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THE MINIMAL HEIGHT OF JACOBIAN FIBRATIONS ON $K3$ SURFACES

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Abstract. We study the Mordell-Weil lattices of elliptic fibrations with sections on algebraic surfaces over complex numbers. In this paper, we obtain the minimum of the height pairing of such fibrations on $K3$ surfaces.

1. Introduction. Let $\Phi: X \to C$ be a Jacobian fibration on an algebraic surface X, i.e., Φ is an elliptic fibration on X with a global section. Then Shioda [8] defined a symmetric bilinear form on the Mordell-Weil group of Φ, which is called the height pairing. Moreover, Oguiso-Shioda [6] classified all Jacobian fibrations on rational surfaces. It follows from this classification that the minimum of the height pairing is not less than $1/30$, and by Shioda [9], it is equal to $1/30$.

In this paper, we consider this problem in the case of $K3$ surfaces. We do not know the classification of all Jacobian fibrations on $K3$ surfaces. However, it follows from Shioda [9] that this value is at least $1/120$. Here, we obtain the following:

MAIN THEOREM 1.1. *The minimal height of all Jacobian fibrations on $K3$ surfaces is equal to* $11/420$.

In the same method as in [5, §6], we prove the existence of such a Jacobian fibration. In fact, we give the following:

THEOREM 1.2. *Let X be a $K3$ surface with the transcendental lattice*

$$T_X = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$$

(*for the existence of X and the representation of T_X, see Shioda-Inose* [10]). *Then, there exists a Jacobian fibration $\Phi: X \to \boldsymbol{P}^1$ whose minimal height is equal to* $11/420$ (*the type of its singular fibres is $D_5 \oplus A_6 \oplus A_4 \oplus A_2$*).

To prove the minimality, suppose that there exists a Jacobian fibration on a $K3$ surface, whose minimal height is less than $11/420$. Let S be a section with $0 < \sigma < 11/420$ where $\sigma := \langle (S), (S) \rangle$ (see Shioda [8]). By using the properties of the root lattices A_m, D_n and E_p, and using computer, we see that there are 30 types of such a section S. It follows from this classification that the minimal height is at least $1/120$. However all S

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are torsions of the Mordell-Weil groups. This contradicts $\sigma \neq 0$.

REMARK. From the above, we give a new lower bound for the minimal height of Jacobian fibrations on algebraic surfaces.

We remark here that the Jacobian fibration with the minimal height 11/420 is not unique.

COROLLARY 1.3. *Let Z be a K3 surface on which there exists a Jacobian fibration with the minimal height* 11/420.

(i) *The Picard number* ρ *of* Z *is equal to* 19 *or* 20.

(ii) *If* $\rho = 19$, *then the type of singular fibres is* $A_6 \oplus A_4 \oplus A_3 \oplus A_2 \oplus A_1$.

(iii) *If* $\rho = 20$, *then the type of singular fibres is* $D_5 \oplus A_6 \oplus A_4 \oplus A_2$, $A_6 \oplus A_4 \oplus A_3 \oplus A_2 \oplus A_1^{\oplus 2}$ *or* $A_6 \oplus A_4 \oplus A_3 \oplus A_2 \oplus A_1$.

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2. Lattices. By a *lattice* (L, b) we mean a finitely generated free $\boldsymbol{Z}$-module L, endowed with a non-degenerate symmetric bilinear form $b: L \times L \to \boldsymbol{Z}$. An *even* lattice is a lattice whose associated quadratic form $x^2 := b(x, x)$ takes even values. Simply, we say a lattice L instead of (L, b) when there is no fear of confusion.

If $\{e_1, e_2, \ldots, e_n\}$ is a $\boldsymbol{Z}$-basis for a lattice L, we define a non-degenerate symmetric matrix $I = (b(e_j, e_k))_{1 \le j,k \le n}$. Then the determinant and the signature of a lattice L are defined as

$$\det L := |\det I| > 0 \quad \text{and} \quad \operatorname{sgn} L := \operatorname{sgn} I .$$

A lattice L is *unimodular* if det $L = 1$. We define the *positive- (negative-) definiteness* of a lattice L by that of the matrix I. Frequently, a lattice L is expressed by the matrix I.

The *hyperbolic lattice* H is defined by

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If (L, b) is a lattice, then $(L, -b)$ is also a lattice. We denote this lattice by $-L$.

A sublattice T of L is a submodule of L such that $(T, b|_{T \times T})$ is a lattice. A lattice M is an *overlattice* of L if L becomes a sublattice of M such that the index $[M: L]$ is finite.

By $S \oplus T$, we denote the orthogonal direct sum of lattices S and T. A lattice is *indecomposable* if it cannot be obtained as an orthogonal direct sum of two non-trivial sublattices. The orthogonal complement $T^{\perp}$ of T is defined as

$$T^{\perp} = \{x \in L \mid b(x, y) = 0 \text{ for all } y \in T\} .$$

A sublattice T of L is said to be *primitive* if the quotient L/T is torsion-free. The *primitive closure* of T in L is:

$$\bar{T}=\{x\in L \mid mx\in T \text{ for some positive integer } m\}\,.$$

The *dual lattice* L^* of L is defined by

$$L^*=\{x\in L\otimes \boldsymbol{Q} \mid b(x, y)\in \boldsymbol{Z} \text{ for all } y\in L\}\,.$$

The canonical bilinear form on L^* induced by b is denoted by the same letter b. Now let $G_L := L^*/L$. If L is an even lattice, then $q_L\colon G_L \to \boldsymbol{Q}/2\boldsymbol{Z}$ is defined by

$$q_L(x+L)\equiv x^2 \bmod 2\boldsymbol{Z}\,.$$

We shall call (G_L, q_L) the *discriminant form of L.*

Recall the following lemma:

LEMMA 2.1 (e.g., Barth-Peters-Van de Ven [1, Lemma I.2.1]). *Let L be a lattice.*

(i) $\det L=[L^*\colon L]=\#G_L$.

(ii) *If M is an overlattice of L, then* $\det L=(\det M)\cdot[M\colon L]^2$.

An *isometry* of a lattice L is an isomorphism as a $\boldsymbol{Z}$-module compatible with the bilinear form b.

Let L be a negative-definite even lattice. We call $e\in L$ a *root* if $e^2=-2$. Put $\Delta(L):=\{e\in L \mid e^2=-2\}$. Then the sublattice of L spanned by $\Delta(L)$ is called the *root type* of L and is denoted by L_{root}.

The lattices A_m $(m\geq 1)$, D_n $(n\geq 4)$, E_p $(p=6, 7, 8)$ defined by the Dynkin diagrams in the Figure are called *root lattices*:

$A_m=\langle a_1, a_2, \dots, a_m\rangle$

$D_n=\langle d_1, d_2, \dots, d_n\rangle$

$E_p=\langle e_1, e_2, \dots, e_p\rangle$

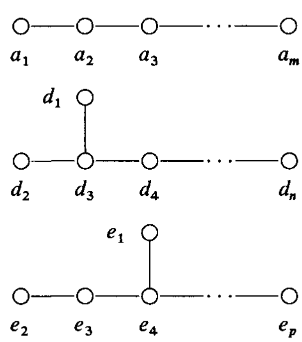

FIGURE.

where the vertices a_j, d_k and e_l satisfy $(a_j)^2=(d_k)^2=(e_l)^2=-2$ and two vertices, for example a_j and $a_{j'}$, are jointed by an r-tuple line if and only if $b(a_j, a_{j'})=r$. We denote by α_j, δ_k or ε_l the dual base associated with a_j, d_k or e_l respectively.

Let us recall the following facts:

LEMMA 2.2 (Bourbaki [2, pp. 250–270]). *Let L be a root lattice.*
(i) *L is an indecomposable negative-definite even lattice.*
(ii) *G_L and q_L are as given in Table 1.*

TABLE 1.

