easily from the definition of the relative extremal function. The second one we can show first on the cross $E_1\times\Omega_2 \cup \Omega_1\times E_2$ and then on $\Omega_1\times\Omega_2$ fixing one of the variables.

If $K_j \subset \Omega_j$ are compact then we can approximate them from above by compacts K_j^l such that $\Omega_j \setminus K_j^l$ are regular in the real sense. Thus, by Proposition 3.1.3.v we may assume that $\Omega_j \setminus K_j$ are regular and, by Theorem 1.4.3, that u_{K_j,Ω_j} are continuous on $\overline{\Omega}_j$ and $u_{K_1\times K_2,\Omega_1\times\Omega_2}$ is continuous on $\overline{\Omega}_1\times\overline{\Omega}_2$. Then the inequality "$\ge$" in (3.1.3) follows immediately. If $z^1 \in K_1$ then $u_{K_1\times K_2,\Omega_1\times\Omega_2}(z^1,\cdot) = -1$ on K_2 and therefore we have "$\le$" on $\overline{\Omega}_1\times K_2 \cup K_1\times\overline{\Omega}_2$. By Theorem 2.3.5 the right hand-side of (3.1.3) is a maximal function in $(\Omega_1\setminus K_1)\times(\Omega_2\setminus K_2)$, which gives (3.1.3). By approximation, we immediately conclude that (3.1.3) holds also for open subsets of Ω_j.

The inequality "$\ge$" in (3.1.4) is clear. For $\varepsilon > 0$ set $U_j^\varepsilon := \{u^*_{E_j,\Omega_j} < -1+\varepsilon\}$. We see that $(1-\varepsilon)^{-1}u^*_{E_j,\Omega_j} \le u_{U_j^\varepsilon,\Omega_j} \le u^*_{E_j,\Omega_j}$, hence $u_{U_j^\varepsilon,\Omega_j} \uparrow u^*_{E_j,\Omega_j}$ as $\varepsilon\downarrow 0$. By (3.1.5) on $U_1^\varepsilon\times U_2^\varepsilon$ we have $u_{E_1\times E_2,\Omega_1\times\Omega_2} \le -(1-\varepsilon)^2$, so $u^*_{E_1\times E_2,\Omega_1\times\Omega_2} \le -(1-\varepsilon)^2$ there. It follows that on $\Omega_1\times\Omega_2$

$$(1-\varepsilon)^2 u^*_{E_1\times E_2,\Omega_1\times\Omega_2} \le u_{U_1^\varepsilon\times U_2^\varepsilon,\Omega_1\times\Omega_2} \le u^*_{E_1\times E_2,\Omega_1\times\Omega_2}.$$

Therefore,

$$\begin{aligned} u^*_{E_1\times E_2,\Omega_1\times\Omega_2}(z^1,z^2) &= \lim_{\varepsilon\downarrow 0} u_{U_1^\varepsilon\times U_2^\varepsilon,\Omega_1\times\Omega_2}(z^1,z^2) \\ &= \lim_{\varepsilon\downarrow 0}\max\{u_{U_1^\varepsilon,\Omega_1}(z^1), u_{U_2^\varepsilon,\Omega_2}(z^2)\} \le \max\{u^*_{E_1,\Omega_1}(z^1), u^*_{E_2,\Omega_2}(z^2)\}, \end{aligned}$$

which shows (3.1.4).

(3.1.5) follows from (3.1.4), Proposition 3.1.10 and Theorem 2.1.11, whereas (3.1.6) can be deduced from (3.1.5) and Theorem 3.1.4. ■

Exercise Show that if $\Omega_j \Subset \mathbb{C}^n$ are hyperconvex and $K_j \subset \Omega$ are such that $\partial K_j \cap K_j$ are compact then (3.1.3) holds.

3.2. The global extremal function

We shall consider the following two families of entire plurisubharmonic functions.

$$\mathcal{L} := \{u \in PSH(\mathbb{C}^n) : \limsup_{|z|\to\infty} (u(z) - \log|z|) < \infty\}$$
$$\mathcal{L}_+ := \{u \in PSH \cap L^\infty_{loc}(\mathbb{C}^n) : \limsup_{|z|\to\infty} |u(z) - \log|z|\,| < \infty\}.$$

Note that Theorem 3.1.1 meant precisely that $P \subset \mathbb{C}^n$ is pluripolar iff $P \subset \{u = -\infty\}$ for some $u \in \mathcal{L}$.

Proposition 3.2.1. *i) If $u \in \mathcal{L}$ then*

$$u(z) \le \max_{\overline{B}(z_0,r)} u + \log\frac{|z-z_0|}{r} \quad \textit{if} \quad |z-z_0| \ge r; \tag{3.2.1}$$

ii) If $\{u_\alpha\}$ is a family of functions from $\mathcal{L}$ and $u = \sup_\alpha u_\alpha$ then either $u^* \in \mathcal{L}$ or $u^* \equiv +\infty$;

iii) Let $B = B(0,1)$ be the unit ball in $\mathbb{C}^n$. There exists $c_n > 0$ such that for every $u \in \mathcal{L}$ we have

$$\max_{\overline{B}} u \le \frac{1}{\sigma(\partial B)}\int_{\partial B} u\,d\sigma + c_n.$$

Proof. i) By v denote the right hand-side of (3.2.1). For every $\alpha < 1$ we have $\alpha u \le v$ on $\partial B(z_0, R)$ for R big enough. Since the same inequality holds on $\partial B(z_0, r)$, (3.2.1) follows from the maximality of v in $\mathbb{C}^n \setminus \{0\}$.

ii) Assume that $u^*(z_0) < +\infty$ for some z_0. Then there is $r > 0$ such that $M = \max_{\overline{B}(z_0,r)} u < +\infty$. By i) for every α and $|z - z_0| \ge r$ we have $u_\alpha(z) \le M + \log|z-z_0|/r$, hence $u^* \in \mathcal{L}$.

iii) We may assume that $\max_{\overline{B}} u = 0$. From i) it easily follows that $\max_{\overline{B}(0,r)} u \ge \log r$, if $0 < r < 1$. From the fact that u is in particular subharmonic we deduce that for $z \in \overline{B}(0,r)$

$$u(z) \le \frac{1}{\sigma(\partial B)}\int_{\partial B}\frac{1-|z|^2}{|z-w|^{2n}}u(w)d\sigma(w) \le \frac{1-r}{(1+r)^{2n-1}}\,\frac{1}{\sigma(\partial B)}\int_{\partial B} u\,d\sigma.$$

Therefore, if $0 < r < 1$,

$$\frac{1}{\sigma(\partial B)}\int_{\partial B} u\,d\sigma \ge \log r\frac{(1+r)^{2n-1}}{1-r}. \blacksquare$$

Proposition 3.2.2. *i) If $u \in \mathcal{L} \cap L^\infty_{\text{loc}}$ then $\int_{\mathbb{C}^n} (dd^c u)^n \le (2\pi)^n$.*

ii) If $u \in \mathcal{L}_+$ then $\int_{\mathbb{C}^n} (dd^c u)^n = (2\pi)^n$.

Proof. Since ii) holds in the particular case of the function $\log^+ |z| \in \mathcal{L}_+$, it is enough to show that if $u \in \mathcal{L} \cap L^\infty_{\text{loc}}$ and $v \in \mathcal{L}_+$ then $\int_{\mathbb{C}^n} (dd^c u)^n \le \int_{\mathbb{C}^n} (dd^c v)^n$. Take $K \Subset \mathbb{C}^n$. By adding positive constants if necessary, we may assume that $0 \le 2v \le u$ on K. For every α, $1 < \alpha < 2$, there is R big enough such that $u \le \alpha v$ in a neighborhood of $\partial B(0, R)$. By Theorem 2.2.6 we have therefore

$$\int_K (dd^c u)^n \le \int_{\{\alpha v < u\}} (dd^c u)^n \le \alpha^n \int_{\{\alpha v < u\}} (dd^c v)^n \le \alpha^n \int_{\mathbb{C}^n} (dd^c v)^n$$

and the proposition follows. ∎

Let E be a bounded subset of $\mathbb{C}^n$. The *global extremal function* of E is defined by

$$V_E := \sup\{u \in \mathcal{L} : u|_E \le 0\}.$$

(Sometimes V_E is called the *Siciak extremal function*.)

Exercise Show that $V_{B(z_0,r)}(z) = V_{\overline{B}(z_0,r)}(z) = \log^+ \dfrac{|z - z_0|}{r}$.

Here are the basic properties of V_E.

Theorem 3.2.3. *i) If $E_1 \subset E_2$ then $V_{E_1} \ge V_{E_2}$;*

ii) P is pluripolar iff $V_P^* \equiv +\infty$;

iii) If E is not pluripolar then $V_E^* \in \mathcal{L}_+$;

iv) If E is not pluripolar then V_E^* is maximal in $\mathbb{C}^n \setminus \overline{E}$;

v) $V_{E \cup P}^* = V_E^*$ if P is pluripolar;

vi) If $K_j \downarrow K$ and K_j are compact then $V_{K_j} \uparrow V_K$;

vii) If $E_j \uparrow E$ then $V_{E_j}^* \downarrow V_E^*$;

viii) If K is compact then the supremum in the definition of V_K can be taken only over smooth functions. In particular, V_K is lower semicontinuous.

Proof. i) is obvious.

ii) If P is pluripolar then there is $u \in \mathcal{L}$ with $P \subset \{u = -\infty\}$. For every $M > 0$ we have $u + M \le V_P$, hence $V_P = +\infty$ on $\{u > -\infty\}$, so $V_P^* \equiv +\infty$. Now assume that $V_P^* \equiv +\infty$ and let B be the unit ball. For every $j = 1, 2, \ldots$ there is $u_j \in \mathcal{L}$ with $u_j|_P \le 0$ and $M_j = \max_{\overline{B}} u_j \ge 2^j$. Set $u := \sum_{j=1}^{\infty} 2^{-j}(u_j - M_j)$. By Proposition 3.2.1.i

$u_j(z) \leq M_j + \log^+|z|$, thus $u(z) \leq \log^+|z|$. From Proposition 3.2.1.iii it follows that

$$\frac{1}{\sigma(\partial B)}\int_{\partial B} u\,d\sigma = \sum_{j=1}^{\infty} 2^{-j}\frac{1}{\sigma(\partial B)}\int_{\partial B}(u_j - M_j)\,d\sigma \geq -c_n.$$

Thus u is not identically $-\infty$ and $u \in \mathcal{L}$, since for every $R > 1$ we may write $u = \sum_{j=1}^{\infty} 2^{-j}(u_j - \log R - M_j) + \log R$. On P we have $u \leq -\sum_j 2^{-j}M_j = -\infty$.

iii) From ii) and Proposition 3.2.1.ii it follows that $V_P^* \in \mathcal{L}$. Since E is contained in some ball, we conclude that $V_P^* \in \mathcal{L}_+$.

Proofs of iv)-vii) are now similar to the proofs of corresponding properties of the relative extremal function in section 3.1.

iv) By iii) V_E^* is locally bounded. By the Choquet lemma there is a sequence $u_j \in \mathcal{L}$, $u_j|_E \leq 0$, increasing to V_E^* almost everywhere. If $B \Subset \mathbb{C}^n \setminus \overline{E}$ is a ball then we can replace u_j by $\widehat{u}_j$ given by Proposition 1.4.10, thus V_E^* is maximal in B by Theorem 2.3.4.

v) We have to show that $v \leq V_{E\cup P}^*$ for every $v \in \mathcal{L}$ with $v|_E \leq 0$. We can find $u \in \mathcal{L}$ such that $P \subset \{u = -\infty\}$ and $u|_E \leq 0$. For every $\varepsilon > 0$ we have therefore $(1-\varepsilon)v + \varepsilon u \leq V_{E\cup P}$, hence $v \leq V_{E\cup P}$ on $\{u > -\infty\}$ and $v \leq V_{E\cup P}^*$ everywhere.

vi) We have $V_{K_j} \uparrow w \leq V_K$. Take $v \in \mathcal{L}$ with $v|_K \leq 0$. For every $\varepsilon > 0$ the set $\{v < \varepsilon\}$ is an open neighborhood of K and so there is j such that $v|_{K_j} \leq \varepsilon$. Thus $v - \varepsilon \leq V_{K_j} \leq w$ and $V_K \leq w$.

vii) We have $V_{E_j}^* \downarrow u \geq V_E^*$. We may assume that E is not pluripolar, hence $u \in \mathcal{L}$ and the set $P = \bigcup_j \{V_{E_j} < V_{E_j}^*\}$ is pluripolar. Then $u = 0$ on $E \setminus P$ and $u \leq V_{E\setminus P} \leq V_{E\setminus P}^* = V_E^*$ by v).

viii) Take $u \in \mathcal{L}$ with $u|_K = 0$. Then for $\varepsilon > 0$ $u_\varepsilon = u * \rho_\varepsilon \in \mathcal{L} \cap C^\infty$ and $\max_K u_\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$. Thus for every $\delta > 0$ there is $\varepsilon > 0$ such that $u - \delta \leq u_\varepsilon - \delta \leq V_K$. ∎

The next result, due essentially to Zahariuta [Zah], shows that for K compact V_K can be defined by means of polynomials in $\mathbb{C}^n$ and that was in fact the original definition of Siciak [Sic1].

