

**THE MINIMAL FREE RESOLUTION OF A
STAR-CONFIGURATION IN \mathbb{P}^n AND THE WEAK
LEFSCHETZ PROPERTY**

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ABSTRACT. We find the Hilbert function and the minimal free resolution of a star-configuration in \mathbb{P}^n . The conditions are provided under which the Hilbert function of a star-configuration in \mathbb{P}^2 is generic or non-generic. We also prove that if \mathbb{X} and \mathbb{Y} are linear star-configurations in \mathbb{P}^2 of types t and s , respectively, with $s \geq t \geq 3$, then the Artinian k -algebra $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property.

1. Introduction

Throughout the paper, $R = k[x_0, x_1, \dots, x_n]$ will be an $(n + 1)$ -variable polynomial ring over an algebraically closed field k of characteristic 0, and the symbol \mathbb{P}^n will denote the projective n -space over a field k . Let I be a homogeneous ideal of R . Then the numerical function

$$\mathbf{H}_{R/I}(t) := \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the ring R/I . If $I := I_{\mathbb{X}}$ is the ideal of a subscheme \mathbb{X} in \mathbb{P}^n , then we denote

$$\mathbf{H}_{R/I_{\mathbb{X}}}(t) := \mathbf{H}_{\mathbb{X}}(t) \quad \text{for } t \geq 0$$

and call it the *Hilbert function* of \mathbb{X} . Many interesting problems in the study of Hilbert functions and minimal free resolutions of standard graded algebras have been studied (see [12, 13, 14, 15, 16, 19]).

A graded Artinian k -algebra $A = \bigoplus_{i=0}^s A_i$ ($A_s \neq 0$) has the *weak Lefschetz property* if the homomorphism $(\times L) : A_i \rightarrow A_{i+1}$ induced by multiplication by a general linear form L has maximal rank for all i . In this case, we call

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L a Lefschetz element. This fundamental property has been studied by many authors (see [3, 6, 12, 17, 21, 22, 24, 25]).

In [2], the following interesting result has been proved.

Proposition 1.1 (Proposition 3.4, [2]). *Let F_1, F_2, \dots, F_r be general forms in $R = k[x_0, x_1, \dots, x_n]$ with $r \geq 3$. Then*

$$\bigcap_{1 \leq i < j \leq r} (F_i, F_j) = \sum_{i=1}^r (F_1 \cdots \hat{F}_i \cdots F_r),$$

where $\hat{*}$ means that we omit $*$.

The variety \mathbb{X} in \mathbb{P}^n of the ideal

$$\bigcap_{1 \leq i < j \leq r} (F_i, F_j) = \sum_{i=1}^r (F_1 \cdots \hat{F}_i \cdots F_r)$$

in Proposition 1.1 is called a *star-configuration* in \mathbb{P}^n of type r . Furthermore, if the F_i are all general linear forms in R , the star-configuration \mathbb{X} is called a *linear star-configuration* in \mathbb{P}^n .

The Terracini Lemma in [27] says that the Hilbert function of the union of star-configurations in \mathbb{P}^n gives the dimensions of *the secant varieties of the varieties of reducible forms* (see also [2, 4, 5, 9, 10, 23]). In [18], Geramita, Migliore, and Sabourin showed that a linear star-configuration in \mathbb{P}^2 has a generic Hilbert function. In this paper we study Hilbert functions and minimal free resolutions of star-configurations in \mathbb{P}^n , and give answers to the following two interesting questions.

Question 1.2. Let F_1, \dots, F_r be general forms in $R = k[x_0, x_1, \dots, x_n]$ of degrees $1 \leq d_1 \leq \dots \leq d_r$, respectively.

- (a) What is the Hilbert function of the ideal of a star-configuration defined by F_1, \dots, F_r ?
- (b) What is the minimal free resolution of the ideal of a star-configuration defined by F_1, \dots, F_r ?

Question 1.3. Let \mathbb{X} and \mathbb{Y} be star-configurations in \mathbb{P}^2 defined by general forms of degree $d \geq 1$. Does the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the weak Lefschetz property?

