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William H. Meeks, Harold Rosenberg

Institutions: University of Massachusetts Amherst, Institut de Mathématiques de Jussieu

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The minimal lamination closure theorem.

William H. Meeks, III* Harold Rosenberg

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Abstract

We prove that the closure of a complete embedded minimal surface M in a Riemannian three-manifold N has the structure of a minimal lamination, when M has positive injectivity radius. When N is $\mathbb{R}^3$, we prove that such a surface M is properly embedded. Since a complete embedded minimal surface of finite topology in $\mathbb{R}^3$ has positive injectivity radius, the previous theorem implies a recent theorem of Colding and Minicozzi: A complete embedded minimal surface of finite topology in $\mathbb{R}^3$ is proper. More generally, we prove that if M is a complete embedded minimal surface of finite topology and N has nonpositive sectional curvature (or is the Riemannian product of a homogeneously regular Riemannian surface with $\mathbb{R}$), then the closure of M has the structure of a minimal lamination.

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1 Introduction.

The reader not familiar with the subject of minimal laminations should think about a geodesic on a Riemannian surface. If the geodesic is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination of the surface. When this geodesic has no accumulation points, then it is proper. Otherwise, there pass complete embedded geodesics through the accumulation points forming the leaves of the geodesic lamination of the surface. The similar result is true for a complete embedded minimal surface of locally bounded curvature (curvature is bounded in compact extrinsic balls) in a Riemannian three-manifold [18]. Our main theorem below replaces this bounded curvature hypothesis

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on the minimal surface by the hypothesis of positive injectivity radius. Also, see Theorem 4 in section 2 for a useful generalization of the following main theorem of this paper.

Theorem 1 (Minimal Lamination Closure Theorem). *If M is a complete embedded minimal surface with positive injectivity radius in a Riemannian three-manifold N, then the closure $\overline{M}$ of M has the structure of a $C^{1,\alpha}$-minimal lamination of N.*

In the case $N = \mathbb{R}^3$, we have the following stronger result. This result is a consequence of Theorem 1 and its proof (assuming Theorem 4) appears in section 2, almost immediately after the statement of Theorem 7.

Theorem 2. *A complete embedded connnected minimal surface in $\mathbb{R}^3$ with positive injectivity radius is always properly embedded.*

Theorem 2 implies Colding and Minicozzi's recent theorem that a complete embedded minimal surface of finite topology in $\mathbb{R}^3$ is properly embedded. The reason for this is that a complete embedded minimal surface with finite topology in $\mathbb{R}^3$ has positive injectivity radius, (see the short proof of this fact given immediately after the statement of Theorem 5 of section 2). The strategy of our proof of Theorem 1 is similar to the proof of some of the results in [1]. However, our starting point is positive injectivity and our theorem applies in three-dimensional manifolds. We devote section 2 to the proof of Theorem 1 and some other closely related results.

In section 3 we give some other interesting applications of the Minimal Lamination Closure Theorem which include the following results given below in Theorem 3. Some of these applications also depend on the Local Picture on the Scale of Topology Theorem from [13], whose statement is presented for the readers convenience in Theorem 14 in section 3. We remark that the manifolds N and Δ in Theorem 3 below need not be complete. We also refer the interested reader to our related recent paper [17] for the theory of properly embedded minimal surfaces in $M \times \mathbb{R}$, where M is a compact Riemannian surface.

Theorem 3. *Suppose M is a complete embedded minimal surface of finite topology in a Riemannian three-manifold N.*

1. *If N has nonpositive sectional curvature, then the closure $\overline{M}$ has the structure of a minimal lamination.*
2. *If $N = \Delta \times \mathbb{R}$ where Δ is a Riemannian surface, then $\overline{M}$ has the structure of a minimal lamination of N.*

Corollary 1. *Suppose M is a complete embedded connected minimal surface of finite topology in $\Delta \times \mathbb{R}$, where Δ is a homogeneously regular Riemannian surface of nonnegative curvature. If M is not properly embedded in $\Delta \times \mathbb{R}$, then Δ is a flat torus and M is a totally geodesic submanifold. (Also, see Theorem 15)*

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2 Proof of the minimal lamination closure theorem.

In this section we prove a general result concerning the limit set of certain complete embedded minimal surfaces in Riemannian three-manifolds.

We recall that if $M \subset N$, then $p \in N$ is a *limit point* of M if $p \in \overline{M} - M$ or if $p \in M$ and there is a sequence of points $p_n \in M$ which converges to p in N but does not converge to p in M in the intrinsic Riemannian topology on M. We let $L(M)$ denote the set of limit points of M in N.

The following theorem is the main result of this section and contains Theorem 1 given in the Introduction.

Theorem 4. *If M is a complete embedded minimal surface of positive injectivity radius in a Riemannian three-manifold N (not necessarily complete), then the closure $\overline{M}$ of M in N has the structure of a $C^{1,\alpha}$-minimal lamination $\mathcal{L}$ with the components of M being leaves of $\mathcal{L}$. In particular, if M is connected, one of the following three statements holds:*

1. *M is properly embedded in N, (hence, the limit set $L(M)$ is empty).*

2. *The limit set $L(M) \subset \mathcal{L}$ has the structure of nonempty sublamination which is disjoint from M and M is properly embedded in the open set $N - L(M)$.*

3. *$L(M) = \mathcal{L}$ and M is not the only leaf in $\mathcal{L}$ (here we consider $\mathcal{L}$ to be a subset of N).*

4. *Furthermore, even without the hypothesis that M is connected,*

   (a) The universal covers of the leaves of $L(M)$ are stable.

   (b) If the metric of N is homogeneously regular, then there exists positive constants C and ε, depending on N and the injectivity radius of M such that the absolute Gaussian curvature of M

   in the ε-neighborhood of any limit leaf of M is less than C. In particular, in this case, the injectivity radius of the minimal lamination $\mathcal{L}$ (the leaves of $\mathcal{L}$) is also positive.

The proof of the above theorem is inspired by papers of Colding and Minicozzi [1, 3, 4, 5, 6, 2], where they study the local and global geometry of embedded minimal surfaces with fixed finite genus in Riemannian three-manifolds. Especially important in our proof of Theorem 4 will be their recent paper [1], where they prove the following theorem.

Theorem 5. *A complete connected embedded minimal surface M of finite topology in $\mathbb{R}^3$ is properly embedded.*

We now explain why this result is a simple consequence of Theorem 2. By Theorem 2, it is sufficient to observe that the injectivity radius of a complete embedded minimal surface M in $\mathbb{R}^3$ of finite topology is positive. To see this, suppose $p_n \in M$ are points where the injectivity radius function satisfies $\lim_{n\to\infty} \mathrm{Inj}_M(p_n) = 0$; in particular, the p_n diverge in M. Let $\gamma(n)$ be an embedded geodesic loop based at p_n, which is smooth except at p_n of length $2 \cdot \mathrm{Inj}_M\,(p_n)$ (note $\gamma(n)$ exists since M has nonpositive curvature). Since M has nonpositive curvature, the Gauss-Bonnet formula implies $\gamma(n)$ cannot bound a disk on M. Since M has finite topology and the $\gamma(n)$ form a divergent sequence, we can choose the $\gamma(n)$ on one annular end E of M. Then $\gamma(1)$ and $\gamma(n)$ bound a compact annulus $E(n) \subset E$. Each $E(n)$ has absolute total curvature at most 4π by the Gauss-Bonnet formula, hence the end E has finite absolute total curvature. Since E is complete, embedded and has finite total curvature, it is asymptotic to a half catenoid or the end of a plane and the injectivity radius of M restricted to E is bounded away from zero. Hence, M has positive injectivity radius. This proves Theorem 5 follows from Theorem 2.

Throughout the paper it will be crucial to distinguish between intrinsic and extrinsic balls centered at points of a surface Σ inside a Riemannian three-manifold N; given $p \in \Sigma$ and $R > 0$, we will denote by $B_\Sigma(p, R)$ (resp. $B_N(p, R)$) the closed intrinsic (resp. extrinsic) ball of center p and radius R. Furthermore:

Definition 1. $\Sigma(p, R)$ will stand for the component of $\Sigma \cap B_N(p, R)$ passing through p.

Our proof of Theorem 4 depends on proving an appropriate version of Proposition 1.1 in [1] for the case of a compact embedded minimal disk of fixed size small geodesic radius, using standard blow-up arguments. However, we feel that our arguments at key points are somewhat different from those in [1] and may illuminate the reader not only in the three-manifold setting but also in their case of minimal disks in $\mathbb{R}^3$.

For the readers convenience, we now state explicitly Proposition 1.1 in [1], which corresponds to our Theorem 6 stated immediately after it.

Proposition 1.1. There exists $\delta_1 \in (0, \frac{1}{2})$ so that if $\Sigma \subset \mathbb{R}^3$ is an embedded minimal disk, then for all intrinsic balls $B_\Sigma(x, R)$ in $\Sigma - \partial\Sigma$:

$$\Sigma(x, \delta_1 R) \subset B_\Sigma(x, \frac{R}{2}).$$

Theorem 6. *Suppose Σ is a compact embedded minimal disk in a homogeneously regular three-manifold N with injectivity radius function $I_\Sigma \colon \Sigma \to [0, \infty)$ equal to the distance*

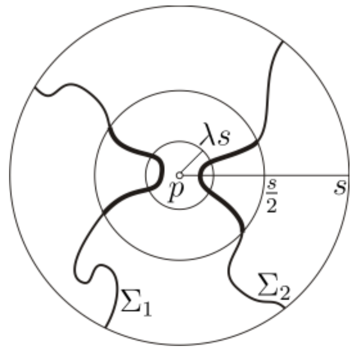

Figure 1: The portions of Σ_1, Σ_2 inside $B_N(p, s/2)$ have curvature estimates.

to the boundary function $d_\Sigma(\cdot, \partial\Sigma)$. There exist numbers $\delta \in (0, \frac{1}{2})$ and $R_0 > 0$, both depending only on N, such that if $B_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ and $R \leq R_0$, then:

$$\Sigma(x, \delta R) \subset B_\Sigma(x, \frac{R}{2}).$$

Furthermore, $\Sigma(x, \delta R)$ is a compact embedded minimal disk in $B_N(x, \delta R)$ with $\partial\Sigma(x, \delta R) \subset \partial B_N(x, \delta R)$.

Notice that $\delta < \frac{1}{2}$ is a natural assumption in the above definition, since the intrinsic distance dominates the extrinsic distance.

Before proving Theorem 6, we discuss some of its consequences including the proofs of Theorem 2 and Theorem 4.

An immediate consequence of the curvature estimates of Colding and Minicozzi in [6] is the following theorem. We remark that these curvature estimates in a homogeneously regular three-manifold can be expressed in terms of the intrinsic or extrinsic curvature by the Gauss equation.

Theorem 7 (Colding, Minicozzi). *Let N be a homogeneously regular three-manifold. There exist constants $\varepsilon > 0, c > 0$ and $\lambda \in (0, 1)$, depending only upon N, such that if Σ_1 and Σ_2 are disjoint embedded minimal disks in $B_N(p, s), s \leq \varepsilon$, with $\partial\Sigma_i \subset \partial B_N(p, s)$, and if each Σ_i meets $B_N(p, \lambda s)$, then each component of $\Sigma_i \cap B_N(p, s/2)$ which intersects $B_N(p, \lambda s)$ satisfies $|A_i|^2 \leq c/s^2$ for $i = 1, 2$. Here, A_i is the second fundamental form of Σ_i, see Figure 1.*

Remark 1. Colding and Minicozzi's idea to prove Theorem 7 is to reduce it to their one-sided curvature estimates in [6] as follows. Using Σ_1 and Σ_2 as barriers, one constructs a stable minimal disk F between Σ_1 and Σ_2 with $F \subset B_N(p, s)$, $\partial F \subset \partial B_N(p, s)$. The surface F is stable and is as close to p as Σ_1 and Σ_2. Then the curvature bounds for the stable F allow one to apply the one-sided curvature estimates in [6].