Theorem 3.2.4. *Assume that $K \subset \mathbb{C}^n$ is compact. Then*

$$V_K = \sup\left\{\frac{1}{d}\log|P| : P \in \mathbb{C}[z_1, \dots, z_n],\ d \geq \deg P,\ |P| \leq 1 \text{ on } K\right\}.$$

Theorem 3.2.4 will easily follow from the following approximation property of functions from $\mathcal{L}$ proved Siciak [Sic4] (see also [Sic3]).

Theroem 3.2.5. *For every $u \in \mathcal{L}$ there is a sequence*

$$u_j = \max_{1\leq k\leq k_j}\frac{1}{d_{jk}}\log|P_{jk}|$$

decreasing to u, where P_{jk}, $j = 1, 2, \ldots$, $1 \le k \le k_j$, are polynomials in $\mathbb{C}^n$ and $d_{jk} \ge \deg P_{jk}$.

Proof. For $\zeta \in \mathbb{C}$, $\zeta \ne 0$, set $h(z, \zeta) := u(z/\zeta) + \log|\zeta|$. Then h is plurisubharmonic in $\mathbb{C}^{n+1} \setminus \{\zeta = 0\}$ and it is locally bounded near the hyperplane $\{\zeta = 0\}$, since $u \in \mathcal{L}$. For $\varepsilon > 0$ the function $h_\varepsilon = h + \varepsilon \log|\zeta|$ is therefore plurisubharmonic in $\mathbb{C}^{n+1}$ and in $\{|\zeta| < 1\}$ we have $h^* = (\sup_\varepsilon h_\varepsilon)^*$, hence h can extended to a plurisubharmonic function in the entire $\mathbb{C}^{n+1}$.

The domain $\Omega := \{h < 0\}$ is a balanced pseudoconvex domain in $\mathbb{C}^{n+1}$, hence Ω is convex with respect to the homogeneous polynomials by Theorem 1.4.8. Therefore there exists a sequence K_j of compact subsets of Ω with $K_j = \widehat{K}_j^H$, $K_j \subset \text{int}K_{j+1}$ and $K_j \uparrow \Omega$. We claim that for every j there are homogeneous polynomials $Q_{j1}, \ldots, Q_{jk_j}$ such that $K_j \subset \bigcap_{1 \le k \le k_j} \{|Q_{jk}| \le 1\} \subset \text{int}K_{j+1}$. Indeed, for every $a \in \partial K_{j+1}$ there exists a homogeneous polynomial Q_a with $|Q_a(a)| > ||Q_a||_{K_j} = 1$. We can choose a finite number of points $a_1, \ldots, a_{k_j}$ with $\partial K_{j+1} \subset \bigcup_k \{|Q_{a_k}| > 1\}$, thus we may take $Q_{jk} = Q_{a_k}$.

Now set $d_{jk} := \deg Q_{jk}$ and $f_j := \max_k |Q_{jk}|^{1/d_{jk}}$. Then f_j is homogeneous of degree 1, $K_j \subset \{f_j \le 1\} \subset \text{int}K_{j+1}$, hence f_j is increasing to the Minkowski functional of Ω, that is to e^h. It follows that the polynomials $P_{jk}(z) := Q_{jk}(z, 1)$ satisfy the hypothesis of the theorem. ■

Proof of Theorem 3.2.4. The inequality "$\ge$" is clear. To show the converse take $u \in \mathcal{L}$ with $u|_K = 0$ and let u_j be a sequence given by Theorem 3.2.5. Then $\max_K u_j \downarrow 0$ as $j \uparrow \infty$, hence for every $\delta > 0$ there is j such that $u_j - \delta \le u \le V_K$ and the theorem follows. ■

For a different proof of Theorem 3.2.4 making use of the Hörmander L^2-estimates see [Dem3].

Exercise Assume that K is compact. Show that the polynomial hull of K is given by $\widehat{K} = \{V_K \le 0\}$ and $V_{\widehat{K}} = V_K$.

A compact subset K of $\mathbb{C}^n$ is called *L-regular* if $V_K^* = 0$ on K. If K is L-regular then $V_K^* \le V_K$, hence V_K is upper semicontinuous and thus continuous in $\mathbb{C}^n$ by Theorem 3.2.3.viii. It is also easy to see that if $\mathbb{C}^n \setminus K$ is regular (in the real sense) then K is L-regular. Therefore, every compact set in $\mathbb{C}^n$ can be approximated from above by L-regular sets.

Exercise Show that the cube $[-1, 1]^n$ is an L-regular subset of $\mathbb{C}^n$ but it is not regular as a subset of $\mathbb{R}^{2n}$ for $n \ge 2$.

If E is a bounded subset of $\mathbb{C}^n$ then $\mu_E := (dd^c V_E^*)^n$ is called the *equlibrium measure* of E. It is supported on ∂E and, by Proposition 3.2.2.ii, its total mass is equal to $(2\pi)^n$.

We have the following analogues of Propositions 3.1.9, 3.1.10 and Theorem 3.1.11 for the global extremal function:

Proposition 3.2.6. *For every bounded subset E of $\mathbb{C}^n$ there is a decreasing sequence of bounded open sets G_j containing E such that $V_{G_j}^* = V_{G_j} \uparrow V_E^*$ almost everywhere. If E is Borel then one can find an increasing sequence of compact sets $K_j \subset E_j$ such that $V_{K_j}^* =\downarrow V_E^*$.*

Proof. We proceed in the same way as in the proof of Proposition 3.1.9. As a suitable generalized capacity we may take

$$\gamma(A) := \int_{B(0,R)} (V_A^* + 1)^{-1} d\lambda, \quad A \subset B(0,R),$$

where R is big enough. ■

Proposition 3.2.7. *If E is a bounded subset of $\mathbb{C}^n$ then*

$$\int_{\{V_E^*>0\}} (dd^c V_E^*)^n = 0. \ ■$$

Theorem 3.2.8. *If K_j are compact in $\mathbb{C}^{n_j}$, $j = 1, 2$, then*

(3.2.2) $$V_{K_1 \times K_2} = \max\{V_{K_1}, V_{K_2}\}.$$

If E_j are arbitrary bounded subsets of $\mathbb{C}^{n_j}$ then

(3.2.3) $$V_{E_1 \times E_2}^* = \max\{V_{E_1}^*, V_{E_2}^*\}$$

and

(3.2.4) $$\mu_{E_1 \times E_2} = \mu_{E_1} \mu_{E_2}.$$

Proof. It is almost the same as the proof of Theorem 3.1.11. It is left as an **Exercise** to the reader to show that (3.2.2) implies (3.2.3) and (3.2.4). To prove (3.2.2) we may assume that K_j are L-regular. Set $v = \max\{V_{K_1}, V_{K_2}\}$. Then $V_{K_1 \times K_2} \le v$ on $\mathbb{C}^{n_1} \times K_2 \cup K_1 \times \mathbb{C}^{n_2}$ and, since $v \in \mathcal{L}_+$, for every $a > 1$ we have $V_{K_1 \times K_2}(z) \le av(z)$ if $|z|$ big enough. By

Theorem 2.1.10, v is maximal in $(\mathbb{C}^{n_1} \setminus K_1) \times (\mathbb{C}^{n_2} \setminus K_2)$, hence $V_{K_1 \times K_2} \le av$ everywhere and the theorem follows. ■

The next result is a comparison of the relative and global extremal functions (see [Kli2, Proposition 5.3.3]) and their Monge-Ampère measures (this part is due to Levenberg [Lev] in case of compact sets, with a much more complicated proof though).

Theorem 3.2.9. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and E a non-pluripolar, relatively compact subset of Ω such that $\widehat{\overline{E}} \subset \Omega$. Then $0 < \inf_{\partial\Omega} V_{\overline{E}} \le \sup_{\partial\Omega} V_E < +\infty$ and on Ω we have*

$$\inf_{\partial\Omega} V_{\overline{E}}(u_{E,\Omega} + 1) \le V_E \le \sup_{\partial\Omega} V_E(u_{E,\Omega} + 1) \tag{3.2.5}$$

and

$$(\inf_{\partial\Omega} V_{\overline{E}})^n (dd^c u^*_{E,\Omega})^n \le (dd^c V^*_E)^n \le (\sup_{\partial\Omega} V_E)^n (dd^c u^*_{E,\Omega})^n. \tag{3.2.6}$$

Proof. The first statement is clear, since $V_{\overline{E}} > 0$ on $\mathbb{C}^n \setminus \widehat{\overline{E}} \supset \partial\Omega$, $V_{\overline{E}}$ is lower semicontinuous and $V^*_E \in \mathcal{L}$. If $u \in \mathcal{L}$ is such that $u|_E \le 0$ then $u/\sup_{\partial\Omega} V_E - 1 \le u_E$ which implies the second inequality in (3.2.5).

Take $v \in PSH(\Omega)$ with $v \le 0$, $v|_E \le -1$ and let $0 < \varepsilon < m := \inf_{\partial\Omega} V_{\overline{E}}$. If K_j is a sequence of L-regular sets decreasing to $\overline{E}$, from the lower semicontinuity of V_{K_j} it follows that $\inf_{\partial\Omega} V_{K_j} \uparrow m$. It we set $u := V_{K_j}$ for j big enough, then $u \in \mathcal{L} \cap C$, $u|_E \le 0$ and $u|_{\partial\Omega} \ge m - \varepsilon$. Therefore the function

$$w := \begin{cases} \max\{(m-\varepsilon)(v+1), u\} & \text{on } \Omega, \\ u & \text{on } \mathbb{C}^n \setminus \Omega \end{cases}$$

belongs to $\mathcal{L}$ and $w|_E \le 0$. Hence $(m-\varepsilon)(v+1) \le V_E$ on Ω and (3.2.5) follows.

To show (3.2.6) observe that if $u_1, u_2 \in PSH \cap L^\infty_{\text{loc}}$ and $u_1 \ge u_2$ then by Theorem 2.2.10 on $\{u_1 = u_2\}$ we have $(dd^c u_1)^n = (dd^c \max\{u_1, u_2\})^n \ge (dd^c u_2)^n$. By Propositions 3.1.11 and 3.2.7, $(dd^c u^*_E)^n = (dd^c V^*_E)^n = 0$ outside the set $\{V^*_E = u^*_E + 1 = 0\}$ and (3.2.6) follows directly from (3.2.5). ■

Exercise Assume that E is a bounded subset of $\mathbb{C}^n$ and let R be so big that E is contained in a connected component Ω of the open set $\{V^*_E < R\}$. Show that

$$V_E \le R(u_{E,\Omega} + 1) \le R(u^*_{E,\Omega} + 1) = V^*_E.$$

3.3. The pluricomplex Green function

Let Ω be a domain in $\mathbb{C}^n$ and $w \in \Omega$. The *pluricomplex Green function* of Ω with a pole at w was first defined in [Kli1] as follows

$$g_{\Omega,w} = g_\Omega(w,\cdot) := \sup\{u \in PSH(\Omega) : u \le 0, \limsup_{z\to w}(u(z) - \log|z-w|) < \infty\}.$$

Exercise Show that $g_{B(w,r)}(w,z) = \log\frac{|z-w|}{r}$.

The next proposition lists the basic properties of g_Ω and the proofs are left as an **Exercise** to the reader.

Proposition 3.3.1. *i) If $\Omega_1 \subset \Omega_2$ then $g_{\Omega_1} \ge g_{\Omega_2}$ on $\Omega_1 \times \Omega_1$;*

ii) If $B(w,r) \subset \Omega$ then $g_\Omega(w,z) \le \log\frac{|z-w|}{r}$, $z \in \Omega$;

iii) If $\Omega \subset B(w,R)$ then $\log\frac{|z-w|}{R} \le g_\Omega(w,z)$, $z \in \Omega$;

iv) Either $g_{\Omega,w} \equiv -\infty$ or $g_{\Omega,w} \in PSH(\Omega)$; in the latter case $g_{\Omega,w}$ is maximal in $\Omega \setminus \{w\}$;

v) If $\Omega_j \uparrow \Omega$ then $g_{\Omega_j} \downarrow g_\Omega$. ■

The following result can be used in constructing explicit examples of the Green function with a pole at the origin.

Proposition 3.3.2. *Let Ω be a balanced pseudoconvex domain in $\mathbb{C}^n$. Then $g_{\Omega,0} = \log f_\Omega$, where f_Ω is the Minkowski functional of Ω.*

Proof. From Theorem 1.4.8 it follows that $\log f_\Omega$ is psh in Ω. From the definition of the Minkowski functional one can deduce that if $B(0,r) \subset \Omega$ then $f_\Omega(z) \le |z|/r$, hence $\log f_\Omega \le g_{\Omega,0}$. On the other hand, if we fix $z_0 \in \partial B(0,1)$, then $\{\lambda \in \mathbb{C} : \lambda z_0 \in \Omega\} = \Delta(0,\rho)$ for some $\rho \in (0,+\infty]$ and $\log f_\Omega(\lambda z_0) = \log|\lambda|/r = g_{\Delta(0,r)}(0,\lambda)$ and it follows that $g_{\Omega,0} \le \log f_\Omega$. ■

The following comparison of the Green function with the relative extremal function of a ball will turn out handy.