In Section 2, we introduce preliminary results and definitions and then find the Hilbert function and the minimal free resolution of a star-configuration in \mathbb{P}^n (Theorem 2.1 and Corollary 2.5), which is the complete answer to Question 1.2. In Section 3 we show that the star-configuration \mathbb{X} in \mathbb{P}^2 , defined by general forms F_1, \dots, F_r of the same degree d with $d = 1, 2$, has a generic Hilbert function (see Proposition 3.1), which slightly generalizes the result of [18]. In other words,

$$\mathbf{H}_{\mathbb{X}}(t) = \min \left\{ \binom{t+2}{2}, \deg(\mathbb{X}) \right\} \quad \text{for } t \geq 0.$$

However, if the star-configuration \mathbb{X} is defined by general forms F_1, \dots, F_r of the same degree d with $d \geq 3$, then the Hilbert function of the star-configuration is NEVER generic (see Example 3.2, Proposition 3.3, and Remark 3.4). In Section 4, we show that if \mathbb{X} and \mathbb{Y} are linear star-configurations in \mathbb{P}^2 of types t and s with $s \geq t \geq 3$, then the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property (see Theorem 4.2), which is the answer to Question 1.3 for $d = 1$. However, Question 1.3 is still open for $d > 1$.

2. Star-configurations in \mathbb{P}^n

Let \mathbb{X} be a star-configuration in \mathbb{P}^n defined by general forms F_1, \dots, F_r in $R = k[x_0, \dots, x_n]$ of degrees $1 \leq d_1 \leq \dots \leq d_r$, respectively. By Proposition 1.1, a star-configuration \mathbb{X} in \mathbb{P}^n is a closed subscheme of codimension 2, and the ideal of a linear star-configuration \mathbb{X} has r generators of degree $r - 1$. For any matrix M with entries in an arbitrary ring R we write $I_t(M)$ for the ideal generated by the $t \times t$ minors of M . We begin with the following theorem, which gives an answer to Question 1.2.

Theorem 2.1. *Let \mathbb{X} be a star-configuration in \mathbb{P}^n defined by general forms F_1, \dots, F_r in $R = k[x_0, \dots, x_n]$ of degrees $1 \leq d_1 \leq \dots \leq d_r$, respectively, and let $d = d_1 + d_2 + \dots + d_r$. Then the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \rightarrow R^{r-1}(-d) \rightarrow \bigoplus_{i=1}^r R(-(d-d_i)) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0.$$

Proof. For the proof, we first introduce some notations.

- Let $\mathbf{e}_i = [0, \dots, \overset{i\text{-th}}{1}, \dots, 0]^T$ be an i -th standard vector in R^r for $i = 1, \dots, r$.
- Define $\sigma_{i,j} = F_i \mathbf{e}_i - F_j \mathbf{e}_j$ for $1 \leq i < j \leq r$.
- Let M be an $r \times (r-1)$ matrix whose column vectors are $\sigma_{1,2}, \sigma_{2,3}, \dots, \sigma_{r-1,r}$, that is,

$$M := \begin{pmatrix} F_1 & 0 & 0 & \cdots & 0 & 0 \\ -F_2 & F_2 & 0 & \cdots & 0 & 0 \\ 0 & -F_3 & F_3 & \cdots & 0 & 0 \\ 0 & 0 & -F_4 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -F_{r-1} & F_{r-1} \\ 0 & 0 & \cdots & \cdots & 0 & -F_r \end{pmatrix}.$$

- Let $\delta_i = F_1 \cdots \hat{F}_i \cdots F_r$ be a homogeneous polynomial of degree $d - d_i$ for $i = 1, \dots, r$.
- Define two maps ψ and φ as

$$\begin{aligned} \psi &: \bigoplus_{i=1}^r R(-(d-d_i)) \xrightarrow{[\delta_1, \dots, \delta_r]} R, \quad \text{and} \\ \varphi &: R^{r-1}(-d) \xrightarrow{M} \bigoplus_{i=1}^r R(-(d-d_i)). \end{aligned}$$