We now apply Theorem 4 to prove Theorem 2.

Proof of Theorem 2. Theorem 2 states that any complete embedded minimal surface $M \subset \mathbb{R}^3$ with positive injectivity radius is properly embedded. If this were not the case for M, then either Theorem 1 or Theorem 4 shows that the closure of M has the structure of a minimal lamination of $\mathbb{R}^3$ with some limit leaf L and the last statement of Theorem 4 shows that M has bounded curvature in some ε-neighborhood of L. Theorem 1.6 in [18] states that the limit leaves of a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ are planes and Lemma 1.3 in [18] used in its proof states that a nonflat leaf of $\mathcal{L}$ does not have bounded curvature in any ε-neighborhood of a limit leaf of $\mathcal{L}$. Therefore, M is proper in $\mathbb{R}^3$ which proves Theorem 2. $\square$

We next apply Theorems 6 and 7 to prove Theorem 4.

Proof of Theorem 4. Suppose M is a complete embedded minimal surface satisfying the positive injectivity radius hypothesis of Theorem 4 in a Riemannian three-manifold N. By the proof of Lemma 1.1 in [18], the closure $\overline{M}$ of M will be a minimal lamination of N if M has *locally bounded curvature* in the following sense: for every $p \in N$, there exists a neighborhood of p in N where M has bounded curvature. So it remains to prove that M has locally bounded curvature.

If M does not have locally bounded curvature in N, then there exists a point $p \in N$ such that in arbitrarily small extrinsic balls centered at p, M does not have bounded curvature. Embed a small 2ε-neighborhood $B_N(p, 2\varepsilon)$ of p isometrically into a homogeneously regular three-manifold $\widetilde{N}$. We will apply Theorem 6 in $\widetilde{N}$. Also choose ε smaller than the injectivity radius of M. By assumption, there exists a sequence $p_n \in M$ diverging in M, $p_n \to p$ in $\widetilde{N}$, where p_n has absolute curvature at least n and such that the intrinsic geodesic disks $B_M(p_n, \varepsilon)$ of radius ε centered at the points p_n lie in $B_N(p, 2\varepsilon)$ and are pairwise disjoint (these $B_M(p_n, \varepsilon)$ are topologically disks since ε has been chosen smaller than the injectivity radius of M, and can be assumed pairwise disjoint because p is a limit point of M). Also, we will assume that ε is sufficiently small so that the geodesic spheres in $\widetilde{N}$ centered at p of radius less than or equal to ε are mean convex. This assumption guarantees that if $r < \varepsilon$ and D is a compact minimal disk in $B_{\widetilde{N}}(p, r)$ and its boundary is contained outside of $B_{\widetilde{N}}(p, \frac{1}{4}r)$, then D intersected with the interior of $B_{\widetilde{N}}(p, \frac{1}{4}r)$ consists of disk components (or is empty) with their boundaries in $\partial B_{\widetilde{N}}(p, \frac{1}{4}r)$.

By Theorem 6 (the disks $B_M(p_n, \varepsilon)$ satisfy the injectivity radius function hypothesis of Theorem 6), there exist a $\delta \in (0, \frac{1}{2})$ and an $R_0 > 0$ (which we may assume is less than ε), so that for $R \leq R_0$, the component $M(p_n, \delta R)$ of $M \cap B_{\widetilde{N}}(p_n, \delta R)$ that contains p_n is a compact disk, whose boundary is on $\partial B_{\widetilde{N}}(p_n, \delta R)$. Hence for n large, $B_{\widetilde{N}}(p, \frac{\delta}{4}R) \cap M(p_n, \delta R)$ contains a disk containing p_n, whose boundary is on the boundary of $B_{\widetilde{N}}(p, \frac{\delta}{4}R)$, see Figure 2. This yields an infinite number of disjoint embedded minimal disks in this ball, with boundaries on the boundary of the ball, and, since $p_n \to p$, one obtains a

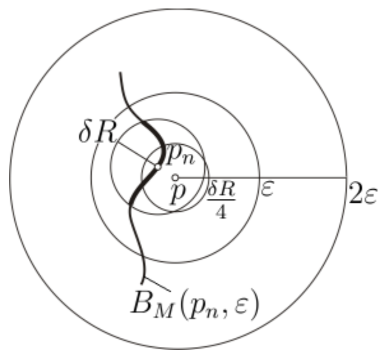

Figure 2: The thickest line represents the component $M(p_n, \delta R)$.

contradiction to the curvature estimates given in Theorem 7. This completes the proof of the fact that $\mathcal{L} = \overline{M}$ is a minimal lamination, assuming Theorem 6.

We now verify points 1, 2, 3 in the statement of Theorem 4 in the case M is connected. Note that if M is properly embedded, then, by definition of proper, $L(M) = \emptyset$ and $M = \overline{M}$, which is the possibility given in point 1. Suppose now that M is not properly embedded and let $L(M)$ be the nonempty limit set of M. Suppose $p \in L(M)$ and $\{p_n\}_n$ is a divergent sequence of points in M that converge to p in N. Since the curvature of M is bounded in a small neighborhood of p, for some small $\varepsilon > 0$, we may assume $\{B_M(p_n, \varepsilon)\}_n$ is a collection of disjoint disks that converge to a disk $D_p \subset L(M)$. Now it is clear (by analytic continuation, i.e., the holonomy of the lamination) that a compact arc in L starting at p can be lifted into the leaf of M through p_n, for n large. Thus, each point of L is a limit point of M. Also, by the same reasoning, $\overline{L} \subset L(M)$. Hence, $L(M)$ is a sublamination of $\mathcal{L}$.

If $M \cap L(M) \neq \emptyset$, then, by connectedness, M is a leaf of $L(M)$, and so, M is contained in the closed set $L(M)$ of N (where we consider $L(M)$ to be a subset of N with the structure of a lamination). Hence, if $M \cap L(M) \neq \emptyset$, then $M \subset L(M)$, and so, $L(M) = \mathcal{L}$ (both considered to be laminations). By the definitions of limit points and properness, if $M \cap L(M) = \emptyset$ and $L(M) \neq \emptyset$, then M is properly embedded in the open set $N - L(M)$. This completes the proof of points 1, 2, 3.

We now prove the statement 4(a) at the end of Theorem 4. Let L be a limit leaf of $\mathcal{L}$. By statement 1 of Lemma 18 in the Appendix, if $\pi\colon \widetilde{L} \to L$ is the universal cover of L, then $\widetilde{L}$ is stable. Assume now that N is homogeneously regular (and in particular N is complete) and we will prove statement 4(b). It follows from Schoen's curvature estimates for stable minimal surfaces immersed in a homogeneously regular three-manifold, that $\widetilde{L}$ (and hence L) has bounded curvature with the bound being independent of the limit leaf L. It follows that there is a lower bound on the injectivity radius of the leaves of $\mathcal{L}$, which consists of components of M and leaves of $L(M)$. Now by Theorem 6 and Theorem 7, we obtain a uniform estimate C on the absolute curvature of M in some fixed ε-neighborhood of $L(M)$. This completes the proof of the last statement of Theorem 4, and so, the theorem now follows. □

The proof of Theorem 6 will be carried out in the following subsections.

2.1 A chord arc property for embedded minimal disks with boundary on the boundary of a ball.

We now assume that N is a homogeneously regular three-manifold. Note that after scaling the metric of N by a fixed large constant, we may assume:

1. The injectivity radius of N is at least 10 at every point.

2. The geodesic spheres of radius less than or equal to 10 are mean convex.

3. Conditions 1 and 2 give rise to the following *convex hull property* for compact minimal surfaces Σ: If Σ is a compact minimal surface in $B_N(p, 10)$ whose boundary lies in $B_N(p, \varepsilon)$, $\varepsilon \leq 10$, then $\Sigma \subset B_N(p, \varepsilon)$. Furthermore, if also $\eta \in (0, \varepsilon)$ and Σ is a disk whose boundary lies outside of $B_N(p, \eta)$, then the interior of $B_N(p, \eta)$ intersects Σ in open disk components (possibly empty) with their boundaries in $\partial B_N(p, \eta)$.

The next proposition corresponds to the similar Proposition 2.1 in [1]. We remark that their Proposition 2.1 is analogous to the earlier stated Proposition 1.1; there is an additional boundary hypothesis in Proposition 2.1, as in our Proposition 8 below.

Proposition 8. *Let $\Sigma \subset N$ be a compact embedded minimal disk. There exists a $\delta_2 \in (0, \frac{1}{2})$ independent of Σ such that if $x \in \Sigma$, $\Sigma \subset B_N(x, R)$ with $R \leq 1$ and $\partial\Sigma \subset \partial B_N(x, R)$, then $\Sigma(x, \delta_2 R) \subset B_\Sigma(x, \frac{R}{2})$.*

Proof. We will give a variant of the proof of Proposition 2.1 in [1]. Our proof uses Meeks' Lamination Metric Theorem (Theorem 2 in [11]) to shorten and clarify Colding and Minicozzi's argument.

We will give a proof by contradiction. Suppose there is no such universal δ_2. Then, for each $n \in \mathbb{N}$, we can find an embedded minimal disk $\Sigma(n) \subset B_N(x_n, R_n)$, with $x_n \in \Sigma(n)$ and $R_n \leq 1$, such that $\partial\Sigma(n) \subset \partial B_N(x_n, R_n)$ but $\Sigma(n)(x_n, \frac{R_n}{n})$ is not contained in $B_{\Sigma(n)}(x_n, \frac{R_n}{2})$. Multiplying the metric of $B_N(x_n, R_n)$ by $\frac{n}{R_n}$, we obtain new balls $B(n)$ of radius n, which, when we view in geodesic coordinates centered at the corresponding origins $\widetilde{x}_n$ (which we identify with the origin in $\mathbb{R}^3$), are arbitrarily close for n large enough to balls of radius n in $\mathbb{R}^3$ centered at the origin $\vec{0}$. Let $\widetilde{\Sigma}(n)$ be the associated minimal surfaces in $B(n)$, see Figure 3. Note that by assumption, $\widetilde{\Sigma}(n)(\vec{0}, 1)$ is not contained in $B_{\widetilde{\Sigma}(n)}(\vec{0}, \frac{n}{2})$, which implies that the diameters of the $\widetilde{\Sigma}(n)(\vec{0}, 1)$ are unbounded; we

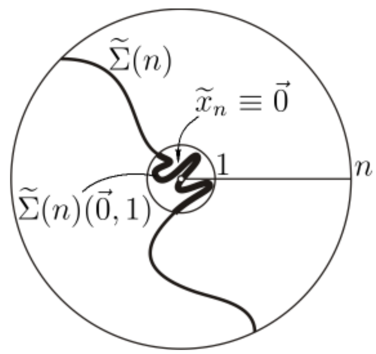

Figure 3: The component $\widetilde{\Sigma}(n)(\vec{0},1)$ inside the rescaled surface $\widetilde{\Sigma}(n)$.

will obtain a contradiction by proving the surfaces $\widetilde{\Sigma}(n)(\vec{0},1)$ have uniformly bounded diameter.

Assume for the moment that for all $R > 0$, the Gaussian curvatures of the surfaces $\widetilde{\Sigma}(n) \cap B_{\mathbb{R}^3}(\vec{0}, R)$ are uniformly bounded. It is a standard result that a subsequence of these surfaces converges to a minimal lamination of $\mathbb{R}^3$ with empty singular set of convergence. More precisely, given any sequence of points $p_n \in \widetilde{\Sigma}(n)$ that converges to a point $p \in \mathbb{R}^3$ (limit geodesic coordinates), then there exists an $\varepsilon > 0$ depending on the uniform local curvature bound of the $\widetilde{\Sigma}(n)$ around p, such that the intrinsic ball $B_{\widetilde{\Sigma}(n)}(p_n, \varepsilon) \subset \widetilde{\Sigma}(n)$ is a graph of uniformly bounded gradient over the tangent plane $T_{p_n}\widetilde{\Sigma}(n)$. A subsequence of these graphs converges to a minimal disk $G(p)$ of geodesic radius ε centered at p which is also a graph over its projection to the tangent space at p. Furthermore, $G(p)$ satisfies the same uniform curvature estimates as the $\widetilde{\Sigma}(n)$. A standard diagonal argument coupled with the proof of Lemma 1.1 on limit laminations in [18] implies that a subsequence of the $\widetilde{\Sigma}(n)$ converges to a $C^{1,\alpha}$ minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with empty singular set of convergence.