Proposition 3.3.3. *Assume that $r, R > 0$ are such that $B(w,r) \Subset \Omega \subset B(w,R)$. Then for $\rho \in (0,r]$ we have*

$$\log\frac{R}{\rho}\, u_{\overline{B}(w,\rho),\Omega}(z) \le g_\Omega(w,z) \le \log\frac{r}{\rho}\, u_{\overline{B}(w,\rho),\Omega}(z), \quad z \in \Omega \setminus B(w,\rho). \tag{3.3.1}$$

Proof. If $z \in \partial B(w,\rho)$ then (3.3.1) follows from Proposition 3.3.1.ii,iii. The second inequality in (3.3.1) can be then deduced from Proposition 3.3.1.iv and the definition of the relative function. On the other hand, the function

$$v(z) := \begin{cases} \max\{\log \frac{|z-w|}{R}, \log \frac{R}{\rho}\, u_{\overline{B}(w,\rho),\Omega}(z)\}, & \text{if } z \in \Omega \setminus B(w,\rho) \\ \log \frac{|z-w|}{R}, & \text{if } z \in B(w,\rho) \end{cases}$$

is plurisubharmonic and $v \le g_{\Omega,w}$ which gives the first inequality in (3.3.1). ∎

Exercise Let Ω be a bounded balanced domain in $\mathbb{C}^n$. Prove that Ω is hyperconvex iff $\log f_\Omega$ is plurisubharmonic and continuous in $\mathbb{C}^n \setminus \{0\}$.

The next two results are due to Demailly [Dem1].

Theorem 3.3.4. *Assume that Ω is bounded and hyperconvex. Then e^{g_Ω} is continuous on $\Omega \times \overline{\Omega}$ (on $\Omega \times \partial\Omega$ we set $g_\Omega := 0$).*

Proof. The continuity at the points from the diagonal of $\Omega \times \Omega$ follows directly from Proposition 3.3.1.ii. It therefore remains to show the continuity of g_Ω on $\Omega' \times (\overline{\Omega} \setminus \Omega'')$, if $\Omega' \Subset \Omega'' \Subset \Omega$. Let $r, R > 0$ be such that $B(w,r) \Subset \Omega'' \Subset \Omega \subset B(w,R)$ for every $w \in \Omega'$. For $\rho \in (0,r)$ set

$$u_\rho(w,z) := u_{\overline{B}(w,\rho),\Omega}(z), \quad w \in \Omega',\ z \in \overline{\Omega}.$$

By Proposition 3.3.3 on $\Omega' \times (\overline{\Omega} \setminus \Omega'')$ we have

$$\log \frac{R}{\rho}\, u_\rho \le g_\Omega \le \log \frac{r}{\rho}\, u_\rho,$$

and, since $\log \frac{R}{\rho} / \log \frac{r}{\rho} \downarrow 1$ as $\rho \downarrow 0$, it remains to show that for a fixed ρ the function u_ρ is continuous on $\Omega' \times \overline{\Omega}$. Observe that, since Ω is hyperconvex, for a fixed w, the function $u_\rho(w,\cdot)$ is continuous on $\overline{\Omega}$ by Proposition 3.1.3.vii. Therefore, it is enough to show that, if $w_j \longrightarrow w$ then $u_\rho(w_j,\cdot) \longrightarrow u_\rho(w,\cdot)$ uniformly in $\overline{\Omega}$. Fix small $\varepsilon > 0$. Then for j big enough we have $\overline{B}(w_j,\rho-\varepsilon) \subset \overline{B}(w,\rho) \subset \overline{B}(w_j,\rho+\varepsilon)$ and

$$u_{\overline{B}(w_j,\rho)} - \frac{\log \frac{\rho+\varepsilon}{\rho}}{\log \frac{r}{\rho}} \le u_{\overline{B}(w_j,\rho+\varepsilon)} \le u_{\overline{B}(w,\rho)} \le u_{\overline{B}(w_j,\rho-\varepsilon)} \le u_{\overline{B}(w_j,\rho)} + \frac{\log \frac{\rho}{\rho-\varepsilon}}{\log \frac{r}{\rho-\varepsilon}}$$

on $\overline{\Omega}$. This completes the proof of the theorem. ∎

Theorem 3.3.5. *Let $w \in \Omega$, where Ω is a bounded domain in $\mathbb{C}^n$. Assume that u_j is a sequence of locally bounded plurisubharmonic functions defined in a neighborhood U of w, $U \subset \Omega$, decreasing to $g_{\Omega,w}$. Then $(dd^c u_j)^n$ tends weakly to $(2\pi)^n \delta_w$.*

Proof. Let r, R and ρ be as in Proposition 3.3.3 with $B(w,r) \Subset U$. For simplicity we may assume that $w = 0$ and denote $g = g_{\Omega,w}$, $\overline{B}_\rho = \overline{B}(w,\rho)$. In view of Proposition 3.3.1.iv it is enough to show that

$$\lim_{j\to\infty} \int_{\overline{B}_\rho} (dd^c u_j)^n = (2\pi)^n. \tag{3.3.2}$$

The idea will be similar as in the proof of Proposition 3.2.2. We have $\log \frac{|z|}{R} \le g(z) \le \log \frac{|z|}{r}$, $z \in \overline{B}_r$. Fix $\alpha < 1$ and let $C > 0$ be such that $\alpha \log \rho - C < \log \frac{\rho}{R}$. We can find $\varepsilon > 0$ so small that $\alpha \log \varepsilon - C > \log \frac{\varepsilon}{r}$. Then, since $\max_{\overline{B}_\varepsilon} u_j \downarrow \max_{\overline{B}_\varepsilon} g$ as $j \uparrow \infty$, for j big enough we have $\overline{B}_\varepsilon \Subset \{u_j < f\} \Subset \overline{B}_\rho$, where $f(z) = \max\{\alpha \log|z| - C, \log \frac{\varepsilon}{r}\}$. By Theorem 2.2.6 we have

$$\int_{\overline{B}_\rho} (dd^c u_j)^n \ge \int_{\{u_j<f\}} (dd^c u_j)^n \ge \int_{\{u_j<f\}} (dd^c f)^n = \int_{\overline{B}_\varepsilon} (dd^c f)^n = \alpha^n (2\pi)^n.$$

On the other hand, fix $\beta > 1$. Then $u_j(z) \ge \log \frac{|z|}{R} \ge \beta \log \frac{|z|}{r}$ for $z \in \overline{B}_\delta$, where $\delta > 0$ depends on β, thus

$$\int_{\overline{B}_\delta} (dd^c u_j)^n \le \int_{\overline{B}_r} \left(dd^c \max\{u_j, \beta \log \frac{|z|}{r}\} \right)^n = \beta^n (2\pi)^n.$$

Now, since $(dd^c u_j)^n \longrightarrow 0$ on $U \setminus \{0\}$, we have

$$\lim_{j\to\infty} \int_{\overline{B}_\rho \setminus \overline{B}_\delta} (dd^c u_j)^n = 0$$

and (3.3.2) follows. ∎

Exercise Prove that g_Ω is symmetric if $n = 1$.

Exercise Let $\Omega = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1 z_2| < 1\}$. Show that for $w = (w_1,w_2) \in \Omega$ one has

$$g_\Omega((w_1,w_2),(z_1,z_2)) = \begin{cases} \log \left| \dfrac{z_1 z_2 - w_1 w_2}{1 - \overline{w}_1 \overline{w}_2 z_1 z_2} \right| & \text{if } (w_1,w_2) \ne (0,0), \\ \frac{1}{2} \log |z_1 z_2| & \text{if } (w_1,w_2) = (0,0). \end{cases}$$

In particular, g_Ω is not symmetric.

In view of Propositon 3.3.1.v it means in particular that if $n \ge 2$ then g_Ω need not be symmetric even if Ω is a very regular, bounded hyperconvex domain. A domain with non-symmetric Green function was constructed for the first time by Bedford and Demailly [BD] and the above simple example is due to Klimek.

The product property for the Green function was first proved in [JP1]:

Theorem 3.3.6. *Let Ω_j be bounded hyperconvex domains in $\mathbb{C}^{n_j}$, $j = 1, 2$. Then for $w^j \in \Omega_j$ we have*

$$g_{\Omega_1 \times \Omega_2, (w^1, w^2)} = \max\{g_{\Omega_1, w^1}, g_{\Omega_2, w^2}\}.$$

Proof. We may assume that $w^j = 0$. The inequality "$\geq$" follows directly from the definition of the Green function. Theorem 2.1.10 implies that the function $u := \max\{g_{\Omega_1,0}, g_{\Omega_2,0}\}$ is maximal in $(\Omega_1 \setminus \{0\}) \times (\Omega_2 \setminus \{0\})$ and it easily follows that it is in fact maximal in $\Omega_1 \times \Omega_2 \setminus \{(0,0)\}$. For every $\alpha > 1$ there exists $\varepsilon > 0$ such that $\alpha g_{\Omega_1 \times \Omega_2} \leq u$ on $\overline{B}((0,0), \varepsilon)$. Since Ω_j are hyperconvex, we have $u = 0$ on $\partial(\Omega_1 \times \Omega_2)$ and from the maximality of u it follows that $\alpha g_{\Omega_1 \times \Omega_2} \leq u$ in $\Omega_1 \times \Omega_2 \setminus \overline{B}((0,0), \delta)$ for every $\delta \in (0, \varepsilon)$. ∎

We finish this section with a result describing the behavior of a Green function in hyperconvex domains when a pole approaches the boundary:

Theorem 3.3.7. *Let Ω be a bounded hyperconvex domain. Then for $p < \infty$ we have*

$$\lim_{w \to \partial\Omega} ||g_{\Omega,w}||_{L^p(\Omega)} = 0.$$

Proof. By Proposition 3.3.1.iii we have

$$||g_{\Omega,w}||_{L^p(\Omega)} \leq C(p, \Omega). \tag{3.3.3}$$

In particular, it is enough to show that for $\Omega' \Subset \Omega$

$$\lim_{w \to \partial\Omega} ||g_{\Omega,w}||_{L^p(\Omega')} = 0.$$

By Theorem 1.4.6 there exists $\psi \in PSH(\Omega) \cap C(\overline{\Omega})$ with $\psi|_{\partial\Omega} = 0$ and $(dd^c\psi)^n \geq d\lambda$ on Ω'. Then, using Theorems 2.1.8 and 3.3.5

$$\begin{aligned}
||g_{\Omega,w}||^n_{L^n(\Omega')} &\leq \int_\Omega (-g_{\Omega,w})^n (dd^c\psi)^n \\
&= \lim_{k\to\infty} \int_\Omega (-\max\{g_{\Omega,w}, -k\})^n (dd^c\psi)^n \\
&\leq n! ||\psi||^{n-1}_{L^\infty(\Omega)} \lim_{k\to\infty} \int_\Omega |\psi| (dd^c \max\{g_{\Omega,w}, -k\})^n \\
&= n!(2\pi)^n ||\psi||^{n-1} |\psi(w)|.
\end{aligned}$$

This proves the theorem for $p = n$. Now, the general case follows easily from the Hölder inequality and (3.3.3). ∎

IV. Some applications of pluripotential theory

4.1. The Bergman metric

Throughout this section let Ω be a bounded domain in $\mathbb{C}^n$. By $H^2(\Omega)$ we denote the set of holomorphic functions in Ω that are in $L^2(\Omega)$. If f is holomorphic in Ω then in particular $|f|^2$ is a subharmonic function and we have

$$|f(z)|^2 \leq \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |f|^2 d\lambda, \quad 0 < r < \operatorname{dist}(z, \partial\Omega).$$

Therefore

$$|f(z)| \leq \frac{c_n}{(\operatorname{dist}(z,\partial\Omega))^n} ||f||, \quad z \in \Omega,\ f \in H^2(\Omega), \tag{4.1.1}$$

where by $||f||$ we denote the norm of f in $L^2(\Omega)$. It follows that

$$\sup_K |f| \leq c(K,\Omega)\, ||f||, \quad K \Subset \Omega,\ f \in H^2(\Omega).$$

From this one can easily deduce that $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, and thus a separable Hilbert space with a scalar product

$$\langle f, g\rangle = \int_\Omega f\overline{g} d\lambda, \quad f, g \in H^2(\Omega).$$

For a fixed $z \in \Omega$ we have a functional

$$H^2(\Omega) \ni f \longmapsto f(z) \in \mathbb{C}$$

which is continuous by (4.1.1). It follows that there is $g_z \in H^2(\Omega)$ such that $f(z) = \langle f, g_z\rangle$ for every $f \in H^2(\Omega)$. For $\zeta, z \in \Omega$ we set $K_\Omega(\zeta, z) = g_z(\zeta)$, so that

$$f(z) = \langle f, K_\Omega(\cdot, z)\rangle, \quad z \in \Omega,\ f \in H^2(\Omega).$$

In particular,

$$K_\Omega(z, w) = \langle K_\Omega(\cdot, w), K_\Omega(\cdot, z)\rangle = \overline{K_\Omega(w, z)}, \quad z, w \in \Omega.$$

We conclude that for $z, w \in \Omega$ $K_\Omega(\cdot, w)$ is holomorphic and $K_\Omega(z, \cdot)$ antiholomorphic in Ω. Hence, $K_\Omega(z, \bar{\cdot})$ is holomorphic in a domain $\Omega^* := \{\overline{w} : w \in \Omega\}$. From the Hartogs theorem on separate analyticity it follows that $K_\Omega(\cdot, \bar{\cdot})$ is holomorphic in $\Omega \times \Omega^*$ and so $K_\Omega \in C^\infty(\Omega \times \Omega)$. K_Ω is called a *Bergman kernel* of Ω.