	A_m	D_n n: even	D_n n: odd	E_6	E_7	E_8
G_L	$\mathbf{Z}/(m+1)\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0
generators	α_m	δ_1, δ_n	δ_1	ε_6	ε_7	
q_L	$(-m/(m+1))$	$\begin{pmatrix} -n/4 & -1/2 \\ -1/2 & -1 \end{pmatrix}$	$(-n/4)$	$(-4/3)$	$(-3/2)$	

LEMMA 2.3 (Nikulin [4, Proposition 1.4.1]). *Let L be an even lattice. Then, for an even overlattice M of L, we have an isotropic subgroup M/L of $G_L = L^*/L$ with respect to q_L, i.e., $q_L|_{M/L} = 0$. This determines a bijective correspondence between even overlattices of L and isotropic subgroups of G_L with respect to q_L.*

LEMMA 2.4 (e.g., Nikulin [4, Proposition 1.6.1]). *Let L be an even unimodular lattice and let T be a primitive sublattice. Then we have*

$$G_T \cong G_{T^\perp} \cong L/(T \oplus T^\perp), \qquad q_{T^\perp} = -q_T.$$

In particular, $\det T = \det T^\perp = [L : T \oplus T^\perp]$.

THEOREM 2.5 (Nikulin [4, Corollary 1.6.2]). *Let S and T be even lattices such that $G_S \cong G_T$ and $q_S = -q_T$. Then there exists an even unimodular lattice L which satisfies the following*:
(1) *L is an overlattice of $S \oplus T$.*
(2) *The embeddings of both S and T into L are primitive.*
(3) *$S = T^\perp$ in L, $T = S^\perp$ in L.*

3. Jacobian fibrations on $K3$ surfaces.

3.1. Jacobian fibrations and the Mordell-Weil groups. Let X be an algebraic $K3$ surface, i.e., $\mathscr{K}_X = \mathscr{O}_X$ and $\dim H^1(X, \mathscr{O}_X) = 0$. Then the second cohomology group $H^2(X, \mathbf{Z})$ equipped with the cup product is an even unimodular lattice of signature (3, 19) isometric to $H^{\oplus 3} \oplus E_8^{\oplus 2}$ (cf. [1, Proposition VIII.3.2]). The primitive sublattice

$$S_X := H^{1,1}(X, \mathbf{R}) \cap H^2(X, \mathbf{Z})$$

of $H^2(X, \mathbf{Z})$ is called the *Picard lattice* of X. Then, by the definition of $K3$ surfaces, S_X is isomorphic to the Néron-Severi group of X. The sublattice

$$T_X := S_X^\perp \quad \text{in} \quad H^2(X, \mathbf{Z})$$

is called the *transcendental lattice* of X.

THEOREM 3.1 (Shioda-Inose [10]). (i) *Let Q be a positive-definite even lattice of rank 2. Then there exists a singular $K3$ surface X with $T_X \cong Q$.*

(ii) *Suppose $Q \subset H^{\oplus 3} \oplus E_8^{\oplus 2}$ is a primitive sublattice of signature $(2, 20-\rho)$. Then there exists an algebraic $K3$ surface X with $T_X \cong Q$.*

(ii) follows from the surjectivity of the period map.

Let $\Phi\colon X \to \boldsymbol{P}^1$ be a Jacobian fibration on X, i.e., Φ is an elliptic fibration on X with a global section O. Let $F_v = \Phi^{-1}(v)$ denote the fibre over $v \in \boldsymbol{P}^1$. For each $v \in \boldsymbol{P}^1$, let

$$F_v = \Theta_{v,0} + \sum_{j=1}^{m_v - 1} \mu_{v,j} \Theta_{v,j} ,$$

where $\Theta_{v,j}$ $(0 \le j \le m_v - 1)$ are the irreducible components of F_v, being m_v in number, such that $\Theta_{v,0}$ is the unique component of F_v meeting O. Then we define the following sublattices of S_X:

$$U := \langle c_1(O), c_1(F) \rangle \qquad (F\colon \text{the fibre of } \Phi)$$

$$T_v := \langle c_1(\Theta_{v,j}) \mid 1 \le j \le m_v - 1 \rangle \qquad (v \in \boldsymbol{P}^1)$$

$$T := \bigoplus_{v \in \boldsymbol{P}^1} T_v .$$

We shall call T the *type* (of singular fibres) of the Jacobian fibration Φ.

In view of this embedding, let $W := U^\perp$ in S_X. Then W becomes a negative-definite even lattice. Since U is unimodular, $S_X = U \oplus W$.

LEMMA 3.2 (Nishiyama [5]). (i) *U is isometric to the hyperbolic lattice H.*

(ii) *The type T of a Jacobian fibration is isometric to the root type W_{root}.*

The *Mordell-Weil group* of a Jacobian fibration Φ is the subgroup of $\operatorname{Aut}(X)$ consisting of all automorphisms acting on a general fibre as translations. In this paper, we use the following description of the Mordell-Weil group of Φ.

THEOREM 3.3 (Shioda [8, Theorem 1.3]). *The Mordell-Weil group of a Jacobian fibration Φ is isomorphic to the quotient $S_X/(U \oplus T) = W/T$.*

3.2. The Mordell-Weil lattices. In order to define a good pairing on the Mordell-Weil group W/T, we first define a homomorphism $\varphi\colon W \to W \otimes \boldsymbol{Q}$, which satisfies $\varphi(w) \equiv w \pmod{T \otimes \boldsymbol{Q}}$ and $\varphi(w) \in T^\perp$ in $W \otimes \boldsymbol{Q}$. (These conditions guarantee the uniqueness of φ.)

In particular, let $\{t_j\}$ be a $\boldsymbol{Z}$-basis for a lattice T, and let $\{\tau_j\}$ be the associated dual basis. Then

$$\varphi(w) = w - \sum_j b(w, t_j)\tau_j .$$

LEMMA 3.4 (Shioda [8, Lemma 8.3]). *Let $N := T^{\perp}$ in W, and let*

$$l := \text{L.C.M.}\{\det T_v \mid T_v \neq 0\}$$

be the least common multiple of $\det T_v$. *Then φ induces an injection*

$$\varphi' : (W/T)_{\text{free}} \hookrightarrow \frac{1}{l}N := \left\{\frac{1}{l}w \,\middle|\, w \in N\right\} \subset N \otimes \boldsymbol{Q}.$$

The non-degenerate symmetric bilinear form $\langle *, * \rangle := -b(\varphi'(*), \varphi'(*))$ is called the *height pairing*. We now define the *Mordell-Weil lattice* of a Jacobian fibration Φ as the pair $((W/T)_{\text{free}}, \langle *, * \rangle)$. Similarly, for $w, w' \in W$, we use the notation $\langle w, w' \rangle$.

THEOREM 3.5. *The determinant of the Mordell-Weil lattice is equal to*

$$\det W/\det \bar{T} = (\det T_X) \cdot [\bar{T} : T]^2/\det T,$$

where $\bar{T}$ is the primitive closure of T in W.

PROOF. In the first place, we have $G_{S_X} = G_{T_X}$ by Lemma 2.4. Since $S_X = U \oplus W$, $\det W = \det T_X$. By Lemma 2.1, $\det T = \det \bar{T} \cdot [\bar{T} : T]^2$. Therefore $\det W/\det \bar{T} = (\det T_X) \cdot [\bar{T} : T]^2/\det \mathrm{T}$.

