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The minimal number of generators of a Togliatti system

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Abstract We compute the minimal and the maximal bound on the number of generators of a minimal smooth monomial Togliatti system of forms of degree d in $n+1$ variables, for any $d \geq 2$ and $n \geq 2$. We classify the Togliatti systems with number of generators reaching the lower bound or close to the lower bound. We then prove that if $n=2$ (resp. $n=2,3$) all range between the lower and upper bound is covered, while if $n \geq 3$ (resp. $n \geq 4$) there are gaps if we only consider smooth minimal Togliatti systems (resp. if we avoid the smoothness hypothesis). We finally analyze for $n=2$ the Mumford–Takemoto stability of the syzygy bundle associated with smooth monomial Togliatti systems.

Keywords Osculating space · Weak Lefschetz property · Laplace equations · Toric varieties

Mathematics Subject Classification 13E10 · 14M25 · 14N05 · 14N15 · 53A20

Contents

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1 Introduction

The classification of the smooth projective varieties satisfying at least one Laplace equation is a classical problem, still very far from being solved. We recall that a projective variety $X \subset \mathbb{P}^N$ is said to satisfy a Laplace equation of order d, for an integer $d \geq 2$, if its d-osculating space at a general point has dimension strictly less than expected. The most famous example is the Togliatti surface, a rational surface in $\mathbb{P}^5$ parametrized by cubics, obtained from the third Veronese embedding $V(2, 3)$ of $\mathbb{P}^2$ by a suitable projection from four points: the Del Pezzo surface obtained projecting $V(2, 3)$ from three general points on it admits a point which belongs to all its osculating spaces, so projecting further from this special point one obtains a surface having all osculating spaces of dimension ≤ 4 instead of the expected 5. This surface is named from Eugenio Togliatti who gave a classification of rational surfaces parametrized by cubics and satisfying at least one Laplace equation of order 2. For more details, see the original articles of Togliatti [25,26] or [10,14,27] for discussions of this example. In [16], the two authors of this note and Ottaviani described a connection, due to apolarity, between projective varieties satisfying at least one Laplace equation and homogeneous artinian ideals in a polynomial ring, generated by polynomials of the same degree, and failing the weak Lefschetz property (WLP for short). Let us recall that a homogeneous ideal $I \subset R := K[x_0, \ldots, x_n]$ fails the weak Lefschetz property in some degree j if, for any linear form L, the map of multiplication by L from $(R/I)_j$ to $(R/I)_{j+1}$ is not of maximal rank (see [18]). Thanks to this connection, explained in detail in Sect. 2, they obtained in the toric case the classification of the smooth rational threefolds parametrized by cubics and satisfying a Laplace equation of order 2, and gave a conjecture to extend it to varieties of any dimension. This conjecture has been recently proved in [17]. Note that the assumption that the variety is toric translates in the fact that the related ideals are generated by monomials, which simplifies apolarity and allows to exploit combinatorial methods. This point of view had been introduced by Perkinson in [22] and applied to the classification of toric surfaces and threefolds satisfying Laplace equations under some rather strong additional assumptions on the osculating spaces.

In this note, we begin the study of the analogous problems for smooth toric rational varieties parametrized by monomials of degree $d \geq 4$, or equivalently for artinian ideals of R generated by monomials of degree d. The picture becomes soon much more involved than in the case of cubics, and, for the moment, a complete classification appears out of reach. We consider mainly minimal smooth toric Togliatti systems of forms of degree d in R, i.e., homogeneous artinian ideals generated by monomials failing the WLP, minimal with respect to this property, and such that the apolar linear system parametrizes a smooth variety.

The first goal of this note is to establish minimal and maximal bounds, depending on n and $d \geq 2$, for the number of generators of Togliatti systems of this form and to classify the systems reaching the minimal bound or close to reach it. We then investigate whether all values comprised between the minimal and the maximal bound can be obtained as number of generators of a minimal smooth Togliatti system. We prove that the answer is positive if $n = 2$, but negative if $n \geq 3$. If we avoid smoothness assumption, the answer becomes positive for

$n = 3$ but is still negative for $n \geq 4$, even though we detect some intervals and sporadic values that are reached. Finally, as applications of our results, we study the Mumford–Takemoto stability of the syzygy bundle associated with a minimal smooth Togliatti system with $n = 2$.

Next we outline the structure of this note. In Sect. 2, we fix the notation and we collect the basic results on Laplace equations and the Weak Lefschetz Property needed in the sequel. Section 3 contains the main results of this note. Precisely, after recalling the results for degree 2 and 3, in Theorem 3.9 we prove that the minimal bound $\mu^s(n, d)$ on the number of generators of a minimal smooth Togliatti system of forms of degree d in $n + 1$ variables, for $d \geq 4$, is equal to $2n + 1$, and classify the systems reaching the bound. Then in Theorem 3.17, we get the complete classification for systems with number of generators $\mu^s(n, d) + 1$. We also compute the maximal bound $\rho^s(n, d)$ and give various examples. In Sect. 4, we prove that for $n = 2$ and any $d \geq 4$ all numbers in the range between $\mu^s(n, d)$ and $\rho^s(n, d)$ are reached (Proposition 4.1), while for $n \geq 3$ the value $2n + 3$ is a gap (Proposition 4.4). We then prove that, avoiding smoothness, for $n = 3$ the whole interval is covered. Finally, Sect. 5 contains the results about stability of the syzygy bundle for minimal smooth monomial Togliatti systems in 3 variables.

Notation Throughout this work, k will be an algebraically closed field of characteristic zero and $\mathbb{P}^n = \mathrm{Proj}(k[x_0, x_1, \ldots, x_n])$. We denote by $V(n, d)$ the Veronese variety image of the projective space $\mathbb{P}^n$ via the d-tuple Veronese embedding. $(F_1, \ldots, F_r)$ stands for the ideal generated by $F_1, \ldots, F_r$, while $\langle F_1, \ldots, F_r \rangle$ denotes the k-vector space they generate.

2 Background and preparatory results

In this section, we recall some standard terminology and notation from commutative algebra and algebraic geometry, as well as some results needed later on. In particular, we briefly recall the relationship between the existence of homogeneous artinian ideals $I \subset k[x_0, x_1, \ldots, x_n]$ which fail the weak Lefschetz property and the existence of (smooth) projective varieties $X \subset \mathbb{P}^N$ satisfying at least one Laplace equation of order $s \geq 2$. For more details, see [16] and [17].

2.1 The weak Lefschetz property

Let $R := k[x_0, x_1, \ldots, x_n] = \oplus_t R_t$ be the graded polynomial ring in $n + 1$ variables over the field k.

Definition 2.1 Let $I \subset R$ be a homogeneous artinian ideal. We say that R/I has the *weak Lefschetz property* (WLP, for short) if there is a linear form $L \in (R/I)_1$ such that, for all integers j, the multiplication map

$$\times L : (R/I)_j \to (R/I)_{j+1}$$

has maximal rank, i.e., it is injective or surjective. We will often abuse notation and say that the ideal I has the WLP. In this case, the linear form L is called a *Lefschetz element* of R/I. If for the general form $L \in (R/I)_1$ and for an integer number j the map $\times L$ has not maximal rank, we will say that the ideal I fails the WLP in degree j.

The Lefschetz elements of R/I form a Zariski open, possibly empty, subset of $(R/I)_1$. Part of the great interest in the WLP stems from the ubiquity of its presence (See, e.g., [2,4,8,9,15–21]) and the fact that its presence puts severe constraints on the possible Hilbert functions, which can appear in various disguises (see, e.g., [23]). Though many algebras are expected to have the WLP, establishing this property is often rather difficult. For example, it was shown by Stanley [24] and Watanabe [28] that a monomial artinian complete intersection ideal $I \subset R$ has the WLP. By semicontinuity, it follows that a *general* artinian complete intersection ideal $I \subset R$ has the WLP, but it is open whether *every* artinian complete intersection of height ≥ 4 over a field of characteristic zero has the WLP. It is worthwhile to point out that the weak Lefschetz property of an artinian ideal I strongly depends on the characteristic of the ground field k, and in positive characteristic, there are examples of artinian complete intersection ideals $I \subset k[x_0, x_1, x_2]$ failing the WLP (see, e.g., Remark 7.10 in [20]).

In [16], Mezzetti, Miró-Roig and Ottaviani showed that the failure of the WLP can be used to construct (smooth) varieties satisfying at least one Laplace equation of order $s \geq 2$ (see also [1,17]). Let us review the needed concepts from differential geometry in order to state this result.

2.2 Laplace equations

Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n and let $x \in X$ be a smooth point. We choose a system of affine coordinates and an analytic local parametrization ϕ around x where $x = \phi(0, \ldots, 0)$ and the N components of ϕ are formal power series. The *s-th osculating space* $T_x^{(s)}X$ to X at x is the projectivized span of all partial derivatives of ϕ of order $\leq s$. The expected dimension of $T_x^{(s)}X$ is $\binom{n+s}{s} - 1$, but in general $\dim T_x^{(s)}X \leq \binom{n+s}{s} - 1$; if strict inequality holds for all smooth points of X, and $\dim T_x^{(s)}X = \binom{n+s}{s} - 1 - \delta$ for a general point x, then X is said to satisfy δ Laplace equations of order s.

Remark 2.2 It is clear that if $N < \binom{n+s}{s} - 1$ then X satisfies at least one Laplace equation of order s, but this case is not interesting and will not be considered in the following.

Let I be an artinian ideal generated by r homogeneous polynomials $F_1, \ldots, F_r \in R$ of degree d. Associated with I_d there is a morphism

$$\varphi_{I_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}.$$

Note that φ_{I_d} is everywhere regular because I is an artinian ideal. Its image $X_{n,I_d} := \mathrm{Im}(\varphi_{I_d}) \subset \mathbb{P}^{r-1}$ is the projection of the n-dimensional Veronese variety $V(n, d)$ from the linear system $\langle (I^{-1})_d \rangle \subset | \mathcal{O}_{\mathbb{P}^n}(d) |= R_d$ where I^{-1} is the ideal generated by the Macaulay inverse system of I (See [16], Sect. 3 for details). Analogously, associated with $(I^{-1})_d$ there is a rational map

$$\varphi_{(I^{-1})_d} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}.$$

The closure of its image $X_{n,(I^{-1})_d} := \overline{\mathrm{Im}(\varphi_{(I^{-1})_d})} \subset \mathbb{P}^{\binom{n+d}{d}-r-1}$ is the projection of the n-dimensional Veronese variety $V(n, d)$ from the linear system $\langle F_1, \ldots, F_r \rangle \subset | \mathcal{O}_{\mathbb{P}^n}(d) |= R_d$. The varieties X_{n,I_d} and $X_{n,(I^{-1})_d}$ are usually called apolar. In the following $X_{n,(I^{-1})_d}$ will simply be denoted by X.

We have:

Theorem 2.3 *Let $I \subset R$ be an artinian ideal generated by r homogeneous polynomials $F_1, \ldots, F_r$ of degree d. If $r \leq \binom{n+d-1}{n-1}$, then the following conditions are equivalent:*

(1) *the ideal I fails the WLP in degree $d-1$;*

(2) *the homogeneous forms* $F_1, \ldots, F_r$ *become k-linearly dependent on a general hyperplane H of* $\mathbb{P}^n$*;*
(3) *the n-dimensional variety* $X = X_{n,(I^{-1})_d}$ *satisfies at least one Laplace equation of order* $d-1$.