We set $k_\Omega(z) := K_\Omega(z, z)$. Then, by the definition of K_Ω, we have

$$k_\Omega(z) = ||K_\Omega(\cdot, z)||^2 = \sup\{|f(z)|^2 : f \in H^2(\Omega), ||f|| \leq 1\}. \tag{4.1.2}$$

Proposition 4.1.1. *i) If $\Omega_1 \subset \Omega_2$ then $k_{\Omega_1} \geq k_{\Omega_2}$;*

ii) If $\Omega_j \uparrow \Omega$ then $K_{\Omega_j} \to K_\Omega$ locally uniformly on $\Omega \times \Omega$, thus $k_{\Omega_j} \downarrow k_\Omega$.

Proof. i) It follows easily from (4.1.2).

ii) Choose domains Ω' and Ω'' so that $\Omega' \Subset \Omega'' \Subset \Omega$. Then for $z, w \in \Omega'$ and j big enough by the Schwarz inequality and i) we have

$$|K_{\Omega_j}(z, w)|^2 \leq k_{\Omega_j}(z) k_{\Omega_j}(w) \leq k_{\Omega''}(z) k_{\Omega''}(w).$$

Therefore the sequence K_{Ω_j} is locally bounded in $\Omega \times \Omega$. If we apply the Montel theorem to the space of holomorphic functions in $\Omega \times \Omega^*$, we see that K_{Ω_j} has a subsequence converging locally uniformly on $\Omega \times \Omega$. Thus, to complete the proof, it is enough to show that if $K_{\Omega_j} \to K$ locally uniformly, then $K = K_\Omega$.

For $w \in \Omega$ we have

$$\begin{aligned} ||K(\cdot, w)||^2_{L^2(\Omega')} &= \lim_{j\to\infty} ||K_{\Omega_j}(\cdot, w)||^2_{L^2(\Omega')} \\ &\leq \liminf_{j\to\infty} ||K_{\Omega_j}(\cdot, w)||^2_{L^2(\Omega_j)} = \liminf_{k\to\infty} K_{\Omega_j} = K(w, w). \end{aligned}$$

Since the estimate holds for arbitrary $\Omega' \Subset \Omega$, we get

$$||K(\cdot, w)||^2_{L^2(\Omega)} \leq K(w, w)$$

and so $K(\cdot, w) \in H^2(\Omega)$ for every $w \in \Omega$. Fix $w \in \Omega$ and $f \in H^2(\Omega)$. To finish the proof it remains to show that

$$f(w) = \int_\Omega f(z) \overline{K(z, w)} d\lambda(z).$$

For j big enough we have

$$f(w) = \int_{\Omega_j} f(z) \overline{K_{\Omega_j}(z, w)} d\lambda(z)$$

and thus

$$\begin{aligned} f(w) - \int_\Omega f(z)\overline{K(z, w)} d\lambda(z) &= \int_{\Omega'} f(z) \left(\overline{K_{\Omega_j}(z, w)} - \overline{K(z, w)}\right) d\lambda(z) \\ &+ \int_{\Omega_j \setminus \Omega'} f(z) \overline{K_{\Omega_j}(z, w)} d\lambda(z) - \int_{\Omega \setminus \Omega'} f(z) \overline{K(z, w)} d\lambda(z). \end{aligned}$$

The absolute value of the second integral we may estimate by $||f||_{L^2(\Omega\setminus\Omega')}\sqrt{K_{\Omega_j}(w,w)}$, and it follows easily that all three integrals are arbitrarily small if Ω' is sufficiently close to Ω and j is big enough. ∎

Let $\phi_0, \phi_1, \dots$ be an orthonormal basis of $H^2(\Omega)$. Then

$$f = \sum_k \langle f, \phi_k\rangle \phi_k, \quad f \in H^2(\Omega),$$

and by (4.1.2) the convergence is locally uniform. Thus

$$K_\Omega(z,w) = \sum_k \overline{\langle \phi_k, K_\Omega(\cdot, w)\rangle}\phi_k(z) = \sum_k \phi_k(z)\overline{\phi_k(w)}, \quad z, w \in \Omega, \tag{4.1.3}$$

and

$$k_\Omega = \sum_k |\phi_k|^2. \tag{4.1.4}$$

Since K_Ω is smooth, from (4.1.2) it follows that $\log k_\Omega \in PSH \cap C^\infty(\Omega)$. The *Bergman metric* of Ω is defined as the Levi form of $\log k_\Omega$:

$$\beta^2(z,X) := \frac{\partial^2}{\partial\lambda\partial\overline{\lambda}} \log k_\Omega(z+\lambda X)\bigg|_{\lambda=0} = \sum_{j,k=1}^n \frac{\partial^2(\log k_\Omega)}{\partial z_j \partial \overline{z}_k}(z) X_j \overline{X}_k, \quad z \in \Omega,\ X \in \mathbb{C}^n.$$

It can be expressed in the following way:

Theorem 4.1.2. *For $z \in \Omega$ and $X \in \mathbb{C}^n$ we have*

$$\beta^2(z,X) = \frac{1}{k_\Omega(z)} \sup\left\{|D_X f(z)|^2 : f \in H^2(\Omega), ||f|| \le 1, f(z) = 0\right\},$$

where $D_X f := \sum_{j=1}^n \partial f/\partial z_j\, X_j$.

Proof. We may assume that $X \neq 0$. Define the following subspaces of $H^2(\Omega)$:

$$H' := \{f \in H^2(\Omega) : f(z) = 0\},$$
$$H'' := \{f \in H' : D_X f(z) = 0\}.$$

Then $H'' \subset H' \subset H^2(\Omega)$ and in both cases the codimension equals 1, since H' and H'' are defined as kernels of nonzero functionals ($\langle \cdot - z, X\rangle \in H'' \setminus H'$). Let $\phi_0, \phi_1, \dots$ be an orthonormal basis of $H^2(\Omega)$ such that $\phi_1 \in H'$ and $\phi_k \in H''$ for $k \ge 2$. Then using (4.1.4) we may easily compute that

$$k_\Omega(z) = |\phi_0(z)|^2, \quad \beta^2(z,X) = \frac{|D_X\phi_1(z)|^2}{|\phi_0(z)|^2}.$$

This gives "$\le$". To get the inverse inequality take any $f \in H'$ with $||f|| \le 1$. Then $\langle f, \phi_0\rangle = 0$ and

$$f = \sum_{k=1}^{\infty} \langle f, \phi_k\rangle \phi_k.$$

Therefore

$$|D_X f(z)| = |\langle f, \phi_1\rangle D_X \phi_1(z)| \le |D_X \phi_1(z)|$$

and the theorem follows. ∎

We see therefore that $\log k_\Omega$ is strictly plurisubharmonic and thus the Bergman metric is indeed a metric (even a Kähler one). Namely, if $\gamma : [0,1] \to \Omega$ is a continuous and piecewise smooth curve, then its length is defined by

$$l(\gamma) = \int_0^1 \beta(\gamma(t), \gamma'(t))dt$$

and the *Bergman distance* $\mathrm{dist}_\Omega(z,w)$ between two points $z, w \in \Omega$ is the infiumum over the lengths of all such curves joining z and w. If Ω with this distance is complete then we say that it is *Bergman complete*. The next result is due to Bremermann [Bre].

Proposition 4.1.3. *If Ω is Bergman complete then it must be pseudoconvex.*

Proof. If Ω is not pseudoconvex then by the definition of a domain of holomorhpy there are domains Ω_1, Ω_2 such that $\emptyset \ne \Omega_1 \subset \Omega \cap \Omega_2$ and for every f holomorphic in Ω there exists $\widetilde{f}$ holomorphic in Ω_2 such that $f = \widetilde{f}$ on Ω_1. We may assume that Ω_1 is a connected component of $\Omega \cap \Omega_2$ such that the set $\Omega_2 \cap \partial\Omega \cap \partial\Omega_1$ is nonempty. Since $K_\Omega(\cdot, \bar{\cdot})$ is holomorphic in $\Omega \times \Omega^*$, it follows that there exists $\widetilde{K} \in C^\infty(\Omega_2 \times \Omega_2)$ such that $\widetilde{K}(\cdot, \bar{\cdot})$ is holomorphic in $\Omega_2 \times \Omega_2^*$ and $\widetilde{K} = K_\Omega$ in $\Omega_1 \times \Omega_1$. This means that every sequence $z_k \to \Omega_2 \cap \partial\Omega \cap \partial\Omega_1$ is a Cauchy sequence with respect to dist_Ω, which contradicts the completeness of Ω. ∎

The converse is not true and perhaps the simplest example of a pseudoconvex but not Bergman complete domain is a punctured disc on the plane.

Our next goal is to show the following criterion due to Kobayashi ([Kob1], [Kob2], see also [Kob3]).

Theorem 4.1.4. *Let Ω be a bounded domain satisfying*

$$\overline{\lim_{z \to \partial\Omega}} \frac{|f(z)|^2}{k_\Omega(z)} < ||f||^2, \quad f \in H^2(\Omega) \setminus \{0\}.$$

Then Ω is Bergman complete.

Before proving Theorem 4.1.4 we want to look at a general construction of Kobayashi. Define a mapping

$$\iota : \Omega \ni z \longmapsto \langle K_\Omega(\cdot, z)\rangle \in \mathbb{P}(H^2(\Omega)).$$

One can easily check that ι is one-to-one. In the projective space $\mathbb{P}(H^2(\Omega))$ we have the Fubini-Study metric $\mathcal{F}$ (see Section A4). The following fact was first observed by Kobayashi [Kob1]:

Proposition 4.1.5. *One has* $\beta^2 = \iota^*\mathcal{F}^2$, *that is*

$$\beta^2(z, X) = \mathcal{F}^2(\iota(z), \iota'(z).X), \quad z \in \Omega, X \in \mathbb{C}^n.$$

Proof. Let $\gamma : (-\varepsilon, \varepsilon) \to \Omega$ be a smooth curve with $\gamma(0) = z$, $\gamma'(0) = X$. Set $\widetilde{\gamma}(t) = K_\Omega(\cdot, \gamma(t))$. We have to show that

$$\beta^2(z, X) = \frac{||\widetilde{\gamma}(0)||^2||\widetilde{\gamma}'(0)||^2 - |\langle\widetilde{\gamma}(0), \widetilde{\gamma}'(0)\rangle|^2}{||\widetilde{\gamma}(0)||^4}. \tag{4.1.5}$$

Let $\phi_0, \phi_1, \dots$ be an orthonormal basis of $H^2(\Omega)$ chosen in the same way as in the proof of Theorem 4.1.2. Then

$$\widetilde{\gamma}(0) = \overline{\phi_0(z)}\phi_0,$$
$$\widetilde{\gamma}'(0) = \overline{D_X\phi_0(z)}\phi_0 + \overline{D_X\phi_1(z)}\phi_1$$

and one can easily check that both hand-sides of (4.1.5) are equal to

$$\frac{|D_X\phi_1(z)|^2}{|\phi_0(z)|^2}. \ \blacksquare$$

Thus ι is an imbedding of (Ω, β^2) into the space $(\mathbb{P}(H^2(\Omega)), \mathcal{F}^2)$. It is therefore distance decreasing and combining this with Proposition A4.2 we have obtained the following estimate:

Proposition 4.1.6. *For* $z, w \in \Omega$ *we have*

$$\operatorname{dist}_\Omega(z, w) \geq \arccos\frac{|K_\Omega(z, w)|}{\sqrt{k_\Omega(z)k_\Omega(w)}}. \ \blacksquare$$

Proof of Theorem 4.1.4. Let z_k be a Cauchy sequence in Ω (with respect to the Bergman metric). Suppose that z_k has no accumulation point in Ω. It is easy to check that this is equivalent to the fact that $z_k \to \partial\Omega$. Since $\iota(z_k)$ is a Cauchy sequence in $\mathbb{P}(H^2(\Omega))$ which is complete, it follows that there is $f \in H^2(\Omega) \setminus \{0\}$ such that $\iota(z_k) \to \langle f \rangle$. Therefore

$$\frac{|f(z_k)|}{k_\Omega(z_k)} = \big|\langle f, \frac{K_\Omega(\cdot, z_k)}{\sqrt{k_\Omega(z_k)}}\rangle\big| \to ||f||$$

as $k \to \infty$, which contradicts the assumption of the theorem. ■

By the way, Proposition A4.3 gives the following lemma due to Pflug [Pfl]:

Lemma 4.1.7. *Assume that Ω is bounded and let z_k be a Cauchy sequence in Ω with respect to the Bergman metric such that $z_k \to \partial\Omega$. Then one can find $f \in H^2(\Omega)$, $||f|| = 1$, and $\lambda_k \in \mathbb{C}$, $|\lambda_k| = 1$, such that*

$$\lambda_k \frac{K_\Omega(\cdot, z_k)}{\sqrt{k_\Omega(z_k)}} \to f$$

in $H^2(\Omega)$ as $k \to \infty$. ■

Theorem 4.1.4 also easily implies the following:

Corollary 4.1.8. *Let Ω be a bounded domain in $\mathbb{C}^n$ such that $\lim_{z \to \partial\Omega} k_\Omega(z) = \infty$ and for every $z_0 \in \partial\Omega$ the space*

$$\{f \in H^2(\Omega) : \overline{\lim_{z \to z_0}} |f(z)| < \infty\}$$

is dense in $H^2(\Omega)$ (this is for example the case if $H^\infty(\Omega)$ is dense in $H^2(\Omega)$). Then Ω is Bergman complete.