Secondly, let $\{\bar{t}_l\}$ be a $\boldsymbol{Z}$-basis for the lattice $\bar{T}$, and let $\{\bar{\tau}_l\}$ be the associated dual basis. Take $w_j \in W$ so that $\{w_j, \bar{t}_l\}$ is a $\boldsymbol{Z}$-basis for the lattice W. For all j, there exist $\lambda_l^j \in \boldsymbol{Q}$ such that $\sum_m b(w_j, \bar{t}_m)\bar{\tau}_m = \sum_l \lambda_l^j \bar{t}_l$, i.e., $\lambda_l^j := \sum_m b(w_j, \bar{t}_m) b(\bar{\tau}_m, \bar{\tau}_l)$. Then we have

$$\varphi(w_j) = w_j - \sum_m b(w_j, \bar{t}_m)\bar{\tau}_m = w_j - \sum_l \lambda_l^j \bar{t}_l,$$

$$b(w_j, \bar{t}_m) - \sum_l \lambda_l^j b(\bar{t}_l, \bar{t}_m) = b\left(w_j - \sum_l \lambda_l^j \bar{t}_l, \bar{t}_m\right) = b(\varphi(w_j), \bar{t}_m) = 0,$$

$$b(w_j, w_k) - \sum_l \lambda_l^j b(\bar{t}_l, w_k) = b(\varphi(w_j), w_k) = b(\varphi(w_j), \varphi(w_k)).$$

By using these equations, we can transform the matrix W as follows:

$$W = \left(\begin{array}{c|c} (b(w_j, w_k))_{j,k} & (b(w_j, \bar{t}_m))_{j,m} \\ (b(\bar{t}_l, w_k))_{l,k} & \bar{T} = (b(\bar{t}_l, \bar{t}_m))_{l,m} \end{array}\right) \longrightarrow \left(\begin{array}{c|c} V := (b(\varphi(w_j), \varphi(w_k)))_{j,k} & 0 \\ 0 & \bar{T} \end{array}\right).$$

Note that $-V$ is nothing but the Mordell-Weil lattice. Thus the determinant of the Mordell-Weil lattice is equal to $\det W/\det \bar{T}$. q.e.d.

3.3. The minimal height. The *minimal height* of a Jacobian fibration Φ is

$$\mu(\Phi) := \min\{\langle w, w \rangle \mid \varphi(w) \neq 0\}.$$

Therefore the main theorem means

$$\min\{\mu(\Phi) \mid \Phi : \text{Jacobian fibrations on } K3 \text{ surfaces}\} = 11/420.$$

REMARK. On the other hand, for any positive integer m, there exists a Jacobian fibration with $\mu(\Phi)=4+2m$ (cf. [6, Appendix 1]).

4. Proof of the main theorem.

4.1. Existence. To prove the main theorem, we first of all show the existence of a Jacobian fibration with $\mu(\Phi)=11/420$.

THEOREM 4.1. *Let X be a singular K3 surface with*

$$T_X=\begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}.$$

Then, there exists a Jacobian fibration Φ on X the type of whose singular fibres is $D_5 \oplus A_6 \oplus A_4 \oplus A_2$, with $\mu(\Phi)=11/420$.

Theorem 3.1 guarantees that there exists such an X. We use the following lemma to prove Theorem 4.1.

LEMMA 4.2 (Nishiyama [5]). *Let X' be a K3 surface. Let T_0 be a negative-definite even lattice such that $G_{T_0}=G_{T_{X'}}$, $q_{T_0}=q_{T_{X'}}$ and* rank $T_0=$ rank $T_{X'}+4$. *Assume that there exists a primitive embedding of T_0 into a negative-definite even unimodular lattice L of rank 24. Then there exists a Jacobian fibration on X' whose W is isometric to the lattice $T_0^{\perp}$ in L (as for W, see* §3).

PROOF OF THEOREM 4.1. The notation used from now on is as given in §2. Let

$$T_0 := \begin{pmatrix} -4 & 0 & 1 & & & \\ 0 & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 1 & & \\ & 1 & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}.$$

Consider the primitive embedding of T_0 given by $T_0 \cong \langle e_6+e_8, e_1, e_5, e_4, e_3, e_2 \rangle \subset E_8$. An elementary calculation shows that

$$T_0^{\perp} \text{ in } E_8 \supset \langle e_7+e_8, -\varepsilon_7 \rangle = \begin{pmatrix} -2 & -1 \\ -1 & -6 \end{pmatrix} \cong -T_X,$$

where $-\varepsilon_7=6e_1+4e_2+8e_3+12e_4+10e_5+8e_6+6e_7+3e_8$. Note that $\det(-T_X)=11$ is square-free. It follows from Lemma 2.1 that $T_0^{\perp}$ in $E_8=\langle e_7+e_8, -\varepsilon_7\rangle \cong -T_X$. Since E_8 is unimodular, by Lemma 2.4, we have $G_{T_0}=G_{-T_X}=G_{T_X}$, $q_{T_0}=-q_{-T_X}=q_{T_X}$ and rank $T_0=$ rank T_X+4.

Put $L' := D_5^{(1)} \oplus D_5^{(2)} \oplus A_7^{(1)} \oplus A_7^{(2)}$, where $D_5^{(j)}$ and $A_7^{(j)}$ are copies of D_5 or A_7, respectively. Let $\{d_k^{(j)}\}$ be a $\boldsymbol{Z}$-basis for $D_5^{(j)}$ and let $\{\delta_k^{(j)}\}$ be the associated dual basis as in §2. Similarly, we define a $\boldsymbol{Z}$-basis $\{a_k^{(j)}\}$ for $A_7^{(j)}$ and the associated dual basis $\{\alpha_k^{(j)}\}$.

Let $G=\boldsymbol{Z}/4\boldsymbol{Z}\oplus\boldsymbol{Z}/8\boldsymbol{Z}$ be the subgroup of $G_{L'}$ generated by $\delta_2^{(1)}+\delta_1^{(2)}+\alpha_2^{(1)}$ and $\delta_2^{(1)}+\alpha_5^{(1)}+\alpha_1^{(2)}$. Then G is isotropic with respect to $q_{L'}$. Hence from Lemma 2.3, there exists an overlattice L such that $L/L'=G$. Moreover by Lemma 2.1, L is a negative-definite even unimodular lattice of rank 24. It follows from the classification of negative-definite even unimodular lattices of rank 24 (cf. Niemeier [3]) that $L_{\text{root}}=L'$.

Consider the primitive embedding of T_0 into L given by

$$T_0\cong\langle\delta_2^{(1)}+\alpha_5^{(1)}+\alpha_1^{(2)},\ D_5^{(1)}\rangle\subset L\,.$$

Then by Lemma 4.2, there exists a Jacobian fibration Φ such that $W\cong T_0^{\perp}$ in L. Now put $M:=T_0^{\perp}$ in $L_{\text{root}}=D_5^{(2)}\oplus\langle a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, a_4^{(1)}, a_6^{(1)}, a_7^{(1)}, a_2^{(2)}, a_3^{(2)},\dots, a_7^{(2)}, a_5^{(1)}-a_1^{(2)}\rangle$. Thus $T=W_{\text{root}}\cong M_{\text{root}}=D_5^{(2)}\oplus\langle a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, a_4^{(1)}\rangle\oplus\langle a_6^{(1)}, a_7^{(1)}\rangle\oplus\langle a_2^{(2)}, a_3^{(2)},\dots, a_7^{(2)}\rangle=D_5\oplus A_6\oplus A_4\oplus A_2$.

Note that $\{x\in G_T\,|\,q_T(x)\equiv 0 \bmod 2\boldsymbol{Z}\}=0$ (cf. Lemma 2.2). Hence, from Lemma 2.3, $\bar{T}=T$, i.e., the Mordell-Weil group is torsion-free. Therefore the Mordell-Weil group W/T is isomorphic to $\boldsymbol{Z}$.

Moreover, by Theorem 3.5, the determinant of the Mordell-Weil lattice is equal to $\det W/\det\bar{T}=\det T_X/\det T=11/420$. Hence the Mordell-Weil lattice is isometric to the ($\boldsymbol{Q}$-valued) lattice (11/420) of rank 1. Thus, there exists a Jacobian fibration Φ on X with $\mu(\Phi)=11/420$. q.e.d.

