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# The Metric Universal Properties of Period Doubling Bifurcations and the Spectrum for a Route to Turbulence

by

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We formally develop the ideas of the metric universal properties of maps on an interval and then use these results to determine some universal aspects of a route to turbulence. All this material has been published elsewhere in a less condensed fashion.

Start by considering a one parameter family of maps on an interval where for each fixed value of the parameter, $\lambda$, the map f has certain required smoothness properties and possesses a unique extremum within the interval. We are interested in the eventual behavior of the sequence of iterates under f:

$$x_{n+1} = f(\lambda, x_n) \rightarrow \{x_n\}(\lambda) \ .$$

A first possibility is that f possesses a <u>fixed point</u> $x^*$:

$$x^* = f(x^*) \ .$$

(when unrelated to a given context, we suppress the $\lambda$-dependence.) In order for $x^*$ to represent the eventual behavior of $\{x_n\}$, the fixed point must be <u>stable</u> (i.e. $x_n \rightarrow x^*$). Stability is analyzed locally through a linearization about $x^*$:

$$x_n \equiv x^* + \xi_n$$

$$x_{n+1} = x^* + \xi_{n+1} = f(x_n) = f(x^* + \xi_n) = x^* + \xi_n f'(x^*) + 0(\xi^2)$$

or,

$$\xi_{n+1} \cong \xi_n f'(x^*) \qquad \xi_n \cong \xi_0 [f'(x^*)]^n$$

Thus, the criterion for stability is

$$|f'(x^*)| < 1 \ .$$

Should $x^*$ be stable, the approach of $x_n$ to $x^*$ is geometric unless $f'(x^*) = 0$, in which case the approach is faster than any rate of geometric convergence. We term such a fixed point superstable.

The next possible kind of eventual behavior is periodic. For periodicity n this n-cycle $x^*_0, x^*_1, \ldots, x^*_{n-1}$ satisfies

$$x^*_{i+1} = f(x^*_i) \quad i = 0, 1, \ldots \bmod n .$$

In terms of the $n^{th}$ iterate of f, $f^n$:

$$f^n(x) = f(f^{n-1}(x)) \; ; \; f^0(x) = x$$

each $x^*_i$ is a fixed point:

$$f^n(x^*_i) = x^*_i \quad i = 0, \ldots, n-1$$

Accordingly, an n-cycle is stable if

$$|Df^n(x^*_i)| < 1 .$$

It is unnecessary to specify which element $x^*_i$ is intended since

$$Df^n(x^*_i) = \prod_{j=0}^{n-1} f'(x_j) \quad \text{independent of } i .$$

It follows from this chain-rule result that an n-cycle is superstable if one of the elements of the cycle is located at the extremum of f.

Finally there can be an aperiodic eventual behavior which is an infinite sequence of elements that is stable if points in a neighborhood of one of these elements determine sequences that eventually agree with those of the specified set. For our purposes, infinite stable sets (or attractors) shall arise as limits of periodic attractors.

The basic question we are addressing is how the nature of attractors varies with the parameter $\lambda$. For a large class of one-parameter f's it turns out that if at same value of $\lambda$ an r-cycle is stable, then as $\lambda$ is monotonically varied (for definiteness, increased), the r-cycle persists to be stable over same $\lambda$-interval at the endpoint of which it becomes unstable, and a 2r-cycle in turn becomes stable. At this bifurcation point, the elements of the 2r-cycle are infinitesimally separated pairs about the elements of the now unstable r-cycle, with r iterations required to sequentially visit one element of this pair from the other. As $\lambda$ is further increased there is again an interval in which the 2r-cycle is stable with r iterations imaging one element into that element of the cycle nearest to this one. Again, at the end of this $\lambda$-interval, the 2r-cycle becomes unstable, replaced by a stable 4r-cycle. This behavior recurs ad infinitum, so that for each n there is a $\lambda$-interval in which an $r \cdot 2^n$ cycle is stable. In particular within this $\lambda$-interval there is value, $\lambda_n$, such that the $r \cdot 2^n$ cycle is superstable. Moreover, $\lambda_\infty < \infty$ (so that the widths of the $\lambda$-intervals $\to 0$).