Proof. Let $\Omega \ni z_j \to z_0 \in \partial\Omega$. For $f \in H^2(\Omega)$ by the assumption we can find $f_k \in H^2(\Omega)$ such that $\overline{\lim}_{j\to\infty} |f_k(z_j)| < \infty$ and $\lim_{k\to\infty} ||f_k - f|| \to 0$. Then for every k

$$\overline{\lim_{j\to\infty}} \frac{|f(z_j)|}{\sqrt{k_\Omega(z_j)}} \le \overline{\lim_{j\to\infty}} \frac{|f_k(z_j)|}{\sqrt{k_\Omega(z_j)}} + ||f - f_k|| = ||f - f_k||.$$

It follows that

$$\lim_{z\to\partial\Omega} \frac{|f(z)|}{\sqrt{k_\Omega(z)}} = 0$$

and we use Theorem 4.1.4. ■

We will now prove several results relating the Bergman kernel and metric with the pluricomplex Green function. The first one is essentially due to Herbort [Her] (see also

[Che1]).

Theorem 4.1.9. *If Ω is a pseudoconvex domain in $\mathbb{C}^n$ then we have*

$$\frac{|f(w)|}{\sqrt{k_\Omega(w)}} \leq \left(1 + \frac{4}{\int_n^\infty \frac{dx}{xe^x}}\right) ||f||_{L^2(\{g_{\Omega,w} \leq -1\})}, \quad w \in \Omega,\ f \in H^2(\Omega).$$

Proof. We will use Theorem A5.3 with $\varphi := 2ng$ and $\psi := -\log(-g)$, where $g := g_{\Omega,w}$. Since g is a locally bounded plurisubharmonic function in $\Omega \setminus \{w\}$, it follows from Theorem 2.1.9 that $i\partial g \wedge \overline{\partial} g \wedge (i\partial\overline{\partial}|z|^2)^{n-1}$ is a positive Borel measure on $\Omega \setminus \{w\}$ and one can easily deduce that $\overline{\partial} g \in L^2_{loc,(0,1)}(\Omega \setminus \{w\})$. Set

$$\alpha := \overline{\partial}(f \cdot \gamma \circ g) = f \cdot \gamma' \circ g\, \overline{\partial} g \in L^2_{loc,(0,1)}(\Omega),$$

where $\gamma \in C^{0,1}((-\infty, 0))$ with $\gamma'(t) = 0$ near $-\infty$ will be specified later. We have

$$i\alpha \wedge \overline{\alpha} = |f|^2(\gamma' \circ g)^2 i\partial g \wedge \overline{\partial} g \leq |f|^2(\gamma' \circ g)^2 g^2 i\partial\overline{\partial}\psi.$$

By Theorem A5.3 we can find $u \in L^2_{loc}(\Omega)$ such that $\overline{\partial} u = \alpha$ and

$$\int_\Omega |u|^2 e^{-2ng} d\lambda \leq 16 \int_\Omega |f|^2 (\gamma' \circ g)^2 g^2 e^{-2ng} d\lambda.$$

If for $a < -1$ we take

$$\gamma(t) := \begin{cases} \displaystyle\int_{\max\{t,a\}}^{-1} \frac{e^{2ns}}{s} ds, & t \leq -1, \\ 0, & -1 < t < 0, \end{cases}$$

we will get

$$||u||_{L^2(\Omega)} \leq 4||f||_{L^2(\{g \leq -1\})}.$$

The function $f \cdot \gamma \circ g - u$ is equal almost everywhere to a holomorphic $\widetilde{f}$. Moreover, since $e^{-\varphi}$ is not locally integrable near w it follows that $\widetilde{f}(w) = \gamma(a) f(w)$. We also have

$$||\widetilde{f}|| \leq \gamma(a)||f||_{L^2(\{g \leq -1\})} + ||u||_{L^2(\Omega)} \leq (\gamma(a) + 4)||f||_{L^2(\{g \leq -1\})}.$$

Therefore by (4.1.2)

$$\frac{|f(w)|}{\sqrt{k_\Omega(w)}} = \frac{|\widetilde{f}(w)|}{\gamma(a)\sqrt{k_\Omega(w)}} \leq \frac{||\widetilde{f}||}{\gamma(a)}$$

and the desired estimate follows if we let $a \to -\infty$. ∎

Theorem 4.1.10. *For a bounded pseuudoconvex domain Ω in $\mathbb{C}^n$ consider the following conditions*

i) Ω is hyperconvex;

ii) $\lim_{w\to\partial\Omega} \lambda(\{g_{\Omega,w} \leq -1\}) = 0$;

iii) Ω is Bergman complete;

iv) $\lim_{w\to\partial\Omega} k_\Omega(w) = \infty$.

Then i)$\Rightarrow$ii)$\Rightarrow$iii) and ii)$\Rightarrow$iv).

Proof. Theorem 3.3.7 gives i)$\Rightarrow$ii). The implication ii)$\Rightarrow$iii) follows immediately from Theorems 4.1.4 and 4.1.9 and if we use Theorem 4.1.9 for constant functions then we get ii)$\Rightarrow$iv). ∎

The implication i)$\Rightarrow$iv) in Theorem 4.1.10 is due to Ohsawa [Ohs]. i)$\Rightarrow$iii) was proved independently in [BP] and [Her].

For $n \geq 2$ ii) does not imply i), as the following example shows. It is due to Herbort [Her] who used it to show that iii)$\not\Rightarrow$i) in a much more complicated way.

Exercise Set

$$\Omega := \{z \in \Delta_* \times \Delta : |z_2| < e^{-1/|z_1|}\},$$

where Δ stands for the unit disc in $\mathbb{C}$. Show that Ω is pseudoconvex but not hyperconvex. For $w \in \Omega$ denote $g_w := g_{\Omega,w}$. Prove that

$$g_{\Omega,w}(z) \geq \log \frac{|w_1|}{1+2|w_1|^2} \frac{\log|z_2|}{\log e^{-1/(2|w_1|)}}, \quad \text{if } |z_1| \geq 2|w_1|$$

and conclude that i) holds.

The next result, due to Chen [Che2] for $n = 1$, coupled with Corollary 4.1.8 implies that iv)$\Rightarrow$iii) holds in Theorem 4.1.10 for $n = 1$.

Theorem 4.1.11. *Let Ω and U be bounded domains in $\mathbb{C}^n$ such that $\Omega\cup U$ is pseudoconvex with diameter R. Assume that $U \subset B(z_0, r)$. Then for every $f \in H^2(\Omega)$ there exists $F \in H^2(\Omega \cup U)$ such that for every $\lambda > 1$ we have*

$$||F - f||_{L^2(\Omega)} \leq \left(1 + \frac{4}{\log\lambda}\right) ||f||_{L^2(\Omega\cap B(z_0,R(r/R)^{1/\lambda})}.$$

Proof. For $t < R$ set $\eta(t) := -\log(-\log t/R)$ and $\psi(z) := \eta(|z - z_0|)$ for $z \in B(z_0, R)$. Then $-e^{-\psi}$ is plurisubharmonic and $i\partial\psi \wedge \overline{\partial}\psi \leq i\partial\overline{\partial}\psi$. Fix $\rho \in (r, R)$ and define

$$\chi(s) := \begin{cases} 0, & s < \eta(r), \\ \frac{s-\eta(r)}{\eta(\rho)-\eta(r)}, & \eta(r) \leq s < \eta(\rho), \\ 1, & s \geq \eta(\rho). \end{cases}$$

Then $\alpha := \overline{\partial}\widehat{f} \in L^2_{loc,(0,1)}(\Omega \cup B(z_0, r))$, where

$$\widehat{f} := \begin{cases} f \cdot \gamma \circ \psi, & \text{in } \Omega, \\ 0, & \text{in } B(z_0, r). \end{cases}$$

By Theorem A5.3, applied with $\varphi = 0$ and $h = |f|^2(\chi' \circ \psi)^2$ in $\Omega \cup U$, we can find $u \in L^2(\Omega \cup U)$ such that $\overline{\partial} u = \alpha$ and

$$\int_{\Omega \cup U} |u|^2 d\lambda \le \frac{16}{(\eta(\rho) - \eta(r))^2} \int_{\Omega \cap B(z_0,\rho)} |f|^2 d\lambda.$$

It follows that there exists $F \in H^2(\Omega \cup U)$ such that $F = \widehat{f} - u$ almost everywhere. Moreover,

$$\begin{aligned} ||F - f||_{L^2(\Omega)} &\le ||f(1 - \gamma \circ \psi)||_{L^2(\Omega)} + ||u||_{L^2(\Omega)} \\ &\le \left(1 + \frac{4}{\eta(\rho) - \eta(r)}\right) ||f||_{L^2(\Omega \cap B(z_0,\rho))}. \end{aligned}$$

It is now enough to take $\rho := R(r/R)^{1/\lambda}$. ■

The following estimate for the Bergman distance was proven in [Bło3].

Theorem 4.1.12. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $w, \widetilde{w} \in \Omega$ be such that $\{g_{w,\Omega} \le -1\} \cap \{g_{\widetilde{w},\Omega} < -1\} = \emptyset$. Then*

$$\operatorname{dist}_\Omega(w, \widetilde{w}) \ge \frac{\pi}{2} - \arctan\left(1 + \frac{4e^n}{\int_n^\infty \frac{dx}{xe^x}}\right).$$

Proof. Set $f := K_\Omega(\cdot, \widetilde{w})/\sqrt{k_\Omega(\widetilde{w})}$. Define also $\varphi := 2n(g + \widetilde{g})$ and $\psi := -\log(-g)$, where $g := g_{\Omega,w}$ and $\widetilde{g} := \widetilde{g}_{\Omega,w}$. Let a, γ and α be defined in the same way as in the proof of Theorem 4.1.9, then

$$i\alpha \wedge \overline{\alpha} \le |f|^2(\gamma' \circ g)^2 g^2 i\partial\overline{\partial}\psi = \chi_{\{g \le -1\}} |f|^2 e^{2ng} i\partial\overline{\partial}\psi.$$

Theorem A5.3 gives $u \in L^2_{loc}(\Omega)$ with $\overline{\partial} u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le 16 \int_{\{g \le -1\}} |f|^2 e^{2ng} e^{-\varphi} d\lambda \le 16e^{2n},$$

since $\widetilde{g} \ge -1$ on $\{g \le -1\}$ and $||f|| = 1$. Therefore there exists holomorphic $\widetilde{f}$ equal to $(f \cdot \gamma \circ g - u)/\gamma(a)$ almost everywhere such that

$$||\widetilde{f}|| \le 1 + \frac{4e^n}{\gamma(a)}. \tag{4.1.6}$$

Moreover, since $e^{-\varphi}$ is not locally integrable near w and $\widetilde{w}$, we have $\widetilde{f}(w) = f(w)$ and $\widetilde{f}(\widetilde{w}) = 0$ (the latter one because $g(\widetilde{w}) \geq -1$).

We have $\langle \widetilde{f}, f \rangle = \widetilde{f}(\widetilde{w})/\sqrt{k_\Omega(\widetilde{w})} = 0$. It follows that we can find an orthonormal basis $\phi_0, \phi_1, \dots$ of $H^2(\Omega)$ such that $\phi_0 = f$ and $\phi_1 = \widetilde{f}/||\widetilde{f}||$. Then

$$k_\Omega(w) = \sum_{j=0}^{\infty} |\phi_j(w)|^2 \geq |f(w)|^2 (1 + ||\widetilde{f}||^{-2})$$

and

$$\frac{|K_\Omega(w, \widetilde{w})|^2}{k_\Omega(w) k_\Omega(\widetilde{w})} = \frac{|f(w)|^2}{k_\Omega(w)} \geq \frac{||\widetilde{f}||^2}{1 + ||\widetilde{f}||^2}.$$

By Proposition 4.1.6

$$\operatorname{dist}_\Omega(w, \widetilde{w}) \geq \arccos \frac{||\widetilde{f}||}{\sqrt{1 + ||\widetilde{f}||^2}} = \frac{\pi}{2} - \arctan ||\widetilde{f}||.$$

It remains to use (4.1.6) and let $a \to -\infty$. ∎

4.2. Separately analytic functions (being written)

4.3. Approximation of smooth functions (being written)

4.4. Complex dynamics (being written)

Appendix

A1. Lipschitz continuous functions

The following two results can be found in [Łoj]:

Theorem A1.1. (Rademacher) *A Lipschitz continuous function defined on an open set in $\mathbb{R}^n$ is differentiable in the classical sense almost everywhere.* ■

Theorem A1.2. *If u is a Lipschitz continuous function of one variable defined in a neighborhood of an interval $[a, b]$, then*

$$u(b) - u(a) = \int_a^b u'(t)dt. \blacksquare$$

Proposition A1.3. *If $u \in \mathrm{Lip}(\Omega)$ and $\varphi \in C_0^1(\Omega)$ then*

$$\int_\Omega D_j(u\varphi)d\lambda = 0, \quad j = 1, \ldots, n.$$

Proof. By partition of unity we may assume that $\operatorname{supp} \varphi \subset [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \Omega$ and $j = n$. By Theorem A1.2

$$\int_{a_n}^{b_n} \left(D_j(u\varphi)\right)(x_1, \ldots, x_{n-1}, t)dt = 0$$

for all $x_k \in [a_k, b_k]$, $k = 1, \ldots, n-1$. The proposition now follows from the Fubini theorem. ■

Proposition A1.4. *Let f be continuous and u Lipschitz continuous such that $D_n u = f$ almost everywhere. Then $D_n u = f$ everywhere.*

Proof. We may assume that u and f are defined in a neighborhood of $\overline{I}$, where $I = I' \times I_n$ is an open cube. For some $t_0 \in I_n$ set

$$v(x) := \int_{t_0}^{x_n} f(x', t)dt, \quad x = (x', x_n) \in \overline{I},$$

and $w := u - v$. Then w is continuous, $w(x', \cdot)$ is Lipschitz continuous for every $x' \in I'$ and $D_n w = 0$ almost everywhere in I. It is enough to show that w is independent of x_n.