REMARK. Put $w:=\delta_5^{(2)}+(\alpha_2^{(1)}-\alpha_5^{(1)}+\alpha_6^{(1)})+(-\alpha_1^{(2)}+\alpha_2^{(2)})$. Then it is easy to see that $W=\langle w, W_{\text{root}}\rangle$ and $\varphi(w)=\alpha_5^{(1)}/15-\alpha_1^{(2)}/7$. Therefore

$$\langle w, w\rangle=-\varphi(w)^2=\frac{1}{15^2}\cdot\frac{5\cdot 3}{8}+\frac{1}{7^2}\cdot\frac{1\cdot 7}{8}=\frac{1}{120}+\frac{1}{56}=\frac{11}{420}\,.$$

A Jacobian fibration with $\mu(\Phi)=11/420$ is not unique. Here is another example:

PROPOSITION 4.3. *Let Y be a K3 surface with*

$$T_Y=\begin{pmatrix}2&1\\1&6\end{pmatrix}\oplus(-2)\,.$$

Then, there exists a Jacobian fibration Φ' on Y the type of whose singular fibres is $A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1$, with $\mu(\Phi')=11/420$.

PROOF. By Theorem 2.5, there exists a primitive embedding of T_Y into $H^{\oplus 3}\oplus E_8^{\oplus 2}$ such that $(T_Y)^{\perp}=(-T_Y)\oplus E_8^{\oplus 2}$. Then by Theorem 3.1, there exists such a Y.

The notation used from now on is as given in the proof of Theorem 4.1. Let $T_0':=T_0\oplus(-2)$. Then $G_{T_0'}=G_{T_Y}$, $q_{T_0'}=q_{T_Y}$ and rank $T_0'=\text{rank}\,T_Y+4$.

Consider the primitive embedding of T_0' into L given by

$$T_0'\cong\langle\delta_2^{(1)}+\alpha_5^{(1)}+\alpha_1^{(2)},\ D_5^{(1)}\rangle\oplus\langle d_1^{(2)}\rangle\subset L\,.$$

Then by Lemma 4.2, there exists a Jacobian fibration Φ' such that $W\cong(T_0')^{\perp}$ in L. Thus

$T \cong ((T_0')^\perp \text{ in } L_{\text{root}})_{\text{root}} = (\langle d_1^{(2)} \rangle^\perp \text{ in } M)_{\text{root}} = (\langle d_1^{(2)} \rangle^\perp \text{ in } D_5^{(2)})_{\text{root}} \oplus A_6 \oplus A_4 \oplus A_2 = A_6 \oplus A_4 \oplus A_3 \oplus A_2 \oplus A_1$ because $A_1^\perp$ in $D_5 = A_3 \oplus A_1$ (cf. [5, Corollary 4.4]).

In the same method as in the proof of Theorem 4.1, we can show the existence of a Jacobian fibration Φ' with $\mu(\Phi') = 11/420$. q.e.d.

REMARK. Note that $T_Y \cong H \oplus (22)$. Indeed, let $\{t_1, t_2, t_3\}$ be a $\boldsymbol{Z}$-basis for T_Y where $t_1^2 = 2$, $t_2^2 = 6$, $t_3^2 = -2$, $b(t_1, t_2) = 1$ and $b(t_1, t_3) = b(t_2, t_3) = 0$. Then

$$T_Y = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix} \oplus (-2) = \langle t_1, t_2, t_3 \rangle \cong \langle t_1 + t_3, t_2 - 3(t_1 + t_3), 12t_1 - 2t_2 + 11t_3 \rangle$$
$$\cong H \oplus (22) .$$

4.2. Minimality. In the following, we use the notation as in §3. To prove the minimality, we show that there does not exist $w \in W$ such that $\langle w, w \rangle = -\varphi(w)^2 < 11/420$ and $\varphi(w) \neq 0$, for any Jacobian fibration on any $K3$ surface.

Step 1. Let T_v be the orthogonal components of T, i.e., $T = W_{\text{root}} = \bigoplus_v T_v$. Put $n_v := \operatorname{rank} T_v$. Then $T_v = A_{n_v}$, D_{n_v} or E_{n_v}.

DEFINITION 4.4. Let $w \in W$. w is said to be *normalized* if w satisfies, for each v, one of the following conditions:

(i) $w \in T_v^\perp$.

(ii) There exists j_v such that $b(w, t_{j_v}^v) = 1$ and $b(w, t_{j'}^v) = 0$ for all $j' \neq j_v$, where $\{t_k^v\}$ is a $\boldsymbol{Z}$-basis for T_v.

LEMMA 4.5. *For any $w' \in W$, there exists $w \in W$ such that w is normalized and $w \equiv w'$* (mod T). (*In particular*, $\varphi(w) = \varphi(w')$.)

PROOF. Let $\{\tau_k^v\}$ be the dual basis associated with $\{t_k^v\}$. We may assume that $\tau_k^v = \alpha_k$, δ_k or ε_k. From the properties $G_{A_{n_v}} = \{0, \alpha_1, \alpha_2, \ldots, \alpha_{n_v}\}$, $G_{D_{n_v}} = \{0, \delta_1, \delta_2, \delta_{n_v}\}$, $G_{E_6} = \{0, \varepsilon_2, \varepsilon_6\}$, $G_{E_7} = \{0, \varepsilon_7\}$ and $G_{E_8} = 0$ of G_{T_v}, we can directly see that

$$\sum_k b(w', t_k^v)\tau_k^v \equiv 0 \quad \text{or} \quad \tau_{j_v}^v \pmod{T_v} \qquad (\text{for some } j_v) .$$

Let $t^v := \sum_k b(w', t_k^v)\tau_k^v - (0 \text{ or } \tau_{j_v}^v)$. Then $t^v \in T_v$. Thus $w := w' - \sum_v t^v$. q.e.d.

ALTERNATIVE PROOF. From Shioda [8], the sections of a Jacobian fibration become the representative vectors of the Mordell-Weil group. Then there exists a section S such that $c_1(S) \equiv w' (\operatorname{mod} U \oplus T)$. Thus $w = c_1(S) - c_1(O) - (S \cdot O - O^2)c_1(F)$. q.e.d.

Step 2. Suppose that, for a Jacobian fibration on a $K3$ surface, there exists a normalized $w \in W$ such that $\langle w, w \rangle = -\varphi(w)^2 \leq 11/420$ and $\varphi(w) \neq 0$. If w satisfies the condition (i) in Definition 4.4 for all v, then $\varphi(w) = w$. On the other hand, the assumption $\varphi(w) \neq 0$ implies $w \notin T$ and hence $w^2 \neq -2$, i.e., $-w^2 \geq 4$. Therefore there exists v for which w satisfies the condition (ii). Assume that $\{1, 2, \ldots, r\}$ is the set of all v's satisfying the condition (ii). Put $T^w := \bigoplus_{v=1}^r T_v$. Then

$$\sum_{v=1}^{r} n_v = \text{rank } T^w \leq \text{rank } T \leq \text{rank } W - 1 \leq (\text{rank } S_X - 2) - 1 \leq 20 - 3 = 17\,.$$