Let L be the leaf of $\mathcal{L}$ passing through $\vec{0}$. First suppose L is proper in $\mathbb{R}^3$. If the convergence of the $\widetilde{\Sigma}(n)$ to L has multiplicity one, then we can lift the component $L(\vec{0}, 2)$ of $L \cap B_{\mathbb{R}^3}(\vec{0}, 2)$ passing through the origin to a compact domain inside $\widetilde{\Sigma}(n)$ for n large and change distances arbitrarily small amounts. In particular, the diameter of $\widetilde{\Sigma}(n)(\vec{0},1)$ is less than twice the diameter of $L(\vec{0}, 2)$, which gives a contradiction in this case. Hence, the convergence of the $\widetilde{\Sigma}(n)$ to L has higher multiplicity, and so, by Lemma 18, the universal cover of L is stable complete and orientable in $\mathbb{R}^3$; hence, the universal cover of L is flat, which implies that L is a plane. Since $\widetilde{\Sigma}(n)(\vec{0},1)$ converges smoothly to a disk of radius 1 in L, the surfaces $\widetilde{\Sigma}(n)(\vec{0},1)$ are themselves graphs of small gradient over a plane which implies that the diameter of $\widetilde{\Sigma}(n)(\vec{0},1)$ is less than 3 for n large, which again gives a contradiction.

We now know that L cannot be proper in $\mathbb{R}^3$. Since it is a leaf of a minimal lamination, Theorem 1.6 in [13] implies that L is properly embedded in a halfspace or a slab in $\mathbb{R}^3$, it has infinite topology and unbounded Gaussian curvature. Since L is not flat and so not stable, then statement 5 of Lemma 18 in the Appendix implies the convergence of $\widetilde{\Sigma}(n)$ to L is of multiplicity one, so we can lift any simple closed curve α on L to a simple closed curve on $\widetilde{\Sigma}(n)$. This lifted curve bounds a disk $D_n \subset \widetilde{\Sigma}(n)$. By our convex hull property for the metric on $B(n)$ (before scaling, assumption 3 at the beginning of this section), the disks D_n are contained in a fixed ball of $\mathbb{R}^3$. By the isoperimetric inequality, these disks have uniformly bounded area and since they also have uniformly bounded curvature by our assumptions, then they converge to a disk bounded by α. In particular, L is simply connected, which gives a contradiction.

Therefore, we deduce that the curvatures of the surfaces $\widetilde{\Sigma}(n)$ cannot be bounded in some fixed ball in the limit $\mathbb{R}^3$. Hence, after choosing a subsequence, we may assume that there are points $p_n \in \widetilde{\Sigma}(n)$ that stay at uniformly bounded distances from $\vec{0}$ and such that the curvature of $\widetilde{\Sigma}(n)$ is becoming unbounded at p_n. Assume p_n converges to a point q. By Colding and Minicozzi's work in [2,3,4], inside balls centered at q of diverging radius in $B(n)$ (especially see Theorem 0.1 in [4]), after extracting a subsequence, the surfaces $\widetilde{\Sigma}(n)$ converge to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by planes (which after a rotation can be assumed to be horizontal), with singular set of convergence being a connected transverse Lipschitz curve $S(\mathcal{L})$.

By Meeks' $C^{1,1}$-Regularity Theorem [12], $S(\mathcal{L})$ is a vertical line in $\mathbb{R}^3$. If $S(\mathcal{L})$ is disjoint from $B_{\mathbb{R}^3}(\vec{0},1)$, then our previous estimates show that $\widetilde{\Sigma}(n)(\vec{0},1)$ is an almost horizontal disk and so the diameter of $\widetilde{\Sigma}(n)(\vec{0},1)$ is less than 3, which gives a contradiction. So suppose the line $S(\mathcal{L})$ intersects $B_{\mathbb{R}^3}(\vec{0},1)$. In this case, Meeks' Lamination Metric Theorem [11] implies that for n large, and given any two points $p, q \in \widetilde{\Sigma}(n)(\vec{0},1)$, there exists a length minimizing geodesic $\alpha(n,p,q)$ in $\widetilde{\Sigma}(n) \cap B_{\mathbb{R}^3}(\vec{0},2)$ joining p to q with length not more than $d_{\mathbb{R}^3}(p, S(\mathcal{L})) + d_{\mathbb{R}^3}(q, S(\mathcal{L})) + 2 + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$ (one shows that these geodesics converge C^1, away from possibly two points, to the union of two horizontal line segments and a vertical line segment in $S(\mathcal{L})$ as $n \to \infty$ or to some subset of such segments). In particular, the intrinsic diameter of $\widetilde{\Sigma}(n)(\vec{0},1)$ is bounded by 6 for n large, which gives the desired contradiction. This completes the proof of Proposition 8. $\square$

2.2 Expanding the scale of being δ_2 weakly chord arc.

From this point on in this section, Σ will denote a smooth compact embedded minimal disk in N such that its radius of injectivity function I_Σ is equal to the distance function to $\partial\Sigma$. This is important to keep in mind. This property forces all geodesic balls in Σ, disjoint from $\partial\Sigma$, to have geodesic coordinates and to be disks topologically.

We now define the key notion of a subdisk of Σ being δ weakly chord arc.

Definition 2. (Weakly chord arc). Let $\delta \in (0, \frac{1}{2})$. An intrinsic ball $B_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ is said to be δ *weakly chord arc* if for all $s \leq R$, the following condition holds:

$$\Sigma(x, \delta s) \subset B_\Sigma(x, \frac{s}{2}).$$

Notice that the definition that a ball $B_\Sigma(x, R)$ in Σ be δ weakly chord arc requires $B_\Sigma(x, R) \subset \Sigma - \partial\Sigma$.

Definition 3. Given $\delta \in (0, \frac{1}{2})$ and $x \in \Sigma - \partial\Sigma$, define:

$$R(x, \delta) = \sup\{R < \mathrm{dist}(x, \partial\Sigma) \mid \text{the ball } B_\Sigma(x, R) \text{ is } \delta \text{ weakly chord arc}\}.$$

Remark 9. Suppose $B_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ is δ weakly chord arc where $R \leq 10$. Notice that since $\delta \in (0, \frac{1}{2})$, then $\delta R \leq 10$ as well. Then, for all $s \leq R$, $\partial\Sigma(x, \delta s) \subset \partial B_N(x, \delta s)$, since $\Sigma(x, \delta s) \subset B_\Sigma(x, \frac{s}{2})$ and $B_\Sigma(x, \frac{s}{2}) \cap \partial\Sigma = \emptyset$. The convex hull property 3 given at the beginning of section 2.1 implies $\Sigma(x, \delta s)$ is a compact disk.

We note that our definition of the $R(x, \delta)$ function differs somewhat from the related $R_\delta(x)$ function defined in [1]. In their definition they take the supremum over all possible $R < \mathrm{dist}(x, \partial\Sigma)$ such that $\Sigma(x, \delta R) \subset B_\Sigma(x, \frac{R}{2})$. Note that $R(x, \delta) \leq R_\delta(x)$.

We now state and prove a key proposition that in certain cases allows us to prove that if a given ball $B_\Sigma(x, R)$ is δ_2 weakly chord arc, then $B_\Sigma(x, 5R)$ is also δ_2 weakly chord arc. The next result corresponds to Proposition 3.4 in [1] and, as in the previous proof of Proposition 8, we give a somewhat different approach to proving it. Here, δ_2 is the constant that was defined in Proposition 8.

Proposition 10. *There exists a constant $C_b > 5$, independent of Σ, so that if $R_0 > 0$ is such that $C_b R_0 \leq 1$ and $B_\Sigma(y, C_b R_0) \subset \Sigma - \partial\Sigma$ satisfies:*

$$\textit{every intrinsic subball } B_\Sigma(z, R_0) \subset B_\Sigma(y, C_b R_0) \textit{ is } \delta_2 \textit{ weakly chord arc},$$

then, $B_\Sigma(y, 5R_0)$ is δ_2 weakly chord arc. In particular, $R(y, \delta_2) \geq 5R_0$.

Proof. Were Proposition 10 to fail, there would exist $C_n > n$, $R_n > 0$ with $C_n R_n \leq 1$, and embedded minimal disks $\Sigma(n) \subset N$ satisfying: $B_{\Sigma(n)}(y_n, C_n R_n) \subset \Sigma(n) - \partial\Sigma(n)$, every intrinsic subball $B_{\Sigma(n)}(z, R_n) \subset B_{\Sigma(n)}(y_n, C_n R_n)$ is δ_2 weakly chord arc but $B_{\Sigma(n)}(y_n, 5R_n)$ is not δ_2 weakly chord arc. The surface $B_{\Sigma(n)}(y_n, C_n R_n)$ is a disk by our hypothesis that the injectivity radius function of $\Sigma(n)$ is equal to the intrinsic distance function to the boundary of $\Sigma(n)$.

Suppose that, after passing to a subsequence,

$$\Sigma(n)(y_n, 5R_n) \cap \partial B_{\Sigma(n)}(y_n, C_n R_n) = \emptyset \text{ for all } n,$$

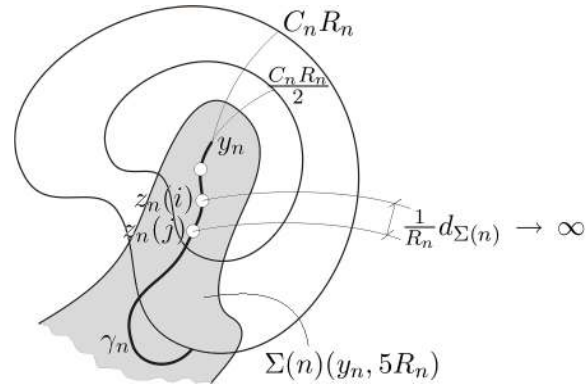

Figure 4: $d_{\Sigma(n)}$ denotes the intrinsic distance in $\Sigma(n)$.

and let us check that for all n, $B_{\Sigma(n)}(y_n, 5R_n)$ is δ_2 weakly chord arc (which is a contradiction). Since $B_{\Sigma(n)}(y_n, C_nR_n) \subset \Sigma(n) - \partial\Sigma(n)$, the above intersection equation implies $\Sigma(n)(y_n, 5R_n) \subset \Sigma(n) - \partial\Sigma(n)$. Hence, for every $s \leq 5R_n$, $\Sigma(n)(y_n, s)$ does not intersect $\partial\Sigma(n)$, and therefore, $\partial\Sigma(n)(y_n, s) \subset \partial B_N(y_n, s)$. By the convex hull property (recall that $R_n \to 0$), $\Sigma(n)(y_n, s)$ is a disk. Since $s \leq 1$, for n large enough, Proposition 8 applies to $\Sigma(n)(y_n, s)$ and $B_{\Sigma(n)}(y_n, 5R_n)$ is δ_2 weakly chord arc.