The proof of Proposition A1.3 gives

$$0 = \int_I \varphi D_n w \, d\lambda = -\int_I w D_n \varphi \, d\lambda, \quad \varphi \in C_0^1(I).$$

For test functions of the form $\varphi(x) = \varphi_1(x')\varphi_2(x_n)$ we obtain

$$\int_{I'} \varphi_1(x') \int_{I_n} w(x', x_n)\varphi_2'(x_n)dx_n d\lambda(x') = 0.$$

Thus for all $x' \in I'$ and $\varphi_2 \in C_0^1(I_n)$

$$\int_{I_n} w(x', x_n)\varphi_2'(x_n)dx_n = 0.$$

By Theorem A1.2 $(w(x', \cdot))' = 0$ for all $x' \in I'$ and thus $w(x', \cdot)$ is constant. ∎

Proposition A1.5. *Assume that* $u \in C^{1,1}(\Omega)$ *and* $|D^2 u| \le M$. *Then*

$$|u(x+h) + u(x-h) - 2u(x)| \le M|h|^2, \quad x \in \Omega, \ |h| < \operatorname{dist}(x, \partial\Omega).$$

Proof. If u is C^2 then it follows easily from the Taylor formula with the Lagrange remainder. The general case may then be obtained by approximation. ∎

A2. Some lemmas on measure theory and topology

Lemma A2.1. *Let* μ_j *be a sequence of Radon measures on an open* $\Omega \subset \mathbb{R}^n$ *converging weakly to a Radon measure* μ. *Then*

i) If $G \subset \Omega$ *is open then* $\mu(G) \le \liminf_{j\to\infty} \mu_j(G)$;

ii) If $K \subset \Omega$ *is compact then* $\mu(K) \ge \limsup_{j\to\infty} \mu_j(K)$;

iii) If $E \Subset \Omega$ *is such that* $\mu(\partial E) = 0$ *then* $\mu(E) = \lim_{j\to\infty} \mu_j(E)$.

Proof. i) Let L be a compact subset of G and let $\varphi \in C_0(G)$ be such that $\varphi \geq 0$, $\varphi = 1$ on L. Then

$$\mu(L) \leq \mu(\varphi) = \lim \mu_j(\varphi) \leq \underline{\lim}\, \mu_j(G).$$

Interior regularity of μ gives i).

ii) Let U be an open neighborhood of K and let $\varphi \in C_0(U)$ be such that $\varphi \geq 0$, $\varphi = 1$ on K. Then

$$\mu(U) \geq \mu(\varphi) = \lim \mu_j(\varphi) \geq \overline{\lim}\, \mu_j(K).$$

Exterior regularity of μ gives ii).

iii) This is an obvious consequence of i) and ii). ■

Lemma A2.2. *Let f_j be a decreasing sequence of upper semicontinuous functions converging to f and let μ_j be a sequence of nonnegative Borel measures converging weakly to μ. If $f_j\mu_j \longrightarrow \nu$ weakly, then $\nu \leq f\mu$.*

Proof. For some j_0 let g_k be a sequence of continuous functions deacreasing to f_{j_0}. Then for $j \geq j_0$ we have $f_j\mu_j \leq f_{j_0}\mu_j \leq g_k\mu_j$, hence $\nu \leq g_k\mu$. The Lebesgue monotone convergence theorem yields $\nu \leq f_{j_0}\mu$ and $\nu \leq f\mu$. ■

The next result is known as the Choquet lemma.

Lemma A2.3. *Let $\{u_\alpha\}$ be a family of upper semicontinuous functions on an open $\Omega \subset \mathbb{R}^n$ locally uniformly bounded above. Then there exists a countable subfamily $\{u_{\alpha_j}\}$ such that $(\sup_\alpha u_\alpha)^* = \left(\sup_j u_{\alpha_j}\right)^*$.*

Proof. Let B_j be a countable basis of topology in Ω and set $u := \sup_\alpha u_\alpha$. For every j we may find a sequence $z_{jk} \in B_j$ such that $\sup_{B_j} u = \sup_k u(z_{jk})$. For every j and k there is a sequence of indices α_{jkl} such that $u(z_{jk}) = \sup_l u_{\alpha_{jkl}}(z_{jk})$. Set $v := \sup_{j,k,l} u_{\alpha_{jkl}}$. Then

$$\sup_{B_j} v \geq \sup_k v(z_{jk}) \geq \sup_{k,l} u_{\alpha_{jkl}}(z_{jk}) = \sup_k u(z_{jk}) = \sup_{B_j} u,$$

thus $v^* \geq u^*$ and the lemma follows. ■

Lemma A2.4. *Assume $f, g : (-\infty, 0) \longrightarrow (-1, 0)$ are continuous and such that $f \leq g < 0$ and $\lim_{t\to 0^-} f(t) = 0$. Then there exists a convex, increasing $\chi : (-\infty, 0) \longrightarrow (0, +\infty)$ such that $\lim_{t\to 0^-} \chi(t) = +\infty$ and $\chi \circ g \leq \chi \circ f + 1$.*

Proof. For $s < 0$ set $h(s) := \min\{f(t) : g(t) \geq s\}$. Then $h(s) \leq s$ and $h(s) \uparrow 0$ as $s \uparrow 0$. One can find $g : (-\infty, 0) \longrightarrow (0, +\infty)$ such that $g(s) > s - h(s) \geq 0$ and $g(s) \downarrow 0$ as $s \uparrow 0$. Set

$$\chi(t) := \begin{cases} 1 & \text{if} \quad t \leq -1, \\ 1 + \displaystyle\int_{-1}^{t} \frac{1}{g(s)} ds & \text{if} \quad -1 < t < 0. \end{cases}$$

Then χ is increasing and, since $\chi'(t) = 1/g(t)$ is increasing for $t > -1$, χ is convex. Therefore for $s = g(t) > -1$ we obtain

$$\chi(g(t)) \leq \chi(h(s)) + \chi'(s)(s - h(s)) \leq \chi(f(t)) + 1$$

and the lemma follows. ∎

A3. The Choquet capacitability theorem

If X is a topological space then a set function c defined on all subsets of X taking its values at $[0, +\infty]$ is called a *generalized capacity* if it satisfies the following conditions

i) If $E_1 \subset E_2$ then $c(E_1) \leq c(E_2)$;

ii) If $E_j \uparrow E$ then $c(E_j) \uparrow c(E)$;

iii) If $K_j \downarrow K$ and K_j are compact then $c(K_j) \downarrow c(K)$.

From this point on we assume that all considered topological spaces are locally compact and have a countable basis of topology. The main goal of this section is to prove the following theorem due to Choquet.

Theorem A3.1. *Let c be a generalized capacity on X. Then for every Borel subset E of X we have*

$$c(E) = c_*(E) = \sup_{K \subset E,\, K \text{ compact}} c(K). \tag{A3.1}$$

A subset of X is called $F_{\sigma\delta}$ if it is a countable intersection of F_σ subsets. The main tool in the proof of Theorem A3.1 will be the following fact.

Theorem A3.2. *Let E be a relatively compact Borel subset of X. Then there exists a compact topological space Y, an $F_{\sigma\delta}$ subset A of Y and a continuous mapping $f : Y \longrightarrow X$ such that $f(A) = E$.*

First we shall show how Theorem A3.2 implies Theorem A3.1.

Proof of Theorem A3.1. There exist compact subsets K_j increasing to X, therefore $c(E \cap K_j) \uparrow c(E)$. Thus we may assume that E is relatively compact. Let A, Y and f be as in Theorem A3.2. One can easily see that $c \circ f$ is a generalized capacity on Y, so we may assume that E is $F_{\sigma\delta}$ and X is a compact space. Write

$$E = \bigcap_{j \geq 1} F_j, \quad F_j = \bigcup_{k \geq 1} K_{jk},$$

where K_{jk} are compact and increasing in k. Fix $a < c(E)$. We may write

$$E = \bigcup_{k\geq 1} \Big(K_{1k} \cap \bigcap_{j\geq 2} F_j \Big),$$

so there is k_1 such that $c(E_1) > a$, where

$$E_1 = K_{1k_1} \cap \bigcap_{j\geq 2} F_j.$$

For every $l \geq 1$ we can find inductively $k_l \geq 1$ such that the sets

$$E_l = K_{1k_1} \cap \dots \cap K_{lk_l} \cap \bigcap_{j\geq l+1} F_j$$

are decreasing and $c(E_l) > a$. If we set $K := \bigcap_l K_{lk_l}$, then

$$c(K) = \lim_{l\to\infty} c\left(K_{1k_1} \cap \dots \cap K_{lk_l}\right) \geq \lim_{l\to\infty} c(E_l) \geq a$$

and the theorem follows. ■

It remains to prove Theorem A3.2. First we need some simple properties of $F_{\sigma\delta}$ subsets.

Proposition A3.3. *i) If A is an $F_{\sigma\delta}$ and B a closed subset of X then $A \cap B$ is $F_{\sigma\delta}$;*
ii) If A_j are $F_{\sigma\delta}$ subsets of compact spaces Y_j, $j = 1, 2, \dots$, then $\prod A_j$ is $F_{\sigma\delta}$ in $\prod Y_j$;

Proof. i) It is enough to observe that

$$\Big(\bigcap_k \bigcup_l K_{kl}\Big) \cap B = \bigcap_k \bigcup_l K_{kl} \cap B.$$

ii) We may write

$$A_j = \bigcap_k \bigcup_l K_{jkl},$$

where K_{jkl} are compact in Y_j. One can easily show that

$$\prod A_j = \bigcap_k \bigcup_{l_1,\dots,l_k} K_{1kl_1} \times \dots \times K_{kkl_k} \times Y_{k+1} \times Y_{k+2} \times \dots \;\blacksquare$$

Proof of Theorem A3.2. The sets $E \Subset X$ satisfying the hypothesis of the theorem we will call *K-analytic*. Let X' be an open, relatively compact subset of X. Set

$$\mathcal{A} := \{E \subset X' : E \text{ and } X' \setminus E \text{ are K-analytic}\}.$$

If $E \subset X'$ is open then E and $X' \setminus E$ are $F_{\sigma\delta}$ in $\overline{X'}$, hence $\mathcal{A}$ contains all open subsets of X'. Therefore, it remains to show that $\mathcal{A}$ is a σ-algebra. Assume that $E_j \subset X'$ is a sequence of K-analytic sets. We have to show that $\bigcup E_j$ and $\bigcap E_j$ are K-analytic.

First, we claim that for $E \Subset X$ to be K-analytic it is sufficient that the mapping f is defined only on A. Indeed, in such a case by A' denote the graph of f (over A) and let Y' be the closure of A' in the compact space $Y \times \overline{E}$. Thus Y' is compact and A' is closed in $A \times \overline{E}$, hence A' is $F_{\sigma\delta}$ in Y' by Proposition A3.3.i and ii. If $f' : Y' \longrightarrow X$ is the projection onto $\overline{E}$, then f' is continuous and $f'(A') = E$ which proves the claim.

Let A_j be $F_{\sigma\delta}$ in a compact space Y_j and let $f_j : Y_j \longrightarrow X$ be continuous and such that $f_j(A_j) = E_j$. We may write

$$A_j = \bigcap_k \bigcup_l K_{jkl},$$

where K_{jkl} are compact in Y_j. The disjoint union $Y_j' := Y_j \sqcup \{y^0\}$ is compact and so is $Y := \prod Y_j'$. Set

$$A := \bigcup_j \{y^0\} \times \cdots \times \{y^0\} \times A_j \times \{y^0\} \times \cdots = \bigcap_k \bigcup_{l,j} \{y^0\} \times \cdots \times \{y^0\} \times K_{jkl} \times \{y^0\} \times \ldots,$$

so that A can be treated as a disjoint union $\coprod A_j$ and A is $F_{\sigma\delta}$ in Y. Now the mapping $f = \coprod f_j : A \longrightarrow X$ defined by

$$f(y^0, \ldots, y^0, y_j, y^0, \ldots) = f_j(y_j), \quad y_j \in A_j,\ j = 1, 2, \ldots$$

is continuous, hence $f(A) = \bigcup E_j$ is K-analytic.

Now set

$$A := \{y = (y_1, y_2, \ldots) \in \prod A_j : f_1(y_1) = f_2(y_2) = \ldots\}.$$

Then A is closed in $\prod A_j$, hence it is an $F_{\sigma\delta}$ subset of $Y := \prod Y_j$. The mapping $f : A \longrightarrow X$ given by

$$f(y) = f_1(y_1) = f_2(y_2) = \ldots$$

is continuous, thus $f(A) = \bigcap E_j$ is K-analytic. ∎

A4. Projective spaces over Hilbert spaces

Let H be an arbitrary Hilbert space over $\mathbb{C}$. Set $H_* := H \setminus \{0\}$. By $\mathbb{P}(H)$ we denote the projective space over H, that is the set of all complex lines in H containing the origin. We have the natural projection

$$\pi : H_* \ni f \longmapsto \langle f \rangle \in \mathbb{P}(H),$$

where by $\langle f\rangle$ we denote the line given by f. Let

$$H_{\langle f\rangle} := \langle f\rangle^{\perp} = \{g \in H : \langle g, f\rangle = 0\}$$

and by U_p denote the set of lines in H that are not perpendicular to $p \in \mathbb{P}(H)$, that is

$$U_{\langle f\rangle} = \{\langle g\rangle \in \mathbb{P}(H) : \langle g, f\rangle \neq 0\}.$$

Every $f \in H_*$ gives a map

$$\Phi_f : H_{\langle f\rangle} \ni w \longmapsto \langle f + w\rangle \in U_{\langle f\rangle}.$$

It is bijective and its inverse is given by

$$\Phi_f^{-1}(\langle g\rangle) = \frac{||f||^2}{\langle g, f\rangle} g - f, \quad \langle g\rangle \in U_{\langle f\rangle}.$$

For $f, g \in H_*$ we also have $\Phi_f = \Phi_g \circ L_{f,g}$ on $\Phi_f^{-1}(U_{\langle f\rangle} \cap U_{\langle g\rangle})$, where the mapping

$$L_{f,g} : H_{\langle f\rangle} \ni w \longmapsto \frac{||g||^2}{\langle w + f, g\rangle}(w + f) - g \in H_{\langle g\rangle}$$

is a smooth diffeomorphism. The mappings Φ_f give therefore $\mathbb{P}(H)$ a structure of a complex manifold.