Let $\{t_k^v\}$ be a $\boldsymbol{Z}$-basis for T_v and let $\{\tau_k^v\}$ be the associated dual basis. Now, for each $v \leq r$, there exists j_v such that $b(w, t_{j_v}^v) = 1$ and $b(w, t_{j'}^v) = 0$ for all $j' \neq j_v$. Then

$$\varphi(w) = w - \sum_{v=1}^{r} \tau_{j_v}^v\,.$$

Moreover, $\tau_{j_v}^v = \sum_k b(\tau_{j_v}^v, \tau_k^v) t_k^v$. Then $b(w, \tau_{j_v}^v) = \sum_k b(\tau_{j_v}^v, \tau_k^v) b(w, t_k^v) = (\tau_{j_v}^v)^2$. Thus

$$\varphi(w)^2 = w^2 - 2 \sum_{v=1}^{r} b(w, \tau_{j_v}^v) + \sum_{v=1}^{r} (\tau_{j_v}^v)^2 = w^2 - \sum_{v=1}^{r} (\tau_{j_v}^v)^2\,.$$

Step 3. Suppose $T^w = \bigoplus_{v=1}^{r} A_{n_v}$. Let $\{a_k(A_{n_v})\}$ be a $\boldsymbol{Z}$-basis for A_{n_v} and let $\{\alpha_k(A_{n_v})\}$ be the associated dual basis as in §2. Then we may assume that $\tau_{j_v}^v = \alpha_{j_v}(A_{n_v})$. Therefore $(\tau_{j_v}^v)^2 = \alpha_{j_v}(A_{n_v})^2 = -j_v(n_v + 1 - j_v)/(n_v + 1)$. Then

$$\langle w, w \rangle = -\varphi(w)^2 = -w^2 - \sum_{v=1}^{r} \frac{j_v(n_v + 1 - j_v)}{n_v + 1} \in \frac{1}{l}\boldsymbol{Z}\,,$$

where l is the least common multiple of $n_v + 1$. Then $\langle w, w \rangle \geq 1/l$ because $\langle w, w \rangle > 0$. Thus

$$l = \text{L.C.M.}\{n_v + 1 \mid 1 \leq v \leq r\} > \frac{420}{11} > 38\,.$$

Since $\alpha_{j_v}(A_{n_v})^2 = \alpha_{n_v + 1 - j_v}(A_{n_v})^2$, we may suppose that $1 \leq j_v \leq [(n_v + 1)/2]$ where $[*]$ is the greatest integer not more than $*$. We now look for a triple $(\{n_v\}, \{j_v\}, -w^2)$ satisfying the following conditions:

(1) $n_1 \geq n_2 \geq \cdots \geq n_r$ and $\sum_{v=1}^{r} n_v \leq 17$.

(2) $l = \text{L.C.M.}\{n_v + 1 \mid 1 \leq v \leq r\} > 38$.

(3) $1 \leq j_v \leq [(n_v + 1)/2]$ such that $0 < -w^2 - \sum_{v=1}^{r} j_v(n_v + 1 - j_v)/(n_v + 1) \leq 11/420$, where $-w^2 \geq 4$.

REMARK. If the triple satisfies (3), then it satisfies (2), too. However, the condition (2) is useful to determine the triples.

Using computer, we have the following lemma (the program is given in the appendix).

LEMMA 4.6. (i) *There are* 337 *types of* $T^w = \bigoplus A_{n_v}$ *satisfying* (1) *and* (2). (*In particular* $2 \leq r \leq 11$.)

(ii) *The pairs* $(T^w = \bigoplus A_{n_v}, \varphi(w))$ *satisfying* (1), (2) *and* (3) *are classified in Table* 2, *where* $\alpha_k^{(m)} := \alpha_k(A_m)$ *and* $\alpha_k^{(m,m')} := \alpha_k(A_m^{(m')})$ ($A_m^{(m')}$ *is a copy of* A_m), *and* $-w^2 = 4$.

TABLE 2.

No.	$\langle w, w\rangle$	$T^w=\bigoplus_{v=1}^{r} T_v$	$\varphi(w)=w-\sum_{v=1}^{r}\tau_{j_v}^{v}$
1	1/120	$A_7\oplus A_4\oplus A_3\oplus A_2\oplus A_1$	$w-(\alpha_1^{(7)}+\alpha_2^{(4)}+\alpha_1^{(3)}+\alpha_1^{(2)}+\alpha_1^{(1)})$
2	1/105	$A_6\oplus A_5\oplus A_4\oplus A_1^{\oplus 2}$	$w-(\alpha_1^{(6)}+\alpha_2^{(5)}+\alpha_1^{(4)}+\alpha_1^{(1,1)}+\alpha_1^{(1,2)})$
3		$A_6\oplus A_4\oplus A_3\oplus A_2^{\oplus 2}$	$w-(\alpha_1^{(6)}+\alpha_1^{(4)}+\alpha_2^{(3)}+\alpha_1^{(2,1)}+\alpha_1^{(2,2)})$
4		$A_6\oplus A_4\oplus A_2^{\oplus 2}\oplus A_1^{\oplus 2}$	$w-(\alpha_1^{(6)}+\alpha_1^{(4)}+\sum_{k=1}^{2}\alpha_1^{(2,k)}+\sum_{k=1}^{2}\alpha_1^{(1,k)})$
5	1/ 84	$A_6^{\oplus 2}\oplus A_3\oplus A_2$	$w-(\alpha_3^{(6,1)}+\alpha_1^{(6,2)}+\alpha_1^{(3)}+\alpha_1^{(2)})$
6	1/ 70	$A_6^{\oplus 2}\oplus A_4\oplus A_1$	$w-(\alpha_2^{(6,1)}+\alpha_1^{(6,2)}+\alpha_2^{(4)}+\alpha_1^{(1)})$
7	1/ 63	$A_8\oplus A_6\oplus A_3$	$w-(\alpha_2^{(8)}+\alpha_2^{(6)}+\alpha_2^{(3)})$
8		$A_8\oplus A_6\oplus A_1^{\oplus 2}$	$w-(\alpha_2^{(8)}+\alpha_2^{(6)}+\alpha_1^{(1,1)}+\alpha_1^{(1,2)})$
9	1/ 60	$A_9\oplus A_5\oplus A_3$	$w-(\alpha_4^{(9)}+\alpha_1^{(5)}+\alpha_1^{(3)})$
10		$A_5\oplus A_4^{\oplus 2}\oplus A_3$	$w-(\alpha_1^{(5)}+\alpha_2^{(4,1)}+\alpha_2^{(4,2)}+\alpha_1^{(3)})$
11	1/ 56	$A_7\oplus A_6\oplus A_3\oplus A_1$	$w-(\alpha_3^{(7)}+\alpha_1^{(6)}+\alpha_1^{(3)}+\alpha_1^{(1)})$
12	1/ 55	$A_{10}\oplus A_4\oplus A_3$	$w-(\alpha_3^{(10)}+\alpha_1^{(4)}+\alpha_2^{(3)})$
13		$A_{10}\oplus A_4\oplus A_1^{\oplus 2}$	$w-(\alpha_3^{(10)}+\alpha_1^{(4)}+\alpha_1^{(1,1)}+\alpha_1^{(1,2)})$
14	2/105	$A_9\oplus A_6\oplus A_2$	$w-(\alpha_2^{(9)}+\alpha_3^{(6)}+\alpha_1^{(2)})$
15		$A_6\oplus A_4^{\oplus 2}\oplus A_2$	$w-(\alpha_3^{(6)}+\alpha_1^{(4,1)}+\alpha_1^{(4,2)}+\alpha_1^{(2)})$
16	1/ 52	$A_{12}\oplus A_3$	$w-(\alpha_6^{(12)}+\alpha_1^{(3)})$
17	3/140	$A_6\oplus A_4\oplus A_3^{\oplus 2}$	$w-(\alpha_2^{(6)}+\alpha_1^{(4)}+\alpha_2^{(3,1)}+\alpha_1^{(3,2)})$
18		$A_6\oplus A_4\oplus A_3\oplus A_1^{\oplus 2}$	$w-(\alpha_2^{(6)}+\alpha_1^{(4)}+\alpha_1^{(3)}+\alpha_1^{(1,1)}+\alpha_1^{(1,2)})$
19	1/ 44	$A_{10}\oplus A_3\oplus A_1$	$w-(\alpha_5^{(10)}+\alpha_1^{(3)}+\alpha_1^{(1)})$
20	1/ 42	$A_6^{\oplus 2}\oplus A_5$	$w-(\alpha_3^{(6,1)}+\alpha_2^{(6,2)}+\alpha_1^{(5)})$
21	1/ 40	$A_9\oplus A_7\oplus A_1$	$w-(\alpha_2^{(9)}+\alpha_3^{(7)}+\alpha_1^{(1)})$
22		$A_9\oplus A_7$	$w-(\alpha_3^{(9)}+\alpha_3^{(7)})$
23		$A_7\oplus A_4^{\oplus 2}\oplus A_1$	$w-(\alpha_3^{(7)}+\alpha_1^{(4,1)}+\alpha_1^{(4,2)}+\alpha_1^{(1)})$
24	1/ 39	$A_{12}\oplus A_3\oplus A_2$	$w-(\alpha_3^{(12)}+\alpha_2^{(3)}+\alpha_1^{(2)})$
25		$A_{12}\oplus A_2\oplus A_1^{\oplus 2}$	$w-(\alpha_3^{(12)}+\alpha_1^{(2)}+\alpha_1^{(1,1)}+\alpha_1^{(1,2)})$
26	2/ 77	$A_{10}\oplus A_6$	$w-(\alpha_4^{(10)}+\alpha_2^{(6)})$
27	11/420	$A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1$	$w-(\alpha_1^{(6)}+\alpha_2^{(4)}+\alpha_1^{(3)}+\alpha_1^{(2)}+\alpha_1^{(1)})$