Proof See [16, Theorem 3.2]. □

In view of Remark 2.2, the assumption $r \leq \binom{n+d-1}{n-1}$ ensures that the Laplace equations obtained in (3) are not obvious. In the particular case $n = 2$, this assumption gives $r \leq d+1$.

The above result motivates the following definition:

Definition 2.4 Let $I \subset R$ be an artinian ideal generated by r forms $F_1, \ldots, F_r$ of degree d, $r \leq \binom{n+d-1}{n-1}$. We introduce the following definitions:

(1) I is a *Togliatti system* if it satisfies the three equivalent conditions in Theorem 2.3.
(2) I is a *monomial Togliatti system* if, in addition, I (and hence I^{-1}) can be generated by monomials.
(3) I is a *smooth Togliatti system* if, in addition, the n-dimensional variety X is smooth.
(4) A monomial Togliatti system I is said to be *minimal* if I is generated by monomials $m_1, \ldots, m_r$ and there is no proper subset $m_{i_1}, \ldots, m_{i_{r-1}}$ defining a monomial Togliatti system.

The names are in honor of Eugenio Togliatti who proved that for $n = 2$ the only smooth Togliatti system of cubics is $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset k[x_0, x_1, x_2]$ (see [2,25,26]). The main goal of our note is to determine a lower bound $\mu(n, d)$ (resp. $\mu^s(n, d)$) for the minimal number of generators $\mu(I)$ of any (resp. smooth) minimal monomial Togliatti system $I \subset k[x_0, x_1, \ldots, x_n]$ of forms of degree $d \geq 2$ and classify *all* (resp. smooth) minimal monomial Togliatti systems $I \subset k[x_0, x_1, \ldots, x_n]$ of forms of degree $d \geq 2$ which reach the bound, i.e., $\mu(I) = \mu(n, d)$ (resp. $\mu(I) = \mu^s(n, d)$). These results will be achieved in the next section.

3 The minimal number of generators of a smooth Togliatti system

From now on, we restrict our attention to monomial artinian ideals $I \subset k[x_0, \ldots, x_n]$ (i.e., the ideals invariants for the natural toric action of $(k^*)^n$). Recall that when $I \subset R$ is an artinian monomial ideal, the homogeneous part I_d^{-1} of degree d of the inverse system I^{-1} is spanned by the monomials in R_d not in I. It is also worthwhile to recall that for monomial artinian ideals to test the WLP there is no need to consider a general linear form. In fact, we have

Proposition 3.1 *Let* $I \subset R := k[x_0, x_1, \ldots, x_n]$ *be an artinian monomial ideal. Then* R/I *has the WLP if and only if* $x_0 + x_1 + \cdots + x_n$ *is a Lefschetz element for* R/I.

Proof See [20], Proposition 2.2. □

Given an artinian ideal $I \subset k[x_0, x_1, \ldots, x_n]$, we denote by $\mu(I)$ the minimal number of generators of I. We define

$$\begin{aligned}
\mu(n, d) &:= min\{\mu(I) \mid I \in \mathcal{T}(n, d)\},\\
\mu^s(n, d) &:= min\{\mu(I) \mid I \in \mathcal{T}^s(n, d)\},\\
\rho(n, d) &:= max\{\mu(I) \mid I \in \mathcal{T}(n, d)\} \text{ and}\\
\rho^s(n, d) &:= max\{\mu(I) \mid I \in \mathcal{T}^s(n, d)\}
\end{aligned}$$

where $\mathcal{T}(n,d)$ is the set of all minimal monomial Togliatti systems $I \subset k[x_0, x_1, \dots, x_n]$ of forms of degree d and $\mathcal{T}^s(n,d)$ is the set of all minimal smooth monomial Togliatti systems $I \subset k[x_0, x_1, \dots, x_n]$ of forms of degree d. By definition, we have $\mathcal{T}^s(n,d) \subset \mathcal{T}(n,d)$.

Our first goal is to provide a lower bound for $\mu(n,d)$ and $\mu^s(n,d)$. First, we observe that all artinian monomial ideals $I \subset k[x_0, x_1, \dots, x_n]$ generated by forms of degree $d \geq 2$ contain x_i^d for $i = 0, \dots, n$ and the ideals $(x_0^d, \dots, x_n^d)$ do satisfy WLP. Therefore, we always have

$$n+2 \leq \mu(n,d) \leq \mu^s(n,d) \leq \rho^s(n,d) \leq \rho(n,d) \leq \binom{n+d-1}{n-1}. \tag{1}$$

Let us start analyzing the cases $d = 2, 3$.

Remark 3.2 The minimal smooth monomial Togliatti systems $I \subset k[x_0, x_1, \dots, x_n]$ of quadrics were classified in [17], Proposition 2.8. It holds:

(1) $\mathcal{T}^s(2,2) = \emptyset$.
(2) For $n \geq 3$, we have

$$\mu^s(n,2) = \begin{cases} \lambda^2 + 2\lambda + 1 & \text{if } n = 2\lambda \\ \lambda^2 + 3\lambda + 2 & \text{if } n = 2\lambda + 1. \end{cases}$$

(3) For $n \geq 3$, $\rho^s(n,2) = \binom{n}{2} + 3$.

In particular, for $n = 3$ we have $n + 2 < \mu^s(n,2) = \rho^s(n,2) = \binom{n+1}{2}$; for $n = 4$ we have $n+2 < \mu^s(n,2) = \rho^s(n,2) < \binom{n+1}{2}$; and for all $n > 4$ the inequalities in (1) are strict, i.e.,

$$n+2 < \mu^s(n,2) < \rho^s(n,2) < \binom{n+1}{2}.$$

We also have $\mu(n,2) = 2n+1$ for $n \geq 4$ (since we easily check that $\mu(n,2) \geq 2n+1$ and $I = (x_0^2, x_1^2, \dots, x_n^2, x_0x_1, x_0x_2, \dots, x_0x_n)$ fails weak Lefschetz property from degree 1 to degree 2) and $\mu(3,2) = 6$ (since $\mu(3,2) > 5$ and $I = (x_0^2, x_1^2, x_2^2, x_3^2, x_0x_1, x_2x_3)$ fails weak Lefschetz property in degree 1).

Remark 3.3 The minimal smooth monomial Togliatti systems $I \subset k[x_0, x_1, \dots, x_n]$ of cubics were classified in [16], Theorem 4.11 and [17], Theorem 3.4. It holds:

(1) $\rho^s(2,3) = \mu^s(2,3) = 4$,
(2) $\rho^s(3,3) = \mu^s(3,3) = 8$,
(3) $13 = \mu^s(4,3) < 15 = \rho^s(4,3)$, and
(4) For all $n \geq 4$, we have $\rho^s(n,3) = \binom{n+1}{3} + n + 1$,

$$\begin{aligned} \mu^s(n,3) &= min \left\{ \sum_{i=1}^{s} \binom{a_i+2}{3} + \sum_{1 \leq i<j<k \leq s} a_i a_j a_k \mid n+1 = \sum_{i=1}^{s} a_i \quad \text{and} \right. \\ &\qquad n-1 \geq a_1 \geq \cdots \geq a_s \geq 1 \} \\ &= \begin{cases} 2\binom{\lambda+3}{3} & \text{if } n = 2\lambda+1 \\ \binom{\lambda+2}{3} + 2\binom{\lambda+3}{3} & \text{if } n = 2\lambda \end{cases} \end{aligned}$$

and, hence

$$n+2 < \mu^s(n,3) < \rho^s(n,3) < \binom{n+2}{3}.$$

We may also check that $\mu(n,3) = 2n+1$ for $n \geq 3$ (since $\mu(n,3) \geq 2n+1$ and $I = (x_0^3, x_1^3, \dots, x_n^3, x_0^2x_1, x_0^2x_2, \dots, x_0^2x_n)$ fails weak Lefschetz property in degree 2) and $\mu(2,3) = 4$ (since $\mu(2,3) \geq 4$ and $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ fails weak Lefschetz property from degree 2 to degree 3). Notice that $\mu^s(n,2) \geq 2n+1$ unless $n = 2, 3$ and $\mu^s(n,3) \geq 2n+1$ unless $n = 2, 3$.

From now on, we assume $d \geq 4$ and $n \geq 2$. We will prove that $\mu^s(n,d) = \mu(n,d) = 2n+1$. In addition, we will classify all (resp. smooth) minimal monomial Togliatti systems $I \subset k[x_0, x_1, \dots, x_n]$ of forms of degree $d \geq 4$ with $\mu(I) = 2n+1$ and all smooth minimal monomial Togliatti systems $I \subset k[x_0, x_1, \dots, x_n]$ of forms of degree $d \geq 4$ with $\mu(I) = \mu^s(n,d) + 1 = 2n+2$, revealing how the power of combinatorics tools can allow us to deduce pure geometric properties of projections of n-dimensional Veronese varieties $V(n,d)$. To prove it, we will associate with any artinian monomial ideal a polytope and the toric variety $X = X_{n,(I^{-1})_d}$ introduced in Sect. 2.2. Hence, we will be able to tackle our problem with tools coming from combinatorics. In fact, when we deal with artinian monomial ideals $I \subset k[x_0, x_1, \dots, x_n]$, the failure of the WLP can be established by fairly easy combinatoric properties of the associated polytope P_I. To state this result, we need to fix some extra notation.

Let $I \subset k[x_0, x_1, \dots, x_n]$ be an artinian monomial ideal generated by monomials of degree d and let I^{-1} be its inverse system. We denote by Δ_n the standard n-dimensional simplex in the lattice $\mathbb{Z}^{n+1}$, we consider $d\Delta_n$, and we define the polytope P_I as the convex hull of the finite subset $A_I \subset \mathbb{Z}^{n+1}$ corresponding to monomials of degree d in I^{-1}. As usual we define the sublattice $\operatorname{Aff}_{\mathbb{Z}}(A_I)$ in $\mathbb{Z}^{n+1}$ generated by A_I as follows:

$$\operatorname{Aff}_{\mathbb{Z}}(A_I) := \left\{ \sum_{x \in A_I} n_x \cdot x \mid n_x \in \mathbb{Z}, \quad \sum_{x \in A_I} n_x = 1 \right\}.$$

We have:

Proposition 3.4 *Let $I \subset k[x_0, x_1, \dots, x_n]$ be an artinian monomial ideal generated by r monomials of degree d. Assume $r \leq \binom{n+d-1}{n-1}$. Then, I is a Togliatti system if and only if there exists a hypersurface of degree $d-1$ containing $A_I \subset \mathbb{Z}^{n+1}$. In addition, I is a minimal Togliatti system if and only if any such hypersurface F does not contain any integral point of $d\Delta_n \setminus A_I$ except possibly some of the vertices of $d\Delta_n$.*

Proof It follows from Theorem 2.3 and [22], Proposition 1.1. □

Let us illustrate the above proposition with a precise example.