For simplicity we consider $r = 1$ (everything to follow is true for $r \neq 1$ simply by replacing $f \to f^r$) and imagine a $\lambda$-dependent coordinate transformation (conjugacy), if necessary, that maintains the extremum of f at $x = 0$ independent of $\lambda$. Then, $\lambda_n$ is determined by the root of

$$f^{2^n}(\lambda,0) = 0 \ .$$

This equation, for $n > 0$, possesses many roots - in particular $\lambda_{n-1}, \ldots, \lambda_0$. $\lambda_n$ itself is then defined recursively as the smallest root greater than the previously determined one.

At this point the first metrically universal feature emerges. Thus, it turns out that $\lambda_n \to \lambda_\infty$ geometrically: should $\delta_n \to \delta$ where

$$\delta_n \equiv \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1} - \lambda_{n+2}} \tag{1}$$

then asymptotically, $\lambda_\infty - \lambda_n \propto \delta^{-n}$. It is in fact true that $\delta_n \to \delta$ but much more interestingly, <u>$\delta$ is a universal constant</u> independent of the f's considered. (Actually, $\delta$ depends on one detail of f alone, and this is the order of f's extremum. We restrict discussion to those f's with a normal quadratic extremum.) Numerically, as discovered by the author in Nov., 1975,

$$\delta = 4.6692016091029909 \ldots$$

In order to understand this universality and determine an equation for $\delta$, another more fundamental, universal feature had to be unearthed. This feature is a scaling phenomenon, relating the separation of $x = 0$ from its nearest element from cycle to cycle. This spacing in a $2^n$ superstable cycle is

$$d_n \equiv f^{2^{n-1}}(\lambda_n, 0)$$

and the scaling result is

$$-\frac{d_n}{d_{n+1}} \to \alpha \quad \text{with } \alpha \text{ also universal .} \tag{2}$$

Numerically,

$$\alpha = 2.5029078750958928 4 \ldots$$

Clearly

$$d_n \sim (-\alpha)^{-n}$$

where $\lambda_n$ is required to evaluate $d_n$. The special meaning of $\lambda_\infty$ now emerges as that isolated parameter value for which

$$f^{2^n}(\lambda_\infty,0) \sim (-\alpha)^{-n} \text{ for all n sufficiently large .}$$

Since

$$(-\alpha)^n f^{2^n}(\lambda_\infty,0) \to \nu \text{ (constant, dependent upon f)}$$

and since coordinate magnifications are preserved by iterations, we are led to construct

$$(-\alpha)^n f^{2^n}(\lambda_\infty, x/(-\alpha)^n) \to \nu g(x/\nu) \ ; \quad g(0) \equiv 1$$

Indeed, a function g is converged to, and is <u>also universal</u>.

It is now straightforward to deduce an equation for $\alpha$ and g. Denoting the left-hand side of the above equation by $G_n(x)$, it is trivially verified that

$$-\alpha\, G_n(G_n(-x/\alpha)) = G_{n+1}(x) \ .$$

However, apart from an irrelevant magnification by $\nu$, $G_n \to g$. Also, had f been a symmetric function of x, so too would all the $G_n$. Since g is universal, then

$$g(x) = -\alpha g(g(x/\alpha)) \ . \tag{3}$$

This equation is solved by requiring g to be an analytic function of $x^2$ with $g(0) = 1$. This expression is truncated at same order N and the above equation evaluated at N points in the unit interval. The resulting system is then numerically solved and yields an approximation for both $\alpha$ and g. (The author obtained a solution in this fashion satisfied to 20 significant figures on the unit interval with agreement of $\alpha$ and g to their recursive definitions to full available precision in May, 1976).

It is now possible to determine $\delta$ as the $\lambda$ convergence rate as well as to establish the strongest universality property - namely the locations of elements of the attractors. To this end, write

$$\begin{aligned} f(\lambda,x) &= f(\lambda_\infty,x) + (\lambda-\lambda_\infty)\, f_\lambda(\lambda_\infty,x) + \ldots \\ &\equiv G_0(x) + \mu H_0(x) + O(\mu^2) \end{aligned}$$

and define

$$(-\alpha)^n\, f^{2^n}(\lambda,x/(-\alpha)^n) \equiv G_n(x) + \mu H_n(x) + O(\mu^2)\ . \tag{4}$$