We shall show that the following sequence is the minimal free resolution of $R/I_{\mathbb{X}}$.

$$0 \rightarrow R^{r-1}(-d) \xrightarrow{\varphi} \bigoplus_{i=1}^r R(-(d-d_i)) \xrightarrow{\psi} R \rightarrow R/I_{\mathbb{X}} \rightarrow 0.$$

First, we prove that $\text{Im } \varphi = \text{Ker } \psi$. It is obvious that $\text{Im } \varphi \subseteq \text{Ker } \psi$. Conversely, suppose that

$$(a_1, \dots, a_r) \in \text{Ker } \psi, \text{ where } a_i \in R.$$

Since $a_1\delta_1 + \dots + a_r\delta_r = 0$, we have that, for $i = 1, \dots, r$,

$$F_i \mid (a_1\delta_1 + \dots + a_{i-1}\delta_{i-1} + a_{i+1}\delta_{i+1} + \dots + a_r\delta_r) = -a_i\delta_i, \quad \text{i.e., } F_i \mid a_i.$$

Let $a_i = b_i F_i$ for such i . Then we have

$$a_1\delta_1 + \dots + a_r\delta_r = (b_1 + \dots + b_r)F_1 \cdots F_r = 0, \quad \text{that is, } b_1 + \dots + b_r = 0,$$

and so

$$\begin{aligned} (a_1, \dots, a_r) &= (b_1 F_1, b_2 F_2, \dots, b_{r-1} F_{r-1}, b_r F_r) \\ &= (b_1 F_1, b_2 F_2, \dots, b_{r-1} F_{r-1}, -(b_1 + \dots + b_{r-1}) F_r) \\ &= b_1 \sigma_{1,r} + b_2 \sigma_{2,r} + \dots + b_{r-1} \sigma_{r-1,r} \\ &= b_1(\sigma_{1,2} + \dots + \sigma_{r-1,r}) + b_2(\sigma_{2,3} + \dots + \sigma_{r-1,r}) + \dots + b_{r-1} \sigma_{r-1,r} \\ &= b_1 \sigma_{1,2} + (b_1 + b_2) \sigma_{2,3} + \dots + (b_1 + \dots + b_{r-1}) \sigma_{r-1,r} \\ &= M[b_1, b_1 + b_2, \dots, b_1 + \dots + b_{r-1}]^T \\ &\in \text{Im } \varphi, \end{aligned}$$

as we wished.

Second, we show that the map φ is injective. If $\varphi(a_1, \dots, a_{r-1}) = (0, \dots, 0)$ in R^r , then we have

$$\begin{aligned} &\varphi(a_1, \dots, a_{r-1}) \\ &= M[a_1, \dots, a_{r-1}]^T \\ &= a_1 \sigma_{1,2} + a_2 \sigma_{2,3} + \dots + a_{r-1} \sigma_{r-1,r} \\ &= a_1 F_1 \mathbf{e}_1 + (a_2 - a_1) F_2 \mathbf{e}_2 + \dots + (a_{r-2} - a_{r-1}) F_{r-1} \mathbf{e}_{r-1} - a_r F_r \mathbf{e}_{r-1} \\ &= (0, \dots, 0). \end{aligned}$$

This implies that $a_1 F_1 = (a_2 - a_1) F_2 = \dots = (a_{r-2} - a_{r-1}) F_{r-1} = -a_r F_r = 0$. Therefore

$$a_1 = \dots = a_r = 0,$$

which completes the proof. \square

Remark 2.2. Let \mathbb{X} be a star-configuration in \mathbb{P}^n defined by general forms F_1, \dots, F_r in $R = k[x_0, \dots, x_n]$ of degrees $1 \leq d_1 \leq \dots \leq d_r$, respectively.