Hence, we can suppose that $\Sigma(n)(y_n, 5R_n) \cap \partial B_{\Sigma(n)}(y_n, C_nR_n) \neq \emptyset$ for all n. Since $\Sigma(n)(y_n, 5R_n)$ is connected, we can find a path $\gamma_n \subset \Sigma(n)(y_n, 5R_n)$ starting at y_n and ending at a point of $\partial B_{\Sigma(n)}(y_n, C_nR_n)$. After possibly shrinking $\Sigma(n)$, we can assume each disk $\Sigma(n)$ is contained in the ambient ball $N(n) = B_N(y_n, 2)$, since $C_nR_n \leq 1$. Scale the metric of $N(n)$ by $\frac{1}{R_n}$ to obtain $\widetilde{N}(n)$, denote the surfaces related to $\Sigma(n)$ in the new metric by $\widetilde{\Sigma}(n)$, and other related objects denote with tilde as well. Balls of radius C_nR_n then become balls of radius $C_n > n$, and extrinsic balls of radius R_n become balls of radius one. We do the scaling in geodesic coordinates with origin at y_n. The corresponding expanded path $\widetilde{\gamma}_n$ joins $\vec{0}$ with a point at intrinsic distance C_n from $\vec{0}$. Since $C_n \to \infty$, there exists a subset $\widetilde{S}_n = \{\widetilde{z}_n(1), \ldots, \widetilde{z}_n(k(n))\} \subset \widetilde{\gamma}_n \cap B_{\widetilde{\Sigma}(n)}(\vec{0}, \frac{C_n}{2})$ with

1. $k(n) \to \infty$ as $n \to \infty$.
2. The intrinsic distance in $\widetilde{\Sigma}(n)$ between any two of the points of $\widetilde{S}_n$ tends to infinity as $n \to \infty$.

In the original scale, we have corresponding finite sets $S_n \subset \gamma_n \cap B_{\Sigma(n)}(y_n, \frac{C_nR_n}{2})$, see Figure 4. We claim that for any $\widetilde{z} \in \widetilde{S}_n$, $\widetilde{\Sigma}(n)(\widetilde{z}, \delta_2)$ is a compact disk with $\partial\widetilde{\Sigma}(n)(\widetilde{z}, \delta_2) \subset \partial B_{\widetilde{N}(n)}(\widetilde{z}, \delta_2)$. To prove this, it suffices to check that the corresponding disk $B_{\Sigma(n)}(z, R_n)$ is δ_2 weakly chord arc. This follows from our δ_2 weakly chord arc hypothesis together with

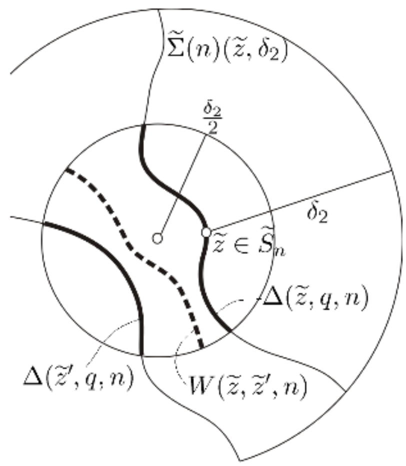

Figure 5: $W(\widetilde{z},\widetilde{z}',n)$ separates $\Delta(\widetilde{z},q,n)$ from $\Delta(\widetilde{z}',q,n)$.

the inclusion $B_{\Sigma(n)}(z,R_n)\subset B_{\Sigma(n)}(y_n,C_nR_n)$ (which in turn follows from the intrinsic triangle inequality).

By the last claim and point 2 above, the disks $\widetilde{\Sigma}(\widetilde{z},\delta_2)$ are pairwise disjoint for distinct points $\widetilde{z}$ of $\widetilde{S}_n$ for n large. Note that for any $\widetilde{z}\in\widetilde{S}_n$, $\widetilde{\Sigma}(n)(\widetilde{z},\delta_2)$ is contained in the the ambient ball $\widetilde{B}_n(6)$ of $\widetilde{N}(n)$ of radius 6 centered at the origin $\widetilde{y}_n$. This follows because given $x\in\Sigma(n)(z,\delta_2R_n)$, $d_N(x,y_n)\leq d_N(x,z)+d_N(z,y_n)<\delta_2R_n+5R_n<6R_n$ where d_N stands for the extrinsic distance in N. Also note that $\widetilde{B}_n(6)$ is arbitrarily close in the metric sense to the usual ball $B_{\mathbb{R}^3}(\vec{0},6)$. The number of disks $\widetilde{\Sigma}(n)(\widetilde{z},\delta_2)$ centered at points $\widetilde{z}\in\widetilde{S}_n$ diverges as $n\to\infty$, by condition 1 in our choice of the points. Hence, after replacing by a subsequence, there is a point $q\in B_{\mathbb{R}^3}(\vec{0},6)$ (in the fixed coordinates $B_{\mathbb{R}^3}(\vec{0},6)$ for all of the $B_{\widetilde{N}(n)}(\widetilde{y}_n,6)$) where the number of points in $B_{\widetilde{N}(n)}(q,\frac{1}{n})\cap\widetilde{S}_n$ is diverging as $n\to\infty$ (in particular, in $B_{\widetilde{N}(n)}(q,\frac{\delta_2}{2})$ as well as $n\to\infty$). The disk $\Delta(\widetilde{z},q,n)$ in $\widetilde{\Sigma}(n)(\widetilde{z},\delta_2)\cap B_{\widetilde{N}(n)}(q,\frac{\delta_2}{2})$ containing $\widetilde{z}$ is nonempty, assuming $\widetilde{z}\in\widetilde{S}_n$ is chosen close enough to q, e.g., $\widetilde{z}\in B_{\widetilde{N}(n)}(q,\frac{\delta_2}{3})$. For any pair of such disjoint such disks $\Delta(\widetilde{z},q,n),\Delta(\widetilde{z}',q,n)$, we can put a stable minimal disk $W(\widetilde{z},\widetilde{z}',n)\subset B_{\widetilde{N}(n)}(q,\frac{\delta_2}{2})$ such that $\partial W(\widetilde{z},\widetilde{z}',n)\subset\partial B_{\widetilde{N}(n)}(q,\frac{\delta_2}{2})$ and $W(\widetilde{z},\widetilde{z}',n)$ separates $\Delta(\widetilde{z},q,n)$ from $\Delta(\widetilde{z}',q,n)$, see Figure 5. A subsequence of the $W(\widetilde{z},\widetilde{z}',n)$ converges as $n\to\infty$ to a stable minimal disk $W(\widetilde{z},\widetilde{z}')\subset B_{\mathbb{R}^3}(q,\frac{\delta_2}{2})$ with $\partial W(\widetilde{z},\widetilde{z}')\subset\partial B_{\mathbb{R}^3}(q,\frac{\delta_2}{2})$ and $q\in W(\widetilde{z},\widetilde{z}')$.

Let $F(q)$ be the component of $W(\widetilde{z},\widetilde{z}')\cap B_{\mathbb{R}^3}(q,\frac{\delta_2}{4})$ that contains q. By Theorem 7, the disks $\widetilde{\Sigma}(n)(\widetilde{z}_n,\frac{\delta_2}{4})$ with $\widetilde{z}_n\in\widetilde{S}_n$ have uniform curvature estimates. Since these disks are trapped by the disks $W(\widetilde{z},\widetilde{z}',n)$ (here we take $\widetilde{z},\widetilde{z}'$ converging to q), then the disk $F(q)$ is the limit of $\widetilde{\Sigma}(n)(\widetilde{z}_n,\frac{\delta_2}{4})$ for some sequence $\widetilde{z}_n\in\widetilde{S}_n$. Furthermore, the property that defines q implies that the number of distinct disks $\widetilde{\Sigma}(n)(\widetilde{z}, \frac{\delta_2}{4})$, $\widetilde{z} \in \widetilde{S}_n$, which are converging to $F(q)$, is diverging as $n \to \infty$.

Let q_1 be a point of $\partial F(q)$. As before, q_1 is a limit of points $\widetilde{w}_n$ on $\widetilde{\Sigma}(n)(\widetilde{z}_n, \frac{\delta_2}{4})$, $\widetilde{z}_n \in \widetilde{S}_n$. We claim that the argument in the last paragraph can be applied exchanging $\widetilde{z}_n$ by $\widetilde{w}_n$. To insert stable minimal disks separating pieces of $\widetilde{\Sigma}(n)$ around $\widetilde{w}_n$ we need intrinsic neighborhoods of $\widetilde{w}_n$ to be disks with boundary in the boundary of an appropriate extrinsic ball; this holds because $B_{\widetilde{\Sigma}(n)}(\widetilde{w}_n, 1)$ is δ_2 weakly chord arc (for this last property to be true it suffices to prove that $B_{\Sigma(n)}(w_n, R_n)$ is δ_2 weakly chord arc, where $w_n \in \Sigma(n)$ is the point corresponding to $\widetilde{w}_n$ in the original scale; this in turns follows from our weakly chord arc hypothesis and from the inclusion $B_{\Sigma(n)}(w_n, R_n) \subset B_{\Sigma(n)}(y_n, C_n R_n)$). Putting new stable minimal disks that separate $\widetilde{\Sigma}(n)(\widetilde{w}_n, \frac{\delta_2}{4})$ and taking limits as before, a subsequence of the disks $\widetilde{\Sigma}(n)(\widetilde{w}_n, \frac{\delta_2}{4})$ converges to another stable minimal disk $F(q_1) \subset B_{\mathbb{R}^3}(q_1, \frac{\delta_2}{4})$ with $\partial F(q_1) \subset \partial B_{\mathbb{R}^3}(q_1, \frac{\delta_2}{4})$. This subsequence can be chosen so that $F(q_1)$ is an analytic continuation of $F(q)$.

Let $F(q, 2)$ denote the analytic continuation of $F(q)$ that we obtain by adding on $F(q_1)$ for all $q_1 \in \partial F(q)$. Each point $q_2 \in \partial F(q, 2)$ is again the limit of points $\widetilde{v}_n \in \Sigma(n)(\widetilde{w}_n, \frac{\delta_2}{4})$, so a subsequence of the disks $\widetilde{\Sigma}(n)(\widetilde{v}_n, \frac{\delta_2}{4})$ converges to a stable disk $F(q_2) \subset B_{\mathbb{R}^3}(q_2, \frac{\delta_2}{4})$ with $\partial F(q_2) \subset \partial B_{\mathbb{R}^3}(q_2, \frac{\delta_2}{4})$. Again the subsequence is chosen so that $F(q_2)$ is an analytic continuation of $F(q, 2)$ and as before we obtain a similar minimal surface $F(q, 3)$.

Continuing these analytic extensions of $F(q)$, we obtain an infinite sequence $F(q) \subset F(q, 2) \subset F(q, 3) \subset \ldots$ of compact minimal surfaces. Since $C_n \to \infty$, every $F(q, k)$ is a limit of disks in $B_{\widetilde{\Sigma}(n)}(\widetilde{y}_n, \frac{3}{4}C_n)$, for n large depending on k. By construction, $F = \bigcup_{k=1}^{\infty} F(q, k)$ is a complete embedded minimal surface in $\mathbb{R}^3$. Any compact domain on F has infinite multiplicity as a limit of disjoint compact domains in $\widetilde{\Sigma}(n)$ for n sufficiently large. Hence, statements 1 and 5 in Lemma 18 imply the universal cover of F is stable and thus, F is a plane.

Let $\widetilde{F}$ be the intersection of the plane F with the ball $B_{\mathbb{R}^3}(\vec{0}, 6)$. The flat disk $\widetilde{F}$ is the uniform limit of disks contained in $\widetilde{\Sigma}(n)$, and for n large, we can approximate $\widetilde{F}$ by at least two disjoint disks D_1, D_2 of $\widetilde{\Sigma}(n)$ with $\partial D_i \subset \partial B_{\mathbb{R}^3}(\vec{0}, 6)$.