For $p \in \mathbb{P}(H)$ the tangent space $T_p\mathbb{P}(H)$ is defined as $\mathcal{S}/\sim$, where $\mathcal{S}$ is the class of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{P}(H)$ with $\gamma(0) = p$ and $\gamma \sim \eta$ if and only if $(\Phi_f^{-1} \circ \gamma)'(0) = (\Phi_f^{-1} \circ \eta)'(0)$, where $f \in H \setminus H_p$ (then $p \in U_{\langle f\rangle}$). The definition of the relation $\sim$ is independent of the choice of such an f because

$$(\Phi_g^{-1} \circ \gamma)'(0) = (L_{f,g} \circ \Phi_f^{-1} \circ \gamma)'(0) = L'_{f,g}(\Phi_f^{-1}(p)) \circ (\Phi_f^{-1} \circ \gamma)'(0), \quad f, g \in H \setminus H_p.$$

We then have

$$\pi'(f) : H \ni X \longmapsto [\langle f + tX\rangle] \in T_{\langle f\rangle}\mathbb{P}(H), \quad f \in H_*.$$

On H_* we define the metric form $\mathcal{P}^2$ by

$$\mathcal{P}^2(f, X) := \left. \frac{\partial^2}{\partial\lambda\partial\bar{\lambda}} \log ||f + \lambda X||^2 \right|_{\lambda=0} = \frac{||X||^2}{||f||^2} - \frac{|\langle X, f\rangle|^2}{||f||^4}, \quad f \in H_*,\ X \in H.$$

The Fubini-Study metric form $\mathcal{F}^2$ on $\mathbb{P}(H)$ is then given as the push-forward of $\mathcal{P}^2$ by π:

$$\mathcal{F}^2(\pi(f), \pi'(f).X) := \mathcal{P}^2(f, X), \quad f \in H_*,\ X \in H.$$

Proposition A4.1. *The form $\mathcal{F}^2$ is well defined.*

Proof. Assume that $\pi(f) = \pi(g)$ and $\pi'(f).X = \pi'(g).Y$. Then $g = \lambda f$ for some $\lambda \in \mathbb{C}_*$ and thus $\pi'(g) = \lambda^{-1}\pi'(f)$, hence $\pi'(f).(Y - \lambda X) = 0$. Since

$$(\Phi_f^{-1})'(\pi(f)).(\pi'(f).X) = (\Phi_f^{-1} \circ \pi)'(f).X = X - \frac{\langle X, f\rangle}{||f||^2} f, \quad f \in H_*,\ X \in H,$$

it follows that

$$\ker \pi'(f) = \langle f \rangle, \quad f \in H_*.$$

We can therefore find $a \in \mathbb{C}$ such that $Y = \lambda X + af$. Then

$$\begin{aligned}\mathcal{P}^2(g, Y) &= \frac{||\lambda X + af||^2}{|\lambda|^2||f||^2} - \frac{|\lambda\langle X, f\rangle + a||f||^2|^2}{|\lambda|^2||f||^4} \\ &= \frac{||X||^2}{||f||^2} - \frac{|\langle X, f\rangle|^2}{||f||^4} \\ &= \mathcal{P}^2(f, X). \blacksquare\end{aligned}$$

Since

$$\Phi'_f(w) : H_{\langle f\rangle} \ni X \longmapsto [\langle f + w + tX\rangle] \in T_{\langle f\rangle}\mathbb{P}(H), \quad f \in H_*,\ w \in H_{\langle f\rangle},$$

in the local coordinates given by Φ_f we have

$$\begin{aligned}\Phi_f^*\mathcal{F}^2(w, Y) &= \mathcal{F}^2(\Phi_f(w), \Phi'_f(Y)) = \mathcal{P}^2(f + w, Y) \\ &= \frac{\partial^2}{\partial\lambda\partial\overline{\lambda}} \log(||f||^2 + ||w + \lambda Y||^2)\Big|_{\lambda=0} \\ &= \frac{||Y||^2}{||f||^2 + ||w||^2} - \frac{|\langle Y, w\rangle|^2}{(||f||^2 + ||w||^2)^2}, \quad f \in H_*,\ w, Y \in H_{\langle f\rangle}.\end{aligned}$$

If $\eta : [0, 1] \to \mathbb{P}(H)$ is a continuous and piecewise smooth curve then its length is given by

$$l(\eta) = \int_0^1 \mathcal{F}(\eta(t), \eta'(t))dt$$

and the distance $d(x, y)$ between $x, y \in \mathbb{P}(H)$ is defined as the infimum of $l(\eta)$ taken over all such η with $\eta(0) = x$, $\eta(1) = y$.

Proposition A4.2. *For $f, g \in H_*$ we have*

$$d(\langle f\rangle, \langle g\rangle) = \arccos \frac{|\langle f, g\rangle|}{||f||\,||g||}.$$

Proof. We may assume that $||f|| = ||g|| = 1$. Let $\eta : [0, 1] \to \mathbb{P}(H)$ be a continuous, piecewise smooth curve with $\eta(0) = \langle f\rangle$, $\eta(1) = \langle g\rangle$. First assume that $\eta([0, 1]) \subset U_{\langle f\rangle}$. Set $\widetilde{\eta} := \Phi_f^{-1} \circ \eta : [0, 1] \to H_{\langle f\rangle}$, so that $\widetilde{\eta}(0) = 0$, $\widetilde{\eta}(1) =: w$. Then

$$l(\eta) = \int_0^1 \left(\frac{||\widetilde{\eta}'(t)||^2}{1 + ||\widetilde{\eta}(t)||^2} - \frac{|\langle\widetilde{\eta}(t), \widetilde{\eta}'(t)\rangle|^2}{(1 + ||\widetilde{\eta}(t)||^2)^2}\right)^{1/2} dt \geq \int_0^1 \frac{||\widetilde{\eta}'(t)||\,dt}{1 + ||\widetilde{\eta}(t)||^2}.$$

Set $\chi(t) := ||\widetilde{\eta}'(t)||$. Then

$$\chi'(t) = \frac{\mathrm{Re}\,\langle \widetilde{\eta}(t), \widetilde{\eta}'(t)\rangle}{||\widetilde{\eta}(t)||} \leq ||\widetilde{\eta}'(t)||.$$

Therefore

$$l(\eta) \geq \int_0^1 \frac{\chi'(t)\,dt}{1+\chi(t)^2} = \arctan||w|| = \arccos|\langle f, g\rangle|$$

(the latter inequality follows from the fact that

$$|\langle f, g\rangle| = (1 + ||w||^2)^{-1/2}).$$

If $\eta([0,1]) \not\subset U$ then let $t_0 \in (0,1]$ be a maximal number satisfying $[0, t_0) \subset U$. Then $\eta(t_0) \notin U$ and similarly as before

$$l(\eta) \geq \int_0^{t_0} \mathcal{F}(\eta(t), \eta'(t))dt \geq \frac{\pi}{2} \geq \arccos|\langle f, g\rangle|.$$

On the other hand, one can easily check that the curve

$$\eta(t) = [(1-t)\langle g, f\rangle f + tg], \quad t \in [0,1],$$

has the required length. ■

Proposition A4.3. *Assume that* $\langle f_k\rangle \to \langle f\rangle$ *in* $\mathbb{P}(H)$, $f_k, f \in H_*$. *Then we can find* $\lambda_k \in \mathbb{C}_*$ *such that* $\lambda_k f_k \to f$ *in* H.

Proof. Set

$$\lambda_k := \frac{|\langle f_k, f\rangle|\,||f||}{\langle f_k, f\rangle\,||f_k||}.$$

Then

$$||\lambda_k f_k - f||^2 = |\lambda|^2||f_k||^2 - 2\mathrm{Re}\,(\lambda_k\langle f_k, f\rangle) + ||f||^2 = 2||f||^2 - 2\frac{|\langle f_k, f\rangle|\,||f||}{||f_k||} \to 0$$

by Proposition A4.2. ■

A5. Some variations of the Hörmander L^2-estimate

First we will sketch the proof of a generalization of [Hör1, Lemma 4.4.1] in case $p = q = 0$. As we will see, one needs to follow [Hör1] with only slight modifications.

Theorem A5.1. *Assume that $\varphi \in PSH(\Omega)$, where Ω is a pseudoconvex domain in $\mathbb{C}^n$. Let $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be such that $\overline{\partial}\alpha = 0$ and*

$$i\alpha \wedge \overline{\alpha} \leq hi\partial\overline{\partial}\varphi \tag{A5.1}$$

for some nonnegative function $h \in L^1_{loc}(\Omega)$ such that the right hand-side of (A5.1) makes sense as a current of order 0 (this is always the case if h is locally bounded). Then there exists $u \in L^2_{loc}(\Omega)$ with $\overline{\partial}u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int_\Omega h e^{-\varphi} d\lambda. \tag{A5.2}$$

Sketch of proof. If the right hand-side of (A5.2) is not finite it is enough to apply [Hör1, Theorem 4.2.2], we may thus assume that it is finite and even equal to 1. We first consider the case when φ is smooth (then of course the right hand-side of (A5.1) is a current of order 0 for every $h \in L^1_{loc}(\Omega)$). We follow the proof of [Hör1, Lemma 4.4.1] and its notation: the function s is smooth, strongly plurisubharmonic in Ω and such that $\Omega_a := \{s < a\} \Subset \Omega$ for every $a \in \mathbb{R}$. We fix $a > 0$ and choose $\eta_\nu \in C_0^\infty(\Omega)$, $\nu = 1, 2, \dots$, such that $0 \leq \eta_\nu \leq 1$ and $\Omega_{a+1} \subset \{\eta_\nu = 1\} \uparrow \Omega$ as $\nu \uparrow \infty$. Let $\psi \in C^\infty(\Omega)$ vanish in Ω_a and satisfy $|\partial\eta_\nu|^2 \leq e^\psi$, $\nu = 1, 2, \dots$, and let $\chi \in C^\infty(\mathbb{R})$ be convex and such that $\chi = 0$ on $(-\infty, a)$, $\chi \circ s \geq 2\psi$ and $\chi' \circ s\, i\partial\overline{\partial}s \geq (1+a)|\partial\psi|^2 i\partial\overline{\partial}|z|^2$. This implies that with $\varphi' := \varphi + \chi \circ s$ we have in particular

$$i\partial\overline{\partial}\varphi' \geq i\partial\overline{\partial}\varphi + (1+a)|\partial\psi|^2 i\partial\overline{\partial}|z|^2. \tag{A5.3}$$

The $\overline{\partial}$-operator gives the densely defined operators T and S between Hilbert spaces:

$$L^2(\Omega, \varphi_1) \xrightarrow{T} L^2_{(0,1)}(\Omega, \varphi_2) \xrightarrow{S} L^2_{(0,2)}(\Omega, \varphi_3),$$

where $\varphi_j := \varphi' + (j-3)\psi$, $j = 1, 2, 3$. (Recall that, if

$$F = {\sum_{\substack{|J|=p\\|K|=q}}}' F_{JK} dz_J \wedge d\overline{z}_K \in L^2_{loc,(p,q)}(\Omega),$$

then

$$|F|^2 = {\sum_{J,K}}' |F_{JK}|^2,$$

$$L^2_{(p,q)}(\Omega, \varphi) = \{F \in L^2_{loc,(p,q)}(\Omega) : ||F||^2_\varphi := \int_\Omega |F|^2 e^{-\varphi} d\lambda < \infty\},$$

$$\langle F, G\rangle_\varphi := \int_\Omega {\sum_{J,K}}' F_{JK} \overline{G}_{JK} e^{-\varphi} d\lambda, \quad F, G \in L^2_{(p,q)}(\Omega, \varphi).)$$

For $f=\sum_j f_j d\overline{z}_j \in C^\infty_{0,(0,1)}(\Omega)$ one can then compute

$$|Sf|^2 = \sum_{j<k}\left|\frac{\partial f_j}{\partial \overline{z}_k} - \frac{\partial f_k}{\partial \overline{z}_j}\right| = \sum_{j,k}\left|\frac{\partial f_j}{\partial \overline{z}_k}\right| - \sum_{j,k}\frac{\partial f_j}{\partial \overline{z}_k}\frac{\partial \overline{f}_k}{\partial z_j} \tag{A5.4}$$

and

$$e^{\psi}T^*f = -\sum_j \delta_j f_j - \sum_j f_j \frac{\partial \psi}{\partial z_j},$$

where

$$\delta_j w := e^{\varphi'}\frac{\partial}{\partial z_j}(we^{-\varphi'}) = \frac{\partial w}{\partial z_j} - w\frac{\partial \varphi'}{\partial z_j}.$$