From Theorem 2.1 and Hilbert-Burch theorem (see [11]), we have that $I_{\mathbb{X}}$ is generated by maximal minors of the matrix

$$M := \begin{pmatrix} F_1 & 0 & 0 & \cdots & 0 & 0 \\ -F_2 & F_2 & 0 & \cdots & 0 & 0 \\ 0 & -F_3 & F_3 & \cdots & 0 & 0 \\ 0 & 0 & -F_4 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -F_{r-1} & F_{r-1} \\ 0 & 0 & \cdots & \cdots & 0 & -F_r \end{pmatrix},$$

and $I_{\mathbb{X}}$ has depth exactly 2. Moreover, by Auslander-Buchsbaum formula (see [11] again), we see that

$$\text{depth}(R/I_{\mathbb{X}}) = \text{depth}(R) - \text{pd}(R/I_{\mathbb{X}}) = (n + 1) - 2 = n - 1.$$

Since \mathbb{X} is a closed subscheme in \mathbb{P}^n of codimension 2, we get that

$$\dim(R/I_{\mathbb{X}}) = \dim \mathbb{X} + 1 = n - 1 = \text{depth}(R/I_{\mathbb{X}}).$$

This implies that $R/I_{\mathbb{X}}$ is a Cohen-Macaulay ring, i.e., \mathbb{X} is an arithmetically Cohen-Macaulay subscheme in \mathbb{P}^n .

The following two corollaries are the special cases of Theorem 2.1 when $d_1 = \cdots = d_r = d$ and $d_1 = \cdots = d_r = 1$, respectively.

Corollary 2.3. *With notations as in Theorem 2.1 for $d_1 = \cdots = d_r = d$, the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \rightarrow R^{r-1}(-dr) \rightarrow R^r(-d(r-1)) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0.$$

Corollary 2.4. *Let \mathbb{X} be a linear star-configuration in \mathbb{P}^n of type r with $r \geq 3$. Then the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \rightarrow R^{r-1}(-r) \rightarrow R^r(-(r-1)) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0.$$

From Theorem 2.1, we can immediately find the Hilbert function of a star-configuration in \mathbb{P}^n defined by general forms F_1, \dots, F_r in $R = k[x_0, \dots, x_n]$, and thus we have the following corollary.

Corollary 2.5. *Let \mathbb{X} be a star-configuration in \mathbb{P}^n defined by general forms F_1, \dots, F_r in $R = k[x_0, \dots, x_n]$ of degrees $1 \leq d_1 \leq \cdots \leq d_r$, respectively.*

Then the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases} \binom{n+i}{n}, & 1 \leq i < d - d_r, \\ \binom{n+i}{n} - \binom{n+i-(d-d_r)}{n}, & d - d_r \leq i < d - d_{r-1}, \\ \binom{n+i}{n} - \binom{n+i-(d-d_r)}{n} - \binom{n+i-(d-d_{r-1})}{n}, & d - d_{r-1} \leq i < d - d_{r-2}, \\ \vdots & \vdots \\ \binom{n+i}{n} - \sum_{j=2}^r \binom{n+i-(d-d_j)}{n}, & d - d_2 \leq i < d - d_1, \\ \binom{n+i}{n} - \sum_{j=1}^r \binom{n+i-(d-d_j)}{n}, & d - d_1 \leq i < d, \\ \binom{n+i}{n} - \sum_{j=1}^r \binom{n+i-(d-d_j)}{n} + (r-1)\binom{n+i-d}{n}, & i \geq d. \end{cases}$$

Proof. From the minimal free resolution of $R/I_{\mathbb{X}}$

$$0 \rightarrow R^{r-1}(-d) \rightarrow \bigoplus_{i=1}^r R(-(d-d_i)) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0,$$

the Hilbert function of \mathbb{X} is

$$\begin{aligned} \mathbf{H}_{\mathbb{X}}(i) &= \dim_k R_i - \left[\sum_{j=1}^r \dim_k R_{i-(d-d_j)} \right] + (r-1) \dim_k R_{i-d} \\ &= \binom{n+i}{n} - \left[\sum_{j=1}^r \binom{n+i-(d-d_j)}{n} \right] + (r-1) \binom{n+i-d}{n}, \end{aligned}$$

as needed. □

3. Some properties of Hilbert functions of star-configurations in \mathbb{P}^2

In this section, we introduce a few more interesting results on star-configurations in \mathbb{P}^2 when $d_1 = \dots = d_r = d$.