Since we began our construction of $\widetilde{F}$ by taking limits of the $\widetilde{\Sigma}(n)(\widetilde{z}, \frac{\delta_2}{4})$, $\widetilde{z} \in B_{\widetilde{\Sigma}(n)}(\widetilde{y}_n, \frac{C_n}{2}) \cap \widetilde{S}_n$, we can assume that $D_i \cap \widetilde{S}_n \neq \emptyset$ for $i = 1, 2$ and for all n large enough ($C_n \to \infty$). But a segment of $\widetilde{\gamma}_n \subset \widetilde{\Sigma}(n)(\vec{0}, 5) \subset B_{\mathbb{R}^3}(\vec{0}, 5)$ joins a point of $D_1 \cap \widetilde{S}_n$ to a point of $D_2 \cap \widetilde{S}_n$. This is a contradiction. $\square$

2.3 The function a_δ.

For our smooth compact disk Σ in the statement of Theorem 6 and for $\delta \in (0, \frac{1}{2})$, the function

$$G(z) = \frac{d_\Sigma(z, \partial\Sigma)}{R(z, \delta)}$$

is bounded on $\Sigma - \partial\Sigma$ and is equal to 1 on a neighborhood of $\partial\Sigma$. To see this first note that if for some $\varepsilon > 0$, $p \in \Sigma - \partial\Sigma$ has distance at least ε from $\partial\Sigma$, then $R(p, \delta)$ is greater than some positive constant that only depends on ε and Σ. This is because the second fundamental form of Σ is bounded and so for $\varepsilon' < \varepsilon$ small, $B_\Sigma(p, \varepsilon')$ is a graph of small gradient over its projection on its tangent space and $\Sigma(p, \frac{\varepsilon'}{2}) \subset B_\Sigma(p, \frac{\varepsilon'}{2} + \varepsilon'')$, where $\frac{\varepsilon''}{\varepsilon'} \to 0$ as $\varepsilon' \to 0$. Since $\delta < \frac{1}{2}$, then $R(p, \delta)$ is bounded from below outside of any small ε-regular neighborhood of $\partial\Sigma$. On the other hand, since the geodesic curvature of $\partial\Sigma$ and the second fundamental form are both bounded, then the same argument shows for some sufficiently small $\varepsilon > 0$, $R(p, \delta)$ is equal to $d_\Sigma(p, \partial\Sigma)$, when $p \in \Sigma - \partial\Sigma$ and $d_\Sigma(p, \partial\Sigma) < \varepsilon$. This proves that the function G is bounded.

Definition 4. Let Σ be as in the statement of Theorem 6 and let $\delta \in (0, \frac{1}{2})$. Then we define:

$$a_\delta = \sup_{z \in \Sigma} \frac{d_\Sigma(z, \partial\Sigma)}{R(z, \delta)}.$$

Lemma 11. *Let Σ be as in the statement of Theorem 6, and let $\delta' \in (0, \frac{1}{2})$. If $a_{\delta'} < c$, $c \in [2, \infty)$, then Theorem 6 holds for Σ with $\delta = \frac{\delta'}{c}$ and for any $R_0 \leq 1$.*

Proof. Suppose $a_{\delta'} < c$, $R_0 \leq 1$, $\delta = \frac{\delta'}{c}$ and $B_\Sigma(x, R) \subset \Sigma - \partial\Sigma$, where $R \leq R_0$. By definition of $a_{\delta'}$,

$$a_{\delta'} \geq \frac{d_\Sigma(x, \partial\Sigma)}{R(x, \delta')} \geq \frac{R}{R(x, \delta')},$$

which implies that $R(x, \delta') > \frac{R}{c}$. By definition of $R(x, \delta')$, $\Sigma(x, \delta' \frac{R}{c}) \subset B_\Sigma(x, \frac{1}{2}\frac{R}{c})$. Since $\frac{R}{c} < R$, we have $\Sigma(x, \delta' \frac{R}{c}) = \Sigma(x, \frac{\delta'}{c} R) \subset B_\Sigma(x, \frac{R}{2})$.

By Remark 9, $\Sigma(x, \frac{\delta'}{c} R)$ is a compact embedded disk in $B_N(x, \frac{\delta'}{c} R)$ with $\partial\Sigma(x, \frac{\delta'}{c} R) \subset \partial B_N(x, \frac{\delta'}{c} R)$. Therefore, Theorem 6 holds for Σ with $\delta = \frac{\delta'}{c}$ and for any $R_0 \leq 1$. $\square$

2.4 Locating the smallest scale which is not δ weakly chord arc.

The proof of the next lemma uses a standard technique for finding a smallest scale R_1 for which some property holds on a minimal surface. The property we are considering here is that of being δ weakly chord arc. In this case we take the proof directly from the proof of the similar Lemma 3.39 in [1].

Lemma 12. *Let Σ be as before and let $\delta \in (0, \frac{1}{2})$. Then there exists a point $y \in \Sigma$ and a number $R_1 > 0$ so that:*

1. $a_\delta R_1 < \frac{1}{2} d_\Sigma(y, \partial\Sigma)$.
2. $R(x, \delta) > R_1$ *for every* $x \in B_\Sigma(y, a_\delta R_1)$.
3. $B_\Sigma(y, 5R_1)$ *is* <u>*not*</u> δ *weakly chord arc.*

Proof. Recall the function G on $\Sigma - \partial\Sigma$ is defined by $G(x) = d_\Sigma(x, \partial\Sigma)/R(x, \delta)$ and extends to a bounded function which has a constant value 1 near $\partial\Sigma$. Thus, $a_\delta = \sup_\Sigma G$ is a finite number. Choose y to be a point so that $G(y)$ is greater than $\frac{a_\delta}{2}$. Let $d_\partial = d_\Sigma(y, \partial\Sigma)$. We have

$$\frac{a_\delta}{2} < \frac{d_\partial}{R(y, \delta)}.$$

We now choose $R_1 = R(y, \delta)/4$ and we will show this definition of R_1 satisfies the statements in the lemma. This value of R_1 gives the inequality $a_\delta R_1 < \frac{1}{2} d_\partial$, which is statement 1 in the lemma. By definition of $R_1, R(y, \delta) = 4R_1$ and by the definition of $R(y, \delta)$ as a supremum, the ball $B_\Sigma(y, 5R_1)$ is not δ weakly chord arc, which proves statement 3.

By statement 1, $a_\delta R_1 < \frac{1}{2} d_\partial$, and so, $B_\Sigma(y, a_\delta R_1) \subset B_\Sigma(y, d_\partial/2)$. So if we check that statement 2 holds for points in $B_\Sigma(y, d_\partial/2)$, then statement 2 holds. If $x \in B_\Sigma(y, d_\partial/2)$, then by the triangle inequality, $d_\partial/2 \leq d_\Sigma(x, \partial\Sigma)$. This inequality, the definition of G and the choice of y give the inequalities

$$\frac{d_\partial}{2R(x, \delta)} \leq G(x) < 2G(y) = \frac{2d_\partial}{R(y, \delta)}.$$

Therefore, $R(x, \delta) > R(y, \delta)/4 = R_1$. This completes the proof of statement 2 and the lemma now follows. $\square$

2.5 The proof of Theorem 6: a_{δ_2} is bounded independently of Σ.

We now prove Theorem 6. From the statement of Theorem 6, we may assume that Σ is a geodesic disk of radius at most R_0, where R_0 satisfies $C_b R_0 \leq 1$ and C_b is given by Proposition 10. By Lemma 11, we just need to prove that a_δ is bounded independently of Σ for some fixed constant $\delta \in (0, \frac{1}{2})$.

Let $\delta = \delta_2$, where δ_2 is given Proposition 8. We now prove a_δ is bounded from above by C_b, where C_b is given in Proposition 10. Suppose there exists a Σ with $a_\delta > C_b$.

By the Lemma 12, there exist a point $y \in \Sigma$ and an R_1, such that:

1. $B_\Sigma(y, a_\delta R_1) \subset B_\Sigma(y, \frac{1}{2} d_\Sigma(y, \partial\Sigma))$. (Note this implies $R_1 \leq R_0$.)

2. $R(x, \delta) > R_1$ for every $x \in B_\Sigma(y, a_\delta R_1)$.

3. $B_\Sigma(y, 5R_1)$ is <u>not</u> δ weakly chord arc.

By definition of $R(x, \delta)$ and statement 2, we have that $B_\Sigma(x, R_1)$ is δ weakly chord arc for every $x \in B_\Sigma(y, C_b R_1) \subset B_\Sigma(y, a_\delta R_1)$. Since $R_1 \leq R_0$, Proposition 10 implies that $B_\Sigma(y, 5R_1)$ is δ weakly chord arc, contradicting statement 3 above. This completes the proof of Theorem 6. $\square$

REMARK 2. The reader can easily check that the minimal lamination closure theorem is true if one assumes the injectivity radius is locally bounded away from zero in the following sense: one can cover N by balls such that the injectivity radius of M is bounded away from zero in each ball.

3 Closure laminations for minimal surfaces of finite topology.

In this section we prove two theorems on when the closures of complete embedded minimal surfaces of finite topology have the structure of a minimal lamination. Our first theorem will just depend on Theorem 4. In the case N has a flat metric, Meeks, Perez and Ros [15] also proved this result.

Theorem 13. *Let N be a Riemannian three-manifold of nonpositive sectional curvature. If M is a complete embedded minimal surface of finite topology in N, then $\overline{M}$ has the structure of a minimal lamination.*

Proof. Suppose $p \in N$ is a limit point of M in N. Since a neighborhood of p embeds in a homogeneously regular manifold, Remark 2 implies that we just need to prove that the points of M that enter some neighborhood of p have injectivity radius greater than some ε. Otherwise, we can find a divergent sequence of points $p_n \in E$, where E is one of the annular ends of M, the points p_n converge to p and the injectivity radius of M at p_n is going to zero as $n \to \infty$. Since N has nonpositive sectional curvature and M is minimal, then M has nonpositive curvature. So, there exist closed embedded geodesic loops γ_n based at p_n of length $2 \cdot I_M(p_n)$ which cannot bound disks on M by the Gauss-Bonnet formula. Hence, the loops $\gamma_n \subset E$ represent generators of the fundamental group of E. In particular, the compact annulus in E bounded by $\gamma_1 \cup \gamma_n$ has nonpositive total curvature bounded from below by -4π. This means that E has finite total curvature.

However, note that the second fundamental form and the Gaussian curvature of M near p_n is becoming unbounded as $n \to \infty$. A blow-up argument on the scale of curvature (see, for example, the statement of the Local Picture Theorem on the Scale of Curvature in [13]) shows that the absolute total curvature of $B_M(p_n, 1)$ is greater than 2π for n large (since a complete minimal surface in $\mathbb{R}^3$ that is not flat has absolute total curvature at least 4π). But this implies that the absolute total curvature of E is infinite, which contradicts that E has finite total curvature. This completes the proof of Theorem 13. $\square$

For the proof of our next result we need to invoke not only Theorem 4 but also a blow-up on the scale of the injectivity radius, which is Theorem 11.1 in [13]. Since we will need its precise statement, we now give it for the readers convenience. The statement of the theorem includes the term *minimal parking garage structure on* $\mathbb{R}^3$. Roughly stated, a parking garage structure is a limit object for a sequence of embedded minimal surfaces that converge to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes with singular set of C^1-convergence being a locally finite set $S(\mathcal{L})$ of lines orthogonal to the planes of $\mathcal{L}$, along which the limiting surfaces have the local appearance of a highly sheeted double multigraph; the set of lines in $S(\mathcal{L})$ are called the columns of the parking garage structure. For example, the sequence of homothetic shrinkings $\frac{1}{n}H$ of a vertical helicoid H converges to a minimal parking garage structure of $\mathbb{R}^3$ consisting of the minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ being the x_3-axis. We note that the limit minimal parking garage structures that we will see in our application of this result arise from limits of minimal surfaces of genus zero, and so, by statement 5.4 in the theorem below, will have exactly two columns. The formal definition of minimal parking garage structure is given in [13] but some of the related language, such as columns of a parking garage structure was first suggested by Matthias Weber, based on the paper by Traizet and Weber (also see the section 7 of our paper [17] for this technique). In their paper, they use this structure to produce certain 1-parameter families of complete embedded periodic minimal surfaces in $\mathbb{R}^3$ which converge to a minimal parking garage structure of $\mathbb{R}^3$. Their examples are obtained by analytically untwisting the limit minimal parking garage structure through an application of the implicit function theorem.