Example 3.5 The artinian ideal $I = (x_0, x_1)^3 + (x_2, x_3)^3 \subset k[x_0, x_1, x_2, x_3]$ defines a minimal monomial Togliatti system of cubics. In fact, the set $A_I \subset \mathbb{Z}^4$ is:

$$\begin{aligned} A_I = \{&(2,0,1,0), (1,0,2,0), (2,0,0,1), (1,0,0,2), (0,2,1,0), (0,1,2,0), \\ &(0,2,0,1), (0,1,0,2), (1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1)\}. \end{aligned}$$

There is a hyperquadric, and only one, containing all points of A_I and no integral point of $3\Delta_3 \setminus A_I$, namely

$$Q(x_0, x_1, x_2, x_3) = 2\left(x_0^2 + x_1^2 + x_2^2 + x_3^2\right) + 4(x_0x_1 + x_2x_3) - 5(x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3).$$

For sake of completeness, we also recall the following useful combinatorial criterion which will allow us to check whether a subset A of points in the lattice $\mathbb{Z}^{n+1}$ defines a smooth toric variety X_A or not.

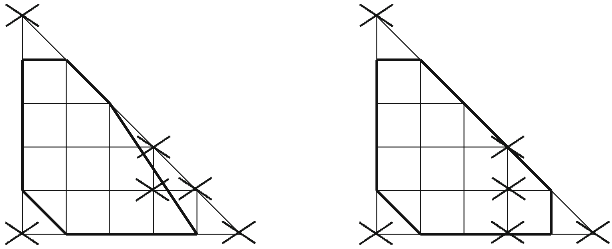


Fig. 1 Non-smooth Togliatti systems with $n = 2$ and $d = 5$

Proposition 3.6 *Let $I \subset k[x_0, x_1, \dots, x_n]$ be an artinian monomial ideal generated by monomials of degree d. Let $A_I \subset \mathbb{Z}^{n+1}$ be the set of integral points corresponding to monomials in $(I^{-1})_d$, S_I the semigroup generated by A_I and 0, P_I the convex hull of A_I and X_{A_I} the projective toric variety associated with the polytope P_I. X_{A_I} is smooth if and only if for any non-empty face Γ of P_I the following conditions hold:*

(1) *The semigroup S_I/Γ is isomorphic to $\mathbb{Z}_+^m$ with $m = \dim(P_I) - \dim \Gamma + 1$.*
(2) *The lattices $\mathbb{Z}^{n+1} \cap Aff_{\mathbb{R}}(\Gamma)$ and $Aff_{\mathbb{Z}}(A_I \cap \Gamma)$ coincide.*

Proof See [6] Chapter 5, Corollary 3.2. Note that in this case $X_{A_I} = X_{n,(I^{-1})_d}$. □

Figure 1 illustrates two examples of minimal Togliatti systems which are non-smooth. The points of the complementary of A_I are marked with a cross.

The condition (1) of Proposition 3.6 is verified if and only if translating each vertex v of the polygon to the origin of $\mathbb{Z}^2$, and considering for each edge coming out of v the first point with integer coordinates, these form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. The condition (2) is equivalent to each point of $\mathbb{Z}^2$ which lies on an edge of the polygon being also a point of A_I. Therefore, the first figure violates condition (1) and the second one violates condition (2).

In order to achieve the classification of minimal (resp. smooth) monomial Togliatti systems $I \subset k[x_0, \dots, x_n]$ of degree $d \geq 4$ with $\mu(I)$ as small as possible, we need to introduce one more definition.

Definition 3.7 A Togliatti system $I \subset k[x_0, x_1, \dots, x_n]$ of forms of degree d is said to be *trivial* if there exists a form F of degree $d - 1$ such that I contains $x_0F, \dots, x_nF$.

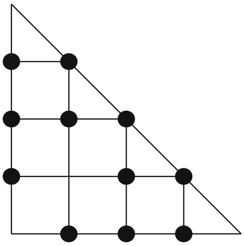
The following remark justifies why we call them trivial.

Remark 3.8 (1) Let F be a homogeneous form of degree $d - 1$. Since $x_0F, x_1F, \dots, x_nF$ become linearly dependent on the hyperplane $x_0 + \cdots + x_n = 0$, using Proposition 3.1, we conclude that any artinian ideal of the form $I = (x_0, \dots, x_n)F + (F_1, \dots, F_s)$ is a (trivial) Togliatti system. In the monomial case, looking at the inverse system that parameterizes the surface X, we can observe that it satisfies a Laplace equation of the simplest form, given by the annihilation of the partial derivative of order $d - 1$ corresponding to the monomial F.

(2) Let $I \subset k[x_0, x_1, \dots, x_n]$ be a monomial Togliatti system of cubics. If I is trivial, then it is not smooth.

Theorem 3.9 *For any integer $n \geq 2$ and $d \geq 4$, we have $\mu^s(n, d) = \mu(n, d) = 2n + 1$. In particular, if $I \subset k[x_0, x_1, \dots, x_n]$ is a minimal (resp. smooth minimal) monomial Togliatti system of forms of degree d, then $\mu(I) \geq 2n + 1$.*

In addition, all *minimal monomial Togliatti systems $I \subset k[x_0, \dots, x_n]$ of forms of degree $d \geq 4$ with $\mu(I) = 2n + 1$ are trivial unless one of the following cases holds:*



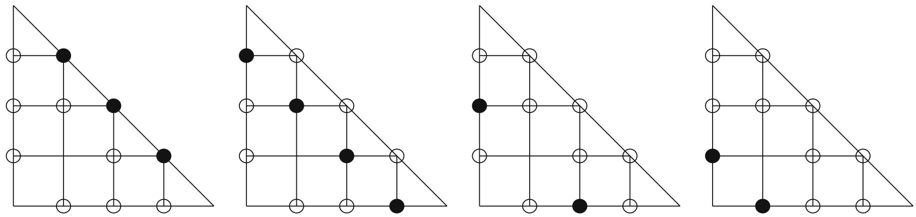


Fig. 2 A^I with $I = (x_0^4, x_1^4, x_2^4, x_0^2 x_1 x_2)$

(1) $(n, d) = (2, 5)$ *and, up to a permutation of the coordinates,* $I = (x_0^5, x_1^5, x_2^5, x_0^3 x_1 x_2, x_0 x_1^2 x_2^2)$.
(2) $(n, d) = (2, 4)$ *and, up to a permutation of the coordinates,* $I = (x_0^4, x_1^4, x_2^4, x_0 x_1 x_2^2, x_0^2 x_1^2)$.

Furthermore, (1) is smooth and (2) is not smooth.

Proof First of all, we observe that $I = (x_0^d, x_1^d, \dots, x_n^d) + x_0^{d-1}(x_1, \dots, x_n) \subset k[x_0, \dots, x_n]$ is a minimal monomial Togliatti system of forms of degree d, and by Proposition 3.6, being $d \geq 4$, it is smooth. Thus, $\mu(n, d) \leq \mu^s(n, d) \leq 2n + 1$.

To prove that $\mu(n, d) = 2n + 1$, we have to check that any monomial artinian ideal $I = (x_0^d, \dots, x_n^d, x_0^{a_0^1} x_1^{a_1^1} \dots x_n^{a_n^1}, \dots, x_0^{a_0^{n-1}} x_1^{a_1^{n-1}} \dots x_n^{a_n^{n-1}})$ with $\sum_{i=0}^n a_i^j = d \geq 4$, $1 \leq j \leq n-1$ has the WLP at the degree $d-1$. According to Proposition 3.4, to prove the last assertion it is enough to prove that no hypersurface of degree $d-1$ contains all points of $A_I \subset \mathbb{Z}^{n+1}$, where, as before, $A_I \subset \mathbb{Z}^{n+1}$ is the set of all integral points corresponding to monomials of degree d in I^{-1}. For any integer $0 \leq i \leq d$, we set $H_i = \{(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \mid a_0 = i\}$ and $A_I^i := A_I \cap H_i$; we have $A_I = \cup_{i=0}^d A_I^d$.

To illustrate this method, in Fig. 2 we show the pictures of the sets A_I, and A_I^0, A_I^1, A_I^2, A_I^3, when $I = (x_0^4, x_1^4, x_2^4, x_0^2 x_1 x_2)$.

We will prove now the theorem proceeding by induction on n. Let us start with the case $n = 2$. We take a monomial artinian ideal $I = (x_0^d, x_1^d, x_2^d, x_0^{a_0^1} x_1^{a_1^1} x_2^{a_2^1})$ with $a_0^1 + a_1^1 + a_2^1 = d \geq 4$, and we show that no plane curve of degree $d-1$ contains all points of $A_I \subset \mathbb{Z}^3$. Since $4 \leq d = a_0^1 + a_1^1 + a_2^1$, we can assume wlog that $2 \leq a_0^1$. We assume that there is a plane curve F_{d-1} of degree $d-1$ containing all points of A_I and we will get a contradiction. Since F_{d-1} contains the d points of A_I^1, it factorizes as $F_{d-1} = L_1 F_{d-2}$. Since F_{d-2} contains the $d-1$ points A_I^0, it factorizes as $F_{d-1} = L_0 L_1 F_{d-3}$. Now, if $a_0^1 = 2$, then A_I^2 contains $d-2$ points, if $a_0^1 > 2$, then A_I^2 contains $d-1$ points; in any case $F_{d-3} = L_2 F_{d-4}$ for a suitable form

F_{d-4} of degree $d-4$. Repeating the argument, we get that $F_{d-1} = L_0L_1 \dots L_{d-2}$, so F_{d-1} does not contain the points of A_I^{d-1}, which is non-empty by assumption. This contradicts the existence of a plane curve of degree $d-1$ containing all integral points of A_I.

Let now $n \geq 3$ and assume that the claim is true for $n-1$. Let us prove that no hypersurface of degree $d-1$ contains all points of $A_I \subset \mathbb{Z}^{n+1}$, where

$$I = \left(x_0^d, \dots, x_n^d, x_0^{a_0^1} x_1^{a_1^1} \dots x_n^{a_n^1}, \dots, x_0^{a_0^{n-1}} x_1^{a_1^{n-1}} \dots x_n^{a_n^{n-1}} \right)$$

with $\sum_{i=0}^n a_i^j = d \geq 4$, $1 \leq j \leq n-1$. Wlog we can assume $a_0^1 \geq a_1^1 \geq \dots \geq a_n^1 \geq 0$ and also $a_0^1 \geq a_0^2$. Therefore $a_0^1 > 0$, so x_0 appears explicitly in the monomial $x_0^{a_0^1} x_1^{a_1^1} \dots x_n^{a_n^1}$ and A_I^0 is equal to $d\Delta_{n-1}$ minus the n vertices and at most $n-2$ other points. By inductive assumption, no hypersurface in n variables of degree $d-1$ contains A_I^0, so F_{d-1} factorizes as L_0F_{d-2}, where F_{d-2} is a hypersurface of degree $d-2$ containing all points of $A_I \setminus A_I^0$.

If the n–1 monomials have $a_0^1 = a_0^2 = \dots = a_0^{n-1} \leq 1$, then $A_I^2 = (d-2)\Delta_{n-1}, \dots, A_I^{d-1} = \Delta_{n-1}$ and we deduce that $F_{d-1} = L_0L_2 \dots L_{d-1}$, because for $j = 2, \dots, d-1$ the simplex $(d-j)\Delta_{n-1}$ is not contained in any hypersurface in $n-1$ variables of degree $d-j$. This gives a contradiction because F_{d-1} misses all points of $A_I^1 \neq \emptyset$. Otherwise, $A_I^1 = (d-1)\Delta_{n-1}$ minus at most $n-2$ points. Then by inductive assumption, there is no hypersurface of degree $d-1$ in $n-1$ variables containing A_I^1. Then we repeat the argument until we reach a contradiction.