Proposition 3.1. *Let \mathbb{X} be a star-configuration in \mathbb{P}^2 defined by general forms F_1, \dots, F_r in $R = k[x_0, x_1, x_2]$ of degree d ($d = 1, 2$) with $r \geq 3$. Then $R/I_{\mathbb{X}}$ has generic Hilbert function, i.e.,*

$$\mathbf{H}_{\mathbb{X}}(-) : 1 \binom{1+2}{2} \binom{2+2}{2} \dots \binom{2+(r-1)d-1}{2} \binom{2+(r-2)d}{2} d^2 \rightarrow \dots$$

Proof. By Proposition 1.1, $I_{\mathbb{X}}$ is the ideal of a set of $\binom{r}{2} \times d^2$ points in \mathbb{P}^2 , and $I_{\mathbb{X}}$ has only r generators in degree $d(r-1)$. Hence it suffices to show that, for $d = 1, 2$,

$$\mathbf{H}_{\mathbb{X}}(d(r-1)) = \deg(\mathbb{X}) = \binom{2+(r-2)d}{2} d^2.$$

Case 1. If $d = 1$, then

$$\begin{aligned} \mathbf{H}_{\mathbb{X}}(r-1) &= \dim_k R_{r-1} - r \\ &= \binom{2+(r-1)}{2} - r \\ &= \binom{2+(r-2)}{2} \cdot 1^2 \end{aligned}$$

(see also Lemma 7.8, [18]).

Case 2. If $d = 2$, then

$$\begin{aligned} \mathbf{H}_{\mathbb{X}}(2(r-1)) &= \dim_k R_{2(r-1)} - r \\ &= \binom{2+2(r-1)}{2} - r \\ &= \binom{2r}{2} - r \\ &= \binom{2+(r-2)}{2} \cdot 2^2. \end{aligned}$$

Therefore, *Cases 1* and *2* complete the proof. □

The following example, however, shows that Proposition 3.1 does not hold for $d = 3$ and $r = 3$.

Example 3.2. Let F_1, F_2, F_3 be general forms in $R = k[x_0, x_1, x_2]$ of degree 3 and let $I = (F_1F_2, F_1F_3, F_2F_3)$. By Corollary 2.5, the Hilbert function of R/I is

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 25 \quad 27 \quad \rightarrow .$$

In other words,

$$\mathbf{H}(R/I, 3(3-1)) = \mathbf{H}(R/I, 6) = 25 \neq 27 = \binom{3}{2} \cdot 3^2,$$

which does not satisfy Proposition 3.1.

The following proposition shows that the Hilbert function of a star-configuration defined by general forms of degree $d \geq 3$ can never be generic (see Remark 3.4).

Proposition 3.3. *Let \mathbb{X} be a star-configuration in \mathbb{P}^2 defined by general forms F_1, \dots, F_r in $R = k[x_0, x_1, x_2]$ of degree $d \geq 3$ with $r \geq 3$. Then the Hilbert function of $R/I_{\mathbb{X}}$ is*

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases} \binom{2+i}{2}, & 0 \leq i \leq d(r-1) - 1, \\ \binom{2+i}{2} - \binom{2+(i-d(r-1))}{2}, & d(r-1) \leq i \leq d(r-1) + (d-2), \\ \deg(\mathbb{X}), & i \geq d(r-1) + (d-2). \end{cases}$$

Proof. By Corollary 2.3, the minimal free resolution of $R/I_{\mathbb{X}}$ is

$$0 \rightarrow R^{r-1}(-dr) \rightarrow R^r(-d(r-1)) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0.$$