Theorem 14 (Local Picture on the Scale of Topology). *Suppose M is a complete embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$ such that the following statements hold.*

1. *For all n, the component M_n of $\mathbb{B}_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact with boundary $\partial M_n \subset \partial \mathbb{B}_N(p_n, \varepsilon_n)$.*

2. *Let $\lambda_n = 1/I_{M_n}(p_n)$, where I_{M_n} denotes the injectivity radius function of M restricted to M_n. Then, $\lambda_n I_{M_n} \geq 1 - \frac{1}{n+1}$ on M_n, and $\lim_{n\to\infty} \varepsilon_n \lambda_n = \infty$.*

3. *The metric balls $\lambda_n \mathbb{B}_N(p_n, \varepsilon_n)$ of radius $\lambda_n \varepsilon_n$ converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n).*

Furthermore, one of the following three possiblities occurs.

4. *The surfaces $\lambda_n M_n$ have bounded curvature on compact subsets of $\mathbb{R}^3$ and for any $k \in \mathbb{N}$, converge C^k on compact subsets of $\mathbb{R}^3$ to a connected properly embedded nonsimply connected minimal surface M_∞ in $\mathbb{R}^3$ with $I_{M_\infty} \geq 1$ on M_∞, $\vec{0} \in M_\infty$ and $I_{M_\infty}(\vec{0}) = 1$.*

5. *The surfaces $\lambda_n M_n$ converge to a limiting minimal parking garage structure of $\mathbb{R}^3$ consisting of a foliation $\mathcal{L}$ by planes and columns $S(\mathcal{L})$, and:*

   5.1 *$S(\mathcal{L})$ contains a line L_1 orthogonal to the planes in $\mathcal{L}$ which passes through the origin.*

   5.2 *$S(\mathcal{L})$ contains a parallel line L_2 of distance 1 from L_1.*

   5.3 *All of the lines in $S(\mathcal{L})$ have distance at least 1 from each other.*

   5.4 *If there exists a bound on the genus of the surfaces $\lambda_n M_n$, then $S(\mathcal{L})$ consists of two components L_1, L_2 with associated limiting double multigraphs being oppositely handed.*

6. *The surfaces $\lambda_n M_n$ converge to a singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$, and*

   6.1 *There exists $R_0 > 0$ such that the surfaces $(\lambda_n M_n) \cap \mathbb{B}(\vec{0}, R_0)$ do not have bounded genus.*

   6.2 *The sublamination $\mathcal{P}$ of $\mathcal{L}$ consisting of planes is nonempty.*

   6.3 *The singular set $S(\mathcal{L})$ of C^1-convergence of $\{\lambda_n M_n\}_n$ to $\mathcal{L}$ is a closed set of $\mathbb{R}^3$ which is contained in $\cup_{P \in \mathcal{P}} P$.*

   6.4 *Every plane in $\mathcal{P}$ intersects $S(\mathcal{L})$ in an infinite set of points, which are at least distance 1 from each other in the plane.*

We now state our final theorem.

Theorem 15. *Let N be a Riemannian three-manifold which is a product $\Delta \times \mathbb{R}$, where Δ is a Riemannian surface. If M is a complete embedded minimal surface of finite topology in N, then:*

1. *$\overline{M}$ has the structure of a minimal lamination.*

2. *Suppose Δ is a homogeneously regular surface and has nonnegative curvature. If M is not properly embedded in N, then Δ is flat and M is totally geodesic in N.*

3. *If Δ is a compact surface of nonnegative curvature, then M has bounded curvature. Furthermore, if M is not totally geodesic in N or Δ is not a flat torus, then M is properly embedded in N and M has linear area growth. (Linear area growth means that for $t \geq 1$ and for any $a \in \mathbb{R}$, the area $A(t)$ of $M \cap (\Delta \times [a-t, a+t])$ satisfies $c_1 t \leq A(t) \leq c_2 t$, for some positive constants c_1, c_2.)*

Proof. Let M be a complete embedded minimal surface of finite topology in $N = \Delta \times \mathbb{R}$. In order to prove that the closure $\overline{M}$ of M in N has the structure of a minimal lamination $\mathcal{L}$ with M as a leaf, it is sufficient to show that M has injectivity radius locally bounded away from zero in N (see Remark 2).

Arguing by contradiction, suppose that there exists a point $p \in N = \Delta \times \mathbb{R}$ and a sequence of points $p_n \in M$ converging to p such that $I_M(p_n)$ converges to zero. Thus, we may also assume that:

1. $I_M(p_n) < \frac{1}{2n}$. In particular, there exists an embedded geodesic loop γ_n of length less than $\frac{1}{n}$ based at p_n or there exists a geodesic ray β_n of length $I_M(p_n)$ starting at p_n and ending at a conjugate point of p_n along β_n.

2. The curves γ_n lie in a fixed annular end representative E of M.

3. $\text{Length}(\gamma_{n+1}) < \frac{1}{n}\text{Length }(\gamma_n)$.

4. Furthermore, the points p_n can be chosen to be blow-up points for the scale of the injectivity radius. This just means that the points p_n can be taken to satisfy the conditions for the points p_n given in the statement of Theorem 14.

We let $h\colon N = \Delta \times \mathbb{R} \to \mathbb{R}$ denote the height function, which restricts to a harmonic function on M. We will refer to the flux of the gradient of h across a simple closed curve on M as the vertical flux of the curve.

Theorem 14 describes the geometry of the local *blow-up picture* $\widetilde{M}(n)$ of M on the scale of the injectivity radius in an intrinsic/extrinsic neighborhood of p_n for n large. Suppose the bounded curvature hypothesis in statement 4 of Theorem 14 holds for the scaled surfaces $\widetilde{M}(n)$ around p_n, then a limit M_∞ of the $\widetilde{M}(n)$ is a properly embedded nonsimply connected minimal surface in $\mathbb{R}^3$. Since outside a compact subset of M the surface has genus zero, the blown-up limit M_∞ has genus zero. By the classification results in [16] for such M_∞, on this scale, the surface M has the appearance of either a catenoid or a two limit end genus zero minimal surface near p_n.

Note that for n large, there are no geodesic rays β_n, as in statement 1 above, since a subsequence of the β_n would give rise to a limit geodesic on M_∞ of length 1 with a conjugate point. This is impossible since the curvature of M_∞ is nonpositive.

Case 1. The surface M_∞ is a catenoid.

Observe that the γ_n can not bound a disk $D(n)$ on E. For then the harmonic height function has an interior maximum or minimum on $D(n)$. Also, observe that M_∞ must have a horizontal axis, where we use coordinates on $\mathbb{R}^3$ so that the notion of horizontal is that induced by the height function h. If not, the vertical flux of the waist circle γ is not zero. Then the curve γ_n has nonzero vertical flux as well. Since all of the γ_n lie on the same annular end E, they have the same vertical flux. But the length of γ_n is an upper bound for the vertical flux and these lengths converge to zero, a contradiction.

Now, γ_n and γ_m for $m > n$, bound a compact annulus $E(n,m)$ on E and, again the height function would have an interior maximum on $E(n,m)$. This is because near each of the two boundary curves of $E(n,m)$, one sees a neighborhood of γ on the horizontal catenoid M_∞. This is a contradiction.

Case 2. The surface M_∞ is a two limit end surface of genus zero.

In this case we remark that the planar ends of M_∞ can not be horizontal. For, were this the case, one can find a horizontal plane that intersects M_∞ in a simple closed curve β with nonzero vertical flux, which is impossible from the similar argument that we gave in the case M_∞ was a catenoid. Hence, the ends of M_∞ are not horizontal. Let β be a planar simple closed curve of M_∞ separating the two limit ends. Let β_n be curves on E, blowing-up to β and note that each of the limit ends of M_∞ bounded by β has points above and below the h-values of β. Clearly the β_n do not bound disks $D(n)$ on M by the maximum principle. Thus, β_n and β_m, $m > n$, bound a compact annulus $E(n,m) \subset E$. For m, n large, there exist simple closed curves on $E(n,m)$ near β_n and near β_m which upon blowing-up, have points higher and lower than the corresponding blow-ups of β_n or β_m (the planar ends of M_∞ are not horizontal). Hence, the height function has an interior maximum on $E(n,m)$, a contradiction.

This concludes the case where the bounded curvature hypothesis in statement 4 of Theorem 14 holds for the scaled surface $\widetilde{M}(n)$ around p_n.

Now assume the curvature is not bounded. By Theorem 14, we have that the surfaces $\lambda_n M_n$ converge either to a minimal parking garage of $\mathbb{R}^3$ or to a singular minimal lamination of $\mathbb{R}^3$. The last case is not possible by item 6.1 in Theorem 14 and by the fact that M_∞ has finite topology. Thus, $\lambda_n M_n$ converge to a minimal parking garage M_∞ of $\mathbb{R}^3$, which has two oppositely oriented columns by item 5.4 of Theorem 14.

We first note that the foliation associated to M_∞ is not by horizontal planes. If not, then consider a sequence of simple closed curves $\widetilde{\beta}_n$ on the surfaces $\lambda_n M_n$ (see the statement of Theorem 14), which start at $\widetilde{p}_n = \lambda_n p_n (= \vec{0})$ on a forming multigraph corresponding to L_1 (equals column 1) and then travel along level 1 from L_1 to L_2 (equals column 2), then up column 2 to level 2, back to column 1 along level 2 and then down to the starting point $\widetilde{p}_n$ on column 1 level 1. The curves $\widetilde{\beta}_n$ can be taken to have length close to 2. Choose n large so that the height function is at most $\frac{1}{4}$ in modulus on $\widetilde{\beta}_n$. Similarly construct such curves $\widetilde{\alpha}_n$ starting near height 1 (above $\widetilde{p}_n$) with height variation less than $\frac{1}{4}$. Let α_n and β_n be curves on M blowing-up to $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$. Notice that neither α_n nor β_n bounds a disk on M by the maximum principle (each curve separates the forming parking garage into two components, each with sheets above or below the curve). The curves α_n and β_n bound an annulus $E(n)$ on E. Clearly, there is a horizontal closed curve on $E(n)$ with nonzero vertical flux (the existence of a horizontal closed curve on $E(n)$ with nonzero vertical flux follows from consideration of the values of the height function along $\alpha_n \cup \beta_n$ together with the height variation that we are assuming for the curves $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$). As before, this is impossible. Hence, the parking garage structure is not horizontal.

Now in the nonhorizontal case, we construct the curves $\widetilde{\alpha}_n$, $\widetilde{\beta}_n$ as before, which now are close to tilted planes. Again the blow-down curves α_n and β_n do not bound disks, so one has the annulus $E(n)$ with boundary components α_n, β_n. In any fixed size tubular neighborhood of β_n, there exist simple closed curves on either side of β_n which upon blowing-up, have points higher and lower than the corresponding blow-up of $\beta_n \cup \alpha_n$ (the planar ends of M_∞ are not horizontal). Hence, the height function has an interior maximum on $E(n)$, a contradiction. This contradiction completes the proof of statement 1 of the theorem.

We now prove statement 2 in the theorem. Suppose M is not proper in $N = \Delta \times \mathbb{R}$, where Δ is a homogeneously regular Riemannian surface and has nonnegative curvature. Note that Δ is complete in this case, since part of the definition of a homogeneously regular manifold states that Δ has positive injectivity radius. Since $\overline{M}$ has the structure of a minimal lamination by statement 1, we know that $\overline{M}$ has a limit leaf L. Then, by statement 1 of Lemma 17 in the Appendix, the universal cover $\widetilde{L}$ of L is a complete stable minimal surface in N. Since N has nonnegative Ricci curvature, Fischer-Colbrie and Schoen's theorem [9] says that $\widetilde{L}$, and hence, also L, is totally geodesic in N and the Ricci curvature of N vanishes along the normal to L.