Finally we will classify all minimal monomial Togliatti systems $I \subset k[x_0, \dots, x_n]$ of forms of degree $d \geq 4$ with $\mu(I) = 2n+1$. First we assume that $n = 2$ and we will show that all of them are trivial unless $d = 5$ and $I = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^2x_2^2)$ or $d = 4$ and $I = (x_0^4, x_1^4, x_2^4, x_0x_1x_2^2, x_0^2x_1^2)$. Take $I = (x_0^d, x_1^d, x_2^d, m_1, m_2) \subset k[x_0, x_1, x_n]$ with $m_i = x_0^{a_0^i} x_1^{a_1^i} x_2^{a_2^i}$ and $\sum_{j=0}^2 a_j^i = d$ a minimal Togliatti system. If there exists $0 \leq i \leq 2$ such that $a_i^1, a_i^2 \geq 2$ (wlog we assume $i = 0$), then the plane curve F_{d-1} containing all integral points of A_I factorizes $F_{d-1} = L_0L_1 \dots L_{d-2}$, and since F_{d-1} cannot miss any point of A_I, we must have $A_I^{d-1} = \emptyset$ which forces $m_1 = x_0^{d-1}x_1, m_2 = x_0^{d-1}x_2$. Assume now that for any $0 \leq i \leq 2$, there exists $1 \leq j \leq 2$ with $a_i^j \leq 1$. Since $d \geq 4$, we may assume $a_0^1, a_1^1 \leq 1$ and $a_2^2 \leq 1$. Therefore, $m_1 \in \{x_0x_1x_2^{d-2}, x_0x_2^{d-1}, x_1x_3^{d-1}\}$ and $m_2 \in \{x_0^a x_1^{d-1-a} x_2, x_0^\alpha x_1^{d-\alpha} \mid 0 \leq a, \alpha \leq d-1\}$. But none gives a minimal Togliatti system because $x_0^d, x_1^d, x_2^d, m_1, m_2$ are linearly independent on a general line of $\mathbb{P}^2$ (see Theorem 2.3) unless $d = 5$ and $m_1 = x_0x_1x_2^3$ and $m_2 = x_0^2x_1^2x_2$ or $d = 4$ and $m_1 = x_0^2x_1x_2$ and $m_2 = x_0x_1^2x_2^2$. Furthermore, applying Proposition 3.6, we easily check that only $I = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^2x_2^2)$ defines a smooth variety.

Assume now $n \geq 3$ and $d \geq 4$ and let $I = (x_0^d, x_1^d, \dots, x_n^d, m_1, \dots, m_n) \subset k[x_0, \dots, x_n]$ with $m_i = x_0^{a_0^i} x_1^{a_1^i} \dots x_n^{a_n^i}$ and $\sum_{j=0}^n a_j^i = d$ be a Togliatti system. There is an integer j, $0 \leq j \leq n$ such that $\#\{i \mid a_j^i \geq 1\} \geq 2$. Therefore, wlog we can assume $a_0^1, a_0^2 \geq 1$. Arguing as in the previous part of the proof, any hypersurface F_{d-1} of degree $d-1$ containing all integral points of A_I factorizes $F_{d-1} = L_0L_1 \dots L_{d-2}$, and since F_{d-1} cannot miss any point of A_I, we must have $A_I^{d-1} = \emptyset$ which forces $m_1 = x_0^{d-1}x_1, m_2 = x_0^{d-1}x_2, \dots, m_n = x_0^{d-1}x_n$ and hence I is trivial, which proves what we want. □

Remark 3.10 Minimal monomial Togliatti systems $I \subset k[x_0, x_1, x_2]$ of forms of degree $d \geq 4$ with $\mu(I) = 5$ were also classified by Albini in [1], Theorem 3.5.1. So, our results can be seen as a generalization of his result to the case of an arbitrary number of variables.

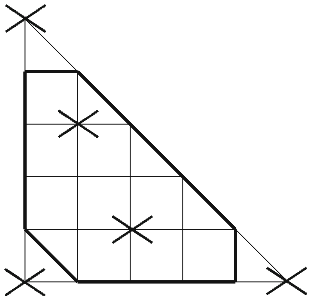


Fig. 3 Smooth non-trivial Togliatti system with $n = 2$ and $d = 5$

Remark 3.11 Up to permutation of the variables, the trivial Togliatti systems with $\mu(I) = 2n + 1$ are of the form $(x_1^d, \ldots, x_n^d) + x_0^{d-1}(x_0, \ldots, x_n)$.

Figure 3 illustrates the only smooth non-trivial example of minimal Togliatti system of forms of degree 5 with $\mu(I) = 5$.

Remark 3.12 In the case of the non-trivial minimal smooth monomial Togliatti system

$$I = \left(x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^2x_2^2\right)$$

all 4-osculating spaces to X have dimension lower than 14, which is the expected dimension, but the dimension of the previous osculating spaces is not constant. Some points of X have 2-osculating space or 3-osculating space of dimension less than the general one (they are *flexes* of X).

This follows from [22], where it is proved that the dimension of the s-osculating space at a point $x \in X$, corresponding to a vertex v_x of the polytope P_I, is maximal if and only if $(P_I \cap \mathbb{Z}^2) \setminus A_I$ contains all points out to level $s - 1$ with respect to v_x. This means that, after translating v_x to the origin and using the first lattice points lying along the two edges of P_I emanating from v_x as basis for the lattice, $(P_I \cap \mathbb{Z}^2) \setminus A_I$ contains all points (a, b) with $a+b \leq s-1$. This remark explains why this example is not included in the list of Perkinson [22], Theorem 3.2.

To better understand its geometry, let us note that the surface X is the projection, from a line L, of the blowing up of $\mathbb{P}^2$ at three general points E_0, E_1, E_2, embedded in $\mathbb{P}^{17}$ by the linear system of the quintics through them. The line L is chosen so to meet all 4-osculating spaces of this surface. We observe that there are three lines of this type, obtained by interchanging the variables. Every such line meets also the 3-osculating space at one of the three points E_i and the 2-osculating spaces at the other two. This gives rise to the flexes. Any curve on X corresponding to a general line through one of the blown up points is a smooth rational quartic. One can check that the flexes result to be singular points of intersection of two irreducible components of some reducible quartics obtained after the projection from L. It would be nice to have a precise geometric description of the inflectional loci of X, but this goes beyond the scope of this article, and we plan to return on this topic in a forthcoming paper.

Remark 3.13 The hypersurface F_{d-1} of degree $d - 1$ that contains the integral points A_I of a minimal monomial Togliatti system

$$I = \left(x_0^d, x_1^d, \ldots, x_n^d, m_1, \ldots, m_n\right) \subset k[x_0, \ldots, x_n]$$

with $\mu(I) = 2n + 1$ can be described. It turns out that if I is trivial then F_{d-1} is the union of $d - 1$ hyperplanes.

If $n = 2$, $d = 4$ and $I = (x_0^4, x_1^4, x_2^4, x_0x_1x_2^2, x_0^2x_1^2)$, then $F_3 = (x_0 + x_1 - 3x_2)(3x_0^2 - 10x_0x_1 + 3x_1^2 - 4x_0x_2 - 4x_1x_2 + x_2^2)$. In this example, the surface $X \subset \mathbb{P}^9$ is the closure of the image of the parametrization $\phi = \phi_{(I^{-1})_4}$ defined by the monomials of degree 4 not in I, i.e.,

$$\left(x_0^3x_1, x_0^3x_2, x_0^2x_1x_2, x_0^2x_2^2, x_0x_1^3, x_0x_1^2x_2, x_0x_2^3, x_1^3x_2, x_1^2x_2^2, x_1x_2^3\right).$$

One computes that its partial derivatives of order 3 satisfy the Laplace equation

$$\left(x_0\phi_{x_0} + x_1\phi_{x_1} - x_2\phi_{x_2}\right)\left(x_0^2\phi_{x_0^2} - 2x_0x_1\phi_{x_0x_1} + x_1^2\phi_{x_1^2} + x_2^2\phi_{x_2^2}\right) = 0.$$

If $n = 2$, $d = 5$ and $I = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^2x_2^2)$, then $F_4(x_0, x_1, x_2) = 24(x_0^4 + x_1^4 + x_2^4) - 154(x_0^3x_1 + x_0x_1^3 - x_0^3x_2 + x_1^3x_2 + x_0x_2^3 + x_1x_2^3) + 269(x_0^2x_1^2 + x_0^2x_3^2 + x_1^2x_2^2) + 288(x_0^2x_1x_2 + x_0x_1x_2^2) - 337x_0x_1^2x_2$ which is irreducible.

Similarly, the Laplace equation satisfied by the parametrization of the surface $X \subset \mathbb{P}^{15}$ is

$$\begin{aligned} &x_0^4\phi_{x_0^4} + x_1^4\phi_{x_1^4} + x_2^4\phi_{x_2^4} - x_0^3x_1\phi_{x_0^3x_1} - x_0^3x_2\phi_{x_0^3x_2} - x_0x_1^3\phi_{x_0x_1^3} - x_0x_2^3\phi_{x_0x_2^3} - x_1^3x_2\phi_{x_1^3x_2} \\ &-x_1x_2^3\phi_{x_1x_2^3} + x_0^2x_1^2\phi_{x_0^2x_1^2} + x_0^2x_2^2\phi_{x_0^2x_2^2} + x_1^2x_2^2\phi_{x_1^2x_2^2} - 3x_0^2x_1x_2\phi_{x_0^2x_1x_2} + 2x_0x_1^2x_2\phi_{x_0x_1^2x_2} \\ &+2x_0x_1x_2^2\phi_{x_0x_1x_2^2} = 0. \end{aligned}$$

Corollary 3.14 *Fix integers $d \geq 4$ and $n \geq 2$. Let $I = (F_1, \ldots, F_r) \subset k[x_0, \ldots, x_n]$ be a monomial artinian ideal of forms of degree d. If $r \leq 2n$ then, for any $s \leq d - 1$, the s-osculating space to X at a general point $x \in X$ has the expected dimension, namely $\binom{n+s}{s} - 1$.*

In next Theorem, we will classify all smooth minimal monomial Togliatti systems $I \in \mathcal{T}^s(n, d)$ whose minimal number of generators exceeds by one the possible minimum. We start with a lemma.