Therefore

$$|\sum_j \delta_j f_j|^2 \le (1+a^{-1})e^{2\psi}|T^*f|^2 + (1+a)|f|^2|\partial\psi|^2. \tag{A5.5}$$

Integrating by parts we get

$$\int_\Omega |\sum_j \delta_j f_j|^2 e^{-\varphi'}d\lambda = \int_\Omega \sum_{j,k}\left(\frac{\partial^2\varphi'}{\partial z_j \partial\overline{z}_k}f_j\overline{f}_k + \frac{\partial f_j}{\partial \overline{z}_k}\frac{\partial \overline{f}_k}{\partial z_j}\right)e^{-\varphi'}d\lambda.$$

Combining this with (A5.3)-(A5.5) we arrive at

$$\int_\Omega \sum_{j,k}\frac{\partial^2\varphi'}{\partial z_j\partial\overline{z}_k}f_j\overline{f}_k e^{-\varphi'}d\lambda \le (1+a^{-1})||T^*f||^2_{\varphi_1} + ||Sf||^2_{\varphi_3}. \tag{A5.6}$$

If we write $\alpha = \sum_j \alpha_j d\overline{z}_j$ then

$$i\alpha\wedge\overline{\alpha} = \sum_{j,k}\overline{\alpha}_j\alpha_k i dz_j \wedge d\overline{z}_k$$

and by (A5.1)

$$|\sum_j \overline{\alpha}_j f_j|^2 \le h\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}f_j\overline{f}_k.$$

Hence, from the Schwarz inequality, (A5.6) and from the fact that $\varphi - 2\varphi_2 \le -\varphi'$ we obtain

$$|\langle\alpha, f\rangle_{\varphi_2}|^2 \le (1+a^{-1})||T^*f||^2_{\varphi_1} + ||Sf||^2_{\varphi_3}$$

for all $f\in C^\infty_{0,(0,1)}(\Omega)$ and thus also for all $f\in D_{T^*}\cap D_S$ (recall that we have assumed that the right hand-side of (A5.2) is 1). If $f'\in L^2_{(0,1)}(\Omega,\varphi_2)$ is orthogonal to the kernel of S then it is also orthogonal to the range of T and thus $T^*f'=0$. Moreover, since $S\alpha=0$, we then also have $\langle\alpha, f'\rangle_{\varphi_2}=0$. Therefore

$$|\langle\alpha, f\rangle_{\varphi_2}| \le \sqrt{1+a^{-1}}||T^*f||_{\varphi_1}, \quad f\in D_{T^*}.$$

By the Hahn-Banach theorem there exists $u_a \in L^2(\Omega, \varphi_1)$ with $||u_a||_{\varphi_1} \leq \sqrt{1+a^{-1}}$ and

$$\langle \alpha, f\rangle_{\varphi_2} = \langle u_a, T^* f\rangle_{\varphi_1}, \quad f \in D_{T^*}.$$

This means that $Tu_a = \alpha$ and, since $\varphi_1 \geq \varphi$ with equality in Ω_a, we have

$$\int_{\Omega_a} |u_a|^2 e^{-\varphi} d\lambda \leq 1 + a^{-1}.$$

We may thus find a sequence $a_j \uparrow \infty$ and $u \in L^2_{loc}(\Omega)$ such that u_{a_j} converges weakly to u in $L^2(\Omega_a, \varphi) = L^2(\Omega_a)$ for every a. This proves the theorem for smooth φ.

Now assume that φ is strongly plurisubharmonic (but otherwise arbitrary, that is possibly even not locally bounded). By the Radon-Nikodym theorem there exists $\beta = \sum_{j,k} \beta_{jk} i dz_j \wedge d\overline{z}_k \in L^1_{loc,(1,1)}(\Omega)$ such that $0 < \beta \leq i\partial\overline{\partial}\varphi$ and $i\alpha \wedge \overline{\alpha} \leq h\beta$. For $\varepsilon > 0$ let $a(\varepsilon)$ be such that $\varphi_\varepsilon := \varphi * \rho_\varepsilon \in C^\infty(\overline{\Omega}_{a(\varepsilon)})$. If $(\varphi_\varepsilon^{jk})$ denotes the inverse matrix of $(\partial^2\varphi_\varepsilon/\partial z_j \partial\overline{z}_k)$ then $h_\varepsilon := \sum_{j,k} \varphi_\varepsilon^{jk} \overline{\alpha}_j \alpha_k$ is the least function satisying $i\alpha \wedge \overline{\alpha} \leq h_\varepsilon i\partial\overline{\partial}\varphi_\varepsilon$. By the previous part we can find $u_\varepsilon \in L^2_{loc}(\Omega_{a(\varepsilon)})$ such that $\overline{\partial} u_\varepsilon = \alpha$ in $\Omega_{a(\varepsilon)}$ and

$$\int_{\Omega_{a(\varepsilon)}} |u_\varepsilon|^2 e^{-\varphi_\varepsilon} d\lambda \leq \int_{\Omega_{a(\varepsilon)}} h_\varepsilon e^{-\varphi_\varepsilon} d\lambda \leq \int_{\Omega_{a(\varepsilon)}} h_\varepsilon e^{-\varphi} d\lambda.$$

We have $\beta_\varepsilon := \beta * \rho_\varepsilon \leq i\partial\overline{\partial}\varphi_\varepsilon$ and there is a sequence $\varepsilon_l \downarrow 0$ such that the coefficients of β_{ε_l} converge pointwise almost everywhere to the respective coefficients of β. Therefore

$$\overline{\lim_{l\to\infty}}\, h_{\varepsilon_l} \leq \overline{\lim_{l\to\infty}} \sum_{j,k} \beta_{\varepsilon_l}^{jk} \overline{\alpha}_j \alpha_k = \sum_{j,k} \beta^{jk} \overline{\alpha}_j \alpha_k \leq h,$$

where (β^{jk}) and (β_ε^{jk}) denote the inverse matrices of (β_{jk}) and $(\beta_{jk} * \rho_\varepsilon)$, respectively. By the Fatou lemma we thus have

$$\overline{\lim_{l\to\infty}} \int_{\Omega_{a(\varepsilon_l)}} |u_{\varepsilon_l}|^2 e^{-\varphi_{\varepsilon_l}} d\lambda \leq 1.$$

Since φ_{ε_l} is a decreasing sequence, we see that the L^2 norm of u_{ε_l} over Ω_a is bounded for every fixed a. Therefore, replacing ε_l with its subsequence if necessary, we see that u_{ε_l} converges weakly in Ω_a for every a to $u \in L^2_{loc}(\Omega)$. For every a and $\delta > 0$ we then have

$$\int_{\Omega_a} |u|^2 e^{-\varphi_{\varepsilon_l}} d\lambda \leq 1 + \delta$$

which completes the proof for strongly plurisubharmonic φ.

If φ is not necessarily strongly plurisubharmonic then we may approximate it by functions of the form $\varphi + \varepsilon|z|^2$. Note that $i\alpha \wedge \overline{\alpha} \leq h\, i\partial\overline{\partial}(\varphi + \varepsilon|z|^2)$ and the general case easily follows along the same lines as before. ∎

The next result is due to Berndtsson [Ber1] (see also [Ber2]).

Theorem A5.2. *Let Ω, φ, α and h be as in Theorem A5.1. Fix $r \in (0,1)$ and assume in addition that $-e^{-\varphi/r} \in PSH(\Omega)$. Then for any $\psi \in PSH(\Omega)$ we can find $u \in L^2_{loc}(\Omega)$ with $\overline{\partial}u = \alpha$ and*

$$\int_\Omega |u|^2 e^{\varphi-\psi} d\lambda \le \frac{1}{(1-\sqrt{r})^2} \int_\Omega h e^{\varphi-\psi} d\lambda.$$

Proof. Approximating $-e^{-\varphi/r}$ and ψ in the same way as in the proof of Theorem A5.1 we may assume that φ and ψ are smooth up to the boundary. Then we have in particular $L^2(\Omega) = L^2(\Omega, a\varphi + b\psi)$ for real a, b and $-e^{-\varphi/r} \in PSH(\Omega)$ means precisely that

$$i\partial\varphi \wedge \overline{\partial}\varphi \le r\, i\partial\overline{\partial}\varphi.$$

Let u be the solution to $\overline{\partial}u = \alpha$ which is minimal in the $L^2(\Omega, \psi)$ norm. This means that

$$\int_\Omega u\overline{f} e^{-\psi} d\lambda = 0, \quad f \in H^2(\Omega).$$

Set $v := e^\varphi u$. Then

$$\int_\Omega v\overline{f} e^{-\varphi-\psi} d\lambda = 0, \quad f \in H^2(\Omega),$$

thus v is the minimal solution in the $L^2(\Omega, \varphi + \psi)$ norm to $\overline{\partial}v = \beta$, where

$$\beta = \overline{\partial}(e^\varphi u) = e^\varphi(\alpha + u\overline{\partial}\varphi).$$

For every $t > 0$ we have

$$\begin{aligned} i\beta \wedge \overline{\beta} &\le e^{2\varphi}[(1+t^{-1}) i\alpha \wedge \overline{\alpha} + (1+t)|u|^2 i\partial\overline{\partial}\varphi] \\ &\le e^{2\varphi}[(1+t^{-1})h + (1+t)r|u|^2] i\partial\overline{\partial}\varphi \\ &\le e^{2\varphi}[(1+t^{-1})h + (1+t)r|u|^2] i\partial\overline{\partial}(\varphi+\psi). \end{aligned}$$

Therefore by Theorem A5.1

$$\int_\Omega |u|^2 e^{\varphi-\psi} d\lambda = \int_\Omega |v|^2 e^{-\varphi-\psi} d\lambda \le (1+t^{-1}) \int_\Omega h e^{\varphi-\psi} d\lambda + (1+t)r \int_\Omega |u|^2 e^{\varphi-\psi} d\lambda.$$

For $t = r^{-1/2} - 1$ we obtain the required result. ∎

Applying Theorem A5.2 with $r = 1/4$ and φ, ψ replaced with $\varphi/4, \psi + \varphi/4$, respectively, we obtain the following estimate essentially due to Donnelly and Fefferman [DF].

Theorem A5.3. *Let Ω, φ, α and h satisfy the assumptions of Theorem A5.1. Assume moreover that $-e^{-\varphi} \in PSH(\Omega)$. Then for any $\psi \in PSH(\Omega)$ we can find $u \in L^2_{loc}(\Omega)$ with $\overline{\partial}u = \alpha$ and*

$$\int_\Omega |u|^2 e^{-\psi} d\lambda \le 16 \int_\Omega h e^{-\psi} d\lambda. \ ∎$$

Some open problems

Problem 1. Assume that a domain $\Omega \Subset \mathbb{C}^n$ is such that each of its boundary points admits a local weak plurisubharmonic barrier (that is for every $z_0 \in \partial\Omega$ there exists a neighborhood U of z_0 and $u \in PSH(\Omega \cap U)$ such that $u < 0$ and $\lim_{z \to z_0} u(z) = 0$). Is Ω hyperconvex? Note that Theorem 1.4.7 would be a direct consequence of a positive answer.

Problem 2. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ admitting a local weak plurisubharmonic barrier at some $z_0 \in \partial\Omega$. Does there exist a global (defined on Ω) weak plurisubharmonic barrier at z_0? A solution in the affirmative would solve Problem 1, since, using the Hartogs figures, one can show that a domain from Problem 1 is pseudoconvex.

Problem 3. Is being an unbounded maximal plurisubharmonic function a local property?

Problem 4. Let $P = \Delta^2$ be the unit bidisk. For $(z, w) \in \partial P$ set $f(z, w) := (\operatorname{Re} z)^2 (\operatorname{Re} w)^2$ so that f is subharmonic on every analytic disk embedded in ∂P. Therefore, by Theorem 2.3.2, $u := u_{f,P} \in PSH \cap C^{1,1}(P) \cap C(\overline{P})$. One can show that $u = 0$ on the set $\{(z, w) \in \overline{P} : |z + w| \le |1 - zw|\}$. It can also be proved that for every $\varepsilon > 0$ the function

$$v_\varepsilon(z, w) = \frac{\varepsilon^2}{4} \left(\left| \frac{z + w}{\varepsilon + 1 - zw} \right|^2 - 1 \right)$$

satisfies $v_\varepsilon \le u$ in $\overline{P}$. An elementary computation then gives for $t \in (0, 1)$

$$u(t,t) \begin{cases} = 0 & \text{if } t \le \sqrt{2} - 1, \\ \ge 2^{-4} \left((2t)^{2/3} - (1 - t^2)^{2/3} \right)^6 & \text{if } t \ge \sqrt{2} - 1. \end{cases}$$

This means that $u \notin C^6(P)$. We conjecture that in fact $u \notin C^2(P)$.

Problem 5. Are the product formulas (3.1.3) and (3.2.1) (without regularizations) true for arbitrary sets K_j?

Problem 6. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Assume that sequences $z_j, w_j \in \Omega$ tend to $z_0 \in \Omega, w_0 \in \partial\Omega$, respectively. Does it follow that $g_{\Omega, w_j}(z_j) \longrightarrow 0$? In other words, is it true that g_Ω is continuous at the points from $\partial\Omega \times \Omega$ (if we assume that it vanishes there)?

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