If Δ has a flat metric, then a slight modification of the proof we gave of Theorem 5 in section 2 implies M has positive injectivity radius. In this case, the leaves of the lift of the nontrivial minimal lamination $\overline{M}$ to the universal cover $\mathbb{R}^3$ of N have positive injectivity radius, which implies the lifted lamination consists of flat leaves (Theorem 2). Therefore, M is flat and totally geodesic in N.

Assume now that Δ is not flat and so has some point of positive curvature. It follows that we may assume, after possibly lifting to a two-sheeted cover, that Δ is simply connected. Since L is totally geodesic in N and N is not flat, then Lemma 17 at the end of this section implies that L is $\gamma \times \mathbb{R}$ for some geodesic in Δ or $L = \Delta \times \{t_0\}$ for some $t_0 \in \mathbb{R}$.

We now prove that L can not be of the form $\Delta \times \{t_0\}$. In a fixed size closed one-sided neighborhood $N(\delta)$ of L, there is a component Σ of $M \cap N(\delta)$ with $\partial\Sigma = \Sigma \cap \partial N(\delta)$, which is a graph (one uses here that for δ small, the vertical projection $\pi : \Sigma \to L = \Delta \times \{t_0\}$ is a quasi-isometry; this follows from the curvature estimate in the last statement of Theorem 4 and the fact that the projection is injective on $\partial\Sigma$ and then one applies the topological result in Lemma 1.4 of [18]).

We now show that Σ cannot exist with a proof similar to the proof of the Halfspace Theorem in [10]. Note that the graph Σ is proper in $\Delta \times [t_0, t_0 + \delta]$ (we are assuming here that Σ lies above $\Delta \times \{t_0\}$). After a possible downward translation of Σ, we may assume that Σ limits to $\Delta \times \{t_0\}$. For some positive c less that δ, there exists a compact embedded minimal annulus A with one boundary component ∂ at height $t_0 + c$ and the other one at height t_0 and A is disjoint from Σ. After replacing δ by c, we may assume that $\delta = c$. Let X be the closure of the unbounded component of $\Delta \times [t_0, t_0 + \delta] - A$. Since the top boundary curve ∂ of A is homologous to infinity in X and ∂X is a good barrier for solving Plateau type problems in X, the curve ∂ bounds a complete properly embedded least area surface Γ. Since $\Delta \times [t_0, t_0 + \delta]$ is simply connected, then the surface Γ is orientable by separation properties. By a result of Fischer-Colbrie [8], Γ has finite total curvature and so it is parabolic. It follows that the bounded harmonic height function on Γ must have the constant value δ, and so is contained in $\Delta \times \{t_0 + \delta\}$, which is clearly impossible since Γ would then not be contained in X near $\partial\Sigma$.

An appropriate modification of the above "halfspace theorem" proof also shows that the limit leaf L can not be of the form $\gamma \times \mathbb{R}$, γ a geodesic of Δ, as follows.

Passing to a covering space, we can assume γ is diffeomorphic to $\mathbb{R}$. Consider a one-sided closed δ-normal interval bundle $N_\delta(L)$ that submerses to $\Delta \times \mathbb{R}$, with the induced metric and induced lamination. Then $N_\delta(L)$ is homeomorphic to $(\gamma \times \mathbb{R}) \times [0, \delta]$, with a flat submanifold $L = (\gamma \times \mathbb{R}) \times \{0\}$ and $L(\delta) = (\gamma \times \mathbb{R}) \times \{\delta\}$ having mean curvature vector pointing out of $N_\delta(L)$. By statement 4(b) in Theorem 4, for a small choice of δ, we may assume that $\overline{M} \cap N_\delta(L)$ has bounded curvature. Again for δ sufficiently small, we may assume that each component of $L \cap N_\delta(L)$ is a normal graph of bounded gradient over the zero section which is L. Let C be such a component which is a graph over a connected domain L_C of L. Let $L_C(\delta)$ be the part of $L(\delta)$ which is also a graph over L_C and let $L'_C(\delta) = L(\delta) - L_C(\delta)$. Then under normal projection to L, $L'_C(\delta) \cup C$ is quasi-isometric to the flat plane L. Hence, C is a parabolic Riemann surface with boundary. Since L is totally geodesic, the distance function of C to L is a bounded positive superharmonic function which has values in its interior which are smaller than its constant value δ on the boundary of the domain. This is a contradiction and completes the proof of statement 2.

We now prove statement 3. It follows from statement 2 that M and Δ have zero curvature when M is not proper in N, and so M has bounded curvature in this case. When M is properly embedded in N, then the Bounded Curvature Theorem, proved by the authors in [17], states that M has bounded curvature (the hypotheses of this theorem are that M be a properly embedded minimal surface of finite genus in the product of a closed Riemannian surface with $\mathbb{R}$). Thus, the first sentence of statement 3 is established: M has bounded curvature. We also have shown the first part of the second statement: if M is not totally geodesic in N or Δ is not a flat torus, then M is properly embedded in N.

The authors' Linear Area Growth Theorem in [17] states that a properly embedded minimal surface of bounded curvature in $\Delta \times \mathbb{R}$ (here Δ can be any closed Riemannian surface) has linear area growth. Thus, M has linear area growth. $\square$

Remark 16. The hypothesis in statement 2 in Theorem 15 that Δ is a homogeneously regular manifold of nonnegative curvature probably can be weakened to the hypothesis that Δ has nonnegative curvature. First note that the completion of a general Riemannian surface Δ product with $\mathbb{R}$ is essentially a manifold with boundary in which a limit leaf L of $\overline{M}$ should be complete and totally geodesic. Also, the property that Δ is homogeneously regular also does not play as key a role as it might appear in the proof, since L has total curvature at most 2π and so a divergent sequence of small ε-disks converges to a disk which is flat and so is homogeneously regular. These are technical issues which we avoid by assuming Δ is homogeneously regular (which implies Δ is also complete).

We also remark that Meeks, Perez and Ros [14] have conjectured that a version of Proposition 8 holds for minimal surfaces with any fixed finite genus g. If the conjecture is true, then it should follow from their work that the limit set $L(M)$ of a complete embedded finite genus minimal surface M in a Riemannian three-manifold N would be a minimal lamination whose limit leaves have the property that their universal covers are stable. In particular, if the curvature of N is positive and the conjecture holds, then a complete embedded finite genus minimal surface in N would be properly embedded (in particular, there are no complete simply connected stable minimal surfaces in such an N). This last result fails if N only has positive scalar curvature, even in the case N is topologically the three-sphere and the surface is an annulus.

Lemma 17. *Let Σ be a complete totally geodesic surface in $M \times \mathbb{R}$, where M is a connected surface which is not flat. Then Σ is of the form $\alpha \times \mathbb{R}$ where α is a geodesic in M or $\Sigma = M \times \{t\}$ for some $t \in \mathbb{R}$.*

Proof. If Σ is vertical and totally geodesic, then it is clearly of the form $\alpha \times \mathbb{R}$ for some geodesic α of M. So, assume that there is a point $q = (p, t) \in \Sigma \subset M \times \mathbb{R}$, where the tangent space is not vertical. Let v_q be a unit tangent vector to Σ at q which is orthogonal to the horizontal geodesic $h_q \subset \Sigma$ at height t. Let $\gamma_q \colon \mathbb{R} \to M \times \mathbb{R}$ be the unit speed geodesic $\gamma_q(t) = (\gamma_q^M(t), \gamma_q^{\mathbb{R}}(t))$ with $\gamma_q'(0) = v_q$. Note that $\gamma_q \subset \Sigma$.

Since Σ is totally geodesic in $M \times \mathbb{R}$, parallel translation $\eta_q(t)$ along γ_q of a unit tangent vector η_q to h_q at q is a vector field along $\gamma_q \subset \Sigma$ which stays normal to γ_q and tangent to Σ. Note that η_q is a horizontal vector at q in $T_p(M \times \mathbb{R})$ and since $\gamma_q^M(\mathbb{R}) \times \mathbb{R}$ is totally geodesic and vertical, then $\eta_q(t)$ is horizontal for all t. It follows that the set of horizontal geodesics $\{h_q \mid q \in \Sigma\}$ in Σ are orthogonal to the set of geodesics $\{\gamma_q \mid q \in \Sigma \cap (M \times \{t\})\}$ and the same is true of their projections to M. It now easily follows that M has two orthogonal foliations by geodesics which implies M is flat. This completes the proof of the lemma. $\square$

4 Appendix: Stability of the leaves in a minimal lamination.

In this section, we collect several general results on the stability of certain leaves of a minimal lamination that we apply in this paper. Many of the arguments in the following lemma appear in our earlier paper [18] on the uniqueness of the helicoid but are difficult to find and appear there only in the $\mathbb{R}^3$ setting. The $\mathbb{R}^3$ setting is simpler because its curvature is nonnegative and the only stable complete minimal surface in $\mathbb{R}^3$ is a plane (see [7, 9, 19]).

Some of the difficulty that appears in the proof of the next theorem arises from the fact that a compact minimal surface can be unstable, while its universal cover is stable. For example, consider a compact surface M of constant curvature -1, and a small warped product perturbation of the product metric on $M \times \mathbb{R}$, so that $M \times \{0\}$ is totally geodesic and the unit normal vectors to $M \times \{0\}$ have positive Ricci curvature; then this surface is unstable but the universal cover of $M \times \{0\}$ is a stable minimal surface. This example was pointed out to us by Richard Schoen. In a related problem one knows that the operator $-\Delta + aK$ (for $a = 2$, this is the the negative of the stability operator in $\mathbb{R}^3$) on any of the Scherk doubly-periodic minimal surfaces in $\mathbb{R}^3$ is not positive for any $a \in (0, \infty)$ but for any sufficiently small $a \in (0, 1)$, the operator $-\Delta + aK$ is positive when lifted to its universal cover; here, K denotes the Gaussian curvature function (see section 10.3 of [13] for details).

Before stating and proving the next lemma, we make a few comments concerning the stability of a minimal surface Σ in a three-manifold N and the holonomy group G_L of a leaf L of a minimal lamination $\mathcal{L}$ of N. First note that stability of Σ is defined in terms of stability of all smooth compact subdomains $\Delta \subset \Sigma$. If Σ is two-sided, then it has a unit normal vector field and so, stability of Σ is reduced to the existence of a positive Jacobi function; hence, a covering space of a stable two-sided minimal surface is again stable. However, in the flat three-manifold $\mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$, Scherk's nonorientable minimal surface with total curvature -2π is stable but its oriented two-sheeted cover is unstable. On the other hand, if $\Sigma \subset N$ is unstable, then it follows immediately from the definition of stability that any finite cover of Σ is unstable, independent of whether or not the surface is two-sided.