Lemma 3.15 *Let $I = (x_0^d, x_1^d, \ldots, x_n^d, m_1, \ldots, m_h) \subset k[x_0, \ldots, x_n]$ with $h \geq n$, $m_i = x_0^{a_0^i} \ldots x_n^{a_n^i}$ for $i = 1, \ldots, h$, be a minimal Togliatti system of forms of degree $d \geq 3$. Assume $a_0^1 \geq a_0^2 \geq \ldots \geq a_0^h$. If $a_0^{h-n+2} > 0$, then $a_0^i > 0$ for all index i.*

Proof Since I is a Togliatti system, there exists a form F_{d-1} of degree $d-1$ in $x_0, \ldots, x_n$ passing through all points of A_I. Its restriction to H_0, $F_{d-1}(0, x_1, \ldots, x_n)$, vanishes at all points of A_I^0. By assumption, to get A_I^0 we have to remove from the simplex $d\Delta_{n-1}$ the n vertices and at most $n-2$ other points. We denote by $I' \subset K[x_1, \ldots, x_n]$ the ideal generated by $x_1^d, \ldots, x_n^d$ and the monomials not containing x_0 among $m_1, \ldots, m_h$. If $F_{d-1}(0, x_1, \ldots, x_n) \neq 0$, I' is a Togliatti system in n variables with $\mu(I') \leq 2n - 2$, which contradicts Theorem 3.9. Hence $F_{d-1}(0, x_1, \ldots, x_n) = 0$ and $F_{d-1} = L_0F_{d-2}$. But I is minimal, so by Proposition 3.4 L_0 does not contain any point of $d\Delta_n \setminus A_I$ except the vertices, which implies that $a_0^i > 0$ for any index i. □

Remark 3.16 Recall that, when I is a monomial Togliatti system, the projective variety X defined by the apolar linear system of forms of degree d has all $(d-1)$-osculating spaces of dimension strictly less than expected, i.e., X satisfies a Laplace equation of order $d - 1$. Since the $(d-1)$-osculating spaces of $V(n, d)$ have the expected dimension, this means that the space that I determines meets the $(d-1)$-osculating space $\mathbb{T}_x^{(d-1)}V(n, d)$ for all

$x \in V(n,d)$. As pointed out in [16], §4, when I is as in Lemma 3.15, i.e., all monomials in I except $x_0^d, \dots, x_n^d$ are multiple of one variable, there is a point $p \in V(n,d)$ such that the intersection of I with the $(d-1)$-osculating space at p meets all the other $(d-1)$-osculating spaces. These Togliatti systems are called in [16] trivial of type B.

For instance, if $t = \binom{n+d-2}{n-1}$ and $F_1, \dots, F_t$ are any general monomials of degree $d-1$, the ideal

$$I = \left(x_0^d, \dots, x_n^d, x_0(F_1, \dots, F_t)\right)$$

is a minimal Togliatti system of the type just described.

Theorem 3.17 *Let $I \subset k[x_0, x_1, \dots, x_n]$ be a smooth minimal monomial Togliatti system of forms of degree $d \geq 4$. Assume that $\mu(I) = 2n+2$. Then I is trivial unless $n = 2$, and up to a permutation of the coordinates, one of the following cases holds:*

(1) *$d = 5$ and $I = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0^2x_1^2x_2, x_0x_1^3x_2)$ or $I = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^3x_2, x_0x_1x_2^3)$ or $I = (x_0^5, x_1^5, x_2^5, x_0^2x_1^2x_2, x_0^2x_1x_2^2, x_0x_1^2x_2^2)$.*

(2) *$d = 7$ and $I = (x_0^7, x_1^7, x_2^7, x_0^3x_1^3x_2, x_0^3x_1x_2^3, x_0x_1^3x_2^3)$ or $I = (x_0^7, x_1^7, x_2^7, x_0^5x_1x_2, x_0x_1^5x_2, x_0x_1x_2^5)$ or $I = (x_0^7, x_1^7, x_2^7, x_0x_1x_2^5, x_0^3x_1^3x_2, x_0^2x_1^2x_2^3)$.*

Proof Let us first assume that $n = 2$ and let $I = (x_0^d, x_1^d, x_2^d, m_1, m_2, m_3) \subset k[x_0, x_1, x_2]$ with $m_i = x_0^{a_0^i} x_1^{a_1^i} x_2^{a_2^i}$ and $\sum_{j=0}^2 a_j^i = d$ be a minimal smooth Togliatti system. We distinguish several cases:

Case 1 We assume that there is $0 \leq j \leq 2$ such that $a_j^1, a_j^2, a_j^3 \geq 2$. Wlog we can assume $j = 0$ and $a_0^1 \geq a_0^2 \geq a_0^3 \geq 2$. Let F_{d-1} be a plane curve containing all points of A_I. Since F_{d-1} contains the d points of A_I^1 and the $d-1$ points of A_I^0, it factorizes as $F_{d-1} = L_0L_1F_{d-3}$.

Let $2 \leq i < d$, then H_i contains $d-i+1$ integral points of $d\Delta_2$; to get A_I^i, we have to remove three points, the first one from $H_{a_0^3}$, the second one from $H_{a_0^2}$ and the third one from $H_{a_0^1}$. First of all, we want to exclude that $a_0^3 < a_0^2$. Otherwise F_{d-3} has as factors $L_2, \dots, L_{a_0^3}$, but in view of minimality $H_{a_0^3}$ must be contained in A_I, which gives a contradiction. Therefore, $a_0^3 = a_0^2$ and there are two subcases to analyze separately:

(1.1) $a_0^1 = a_0^2 = a_0^3 := s \geq 2$. In this case, F_{d-1} factorizes as $F_{d-1} = L_0 \dots L_{s-1}L_{s+1} \dots L_{d-1}$ and the plane curve F_{d-1} contains all points of A_I if and only if $s = d-2$. But in this case $m_1 = x_0^{d-2}x_1^2$, $m_2 = x_0^{d-2}x_1x_2$, $m_3 = x_0^{d-2}x_2^2$, and applying Proposition 3.6, we deduce that $I = (x_0^d, x_1^d, x_2^d, m_1, m_2, m_3)$ is not a smooth Togliatti system since it violates condition (ii) of Proposition 3.6.

(1.2) $u := a_0^1 > a_0^2 = a_0^3 := s \geq 2$. In this case, $F_{d-1} = L_0L_1F_{d-3}$ and F_{d-3} contains all integral points in $\cup_{\ell=2}^{d-1} A_I^\ell$ if and only if $u = s+1$ and $(m_1, m_2, m_3) = x_0^s x_1^a x_2^{d-1-a-s}(x_0, x_1, x_2)$ for a suitable $a \geq 0$. Therefore, I is a trivial smooth Togliatti system.

Case 2 We assume $a_j^i \geq 1$ for all i, j and that for all $0 \leq j \leq 2$ there exists $1 \leq i_j \leq 3$ such that $a_j^{i_j} = 1$. We distinguish 4 subcases, and a straightforward computation allows us to conclude:

(2.1) $(x_0^d, x_1^d, x_2^d, x_0^{d-2}x_1x_2, x_0x_1^{d-2}x_2, x_0x_1x_2^{d-2})$ is a smooth minimal Togliatti system if and only if $d = 5$ or 7.

(2.2) $(x_0^d, x_1^d, x_2^d, x_0^{d-2}x_1x_2, x_0x_1^{d-2}x_2, x_0^ax_1^bx_2^c)$ with $(a, b, c) \neq (1, 1, d-2)$ is a smooth minimal Togliatti system if and only if $d = 5$ and $(a, b, c) = (2, 2, 1)$.

(2.3) $(x_0^d, x_1^d, x_2^d, x_0x_1x_2^{d-2}, x_0^ax_1^bx_2, x_0^ex_1^fx_2^g)$ with $a, b \geq 2$ and $(e, f, g) \neq (d-2, 1, 1), (1, d-2, 1), (1, 1, d-2)$ is a smooth minimal Togliatti system if and only if $d = 7$, $(a, b) = (3, 3)$ and $(e, f, g) = (2, 2, 3)$.

(2.4) $(x_0^d, x_1^d, x_2^d, x_0x_1^ax_2^b, x_0^cx_1x_2^e, x_0^fx_1^gx_2)$ is a smooth minimal Togliatti system if and only if $d = 5$ and $a = b = c = e = f = g = 2$ or $d = 7$ and $a = b = c = e = f = g = 3$.

Case 3. We assume that there exists $a_{j_0}^{i_0} = 0$, $a_j^i \geq 1$ for all $(i, j) \neq (i_0, j_0)$ and that for all $0 \leq j \leq 2$ there exists $1 \leq i_j \leq 3$ such that $a_j^{i_j} = 1$. The smoothness criterion (Proposition 3.6) implies that, up to permutation of the coordinates, we have $m_1 = x_1^{d-1}x_2$ and we can assume $m_2 = x_0^ax_1x_2^b$ and $m_3 = x_0^ux_1^vx_2^w$ with $a, b, u, v, w \geq 1$ and I is never a smooth minimal Togliatti system.

Case 4. We assume that there exists $a_{j_0}^{i_0} = a_{j_1}^{i_1} = 0$ and that for all $0 \leq j \leq 2$ there exists $1 \leq i \leq 3$ such that $a_j^i \leq 1$. The smoothness criterion (Proposition 3.6) implies that, up to permutation of the coordinates, we have $m_1 = x_1^{d-1}x_2$, $m_2 = x_0^{d-1}x_2$ and $m_3 = x_0^ax_1^bx_2^c$ which does not correspond to a smooth minimal Togliatti system.

Let us now assume that $n \geq 3$. We want to prove that all minimal smooth monomial Togliatti systems $I \subset k[x_0, \dots, x_n]$ of forms of degree $d \geq 4$ with $\mu(I) = 2n+2$ are trivial. This time we distinguish two cases:

Case 1. For all $0 \leq j \leq n$, $\#\{i \mid a_j^i \geq 1\} \leq 2$. This implies that each variable x_j appears explicitly in exactly two of the monomials $m_1, \dots, m_{n+1}$. Equivalently, looking at the simplex, the $n+1$ integral points to remove from $d\Delta_n$ to get A_I are all on the exterior facets, and on each facet, there are exactly $n-1$ points. We consider now the restriction of the hypersurface F_{d-1} to a facet, we apply Theorem 3.9, and we get that the corresponding $n-1$ monomials, together with the dth powers of the corresponding variables, form a trivial Togliatti system in n variables of the form described in Remark 3.11. This gives a contradiction, so this case is impossible.

Case 2. There exists $0 \leq j \leq n$ such that $\#\{i \mid a_j^i \geq 1\} \geq 3$. Wlog we can assume $a_0^1 \geq a_0^2 \geq \cdots \geq a_0^{n+1} \geq 0$ and $a_0^3 \geq 1$. Therefore, in view of Lemma 3.15 $a_0^{n+1} > 0$.

This means that all monomials $m_1, \dots, m_{n+1}$ contain x_0. We consider the restrictions of $x_0^d, \dots, x_n^d, m_1, \dots, m_{n+1}$ to the hyperplane $x_n = x_0 + \cdots + x_{n-1}$, and they are linearly dependent by assumption. But in $(x_0+\cdots+x_{n-1})^d$, there is some monomial not containing x_0 that cannot cancel with the others, so its coefficient in a null linear combination must be 0, and by consequence, also the coefficients of $x_1^d, \dots, x_{n-1}^d$ are 0. This implies that the monomials $m_1, \dots, m_{n+1}$ divided by x_0, together with $x_0^{d-1}, \dots, x_n^{d-1}$, form again a Togliatti system but of degree one less, with the same properties. So we can proceed by induction on the degree, until we arrive to $d = 4$. Now we have to prove that there is no hypersurface F_3 of degree 3 containing all points of A_I unless I is a trivial monomial Togliatti system. Since $a_0^{n+1} > 0$, $F_3 = L_0F_2$ and F_2 contains all points of $A_I \setminus A_I^0$. This is possible if and only if I is trivial of type $(x_0^d, \dots, x_n^d) + (x_0, \dots, x_n)m$ where m is a monomial of degree $d-1$ involving at least 2 variables. □

Remark 3.18 In Theorem 3.17, we did not use the smoothness assumption in the cases with $n \geq 3$.