By iterating (4), it is easy to deduce that

$$G_{n+1}(x) = -\alpha\, G_n(G_n(-x/\alpha)) \equiv T(G_n)$$

$$H_{n+1}(x) = -\alpha[H_n(G_n(-x/\alpha)) + G_n'(G_n(-x/\alpha))\, H_n(-x/\alpha)]$$

Now, by the property of $\lambda_\infty$, $G_n \to g$, so that asymptotically,

$$\begin{aligned} H_{n+1}(x) &\cong -\alpha[H_n(g(x/\alpha)) + g'(g(x/\alpha))\, H_n(-x/\alpha)] \\ &= (DT)_g\, H_n \quad \text{(the derivative map of T at its fixed point g)} \end{aligned}$$

It is now a computer fact and not yet completely proven conjecture that

$DT_g$ has a <u>unique</u> eigenvalue in excess of 1, which, as will become apparent, is $\delta$. Thus,

$$H_n \sim c(f)\ \delta^n\ h(x) \tag{5}$$

where h is the associated eigenfunction to $\delta$ and normalized to $h(0) = 1$. Substituting (5) into (4),

$$(-\alpha)^n\ f^{2^n}(\lambda, x/(-\alpha)^n) \sim g(x) + (\lambda-\lambda_\infty)\ c(f)\ \delta^n\ h(x) + 0(\lambda-\lambda_\infty)^2\ .$$

Setting $x = 0$ and $\lambda = \lambda_n$.

$$0 = (-\alpha)^n\ f^{2^n}(\lambda_n, 0) \sim 1 + (\lambda_n-\lambda_\infty)\ c(f)\ \delta^n + \ldots$$

$$\rightarrow c(f)(\lambda_n-\lambda_\infty) \sim \eta\delta^{-n} \tag{6}$$

for some $\eta$, which as shall follow, is independent of f. Formula (6) establishes the leading eigenvalue of DT as $\delta$ of (1).

Next, set $\lambda = \lambda_{n+r}$ for a fixed $r \geq 0$:

$$(-\alpha)^n\ f^{2^n}(\lambda_{n+r}, x/(-\alpha)^n) \sim g - \eta\delta^{-r}\ h\ . \tag{7}$$

Since $2^r$ iterations of the right hand of (7) must possess $x = 0$ as a fixed point, we see that $\eta$ is in fact determined and, as claimed, independent of f. Moreover, we see that

$$\lim_{n\to\infty} (-\alpha)^n\ f^{2^n}(\lambda_{n+r}, x/(-\alpha)^n) = g_r(x)\ . \tag{8}$$

That is, the limit exists and the functions $g_r$ are all universal. (Actually a magnification by the same $\nu$ for all r is understood.) The functions

$g_r$ serve as a basis on which T is the shift:

$$g_{r-1} = Tg_r \quad ; \quad g = \lim_{r\to\infty} g_r \ .$$

By the chain rule and the definition of $g_0$, it is clear that apart from the f-dependent scale, $\nu$, $g_0$ locates the elements of asymptotic $2^n$ cycles about x = 0 as fixed points at extrema. Thus the elements of the attractors are locally universally distributed. (Numerically discovered by the author, March, 1976.) In order to compute $g_0$, (3) is solved, $(DT)_g$ is computed and $\delta$ and h determined. For sufficiently large r, $g_r$ is constructed according to the asymptotic formula (7), and then $2^r$ applications of T determines $g_0$.

The entirety of this theory was constructed by the author by Nov., 1976[1,2] together with an analysis of the approach to the universal asymptotic regime for an arbitrary one parameter family. The proof of the validity of this theory was also claimed at that time to rest upon two conjectures:

i) The uniqueness and existance of the solution to (3) with the stated requirements on g

and ii) The existance of a unique eigenvalue of DT outside the unit disk.

Since that time, rigorous progress has occured. At the end of 1978 Collet[3] et. al. succeeding in proving i) and ii) for $z = 1+\varepsilon$ where z is the order of the maximum of f, with g then a function of $|x|^z$. Their renormalization group - $\varepsilon$ - expansion method, however, did not extend to the case of real interest (i.e. z=2). However Campinino et. al., early in 1980, have succeeded in proving (i) for z = 2, while the uniqueness of $\delta$ is not yet established.

