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THE MINIMAL FAITHFUL DEGREE OF A FUNDAMENTAL INVERSE SEMIGROUP

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This paper shows that the smallest size of a set for which a finite fundamental inverse semigroup can be faithfully represented by partial transformations of that set is the number of join irreducible elements of its semilattice of idempotents.

1. Introduction

In faithfully representing a given semigroup by partial transformations of a set, it is natural to ask what is the least size of a set for which this is possible. This question, among others, was posed by Schein in [7, Problem 45] .

Here, the problem is solved for finite fundamental inverse semigroups. The special case of a semilattice is considered, from which the general case follows by applying a theorem of Munn, which describes fundamental inverse semigroups in terms of principal ideal isomorphisms of semilattices.

2. Preliminaries

Standard terminology and basic results relating to inverse semigroups, and semilattices in particular, as given by Howie in [1], will be

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assumed. The following result, due to Munn [3, 4], will be used in Section 4. Recall that if E is a semilattice, the Munn semigroup T_E consists of all isomorphisms between principal ideals of E.

THEOREM 1 [1, 4.10]. *An inverse semigroup* S *with semilattice of idempotents* E *is fundamental if and only if it is isomorphic to a full inverse subsemigroup of* T_E.

Hereafter all semigroups will be assumed to be finite. Define the minimal faithful degree $\mu(S)$ of a semigroup S to be the least non-negative integer n such that S can be embedded in PT_X, the semigroup of partial transformations of a set X, where X contains n elements. Note that if T is a subsemigroup of S then $\mu(T) \leq \mu(S)$. If X is a set, let id_X denote the identity mapping on X.

The following result follows from more general results about inverse semigroups (see for example [5, IV.5.9], [2, II.8.4],[8] or [6]), though a proof is included for completeness.

PROPOSITION 2. *Let* E *be a semilattice and* $\phi : E \to PT_X$ *a faithful representation. Then there is a faithful representation* $\overline{\phi} : E \to PT_X$ *such that, for all* $e \in E$,

$$e\overline{\phi} = id_Y \quad \text{for some} \quad Y \subseteq X .$$

Proof. Define $\overline{\phi} : E \to PT_X$ by, for $e \in E$,

$$e\overline{\phi} = e\phi \Big|_{\mathrm{range}(e\phi)} .$$

It is routine to verify that $\overline{\phi}$ is a faithful representation. Moreover, if $x \in \mathrm{domain}(e\overline{\phi})$ then $x \in \mathrm{range}(e\phi)$, so, for some $y \in X$, $x = y(e\phi)$, which yields $x(e\overline{\phi}) = [y(e\phi)](e\overline{\phi}) = [y(e\phi)](e\phi) = y(e\phi) = x$, that is, $e\overline{\phi} = \mathrm{id}_{\mathrm{domain}(e\overline{\phi})}$. This completes the proof.

If E is a semilattice, the symbols $\wedge$ and $\vee$ will be used to denote infimum and supremum respectively, when they exist, with respect to the partial order of E. Thus $\wedge E$ is the least element of E, which exists since E is finite, which will be denoted by 0, the zero of E. Note also that $0 = \vee\emptyset$. Call an element e of E join

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irreducible if $e \neq 0$ and, for $f, g \in E$,

$$e = f \vee g \quad \text{implies} \quad e = f \quad \text{or} \quad e = g .$$

Note that by this definition 0 is not join irreducible. Note also that, since E is finite, any non-zero element of E can be expressed as the join of a set of join irreducible elements. If $e \in E$, define

$$\bar{e} = \{f \in E \mid f \leq e \text{ and } f \text{ is join irreducible}\} .$$

LEMMA 3. *If E is a semilattice then, for $e \in E$, $e = \vee\bar{e}$.*

Proof. Let $e \in E$. If $e = 0$ then $\bar{e} = \emptyset$ and indeed $e = \vee\bar{e}$. If $e \neq 0$ then write e as the join of join irreducible elements $e_1, \ldots, e_n$:

$$e = \vee\{e_1, \ldots, e_n\} .$$

Certainly e is an upper bound for $\bar{e}$. Suppose also f is an upper bound for $\bar{e}$, so in particular f is an upper bound for $\{e_1, \ldots, e_n\}$. Hence $e \leq f$, which shows e is the least upper bound for $\bar{e}$, which completes the proof.

LEMMA 4. *Let E be the semilattice which is the subsemigroup of PT_X consisting of all partial transformations of a set X which are identity mappings on their respective domains. Then, in E, for $Y, Z \subseteq X$,*

(i) $id_Y \leq id_Z$ *if and only if* $Y \subseteq Z$;

(ii) $id_{Y \cap Z} = id_Y \wedge id_Z$;

(iii) $id_{Y \cup Z} = id_Y \vee id_Z$.

Proof. These follow immediately from the fact that $id_Y id_Z = id_{Y \cap Z}$.

LEMMA 5. *Let $(P, \leq)$ be any non-empty finite partially ordered set with n elements. Then there is a listing $P = \{p_1, \ldots, p_n\}$ such that, whenever $1 \leq i < j \leq n$,*

$$p_j \nleq p_i .$$

Proof. Define the listing inductively. Let p_1 be any minimal element of P, which exists by finiteness. Assume $p_1, \ldots, p_k$ have

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been chosen, where $k < n$. Choose p_{k+1} to be any minimal element of $P\backslash\{p_1,\ldots,p_k\}$. The minimality of p_{k+1} ensures that the condition of the lemma holds.