To complete the results of Theorems 3.9 and 3.17, in next Proposition we give a criterion to distinguish the smooth ones among the trivial Togliatti systems. To have a complete picture, we also include systems with number of generators bigger than $\rho(n,d)$.

Proposition 3.19 *Let I be a trivial Togliatti system of the form $(x_0,\dots,x_n)m+(x_0^d,\dots,x_n^d)$, where m is a monomial. Then I is smooth if and only if one of the following happens (up to permutation of the variables):*

(1) *$d=2$ and $n=2$ or $n=3$;*
(2) *$d=3$, $n=2$, $m=x_0^2$;*
(3) *$d\geq 4$, $n=2$, $m=x_0^{d-1}$ or $m=x_0^{i_0}x_1^{i_1}x_2^{i_2}$ with $i_0\geq i_1\geq i_2>0$;*
(4) *$d\geq 4$, $n\geq 3$, $m=x_0^{d-1}$ or $m=x_0^{i_0}x_1^{i_1}\dots x_n^{i_n}$ with $i_0\geq i_1\geq\dots\geq i_n\geq 0$ and $i_2>0$.*

Proof If $d=2$, we may assume that $m=x_0$. If $n=2$, then X is a point. Hence, the system I is smooth. Assume $n\geq 3$. After cutting the points of I from Δ, it remains $A_I=A_I^0$, which is the $(n-1)$-dimensional simplex minus the n vertices. Through each vertex of the polytope P_I, there are $2(n-2)$ edges. Then the system is singular unless $n=3$. Indeed by Proposition 3.6, (1), for X to be smooth the number of edges emanating from each vertex must be equal to $n-1$.

If $d=3$, then m can be x_0^2, or x_0x_1. If $n=2$, the first case is smooth, because P_I is a trapezium, and the second one is singular: indeed, we cut from Δ the whole edge $x_0^3-x_1^3$. So an edge of P_I is $x_0^2x_2-x_1^2x_2$, but the central point $x_0x_1x_2$ does not belong to A_I. Therefore this edge gives a singularity. If $n\geq 3$ both cases are singular: the first one because through the vertices of P_I adjacent to x_i^3 there are more than n edges and the second one because P_I contains the 1-dimensional faces for $n=2$.

Now assume $d\geq 4$ and $n=2$. If $m=x_0^{d-1}$, then the system is clearly smooth. If $m=x_0^{d-2}x_1$, then it is singular because the situation is as in Fig. 1. If $m=x_0^{d-i}x_1^{i-1}$ with $i>2$, the system is singular because in the edge $x_0^d-x_1^d$ of P_I we have to cut two points in the middle. Finally if $m=x_0^{i_0}x_1^{i_1}x_2^{i_2}$, with i_0,i_1,i_2 all strictly positive, we get a smooth system because the points of I are all inner points in P_I.

If $d\geq 4$ and $n\geq 3$, then if m is x_0^{d-1}, the system is smooth; if $m=x_0^{d-2}x_1$ or $m=x_0^{d-i}x_1^{i-1}$ with $i>2$ the system is singular, because P_I has a 2-dimensional face which is singular. Finally if m contains at least 3 of the variables, the system is smooth: indeed, on the 1-dimensional edges of P_I, there are no points of I, while on the faces of P_I of dimension at least 2 the points of I are in the interior. □

4 Number of generators of a minimal Togliatti system

We consider now the range comprised between $\mu^s(n,d)$ and $\rho^s(n,d)$ (resp. $\mu(n,d)$ and $\rho(n,d)$) and ask whether all values are reached.

Next Proposition gives a rather precise picture in the case $n=2$.

Proposition 4.1 *With notation as in Sect. 3, we have:*

(1) *For any $d\geq 4$, $\mu^s(2,d)=\mu(2,d)=5$.*
(2) *For any $d\geq 4$, $\rho^s(2,d)=\rho(2,d)=d+1$.*
(3) *For any $d\geq 4$ and any $5\leq r\leq d+1$, there exists $I\in\mathcal{T}^s(2,d)$ with $\mu(I)=r$.*

Proof (1) It follows from Theorem 3.9. (2) By definition, we have $\rho(2,d) \leq d+1$ for any $d \geq 4$. The inequality $\rho^s(2,d) \geq d+1$ (and, hence, $\rho^s(2,d) = \rho(2,d) = d+1$) will follow from (3). (3) For any $d \geq 4$ and for any $5 \leq r \leq d+1$, we consider the ideals

$$I_5 = \left(x_0^d, x_1^d, x_2^d\right) + x_0^{d-1}(x_1, x_2), \text{ and for } r > 5$$
$$I_r = \left(x_0^d, x_1^d, x_2^d\right) + x_0^{d-r+3}x_1x_2\left(x_0^{r-5}, x_0^{r-6}x_1, \ldots, x_0x_1^{r-6}, x_1^{r-5}, x_2^{r-5}\right).$$

We have $\mu(I_r) = r$ and it follows from Propositions 3.4 and 3.6 that $I_r \in \mathcal{T}^s(2,d) \subset k[x_0, x_1, x_2]$, which proves what we want. □

Remark 4.2 Proposition 4.1 does not generalize to the case $n \geq 3$, i.e., not all values of r, $\mu^s(n,d) \leq r \leq \rho^s(n,d)$, occur as the minimal number of generators of a smooth Togliatti system $I \in \mathcal{T}^s(n,d)$. The first case is illustrated in next Lemma for the case $d = 3$ and next Proposition for the general case $d \geq 4$.

Lemma 4.3 *Assume $n \geq 4$ and let I be a minimal Togliatti system of cubics. Then, $\mu(I) \geq 2n+1$. In addition, we have:*

(1) $\mu(I) = 2n+1$ *if and only if I is trivial, i.e., up to permutations of the coordinates, $I = (x_0^3, \ldots, x_n^3) + x_0^2(x_1, \ldots, x_n)$. In particular, $I \in \mathcal{T}(n,3) \setminus \mathcal{T}^s(n,3)$.*
(2) $\mu(I) = 2n+2$ *if and only if I is trivial, i.e., up to permutations of the coordinates, $I = (x_0^3, \ldots, x_n^3) + x_ix_j(x_0, \ldots, x_n)$ with $i \neq j$. In particular, $I \in \mathcal{T}(n,3) \setminus \mathcal{T}^s(n,3)$.*
(3) $\mu(I) \neq 2n+3$.

Proof We proceed by induction on n. With Macaulay2 ([7]), we easily check that $\mu(I) \geq 9$ for any $I \in \mathcal{T}(4,3)$. Assume now $n \geq 5$ and suppose that the result is true for $n-1$. We take $I = (x_0^3, \ldots, x_n^3, m_1, \ldots, m_{n-1})$ with $m_i = x_0^{a_0^i} \ldots x_n^{a_n^i}$, $a_0^i + \cdots + a_n^i = 3$, and we will see that there is no hyperquadric F_2 containing all points of A_I. Assume it exists and we will get a contradiction. Wlog we can assume that x_0 appears explicitly in the monomial m_1 and A_I^0 is equal to $3\Delta_{n-1}$ minus n vertices and at most $n-2$ other points. By induction, no hyperquadric in $x_1, \ldots, x_n$ contains A_I^0. So F_2 decomposes as $F_2 = L_0F_1$, and since there is no hyperplane F_1 containing all the points of $A_I \setminus A_I^0$, we get a contradiction.

Let us now classify all Togliatti systems $I \in \mathcal{T}(n,3), n \geq 4$, with $2n+1 \leq \mu(I) \leq 2n+3$.

(1) Assume $n = 4$, $I \in \mathcal{T}(4,3)$ and $\mu(I) = 2n+1$. Using Macaulay2 we get that I is trivial. Suppose now $n \geq 5$, let $I = (x_0^3, \ldots, x_n^3, m_1, \ldots, m_n) \in \mathcal{T}(n,3)$ with $m_i = x_0^{a_0^i}x_1^{a_1^i} \ldots x_n^{a_n^i}$ and $\sum_{j=0}^n a_j^i = 3$, and let F_2 be a hyperquadric passing through the points of A_I. Wlog we can assume $a_0^1, a_0^2 \geq 1$. Therefore, F_2 factorizes as $F_2 = L_0L_1$, and since F_2 cannot miss any point of A_I, we must have $A_I^2 = \emptyset$ which forces $m_1 = x_0^2x_1, \ldots, m_n = x_0^2x_n$, and hence, I is trivial.

(2) Using Macaulay2, we prove that if $n = 4$, $I \in \mathcal{T}(4,3)$ and $\mu(I) = 2n+2$ then I is trivial. Suppose now $n \geq 5$ and let $I = (x_0^3, \ldots, x_n^3, m_1, \ldots, m_{n+1})$ with $m_i = x_0^{a_0^i}x_1^{a_1^i} \ldots x_n^{a_n^i}$ and $\sum_{j=0}^n a_j^i = 3$. Wlog we can assume $a_0^1 \geq \ldots \geq a_0^{n+1} \geq 0$ and $a_0^1 > 0$. If $a_0^3 > 0$, then $a_0^{n+1} > 0$ by Lemma 3.15 and $F_2 = L_0F_1$ where F_1 is a hyperplane containing all points of $A_I \setminus A_I^0$. This is possible if and only if I is trivial of type $I = (x_0^3, \ldots, x_n^3) + x_ix_j(x_1, \ldots, x_n)$ with $i \neq j$. If $a_0^3 = 0$, then using hypothesis of induction together with the fact that $a_0^1 > 0$ we get that the restriction of $x_0^3, \ldots, x_n^3, m_1, \ldots, m_{n+1}$ to the hyperplane $x_0 = 0$ is trivial of type $(x_1^3, \ldots, x_n^3) + x_1^2(x_2, \ldots, x_n)$ or $(x_1^3, \ldots, x_n^3) + x_ix_j(x_1, \ldots, x_n)$ with $1 \leq i < j \leq n$. Therefore, either $I = (x_0^3, x_1^3, \ldots, x_n^3) + x_1^2(x_2, \ldots, x_n)$ or

$I=(x_0^3,x_1^3,\dots,x_n^3)+x_ix_j(x_1,\dots,x_n)$ with $1\le i<j\le n$, and none of them belongs to $\mathcal{T}(n,3)$.