Hence the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\begin{aligned} \mathbf{H}_{\mathbb{X}}(i) &= \dim_k R_i - r \cdot \dim_k R(-d(r-1))_i + (r-1) \cdot \dim_k R(-dr)_i \\ &= \binom{2+i}{2} - r \cdot \binom{2+i-d(r-1)}{2} + (r-1) \cdot \binom{2+i-dr}{2}, \end{aligned}$$

and so, for $d(r-1) \leq i \leq d(r-1) + (d-2)$, the Hilbert function of $R/I_{\mathbb{X}}$ is now of the form

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases} \binom{2+i}{2}, & 0 \leq i \leq d(r-1) - 1, \\ \binom{2+i}{2} - r \cdot \binom{2+(i-d(r-1))}{2}, & d(r-1) \leq i \leq d(r-1) + (d-2). \end{cases}$$

In particular, the Hilbert function of $R/I_{\mathbb{X}}$ in degree $d(r-1) + (d-2)$ is

$$\begin{aligned} \mathbf{H}_{\mathbb{X}}(d(r-1) + (d-2)) &= \binom{2+(d(r-1)+(d-2))}{2} - \binom{2+(d-2)}{2} \cdot r \\ &= \binom{dr}{2} - \binom{d}{2} \cdot r \\ &= \frac{dr(dr-1)}{2} - \frac{d(d-1)}{2} \cdot r \\ &= \binom{r}{2} \cdot d^2 \\ &= \deg(\mathbb{X}), \end{aligned}$$

and therefore the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases} \binom{2+i}{2}, & 0 \leq i \leq d(r-1) - 1, \\ \binom{2+i}{2} - r \cdot \binom{2+(i-d(r-1))}{2}, & d(r-1) \leq i \leq d(r-1) + (d-2), \\ \deg(\mathbb{X}), & i \geq d(r-1) + (d-2), \end{cases}$$

which completes the proof. \square

Here is an alternative proof of Proposition 3.3 counting the number of minimal generators of $I_{\mathbb{X}}$ in each degree. Moreover, the following alternative proof shows what the polynomial basis for $(I_{\mathbb{X}})_d$ is precisely in each degree d .

Alternative Proof of Proposition 3.3. Since the ideal $I_{\mathbb{X}}$ has only r generators in degree $d(r-1)$, we have

$$\mathbf{H}_{\mathbb{X}}(d(r-1)) = \binom{2+d(r-1)}{2} - r.$$

Now consider the set of r forms in $I_{\mathbb{X}}$ of degree $d(r-1)$

$$\bigcup_{i=1}^r \{F_1 \cdots \hat{F}_i \cdots F_r\}.$$

Then the set $S := \bigcup_{i=1}^r \{x_0 F_1 \cdots \hat{F}_i \cdots F_r, x_1 F_1 \cdots \hat{F}_i \cdots F_r, x_2 F_1 \cdots \hat{F}_i \cdots F_r\}$ of $3r$ forms of degree $d(r-1) + 1$ in $I_{\mathbb{X}}$ is linearly independent. Assume that $x_0 F_1 \cdots \hat{F}_i \cdots F_r$ is a linear combination of the rest of $3r-1$ forms in S as follows:

$$\begin{aligned} & x_0 F_1 \cdots \hat{F}_i \cdots F_r \\ &= (\alpha_{1,0} x_0 + \alpha_{1,1} x_1 + \alpha_{1,2} x_2) \hat{F}_1 F_2 \cdots F_r + \cdots \\ & \quad + (\alpha_{i-1,0} x_0 + \alpha_{i-1,1} x_1 + \alpha_{i-1,2} x_2) F_1 \cdots \hat{F}_{i-1} \cdots F_r \\ & \quad + (\alpha_{i,1} x_1 + \alpha_{i,2} x_2) F_1 \cdots \hat{F}_i \cdots F_r \\ & \quad + (\alpha_{i+1,0} x_0 + \alpha_{i+1,1} x_1 + \alpha_{i+1,2} x_2) F_1 \cdots \hat{F}_{i+1} \cdots F_r + \cdots \\ & \quad + (\alpha_{r,0} x_0 + \alpha_{r,1} x_1 + \alpha_{r,2} x_2) F_1 \cdots F_{r-1} \hat{F}_r, \end{aligned}$$