TABLE 3.

No.	$\langle w, w\rangle$	$T^w=\bigoplus_{v=1}^r T_v$	$\varphi(w)=w-\sum_{v=1}^r \tau_{j_v}^v$
18.1	3/140	$A_6\oplus A_4\oplus A_3\oplus D_4$	$w-(\alpha_2^{(6)}+\alpha_1^{(4)}+\alpha_1^{(3)}+\delta_1^{(4)})$
18.2		$A_6\oplus A_4\oplus A_1\oplus D_5$	$w-(\alpha_2^{(6)}+\alpha_1^{(4)}+\alpha_1^{(3)}+\delta_1^{(5)})$
18.3		$A_6\oplus A_4\oplus D_7$	$w-(\alpha_2^{(6)}+\alpha_1^{(4)}+\delta_1^{(7)})$
19.1	1/ 44	$A_{10}\oplus D_5$	$w-(\alpha_5^{(10)}+\delta_1^{(5)})$
27.1	11/420	$A_6\oplus A_4\oplus A_2\oplus D_5$	$w-(\alpha_1^{(6)}+\alpha_2^{(4)}+\alpha_1^{(2)}+\delta_1^{(5)})$

Step 4.

PROPOSITION 4.7. *Suppose that T^w has D_n or E_n as an orthogonal component. Then Table 3 lists up the pairs $(T^w=\bigoplus_{v=1}^r T_v, \varphi(w))$ satisfying $\langle w, w\rangle=-\varphi(w)^2\leq 11/420$, where $\delta_1^{(m)}:=\delta_1$ in D_m and $-w^2=4$ (up to symmetries of A_m and D_m).*

PROOF. First, consider the case where $T^w=\bigoplus_{v=1}^{r-1} A_{n_v}\oplus E_6$ $(T_r:=E_6)$. Since $G_{E_6}=\{0, \varepsilon_2, \varepsilon_6\}$ and $\varepsilon_2^2=\varepsilon_6^2=-4/3$, we may assume that $\varphi(w)=w-\sum_{v=1}^{r-1}\alpha_{j_v}(A_{n_v})-\varepsilon_2$ $(\tau_{j_r}^r:=\varepsilon_2)$. Hence

$$\varphi(w)^2=w^2-\sum_{v=1}^{r-1}\alpha_{j_v}(A_{n_v})^2-\varepsilon_2^2=w^2-\sum_{v=1}^{r-1}\alpha_{j_v}(A_{n_v})^2+\frac{4}{3}.$$

Let

$$T':=\left(\bigoplus_{v=1}^{r-1} A_{n_v}\right)\oplus\left(\bigoplus_{k=1}^{2} A_2^{(k)}\right).$$

Consider the abstract vector w' such that $(w')^2=w^2$, $b(w', a_{j_v}(A_{n_v}))=1$ and $b(w', a_{j'}(A_{n_v}))=0$ for all v and $j'\neq j_v$, while $b(w', a_1(A_2^{(k)}))=1$ and $b(w', a_2(A_2^{(k)}))=0$ for all $k=1, 2$. Let W' be the lattice spanned by w' and T'. Thus, in W', we may assume that $T^{w'}=T'$ and

$$\varphi(w')=w'-\sum_{v=1}^{r-1}\alpha_{j_v}(A_{n_v})-\sum_{k=1}^{2}\alpha_1(A_2^{(k)}).$$

Then rank $T^{w'}<$ rank $T^w\leq 17$ and

$$\varphi(w')^2=(w')^2-\sum_{v=1}^{r-1}\alpha_{j_v}(A_{n_v})^2-\sum_{k=1}^{2}\alpha_1(A_2^{(k)})^2=w^2-\sum_{v=1}^{r-1}\alpha_{j_v}(A_{n_v})^2+\frac{4}{3}=\varphi(w)^2.$$

By the hypothesis on w, a pair $(T^{w'}, \varphi(w'))$ satisfies the conditions (1), (2) and (3) except $n_v\geq 2=\text{rank } A_2^{(k)}$ for any $v<r$. Therefore the pair is one of those in Table 2.

Note that only the lattices Nos. 3 and 4 in Table 2 have two orthogonal components A_2. Hence $T^{w'}=A_6\oplus A_4\oplus A'\oplus A_2^{\oplus 2}$ where $A'=A_3$ or $A_1^{\oplus 2}$. Thus $T^w=A_6\oplus A_4\oplus A'\oplus E_6$ and hence rank $T^w\geq 18$. Therefore there does not exist such a w in the case where $T^w=\bigoplus_{v=1}^{r-1} A_{n_v}\oplus E_6$.

TABLE 4.

T_r	$-(\tau^r_{j_r})^2$	$\tau^r_{j_r}$	$\bigoplus_k A_{n(k)}$	$\sum_k \alpha_1(A_{n(k)})$	v
E_8	0	0			
E_7	3/2	ε_7	$A_1^{\oplus 3}$	$\sum_{k=1}^{3} \alpha_1(A_1^{(k)})$	4
E_6	4/3	$\varepsilon_2, \varepsilon_6$	$A_2^{\oplus 2}$	$\alpha_1(A_2^{(1)})+\alpha_1(A_2^{(2)})$	2
$D_{2k'}$	$k'/2$	δ_1, δ_2	$A_1^{\oplus k'}$	$\sum_{k=1}^{k'} \alpha_1(A_1^{(k)})$	k'
	1	$\delta_{2k'}$	$A_1^{\oplus 2}$	$\alpha_1(A_1^{(1)})+\alpha_1(A_1^{(2)})$	$2k'-2$
$D_{2k'+1}$	$(2k'+1)/4$	δ_1, δ_2	$A_3 \oplus A_1^{\oplus k'-1}$	$\alpha_1(A_3)+\sum_{k=1}^{k'-1} \alpha_1(A_1^{(k)})$	$k'-1$
	1	$\delta_{2k'+1}$	$A_1^{\oplus 2}$	$\alpha_1(A_1^{(1)})+\alpha_1(A_1^{(2)})$	$2k'-1$

Next, assume that $T^w = \bigoplus_{v=1}^{r-1} A_{n_v} \oplus T_r$ with $T_r \neq A_{n_r}$. Then we can replace T_r by $\bigoplus_k A_{n(k)}$ as in Table 4 and find a pair $(T^{w'}, \varphi(w'))$ as in Table 2. ($v := \operatorname{rank} T^w - \operatorname{rank} T^{w'}$).