(3) Again using Macaulay 2, we prove that the result is true for $n=4$. Suppose now $n\ge 5$ and let $I=(x_0^3,\dots,x_n^3,m_1,\dots,m_{n+2})$ with $m_i=x_0^{a_0^i}x_1^{a_1^i}\dots x_n^{a_n^i}$ and $\sum_{j=0}^n a_j^i=3$. Wlog we can assume $a_0^1\ge\dots\ge a_0^{n+2}\ge 0$ and $a_0^1>0$. If $a_0^4>0$, then $a_0^{n+1}>0$ by Lemma 3.15 and $F_2=L_0F_1$, but this is impossible since there is no a hyperplane containing all points of $A_I\setminus A_I^0$ and no point of $3\Delta_n\setminus A_I$ a part from the vertices. If $a_0^4=0$, then using hypothesis of induction together with the fact that $a_0^1>0$ we get that the restriction of $x_0^3,\dots,x_n^3,m_1,\dots,m_{n+2}$ to the hyperplane $x_0=0$ is trivial of type $(x_1^3,\dots,x_n^3)+x_1^2(x_2,\dots,x_n)$ or $(x_1^3,\dots,x_n^3)+x_ix_j(x_1,\dots,x_n)$, $1\le i<j\le n$, or $(x_1^3,\dots,x_n^3)+x_1^2(x_2,\dots,x_n)+(x_{i_1}x_{i_2}x_{i_3})$, $1\le i_1\le \imath_2\le i_3\le n$ or $(x_1^3,\dots,x_n^3)+x_ix_j(x_1,\dots,x_n)+(x_{i_1}x_{i_2}x_{i_3})$, $1\le i<j\le n$, $1\le i_1\le \imath_2\le i_3\le n$. Therefore, $I=(x_0^3,x_1^3,\dots,x_n^3)+x_1^2(x_2,\dots,x_n)$ or $I=(x_0^3,x_1^3,\dots,x_n^3)+x_ix_j(x_1,\dots,x_n)$, $1\le i<j\le n$ or $I=(x_0^3,x_1^3,\dots,x_n^3)+x_1^2(x_2,\dots,x_n)+(x_{i_1}x_{i_2}x_{i_3})$, $1\le i_1\le \imath_2\le i_3\le n$ or $I=(x_0^3,x_1^3,\dots,x_n^3)+x_ix_j(x_1,\dots,x_n)+(x_{i_1}x_{i_2}x_{i_3})$, $1\le i<j\le n$, $1\le i_1\le \imath_2\le i_3\le n$, and none of them belongs to $\mathcal{T}(n,3)$. □

Proposition 4.4 *Let $n\ge 3$ and $d\ge 4$. Then there is no $I\in\mathcal{T}^s(n,d)$ with $\mu(I)=2n+3$.*

Proof We distinguish two cases:

(1) For all $0\le j\le n$, $\#\{i\mid a_j^i\ge 1\}\le 3$, i.e., every variable appears in at most three of the monomials $m_1,\dots,m_{n+2}$.
If one of the monomials contains all the variables, the other $n+1$ monomials contain two variables each, and we are in the same situation of Theorem 3.17, Case 1, which is impossible. Therefore no monomial contains all variables, and at least two variables appear in three monomials. Assume that x_0 appears in three monomials; then F_{d-1} passes through the integral points of A_I^0. Recall that A_I^0 is equal to $d\Delta_{n-1}$ minus the n vertices and $n-1$ other points. So the removed points form a Togliatti system I' in the n variables $x_1,\dots,x_n$ with $\mu=2n-1$ and we can apply Theorem 3.9. There are two possibilities:

(1.1) $n=3$ and I' is one of the two special Togliatti systems of degree 5 or 4 of Theorem 3.9. If $d=5$, up to permutation of the variables the only possibility is $I=(x_0^5,\dots,x_3^5,x_0^4x_2,x_0^4x_3,x_1^3x_2x_3,x_1x_2^2x_3^2,x_0^ax_1^b)$ with $a,b>0$. But it is easy to check that this is not a Togliatti system. In the case $d=4$, there are two possibilities: $I=(x_0^4,\dots,x_3^4,x_0^3x_2,x_0^3x_3,x_1x_2x_3^2,x_1^2x_2^2,x_0^ax_1^bx_3^c)$ with $a,b>0$, $c\ge 0$, or $(x_0^4,\dots,x_3^4,x_0^2x_1x_2,x_1x_2x_3^2,x_1^2x_2^2,x_0^ax_3^b,x_0^cx_3^d)$ with $a,b,c,d>0$. Both systems are not Togliatti.

(1.2) I' is of the form $(x_1^d,\dots,x_n^d)+x_1^{d-1}(x_2,\dots,x_n)$. In this case x_1 appears in at least $n-1$ monomials, therefore $n=3$ or $n=4$.
If $n=3$, the other three monomials in I are either of the form $x_0^{d-1}(x_2,x_3)$, $x_0^ax_1^bx_2^c$, or of the form $x_0^{d-1}(x_1,x_3)$, $x_0^ax_2^bx_3^c$, with $a>0$, $b>0$, $c\ge 0$. It is immediate to check that they are not Togliatti systems. If $n=4$, then the six monomials $m_1,\dots,m_6$ are of the form $x_0^{d-1}(x_2,x_3,x_4)$, $x_1^{d-1}(x_2,x_3,x_4)$. Also in this case the system is not Togliatti.

(2) There exists an index j such that $\#\{i\mid a_j^i\ge 1\}\ge 4$, i.e., one of the variables appears in at least 4 monomials. We can assume $j=0$. Therefore, by Lemma 3.15, x_0 appears in all monomials $m_1,\dots,m_{n+2}$. Let $m_i'=m_i/x_0$, $i=1,\dots,n+2$. As in the proof

of Theorem 3.17, case 2, we observe that $m'_1, \dots, m'_{n+2}$, together with $x_0^{d-1}, \dots, x_n^{d-1}$, form a Togliatti system I_1 of degree $d-1$. We distinguish the following possibilities:

(2.1) at least one of the monomials m'_i is the $(d-1)$th power of a variable, so $\mu(I_1) < 2n+3$; or
(2.2) $\mu(I_1) = \mu(I) = 2n+3$.

In case (2.1), if $d > 4$, I_1 is trivial, which implies that I contains a trivial Togliatti system and therefore is non-minimal: contradiction. If $d = 4$, $I_1 \in \mathcal{T}(n, 3)$ and $\mu(I_1) \leq 2n+2$. In case (2.2), we can apply the above argument to I_1, and so on, by induction.
In any case, applying repeatedly this procedure, possibly involving different variables, we arrive at a Togliatti system I_1 of degree $d = 3$ with $\mu \leq 2n+3$, which is obtained from I dividing the monomials $m_1, \dots, m_{n+2}$ by a common monomial factor M. If $n = 3$, we conclude with the help of Macaulay2. If $n \geq 4$, by Lemma 4.3, I_1 is trivial of type $(x_0^3, \dots, x_n^3) + x_0^2(x_1, \dots, x_n)$ or $(x_0^3, \dots, x_n^3) + x_i x_j (x_0, \dots, x_n)$. In both cases, I is not minimal and we are done.

□

Remark 4.5 If $n = 3$ and $d = 4$, one can check with the help of Macaulay2 that there exist two types of minimal Togliatti systems I with $\mu(I) = 2n+3 = 9$, both non-smooth, precisely $(x_0^4, x_1^4, x_2^4, x_3^4) + x_0^2(x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3)$ and $(x_0^4, x_1^4, x_2^4, x_3^4) + x_0^2(x_1^2, x_1x_2, x_2^2, x_0x_3, x_3^2)$.

We note that if $d = 2$ the ideal $I = (x_0, x_1)^2 + (x_2, x_3, x_4, x_5)^2$, with $\mu(I) = 2n+3 = 13$, belongs to $\mathcal{T}^s(5, 2)$, while if $d = 3$ then $2n+3 < \mu^s(n, 3)$ for any $n \geq 4$.

Computations made with Macaulay2 illustrate the complexity of the general case. However, some ranges and some sporadic values can be covered. For example:

Example 4.6 For any $d > n \geq 3$ and for any r, $\binom{d+n-2}{n-2} + n + 2 \leq r \leq \binom{d+n-2}{n-2} + d + 1$, there exists $I \in \mathcal{T}^s(n, d)$ with $\mu(I) = r$. (Notice that when $n = 3$ we have $d + 6 \leq r \leq 2d + 2$). In fact, it is enough to take

$$I = (x_0, x_1, \dots, x_{n-2})^d + (x_{n-1}^d, x_n^d) + (x_{n-1}, x_n)^{d-h} m'$$

where $2 \leq h \leq d - n + 1$ and m' is a monomial of degree h containing only $x_0, \dots, x_{n-2}$.

Nevertheless if we delete the smoothness hypothesis, we can generalize Proposition 4.1 and we get

Proposition 4.7 *With the above notation, we have:*

(1) *For any $d \geq 4$, $\mu(n, d) = 2n + 1$.*
(2) *For any $d \geq 4$, $\rho(n, d) = \binom{n+d-1}{n-1}$.*
(3) *For any $d \geq 4$, $n = 3$ and any integer r with $\mu(3, d) = 7 \leq r \leq \rho(3, d) = \binom{d+2}{2}$, there exists $I \in \mathcal{T}(3, d)$ with $\mu(I) = r$.*

Proof (1) It follows from Theorem 3.9.
(2) By definition we have $\rho(n, d) \leq \binom{n+d-1}{n-1}$ for any $d \geq 4$. Let us prove that $\rho(n, d) \geq \binom{n+d-1}{n-1}$, i.e., there exists $I \in \mathcal{T}(n, d)$ with $\mu(I) = \binom{n+d-1}{n-1}$. Consider

$$I = (x_0^d, x_1^d, \dots, x_n^d) + x_1(x_1, \dots, x_n)^{d-1} + x_2(x_2, \dots, x_n)^{d-1} + \dots + x_{n-2}(x_{n-2}, x_{n-1}, x_n)^{d-1} + x_0^3(x_{n-1}, x_n)^{d-3}.$$

We have

$$\begin{aligned}\mu(I) &= n + 1 + \sum_{i=2}^{n-1} [\binom{d-1+i}{i} - 1] + d - 2 \\ &= d + 1 + \sum_{i=2}^{n-1} \binom{d-1+i}{i} \\ &= \sum_{i=0}^{n-1} \binom{d-1+i}{i} \\ &= \binom{d-1+n}{n-1}.\end{aligned}$$

When we substitute x_0 by $x_1 + x_2 + \cdots + x_n$, the $\mu(I)$ generators of I become k-linearly dependent; so I fails WLP in degree $d-1$ (Theorem 2.3) and I is minimal because no proper subset of the generators of I defines a Togliatti system. Therefore, $I \in \mathcal{T}(n, d)$.

(3) Assume $n = 3$. For $r = 7$ we take $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0^{d-1}(x_1, x_2, x_3)$, for $r = 8$ we take $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0^{d-2}x_1(x_0, x_1, x_2, x_3)$ and for $r = 9$ we take $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0^{d-2}(x_1^2, x_0x_1, x_2^2, x_2x_3, x_3^2)$.

We will now proceed by induction on d. In the case $d = 4$, we exhibit an explicit example for any $10 \le r \le 14$ (note that the case $r = 15$ is covered by the example given in (2)):

- $r = 10$: $(x_0, x_1)^4 + (x_2, x_3)^4$ (smooth);
- $r = 11$: $(x_0, x_1)^4 + (x_2^4, x_2^3x_3, x_2^2x_3^2, x_3^4, x_0x_2x_3^2, x_1x_2x_3^2)$;
- $r = 12$: $(x_0, x_1)^4 + (x_2^4, x_2^3x_3, x_2x_3^3, x_3^4, x_0^2x_3^2, x_0x_1x_3^2, x_1^2x_3^2)$;
- $r = 13$: $(x_0, x_1)^4 + (x_2^4, x_2^3x_3, x_2x_3^3, x_3^4, x_0^3x_3, x_0^2x_1x_3, x_0x_1^2x_3, x_1^3x_3)$;
- if $r = 14$: the systems described in Remark 3.16 work in this case.

We suppose now $d > 4$ and we will prove that for any $7 \le r \le \binom{d+2}{2}$ there exists $I \in \mathcal{T}(3, d)$ with $\mu(I) = r$.