Collet[4] et. al. also demonstrated, by late 1979, that a map from $R^n$ to $R^n$ for arbitrary n generically possesses the one-dimensional fixed point g, although such maps must contract volumes near the fixed point. This result explained the appearance of $\alpha$ and $\delta$ in multidimensional maps and, by implication, in multidimensional differential flows that by mid 1979 were computationally well established.[5,6] Moreover during the summer of 1979 Libchaber et. al.[7] had observed a systematic period doubling (some five times) for a physical Beinard flow comprising its transition to turbulence, although at a resolution consistent with $\delta = 4.669 \ldots$ to less than a significant figure, while obtaining the Fourier time spectrum at the transition which, in fact, determines $\alpha = 2.5029 \ldots$ to better than two significant figures. This extraction of $\alpha$ from the spectrum follows from the author's work of Oct., 1979[8,9] which employs the theory to actually determine the evolution of the spectrum throughout the transition regime. We briefly explore this last point.

Consider a system specified by N first order differential equations which depend upon a parameter $\lambda$. At the value $\lambda_n$ the system is in a $2^n$ cycle, so that if the original period is $T_0$,

$$\underset{\sim}{x}_n(t+2^nT_0) = \underset{\sim}{x}_n(t)$$

where $\underset{\sim}{x}_n(t)$ is the N-dimensional trajectory at $\lambda_n$. At $\lambda_{n+1}$ the period has doubled so that

$$\underset{\sim}{x}_{n+1}(t+2^nT_0) - x_{n+1}(t) \neq 0 \tag{8}$$

while

$$\underset{\sim}{x}_{n+1}(t+2^{n+1}T_0) = \underset{\sim}{x}_{n+1}(t) \ .$$

However the quantity of (8) is the spacing of two elements half a cycle apart so that for some appropriate t along the trajectory (analogous to the maximum point) this quantity scales with $\alpha$ from one period doubling to the next (Eq. (2)). That is, for an appropriate $\bar{t}$,

$$\underset{\sim}{x}_{n+1}(\bar{t}+2^n T_0) - \underset{\sim}{x}_{n+1}(\bar{t}) \sim (-\alpha)^{-1} [\underset{\sim}{x}_n(\bar{t}+2^{n-1}T_0) - \underset{\sim}{x}_n(\bar{t})] \ . \tag{9}$$

If (9) can be extended to all $\bar{t}$ in the trajectory, then a powerful recursion would be established. Accordingly, Eq. (2) is first extended to determine the scaling at any point of a cycle. Specifically, we compute

$$\sigma_n(i/2^{n+1}) = d_i^{(n+1)}/d_i^{(n)}$$

where

$$d_i^{(n)} \equiv x_i^{(n)} - x_{i+2^{n-1}}^{(n)} = x_i^{(n)} - f^{2^{n-1}}(\lambda_n, x_i^{(n)})$$

and

$$x_i^{(n)} = f^{2^n}(\lambda_n, 0) \ .$$

It turns out that

$$\sigma(x) = \lim_{n\to\infty} \sigma_n(x) \quad 0 \leq x < 1$$

is computible through iterates of the universal functions $g_r$ of (8), and so is universally determined. Accordingly (9) can be extended to any t by replacing $(-\alpha)^{-1}$ by $\sigma(t/2^{n+1}T_0)$. Thus if the differences along a trajectory (the right hand bracketed term of (9)) in its $2^n$ cycle are known, then these differences in the $2^{n+1}$ cycle are determined by (9), so that, through a Fourier transform, $\underset{\sim}{x}_{n+1}(t)$ is also determined. (All that is required is a Fourier transform of $\sigma$.) Since (9) is recursive, $\underset{\sim}{x}(t)$ can now be computed for all further period doublings. The outcome of this procedure can be easily approximated:

i) Each spectral component of $\underset{\sim}{x}_{n0}$ remains approximately constant as n doubles

ii) Each time n doubles, new spectral components appear midway between the previous ones.

iii) A smooth interpolation of the last new components shifted down (logarithmically) by a universal 8.2 db defines the interpolation of these new components.

To conclude this discussion, the Bénard flow experimental spectrum is in complete agreement with this process, with a measured shift of 8.3 db, which equivalently is an experimental determination of $\alpha$ to two significant figures.

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