where $\alpha_{i,j} \in k$. Then,

$$F_i \mid (x_0 - \alpha_{i,1} x_1 - \alpha_{i,2} x_2) F_1 \cdots \hat{F}_i \cdots F_r, \quad \text{i.e.,} \quad F_i \mid (x_0 - \alpha_{i,1} x_1 - \alpha_{i,2} x_2),$$

which is impossible since $\deg F_i = d > 1$. Similarly, we see that $x_1 F_1 \cdots \hat{F}_i \cdots F_r$ and $x_2 F_1 \cdots \hat{F}_i \cdots F_r$ cannot be a linear combination of the rest of $3r-1$ forms in S for every $1 \leq i \leq r$. Hence the Hilbert function of $R/I_{\mathbb{X}}$ has the form

$$1 \binom{2+1}{2} \cdots \binom{2+(d(r-1)-1)}{2} \left[\binom{2+d(r-1)}{2} - r \right] \left[\binom{2+(d(r-1)+1)}{2} - 3r \right] \cdots$$

By continuing the same process to the degree up to $d(r-1) + (d-2)$, the Hilbert function of $R/I_{\mathbb{X}}$ is now of the form

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases} \binom{2+i}{2}, & 0 \leq i \leq d(r-1) - 1, \\ \binom{2+i}{2} - r \cdot \binom{2+(i-d(r-1))}{2}, & d(r-1) \leq i \leq d(r-1) + (d-2). \end{cases}$$

The rest of this proof is the same as the previous proof, so we omit it here. \square

Remark 3.4. Let \mathbb{X} be as in Proposition 3.3. Since $d \geq 3$, we see that $d(r-1) < d(r-1) + (d-2)$. Moreover, by Proposition 3.3,

$$\mathbf{H}_{\mathbb{X}}(d(r-1)) = \binom{2+d(r-1)}{2} - \binom{2}{2} < \binom{2+d(r-1)}{2} \quad \text{and}$$

$$\deg(\mathbb{X}) = \mathbf{H}_{\mathbb{X}}(d(r-1) + (d-2))$$

$$\begin{aligned}
&= \binom{2 + d(r-1) + (d-2)}{2} - \binom{2 + (d(r-1) + (d-2) - d(r-1))}{2} \\
&= \binom{2 + d(r-1) + (d-2)}{2} - \binom{d}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\deg(\mathbb{X}) - \mathbf{H}_{\mathbb{X}}(d(r-1)) \\
&= \left[\binom{2 + d(r-1) + (d-2)}{2} - \binom{d}{2} \right] - \left[\binom{2 + d(r-1)}{2} - \binom{2}{2} \right] \\
&= \left[\binom{2 + d(r-1) + (d-2)}{2} - \binom{d}{2} \right] - \left[\binom{2 + d(r-1)}{2} - \binom{2}{2} \right] \\
&= \left[\binom{2 + d(r-1) + (d-2)}{2} - \binom{2 + d(r-1)}{2} \right] - \left[\binom{d}{2} - \binom{2}{2} \right] \\
&= \frac{(d-2)(2d(r-1) + d + 1)}{2} - \left[\binom{d}{2} - \binom{2}{2} \right] \\
&\geq \frac{(d-2)(5d+1)}{2} - \frac{d^2-d}{2} + 1 \quad (\text{since } r \geq 3) \\
&= 2d(d-2) \\
&> 0 \quad (\text{since } d \geq 3).
\end{aligned}$$

This implies that

$$\mathbf{H}_{\mathbb{X}}(d(r-1)) < \min \left\{ \binom{2 + d(r-1)}{2}, \deg(\mathbb{X}) \right\}.$$

Therefore, \mathbb{X} does not have generic Hilbert function, as we wished.

4. The weak Lefschetz property

We start with a proposition on the weak Lefschetz property from [12] and provide an answer to Question 1.3 for $d = 1$.