Indeed, for any $7 \le s \le \binom{d+1}{2}$ we take $J \in \mathcal{T}(3, d-1)$ with $\mu(J) = s$ and we define $I = (x_0^d, x_1^d, x_2^d) + x_3 J$. Note that $I \in \mathcal{T}(3, d)$ and $10 \le \mu(I) = \mu(J) + 3 \le \binom{d+1}{2} + 3$. Observe also that $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0(x_1, x_2, x_3)^{d-1} \in \mathcal{T}(3, d)$ and $\mu(I) = \binom{d+1}{2} + 4$. So, it only remains to cover the values of r, $\binom{d+1}{2} + 4 < r \le \binom{d+2}{2}$. To this end, for any $3 \le i \le d-1$ we define

$$I_i = (x_0^d, x_1^d, x_2^d, x_3^d) + (x_1^{i_1}x_2^{i_2}x_3^{i_3} \mid i_1 + i_2 + i_3 = d,\ 1 \le i_1 < d) + x_0^i(x_2, x_3)^{d-i}.$$

First of all we observe that $\mu(I_i) = \binom{d+2}{2} + 3 - i$. Therefore, when i ranges from $i = 3$ to $d-1$, we sweep the interval $[\binom{d+1}{2} + 5, \binom{d+2}{2}]$. By Proposition 3.4 to prove that $I_i \in \mathcal{T}(3, d)$, it is enough to show that there is a surface F_{d-1} of degree $d-1$ containing all integral points of A_{I_i}. Since $A_{I_i}^1 = (d-1)\Delta_2, \ldots, A_{I_i}^{i-1} = (d-i+1)\Delta_2$, we have $F_{d-1} = L_1 \ldots L_{i-1} F_{d-i}$ where F_{d-i} is a surface of degree $d-i$ containing all integral points of $A_{I_i} \setminus \cup_{j=1}^{i-1} A_{I_i}^j$. The surfaces F_{d-i} of degree $d-i$ are parametrized by a k-vector space of dimension $\binom{d-i+3}{3}$. On the other hand, to contain the aligned $d-1$ points of $A_{I_i}^0$ imposes $d-i+1$ conditions on the surfaces of degree $d-i$, to contain the points of $A_{I_i}^{i+1} = (d-i-1)\Delta_2, \ldots, A_{I_i}^{d-1} = \Delta_2$ imposes $\binom{d-i+1}{2}, \ldots, 3$ conditions, respectively, and finally to contain the points of $A_{I_i}^i$ imposes $\binom{d-i+2}{2} - (d-i+1)$ conditions. Summing up we have $\binom{d-i+3}{3} - 1$ conditions. Therefore, there exists at least a surface F_{d-i} of degree $d-i$ through all integral points of $A_{I_i} \setminus \cup_{j=1}^{i-1} A_{I_i}^j$ and, hence a surface $F_{d-1} = L_1 \ldots L_{i-1} F_{d-i}$ of degree $d-1$ containing all integral points of A_{I_i}. □

Remark 4.8 For $n = 3, d = 4$, with Macaulay2 we have obtained the list of all minimal Togliatti systems with $\mu(I) \le 13$. The computations become too heavy for $\mu = 14, 15$.

5 On the stability of the associated syzygy bundles

In this section, we restrict our attention to the case $n = 2$ and we will analyze whether the syzygy bundle E_I on $\mathbb{P}^2$ associated with a minimal smooth monomial Togliatti system $I \in \mathcal{T}(2, d)$ is μ-(semi)stable.

Definition 5.1 A syzygy bundle $E_{d_1,\dots,d_r}$ on $\mathbb{P}^n$ is a rank $r-1$ vector bundle defined as the kernel of an epimorphism

$$(f_1, \dots, f_r) : \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-d_i) \longrightarrow \mathcal{O}_{\mathbb{P}^n}$$

where $(f_1, \dots, f_r) \subset k[x_0, x_1 \dots, x_n]$ is an artinian ideal, and $d_i = deg(f_i)$. When $d_1 = d_2 = \dots = d_r = d$, we write $E_{d,n}$ instead of $E_{d_1,\dots,d_r}$.

Definition 5.2 Let E be a vector bundle on $\mathbb{P}^n$ and set

$$\mu(E) := \frac{c_1(E)}{rk(E)}.$$

The vector bundle E is said to be μ-semistable in the sense of Mumford–Takemoto if $\mu(F) \leq \mu(E)$ for all nonzero subsheaves $F \subset E$ with $rk(F) < rk(E)$; if strict inequality holds, then E is μ-stable.

Note that for a rank s vector bundle E on $\mathbb{P}^n$, with $(c_1(E), s) = 1$, the concepts of μ-stability and μ-semistability coincide.

Using Klyachko results on toric bundles ([11–13]), Brenner deduced the following nice combinatoric criteria for the (semi)stability of the syzygy bundle $E_{d_1,\dots,d_r}$ in the case where the associated forms $f_1, \dots, f_r$ are all monomials. Indeed, we have

Proposition 5.3 *Let $I = (m_1, \dots, m_r) \subset k[x_0, \dots, x_n]$ be a monomial artinian ideal. Set $d_i = \deg(m_i)$. Then the syzygy bundle $E_{d_1,\dots,d_r}$ on $\mathbb{P}^n$ associated with I is μ-semistable (resp. μ-stable) if and only if for every $J = (m_{i_1}, \dots, m_{i_s}) \subsetneq I$, $s \geq 2$, the inequality*

$$\frac{d_J - \sum_{j=1}^s d_{j_i}}{s-1} \leq \frac{-\sum_{i=1}^r d_i}{r-1} \qquad (resp.\ <) \tag{2}$$

holds, where d_J is the degree of the greatest common factor of the monomials $m_{j_i} \in J$.

Proof See [3] Proposition 2.2 and Corollary 6.4. □

Example 5.4 (1) If we consider the monomial artinian ideal $I := (x_0^5, x_1^5, x_2^5, x_0^2x_1^2x_2) \subset k[x_0, x_1, x_2]$, inequality (2) is strictly fulfilled for any proper subset $J \subsetneq \{x_0^5, x_1^5, x_2^5, x_0^2x_1^2x_2\}$. Therefore the syzygy bundle E associated with I is μ-stable.

(2) If we consider the monomial artinian ideal $I := (x_0^5, x_1^5, x_2^5, x_0^4x_1) \subset k[x_0, x_1, x_2]$, then for the subset $J := \{x_0^5, x_0^4x_1\}$ inequality (2) is not fulfilled. Therefore the syzygy bundle E_I associated with I is not μ-stable. In fact, the slope of E_I is $\mu(E_I) = -20/3$ and the syzygy sheaf F associated with J is a subsheaf of E_I with slope $\mu(F) = -6$. Since $\mu(F) \nleq \mu(E_I)$, we conclude that E is not μ-stable.

Remark 5.5 Let I be a monomial artinian ideal generated by r monomials $m_1, \dots, m_r$ of degree d. It easily follows from the above proposition that the syzygy bundle $E_{d,n}$ on $\mathbb{P}^n$ associated to I is μ-(semi)stable if and only if for every subset $J = \{m_{i_1}, \dots, m_{i_s}\} \subsetneq \{m_1, \dots, m_r\}$ with $s := |J| \geq 2$,

$$(d - d_J)r + d_J - sd > 0 \quad (\text{resp. } \geq 0), \tag{3}$$

where d_J is the degree of the greatest common factor of the monomials in J.

Theorem 5.6 *Let $I \subset k[x_0, x_1, x_2]$ be a smooth minimal monomial Togliatti system of forms of degree $d \geq 4$. Assume that $\mu(I) \leq 6$. Let E_I be the syzygy bundle associated with I. We have:*

(a) *E_I is μ-stable if and only if, up to a permutation of the coordinates, one of the following cases holds:*

 (1) *$\mu(I) = 5$, $d = 5$ and $I_1 = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^2x_2^2)$.*
 (2) *$\mu(I) = 6$, $d = 7$ and $I_2 = (x_0^7, x_1^7, x_2^7, x_0^3x_1^3x_2, x_0^3x_1x_2^3, x_0x_1^3x_2^3)$ or $I_3 = (x_0^7, x_1^7, x_2^7, x_0^5x_1x_2, x_0x_1^5x_2, x_0x_1x_2^5)$ or $I_4 = (x_0^7, x_1^7, x_2^7, x_0x_1x_2^5, x_0^3x_1^3x_2, x_0^2x_1^2x_2^3)$.*

(b) *E_I is properly μ-semistable if and only if, up to a permutation of the coordinates, one of the following cases holds:*

 (1) *$\mu(I) = 6$, $d = 5$ and $I_5 = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^3x_2, x_0x_1x_2^3)$.*
 (2) *$\mu(I) = 6$, $d = 5$ and $I_6 = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0^2x_1^2x_2, x_0x_1^3x_2)$ or $I_7 = (x_0^5, x_1^5, x_2^5, x_0^2x_1^2x_2, x_0^2x_1x_2^2, x_0x_1^2x_2^2)$.*

(c) *In all other cases, E_I is unstable.*

Proof First of all, by Theorem 3.9, we have $\mu(I)$ =5 or 6. Using the classification of Togliatti systems $I \in \mathcal{T}(2, d)$ with $5 \leq \mu(I) \leq 6$ given in Theorems 3.9 and 3.17, it is enough to check:

(1) I_i, $1 \leq i \leq 4$ corresponds to μ-stable bundles.
(2) I_i, $5 \leq i \leq 7$ corresponds to properly μ-semistable bundles.
(3) Trivial Togliatti systems $I \in \mathcal{T}(2, d)$ correspond to μ-unstable bundles.

To prove (1) it is enough to observe that inequality (3) is strictly fulfilled for any proper subset $J_i \subsetneq I_i$, $1 \leq i \leq 4$, with $|J_i| \geq 2$.

To prove (2) we check that inequality (3) is satisfied for any proper subset $J_i \subsetneq I_i$, $5 \leq i \leq 7$, with $|J_i| \geq 2$ and there is a subset $J_i^0 \subsetneq I_i$, $5 \leq i \leq 7$, with $|J_i^0| \geq 2$ and verifying $(d - d_{J_i^0})\mu(I_i) + d_{J_i^0} - d\mu(J_i^0) = 0$. For instance, for $I_6 = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0^2x_1^2x_2, x_0x_1^3x_2)$ it is enough to take $J_6^0 = (x_0^3x_1x_2, x_0^2x_1^2x_2) \subset I_6$ since $(d - d_{J_6^0})\mu(I_6) + d_{J_6^0} - d\mu(J_6^0) = (5 - 4)6 + 4 - 2 \times 5 = 0$.

(3) Finally let us check that the syzygy bundle E_I associated with trivial Togliatti systems $I = (x_0, x_1, x_2)m + (m_1, \ldots, m_{r-3}) \in \mathcal{T}(2, d)$ is always μ-unstable. Note that m is a monomial of degree $d - 1$ and m_i, $1 \leq i \leq r - 3$, are monomials of degree d. For the subset $J = (x_0m, x_1m, x_2m) \subset I$ inequality (3) becomes $(d - (d - 1))r + (d - 1) - 3d > 0$ and E_I is μ-unstable. Indeed, the slope of E_I is $\mu(E_I) = \frac{dr}{r-1}$ and the syzygy sheaf F associated with J is a subsheaf of E_I with slope $\mu(F) = \frac{3(d-1)}{2}$. Therefore, $\mu(F) \not\leq \mu(E_I)$ and we conclude that E_I is μ-unstable. □

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