In trying to determine whether or not a limit leaf L of $\mathcal{L}$ is stable, it is essential to understand some basic properties of the holonomy group G_L. For the sake of concreteness, we describe the appropriate picture for understanding this question. Let $\Delta \subset \Sigma$ be a smooth compact subdomain (by the above discussion we may assume that L is two-sided and orientable), let Δ_ε denote a small closed regular ε-neighborhood with coordinates $\Delta \times [-\varepsilon, \varepsilon]$ induced by the exponential map on the closed ε-normal interval bundle to Δ. Let $\mathcal{L}_\varepsilon$ be the induced lamination. Fix a base point $p \in \Delta$ and let $F = (p \times [-\varepsilon, \varepsilon]) \cap \mathcal{L}_\varepsilon$ be the fiber over p. Assume that ε is chosen sufficiently small so that its leaves are graphs or multigraphs over $\Delta = \Delta \times \{0\}$. Given a representative loop of γ an element $[\gamma]$ of the fundamental group of Δ based at p, then for $q \in F$ sufficiently close to $p = p \times \{0\}$, one can vertically lift γ to $\mathcal{L}_\varepsilon$ beginning at q to obtain an arc with end point $\widetilde{H}(\gamma)(q) \in F$. This mapping $\widetilde{H}(\gamma)$ gives rise to an injective map from any sufficiently small neighborhood $N(p) \in F$ of p onto a small neighborhood $W(p) \subset F$: $\widetilde{H}(\gamma)\colon N(p) \to W(p)$. Note that $\widetilde{H}(\gamma)(p) = p$ and the choices of $N(p)$, $W(p)$ are not uniquely determined, but rather we consider $\widetilde{H}(\gamma)$ to be defined on equilavence classes of small neighborhoods of p in F. Also note that there is a natural ordering of the points of the fiber F given by the ordering of their t-coordinates and $\widetilde{H}(\gamma)$ preserves this ordering. Let $N_+(p)$ denote the points in $N(p)$ with positive t-coordinate and similarly define $W^+(q)$. If $N_+(p)$ is discrete, then it is naturally order-preservingly homeomorphic to the negative integers $\mathbb{Z}_-$, where we consider the limit point p to be $-\infty$. Similarly, we can indentify $W(q)$ with an interval of $(-\infty, j) \subset \mathbb{Z}$, and for some $n \in \mathbb{Z}$, $\widetilde{H}(\gamma)(k) = k + n$, where $k + n \in W(q)$ lies n points (above if n is positive, below if n is negative) from $k \in \mathbb{Z} = N(q)$. From this discussion, we obtain rather easily an onto representation $H\colon \pi_1(\Delta, p) \to G$ ($H\colon \pi_1(\Delta, p) \to G_+$ in the one-sided case), called the *holonomy (one-sided holonomy)* representation of fundamental group for the domain Δ which is associated to the lamination $\mathcal{L}$, where G is a group of germs of order-preserving homeomeorphisms of the fiber F which fix p. From our discussion, it follows that when $N_+(p)$ is discrete and G_+ is nontrivial, then G_+ can be identified with the set of integers $\mathbb{Z}$.

Lemma 18 (Stability of Leaves Lemma). *Suppose $\mathcal{L}$ is a minimal lamination of a Riemannian three-manifold N. Then the following statements hold:*

1. *If L is a limit leaf of $\mathcal{L}$, then the universal cover $\widetilde{L}$ of L is a stable minimal surface.*
2. *If M is a leaf of $\mathcal{L}$ and L is a leaf of the sublamination $L(M) \subset \mathcal{L}$ of limit points of M such that the holonomy representation of L on a side containing M has subexponential growth (amenable holonomy group) on compact subdomains of L, then L is stable. (For example, if the holonomy representation has image group isomorphic to a finitely generated abelian group.)*
3. *If M is a leaf of $\mathcal{L}$ and L is a leaf of the sublamination $L(M) \subset \mathcal{L}$ and there is an open set O_L containing L such that $O_L \cap L(M) = L$, then L is stable.*
4. *If N has positive Ricci curvature, then $\mathcal{L}$ has no limit leaves. If N has nonnegative sectional curvature and L is a complete limit leaf of $\mathcal{L}$, then L is simply-connected or 1-connected, totally geodesic and stable.*
5. *If $\{M_n\}_n$ is a sequence of embedded minimal surfaces in N that converge to $\mathcal{L}$ and their convergence to a nonlimit leaf L of $\mathcal{L}$ is of multiplicity greater than one, then L is stable.*

Proof. Let L be a limit leaf of $\mathcal{L}$. An argument given in the proof of Lemma 1.1 in [18] shows that the universal covering $\widetilde{L}$ of L, considered to be an immersed minimal surface in N, is stable. For the sake of completeness, we now give a different argument from the one given in Lemma 1.1 in [18] to show $\widetilde{L}$ is stable.

Let D be a smooth compact simply connected domain in the universal cover $\widetilde{L}$ of L. We first show that there is a positive Jacobi function on the interior of D. Let $N_\varepsilon(D)$ be an ε-normal bundle to the map of D into N that submerses to N under the exponential map. We will consider D to be the zero section of $N_\varepsilon(D)$. Also, note that we can pull back the metric of N to $N_\varepsilon(D)$ as well as the lamination $\mathcal{L}$, giving rise to a lamination $\mathcal{L}(D)$ of $N_\varepsilon(D)$. Note that D is a limit leaf of $\mathcal{L}(D)$. Since D is simply connected, there exist a sequence of leaves D_n of $\mathcal{L}(D)$ which converge to D and which for n large are disjoint positive normal graphs. Fix a point p in the interior of D and normalize the graphing functions f_n of D_n, by dividing by $f_n(p)$, to obtain a sequence of functions $F_n\colon D \to (0,\infty)$ with $F_n(p) = 1$. Standard elliptic theory implies that a subsequence of the F_n restricted to the interior of D converges to a positive Jacobi function f_D. Now take an exhaustion of M by compact domains $D(n)$ containing D with associated positive Jacobi functions $f|_{D(n)}$ defined on their interiors. Again a subsequence of these Jacobi functions converges uniformly on compact subsets of $\widetilde{L}$ to a positive Jacobi function of $\widetilde{L}$, which implies $\widetilde{L}$ is stable. This proves item 1.

We now consider item 2. Suppose L is a limit leaf of M in $\mathcal{L}$ and L is not stable. By the discussion that follows, this property of instability just means that there exists a smooth compact subdomain $D \subset L$ such that the first eigenvalue of the stability operator $\Delta + |A|^2 + Ric$ is negative.

Since we are assuming that D is strictly unstable, we can also assume $D \neq L$ and D has smooth nonempty boundary. Also, since finite covers of unstable domains are unstable, we may assume that both L and N are orientable. By item 1, we may assume D is nonsimply connected. Since D is a nonsimply connected oriented surface of some finite genus, there exist a finite number of pairwise disjoint smooth compact arcs $\alpha_1, ..., \alpha_k$ in D which intersect ∂D orthogonally at their end points and such that upon cutting D along them, we obtain a surface whose Riemannian completion is a compact disk $\mathbb{D}$. The boundary of $\mathbb{D}$ consists of a finite number of simple arcs meeting orthogonally at their end points and with two copies each of the arcs α_i appearing on ∂D.

If $\Pi\colon \widetilde{D} \to D$ is the universal cover of D, then we can lift the interior of $\mathbb{D}$ to $\widetilde{D}$ and consider $\mathbb{D}$ to be embedded in $\widetilde{D}$. Under the action of $\pi_1(D)$ on $\widetilde{D}$, we can also consider the $\pi_1(D)$-orbit of $\mathbb{D}$ to be a tiling of $\widetilde{D}$.

We now define $\mathbb{D}(1) = \overline{\mathbb{D}}$ and inductively define, for $n \in \mathbb{N}$, the domain $\mathbb{D}(n)$ to be the closure of the union of collection of disks in the tiling which are adjacent to the disks in $\mathbb{D}(n-1)$. Let G be holonomy group of D coming from M and let $\sigma\colon \pi_1(L) \to G$ be the associated representation. Let $\widehat{D}$ denote the covering space of D corresponding to the kernel of representation and let $\widehat{\mathbb{D}}(n)$ denote the quotient domain $\mathbb{D}(n)/G \subset \widehat{D} = \widetilde{D}/G$. It is straightforward to check that, under our assumption that the holonomy of D has subexponential growth, as $n \to \infty$, the ratio of the area of $\widehat{\mathbb{D}}(n) - \widehat{\mathbb{D}}(n-1)$ to the area of $\widehat{\mathbb{D}}(n-1)$ converges to zero.

Let $\mathcal{L}_\varepsilon$ denote the induced lamination of a small normal ε-regular neighborhood $D \times [-\varepsilon, \varepsilon] \subset N$ of D. Since the lifted minimal lamination $\mathcal{L}_\varepsilon$ to $\widehat{D} \times [-\varepsilon, \varepsilon]$ has trivial holonomy, the arguments in the proof of item 1 apply to show that $\widehat{D}$ has a positive Jacobi function. So, it remains to prove that $\widehat{D}$ is unstable, thereby giving the desired contradiction.

Consider the first eigenfunction f of the stability operator for D with zero boundary values. Since D is unstable, then, to second order, the associated variation of the area $A(t)$ of the interior of $\mathbb{D}$ in N decreases area at certain nonzero rate $r = A''(0)$ at time $t = 0$. Let f_n be the lifted function defined on $\widehat{\mathbb{D}}(n)$ and note that it decreases the area of $\widehat{\mathbb{D}}(n)$ to second order at the rate r multiplied by the number of fundamental regions in $\widehat{\mathbb{D}}(n)$. After defining cut off functions on a fixed subcollection of pairwise disjoint regular neighborhoods of the arcs of $\partial\mathbb{D}$ not contained in $\partial\widetilde{\mathbb{D}}$, we know that the new normal variation V with zero boundary values increases the area of $\mathbb{D} \subset \widetilde{D}$ to second order by a fixed rate at $t = 0$.

These just referred to cut off functions for $\mathbb{D}$ give rise via the action of $\pi_1(D)$ to cut off functions for f_n, thereby, yielding a global function F_n which is zero along $\partial\widehat{\mathbb{D}}(n)$ and equals f_n on any component of $\widehat{\mathbb{D}}(n) - \widehat{\mathbb{D}}(n-1)$.

For each fundamental domain Δ in $\widehat{\mathbb{D}}(n) - \widehat{\mathbb{D}}(n-1)$, the related normal variation of area of Δ induced by F_n possibly increases area to second order by a rate that is bounded from above, independent of n and Δ. However, the second order rate of decrease of area of a domain Δ in $D(n-1)$ by F_n is the same as the rate given by f which is constant. Since, for n large, the area of $\widehat{\mathbb{D}}(n) - \widehat{\mathbb{D}}(n-1)$ divided by the area of $\widehat{\mathbb{D}}(n-1)$ is arbitrarily small, then the variation F_n of $\widehat{\mathbb{D}}(n)$ decreases area for n large, contradicting that $\widehat{D}$ is stable. This contradiction proves the second statement in the lemma.

Now suppose that M is a leaf of $\mathcal{L}$, L is a leaf of $L(M)$ and L is an isolated limit leaf of $L(M)$ in the sense that there is an open set O_L containing L such that $O_L \cap L(M) = L$. In this case $M \cap (O_L - L)$ is properly embedded in $O_L - L$. Consider $L \cup (M \cap O_L)$ to be a minimal lamination of O_L and note that the holonomy group of this new minimal lamination is isomorphic to $\mathbb{Z}$ or is trivial on the side with M in this case. Hence, item 2 in the lemma implies L is stable, which proves item 3.

The first part of item 4 follows immediately from item 1 that the universal cover $\widetilde{L}$ is stable, and the result that such an N with positive Ricci curvature does not admit a complete stable orientable minimal surface (see [9] and also, see the arguments at the end of our proof of Theorem 15). If N has nonnegative sectional curvature, then the stable universal cover $\widetilde{L}$ is totally geodesic [9]. This means that $\widetilde{L}$ has a complete metric of nonnegative curvature, and so it is conformally $\mathbb{C}$. If L is not simply connected, the orientable cover of L is an annulus with fundamental group $\mathbb{Z}$, and so item 2 implies L is stable. This completes the proof of item 4.

The proof of item 5 is standard. Given a compact domain $D \subset L$, we wish to show that it is stable when the M_n converge to it with multiplicity greater than 1. After lifting to a finite cover of a small regular neighborhood of L, we may assume that everything is orientable, which means that we just need to show that L has a positive Jacobi function. Let $N_\varepsilon(D)$ be a small normal regular neighborhood of D. For n large, the components of $M_n \cap N_\varepsilon(D)$ are, by embeddedness of both M_n and D, small normal graphs over D. Taking differences of two such graphing functions and normalizing the difference to be 1 at some point p in the interior of D, and then taking limits, produces a positive Jacobi function in the interior of D. As in the proof of item 1, we conclude that L also has a positive Jacobi function. This completes the proof of the lemma. $\square$

William H. Meeks, III at bill@math.umass.edu
Mathematics Department, University of Massachusetts, Amherst, MA 01003

Harold Rosenberg at rosen@math.jussieu.fr
Institut de Mathematiques de Jussieu, Paris, France

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