It now follows from Tables 2 and 4 that rank $T^w \geq 18$ except in the cases in Table 3.

Finally, consider the other case where $T^w = (\bigoplus_{v=1}^{r'} A_{n_v}) \oplus (\bigoplus_{v=r'+1}^{r} T_v)$ with $r-r' \geq 2$ and $T_v \neq A_{n_v}$ for all $v > r'$. As above, we can replace $\bigoplus_{v=r'+1}^{r-1} T_v$ by $\bigoplus_k A_{n(k)}$ as in Table 4 and find a lattice $T^{w'}$ as in Table 3. Since the lattices in Table 3 do not have E_p as an orthogonal component, we may assume that $T_v = D_{n_v}$ for all $v > r'$. By Table 4, $T^{w'}$ has $A_1^{\oplus 2}$ or $A_3 \oplus A_1$ as orthogonal components. However there does not exist such a lattice in Table 3. Thus we complete the proof of Proposition 4.7.

Step 5. Let W'' be the lattice spanned by w and T^w. Since $w \in W$ and $T^w \subset T \subset W$, we have $W'' \subset W$. Then $W''_{\text{root}} \subset W_{\text{root}} = T$. In this final step, we shall see that rank $W''_{\text{root}} =$ rank W'' except in the cases No. 27 in Table 2 and No. 27.1 in Table 3. This implies that w is a torsion element in the Mordell-Weil group W/T, i.e., this contradicts the assumption $\varphi(w) \neq 0$.

First, we shall see that rank $W''_{\text{root}} =$ rank W'' in the case No. 1 in Table 2. Let $z := 8\alpha_1^{(7)} + (8\alpha_2^{(4)} - \alpha_1^{(4)}) + 8\alpha_1^{(3)} + (8\alpha_1^{(2)} - \alpha_2^{(2)}) + 8\alpha_1^{(1)}$. Then $z \in T^w$. Moreover, put $y := 8w - z = 8\varphi(w) + \alpha_1^{(4)} + \alpha_2^{(2)}$. Then $y \in W''$ and $y^2 = -8^2(1/120) - 4/5 - 2/3 = -2$. Therefore $\langle y, T^w \rangle = A_7 \oplus \langle y, A_4 \oplus A_2 \rangle \oplus A_3 \oplus A_1 = A_7^{\oplus 2} \oplus A_3 \oplus A_1$. Hence

$$\operatorname{rank} W''_{\text{root}} \geq \operatorname{rank}\langle y, T^w \rangle = \operatorname{rank} W'' .$$

Thus there does not exist w in the case No. 1.

In the same way, we can deal with the other cases and we get the results in Table 5, where $y \in W''$ such that $y \notin T^w$ and $y^2 = -2$, and $x := \varphi(w)$.

From Table 5, we can see that rank $W''_{\text{root}} =$ rank W'' except in the cases Nos. 27 and 27.1. We remark that No. 27 (resp. No. 27.1) corresponds to w in Proposition 4.3 (resp. Theorem 4.1). Hence there does not exist $w \in W$ such that $\langle w, w \rangle < 11/420$ and $\varphi(w) \neq 0$, for any Jacobian fibration on any $K3$ surface. q.e.d.

TABLE 5.

No.	$\langle w, w\rangle$	T^w	y	$\langle y, T^w\rangle$
1	1/120	$A_7 \oplus A_4 \oplus A_3 \oplus A_2 \oplus A_1$	$8x+\alpha_1^{(4)}+\alpha_2^{(2)}$	$A_7^{\oplus 2} \oplus A_3 \oplus A_1$
2 3 4	1/105	$A_6 \oplus A_5 \oplus A_4 \oplus A_1^{\oplus 2}$ $A_6 \oplus A_4 \oplus A_3 \oplus A_2^{\oplus 2}$ $A_6 \oplus A_4 \oplus A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$6x+\alpha_6^{(6)}+\alpha_1^{(4)}$ $6x+\alpha_6^{(6)}+\alpha_1^{(4)}$ $6x+\alpha_6^{(6)}+\alpha_1^{(4)}$	$A_{11} \oplus A_5 \oplus A_1^{\oplus 2}$ $A_{11} \oplus A_3 \oplus A_2^{\oplus 2}$ $A_{11} \oplus A_2^{\oplus 2} \oplus A_1^{\oplus 2}$
5	1/ 84	$A_6^{\oplus 2} \oplus A_3 \oplus A_2$	$7x+\alpha_3^{(3)}+\alpha_1^{(2)}$	$A_6^{\oplus 3}$
6	1/ 70	$A_6^{\oplus 2} \oplus A_4 \oplus A_1$	$7x+\alpha_4^{(4)}+\alpha_1^{(1)}$	$A_6^{\oplus 3}$
7 8	1/ 63	$A_8 \oplus A_6 \oplus A_3$ $A_8 \oplus A_6 \oplus A_1^{\oplus 2}$	$4x+\alpha_8^{(8)}+\alpha_1^{(6)}$ $4x+\alpha_8^{(8)}+\alpha_1^{(6)}$	$A_{15} \oplus A_3$ $A_{15} \oplus A_1^{\oplus 2}$
9 10	1/ 60	$A_9 \oplus A_5 \oplus A_3$ $A_5 \oplus A_4^{\oplus 2} \oplus A_3$	$5x+\alpha_5^{(5)}+\alpha_1^{(3)}$ $5x+\alpha_5^{(5)}+\alpha_1^{(3)}$	$A_9^{\oplus 2}$ $A_9 \oplus A_4^{\oplus 2}$
11	1/ 56	$A_7 \oplus A_6 \oplus A_3 \oplus A_1$	$8x+\alpha_1^{(6)}$	$A_7^{\oplus 2} \oplus A_3 \oplus A_1$
12 13	1/ 55	$A_{10} \oplus A_4 \oplus A_3$ $A_{10} \oplus A_4 \oplus A_1^{\oplus 2}$	$4x+\alpha_1^{(10)}+\alpha_4^{(4)}$ $4x+\alpha_1^{(10)}+\alpha_4^{(4)}$	$A_{15} \oplus A_3$ $A_{15} \oplus A_1^{\oplus 2}$
14 15	2/105	$A_9 \oplus A_6 \oplus A_2$ $A_6 \oplus A_4^{\oplus 2} \oplus A_2$	$5x+\alpha_1^{(6)}+\alpha_2^{(2)}$ $5x+\alpha_1^{(6)}+\alpha_2^{(2)}$	$A_9^{\oplus 2}$ $A_9 \oplus A_4^{\oplus 2}$
16	1/ 52	$A_{12} \oplus A_3$	$2x+\alpha_{12}^{(12)}+\alpha_2^{(3)}$	D_{16}
17 18 18.1 18.2 18.3	3/140	$A_6 \oplus A_4 \oplus A_3^{\oplus 2}$ $A_6 \oplus A_4 \oplus A_3 \oplus A_1^{\oplus 2}$ $A_6 \oplus A_4 \oplus A_3 \oplus D_4$ $A_6 \oplus A_4 \oplus A_1 \oplus D_5$ $A_6 \oplus A_4 \oplus D_7$	$4x+\alpha_1^{(6)}+\alpha_4^{(4)}$ $4x+\alpha_1^{(6)}+\alpha_4^{(4)}$ $4x+\alpha_1^{(6)}+\alpha_4^{(4)}$ $4x+\alpha_1^{(6)}+\alpha_4^{(4)}$ $4x+\alpha_1^{(6)}+\alpha_4^{(4)}$	$A_{11} \oplus A_3^{\oplus 2}$ $A_{11} \oplus A_3 \oplus A_1^{\oplus 2}$ $A_{11} \oplus A_3 \oplus D_4$ $A_{11} \oplus A_1 \oplus D_5$ $A_{11} \oplus D_7$
19 19.1	1/ 44	$A_{10} \oplus A_3 \oplus A_1$ $A_{10} \oplus D_5$	$2x+\alpha_{10}^{(10)}+\alpha_2^{(3)}$ $2x+\alpha_{10}^{(10)}+\delta_5^{(5)}$	$D_{14} \oplus A_1$ D_{16}
20	1/ 42	$A_6^{\oplus 2} \oplus A_5$	$7x+\alpha_1^{(5)}$	$A_6^{\oplus 3}$
21 22 23	1/ 40	$A_9 \oplus A_7 \oplus A_1$ $A_9 \oplus A_7$ $A_7 \oplus A_4^{\oplus 2} \oplus A_1$	$5x+\alpha_7^{(7)}+\alpha_1^{(1)}$ $3x+\alpha_9^{(9)}+\alpha_1^{(7)}$ $5x+\alpha_7^{(7)}+\alpha_1^{(1)}$	$A_9^{\oplus 2}$ A_{17} $A_9 \oplus A_4^{\oplus 2}$
24 25	1/ 39	$A_{12} \oplus A_3 \oplus A_2$ $A_{12} \oplus A_2 \oplus A_1^{\oplus 2}$	$4x+\alpha_{12}^{(12)}+\alpha_2^{(2)}$ $4x+\alpha_{12}^{(12)}+\alpha_2^{(2)}$	$A_{15} \oplus A_3$ $A_{15} \oplus A_1^{\oplus 2}$
26	2/ 77	$A_{10} \oplus A_6$	$3x+\alpha_1^{(10)}+\alpha_6^{(6)}$	A_{17}
27 27.1	11/420	$A_6 \oplus A_4 \oplus A_3 \oplus A_2 \oplus A_1$ $A_6 \oplus A_4 \oplus A_2 \oplus D_5$		