Let \mathbb{X} be a finite set of points in \mathbb{P}^n and define

$$\sigma(\mathbb{X}) = \min\{i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\}.$$

Proposition 4.1 (Proposition 5.15, [12]). *Let \mathbb{X} be a finite set of points in \mathbb{P}^n and let A be an Artinian quotient of the coordinate ring of \mathbb{X} . Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for all $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Then A has the weak Lefschetz property.*

Theorem 4.2. *Let \mathbb{X} and \mathbb{Y} be linear star configurations in \mathbb{P}^2 of types t and s with $s \geq t \geq 3$, respectively. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is an Artinian ring with the weak Lefschetz property.*

Proof. By Proposition 3.1, $R/I_{\mathbb{X}}$ and $R/I_{\mathbb{Y}}$ have generic Hilbert functions. Since $\sigma(\mathbb{X}) \leq \sigma(\mathbb{Y}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$, by Proposition 3.1 again, we have that

$$\mathbf{H}_{\mathbb{X}}(i) = \mathbf{H}_{\mathbb{Y}}(i) = \mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(i) = \binom{i+2}{2}$$

for $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Using the following exact sequence

$$0 \rightarrow R/I_{\mathbb{X} \cup \mathbb{Y}} \rightarrow R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \rightarrow R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \rightarrow 0,$$

we obtain that

$$\begin{aligned} \mathbf{H}_{R/(I_{\mathbb{X}} + I_{\mathbb{Y}})}(i) &= \mathbf{H}_{\mathbb{X}}(i) + \mathbf{H}_{\mathbb{Y}}(i) - \mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(i) \\ &= \mathbf{H}_{\mathbb{X}}(i) \end{aligned}$$

for $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Furthermore, since

$$R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \simeq (R/I_{\mathbb{X}})/((I_{\mathbb{X}} + I_{\mathbb{Y}})/I_{\mathbb{X}})$$

is an Artinian quotient of the coordinate ring $R/I_{\mathbb{X}}$, by Proposition 4.1 $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property, as we wished. \square

Remark 4.3 (CoCoA, [26]). If \mathbb{X} and \mathbb{Y} are not linear star-configurations in Proposition 3.1, then Theorem 4.2 may not hold in general. For example, assume that \mathbb{X} and \mathbb{Y} are star-configurations defined by general forms of degree 2 of the same type 4. Then, by Proposition 3.1 (see also Corollary 2.5), the Hilbert functions of $R/I_{\mathbb{X}}$ and $R/I_{\mathbb{Y}}$ are

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 24 \quad \rightarrow,$$

and thus

$$\sigma(\mathbb{X}) = \sigma(\mathbb{Y}) = 7.$$

Furthermore, the Hilbert function of $R/I_{\mathbb{X} \cup \mathbb{Y}}$, obtained by CoCoA, is

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28 \quad 36 \quad 45 \quad 48 \quad \rightarrow,$$

and thus

$$\begin{aligned} \mathbf{H}(R/I_{\mathbb{X}} + I_{\mathbb{Y}}, 6) &= \mathbf{H}(R/I_{\mathbb{X}}, 6) + \mathbf{H}(R/I_{\mathbb{Y}}, 6) - \mathbf{H}(R/I_{\mathbb{X} \cup \mathbb{Y}}, 6) \\ &= 24 + 24 - 28 \\ &= 20 \\ &\neq \mathbf{H}(R/I_{\mathbb{X}}, 6). \end{aligned}$$

This does not satisfy the conditions in Proposition 4.1, and thus we do not know if Theorem 4.2 still holds for this case when \mathbb{X} and \mathbb{Y} are star-configurations in \mathbb{P}^2 defined by general forms of degree d with $d = 2$.

5. Comments

Theorem 2.1 gives the complete answer to Question 1.2 and Theorem 4.2 gives an answer to Question 1.3 for $d = 1$. In other words, Question 1.3 for $d > 1$ is still open. Thus, we restate Question 1.3 as follows.

Question 5.1 (Restated Question 1.3). Let \mathbb{X} and \mathbb{Y} be star-configurations in \mathbb{P}^2 defined by general forms of degree $d > 1$. Does the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the weak Lefschetz property?

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