3. Semilattices.

THEOREM 6. *Let E be a finite semilattice with n join irreducible elements. Then $\mu(E) = n$.*

Proof. If $E = \{0\}$ then $n = 0$ and $\mu(E) = 0$, so the statement of the theorem holds. Suppose then E contains a non-zero element, so $n \geq 1$.

Let X be the set of all join irreducible elements of E . Define a mapping $\phi : E \to PT_X$ by, for $e \in E$,

$$e\phi = id_{\bar{e}} .$$

For $e,f \in E$, $\overline{ef} = \bar{e} \cap \bar{f}$, so

$$id_{\overline{ef}} = id_{\bar{e}\cap\bar{f}} = id_{\bar{e}}\, id_{\bar{f}} ,$$

which shows ϕ is homomorphic. Also, if $\bar{e} = \bar{f}$ then, by Lemma 3, $e = \vee\bar{e} = \vee\bar{f} = f$, so ϕ is one-one. Hence ϕ is a faithful representation, so $\mu(E) \leq n$.

Suppose now $\psi : E \to PT_Y$ is a faithful representation where Y is a set with m elements. It will be shown that $m \geq n$. Suppose to the contrary that $m < n$.

By Proposition 2, it may be assumed that for each $e \in E$, $e\psi$ is the identity mapping on some subset of Y . Let $X = \{x_1,\ldots,x_n\}$, so that for $i = 1$ to n there is a subset X_i of Y for which $x_i\psi = id_{X_i}$.

By Lemma 5, it may further be assumed that $\{x_1,\ldots,x_n\}$ has been listed so that for $1 \leq i < j \leq n$, $x_j \nleq x_i$, so by Lemma 4, $X_j \nsubseteq X_i$.

Put $Y_i = X_1 \cup \ldots \cup X_i$, for $i = 1$ to n . Thus $Y_1 \subseteq \ldots \subseteq Y_n$ so $|Y_1| \leq \ldots \leq |Y_n| \leq m < n$.

If $n = 1$ then $Y_1 = \emptyset$, so $x_1\psi = id_\emptyset = 0\psi$, which contradicts the fact that ψ is faithful. Hence $n > 1$, and thus $Y_{k+1} = Y_k$ for some

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k , where $1 \le k \le n - 1$, which means that $X_{k+1} \subseteq X_1 \cup \ldots \cup X_k$.

Put $X_i' = X_i \cap X_{k+1}$ for $i = 1$ to k , so

$$X_{k+1} = X_{k+1} \cap (X_1 \cup \ldots \cup X_k)$$
$$= X_1' \cup \ldots \cup X_k' .$$

Thus, by Lemma 4, and since ψ is a faithful representation,

$$x_{k+1}\psi = id_{X_{k+1}} = id_{X_1'} \vee \ldots \vee id_{X_k'}$$
$$= \left(id_{X_1} id_{X_{k+1}}\right) \vee \ldots \vee \left(id_{X_k} id_{X_{k+1}}\right)$$
$$= [(x_1x_{k+1}) \vee \ldots \vee (x_kx_{k+1})]\psi ,$$

so

$$x_{k+1} = (x_1x_{k+1}) \vee \ldots \vee (x_kx_{k+1}) .$$

But x_{k+1} is join irreducible, so $x_{k+1} = x_jx_{k+1}$, for some j where $1 \le j \le k$. Thus $x_{k+1} \le x_j$, so

$$id_{X_{k+1}} = x_{k+1}\psi \le x_j\psi = id_{X_j} .$$

By Lemma 4, $X_{k+1} \subseteq X_j$, which contradicts the fact that $X_{k+1} \nsubseteq X_j$. This shows $m \ge n$, so $\mu(E) \ge n$.

Hence $\mu(S) = n$, which completes the proof.

4. Fundamental inverse semigroups.

THEOREM 7. *Let S be a finite fundamental inverse semigroup with semilattice of idempotents E . Then*

$$\mu(S) = \mu(T_E) = n$$

where n is the number of join irreducible elements of E .

Proof. Clearly $\mu(E) \le \mu(S) \le \mu(T_E)$. By Theorem 6, it remains to prove $\mu(T_E) \le n$.

Let $X = \{x_1, \ldots, x_n\}$ be the set of join irreducible elements of E.

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Define $\phi : T_E \to PT_X$ by, for $\alpha \in T_E$,

$$\alpha\phi : x_i \to \begin{cases} x_i\alpha & \text{if } x_i \in \text{domain}(\alpha) \\ \text{undefined otherwise} . \end{cases}$$

Note that an element of a principal ideal of E is join irreducible in that ideal if and only if it is join irreducible in E. Hence ϕ is well-defined and homomorphic because join irreducible elements are sent to join irreducible elements by semilattice isomorphisms, and one-one because a semilattice isomorphism of a finite semilattice is completely determined by its action on join irreducible elements.

Hence ϕ is a faithful representation, so $\mu(T_E) \le n$, which completes the proof.

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