5. Remark. Here, we classify all possible types of singular fibres of the Jacobian fibrations Φ with $\mu(\Phi)=11/420$.

Let X and Y be $K3$ surfaces with

$$T_X=\begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix} \quad \text{and} \quad T_Y=\begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}\oplus(-2)=H\oplus(22)$$

as in §4.1. For a Jacobian fibration Φ with $\mu(\Phi)=11/420$, put

$$W_X:=U^{\perp} \text{ in } S_X \quad \text{and} \quad W_Y:=U^{\perp} \text{ in } S_Y\,.$$

Then $(W_X)_{\text{root}}=D_5\oplus A_6\oplus A_4\oplus A_2$ and $(W_Y)_{\text{root}}=A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1$.

COROLLARY 5.1. *Let Z be a $K3$ surface on which there exists a Jacobian fibration Φ with $\mu(\Phi)=11/420$.*

(i) *The Picard number $\rho=\operatorname{rank} S_Z$ of Z is at least* 19.

(ii) *If $\rho=19$, then the type of singular fibres is $A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1$. In particular, $S_Z=H\oplus W_Y$.*

(iii) *If Z is a singular $K3$ surface ($\rho=20$), then there exist three types of singular fibres $D_5\oplus A_6\oplus A_4\oplus A_2$, $A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1^{\oplus 2}$ and $A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1$.*

PROOF. For such a Φ, put $W:=U^{\perp}$ in S_Z, i.e., $S_Z=U\oplus W$.

(i) From the classification of T^w in §4.2, $W\supset W_X$ or W_Y. Hence $\rho=2+\operatorname{rank} W\geq 19$.

(ii) If $\rho=19$, then W is an overlattice of W_Y. Since $\det W_Y=\det T_Y=22$ is square-free, by Lemma 2.1, $W=W_Y$. Thus $S_Z=U\oplus W_Y\cong H\oplus W_Y$ (Lemma 3.2).

(iii) Suppose that Z is a singular $K3$ surface. Since $\operatorname{rank} T\leq 17$, the type of singular fibres of such a Φ is isometric to $(W_X)_{\text{root}}$, $(W_Y)_{\text{root}}$ or $(W_Y)_{\text{root}}\oplus A_1$. By Theorem 4.1, there exists such a Φ with the first type. The following lemma shows that there exists such a Φ with the other types.

LEMMA 5.2. *Let Z' be a singular $K3$ surface with $T_{Z'}=(22)\oplus(2n)$ (n is a positive integer). Then $S_{Z'}=H\oplus W_Y\oplus(-2n)$. In particular, there exists a Jacobian fibration with $\mu(\Phi)=11/420$ (the type of its singular fibres is $A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1^{\oplus 2}$ for $n=1$ or $A_6\oplus A_4\oplus A_3\oplus A_2\oplus A_1$ otherwise).*

PROOF. By Lemma 2.4, $q_{W_Y}=q_{S_Y}=-q_{T_Y}=-(1/22)$. Then

$$q_{T_{Z'}}=\left(\frac{1}{22}\right)\oplus\left(\frac{1}{2n}\right)=-q_{W_Y\oplus(-2n)}=-q_{H\oplus W_Y\oplus(-2n)}\,.$$

It follows from [7, Theorem 1 in Appendix to §6] that any primitive embeddings of $T_{Z'}$ into $H^{\oplus 3}\oplus E_8^{\oplus 2}$ are isometric. Therefore, by Theorem 2.5,

$$S_{Z'}=T_{Z'}^{\perp} \text{ in } H^{\oplus 3}\oplus E_8^{\oplus 2}\cong H\oplus W_Y\oplus(-2n)\,.$$

By [5], there exists a Jacobian fibration whose W is isometric to $W_Y\oplus(-2n)$. q.e.d.

Appendix. To prove Lemma 4.6, we used the computer language UBASIC (ver. 8.41 for NEC Personal Computers PC9801 by Yuji Kida). Here is the program we used for Lemma 4.6 (ii).

```
10 T=17
20 dim N(T),R(T),L(T),H(T),I(T),S(T)
30 clr V,No
40 N(0)=T
50 L(0)=1
60                                   *NextN1
70 V=V+1
80 N(V)=N(0)
90                                   *NextN2
100 R(V)=R(V-1)+N(V)
110 L(V)=lcm(L(V-1),N(V)+1)
120 H(V)=int((N(V)+1)/2)
130 N(0)=min(N(V),T-R(V))
140 if N(0)>0 goto *NextN1
150                                  *NextN3
160 if L(V)<38 goto *NextN4
170 clr J
180 I(0)=H(1)
190                                  *NextI1
200 J=J+1
210 if N(J-1)=N(J) then H(J)=I(J-1)
220 I(J)=H(J)
230                                  *NextI2
240 S(J)=S(J-1)+(I(J)*(N(J)+1-I(J)))//(N(J)+1)
250 if J<V goto *NextI1
260 W=int(S(V))+2-int(S(V))@2
270 M=W-S(V)
280 if M>1//38 goto *NextI3
290 No=No+1
300 if V>6 then print M;V:goto *NextI3
310 D=M
320 for P=1 to V
330 D=D*(N(P)+1)
340 next
350 for P=1 to V
360 lprint using(3),N(P);
370 next
380 lprint spc(18-3*V);using(4),M;spc(5)
390 for P=1 to V
400 lprint using(3),I(P);
410 next
420 lprint spc(23-3*V);using(4),R(V);W;D
430                                  *NextI3
440 I(J)=I(J)-1
450 if I(J)>0 goto *NextI2
460 J=J-1
470 if J>0 goto *NextI3
480                                  *NextN4
490 N(V)=N(V)-1
500 if N(V)>0 goto *NextN2
510 if V>1 then V=V-1:goto *NextN3
520 print No
530 end
```

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