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A. JOSEPH

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THE MINIMAL ORBIT IN A SIMPLE LIE ALGEBRA AND ITS ASSOCIATED MAXIMAL IDEAL

By A. JOSEPH

ABSTRACT. — Let $\mathfrak{g}$ be a simple Lie algebra over **C**. If $\mathfrak{g}$ is different from $sl(n+1): n = 1, 2, \ldots$, then $\mathfrak{g}^*$ admits a single non-trivial G-orbit $\mathcal{O}_0$ of minimal dimension. This orbit consists of nilpotent elements, contains $\mathfrak{g}^\beta - \{0\}$, where $\mathfrak{g}^\beta$ is the root subspace of the highest root β, and is not polarizable. Through the study of a certain Heisenberg subalgebra of $\mathfrak{g}$ associated with $\mathfrak{g}^\beta$, it is shown that there exists a unique completely prime, two-sided ideal J_0 of U ($\mathfrak{g}$) whose characteristic variety coincides with $\mathcal{O}_0 \cup \{0\}$. J_0 is shown to be a maximal ideal and that it cannot be induced from any proper subalgebra of $\mathfrak{g}$. The construction of J_0 is very explicit and its central character is computed. For sp (4), J_0 coincides with an ideal constructed by Conze and Dixmier [8] (Ex. 3).

1. Introduction

Let **C** denote the complex numbers and $\mathfrak{g}$ a finite dimensional Lie algebra over **C**. This work arose out of an attempt to construct so-called minimal realizations [20] of $\mathfrak{g}$ from quantum canonical variables. Now as B. Kostant points out to me the companion problem in classical mechanics is implicity solved through the existence of a non-degenerate, closed, antisymmetric two-form [2] (Chap. II), defined on any given G-orbit (G = exp ad $\mathfrak{g}$) of the dual $\mathfrak{g}^*$. Furthermore when the given orbit $\mathcal{O}$ is polarizable [27] (Remarks 4.3.1 and 4.3.2), the corresponding classical realization admits "quantization," a process which associates with $\mathcal{O}$ a two-sided ideal in the enveloping algebra U ($\mathfrak{g}$) of $\mathfrak{g}$.

If $\mathfrak{g}$ is solvable, all orbits in $\mathfrak{g}^*$ are polarizable [11] (Prop. 1.12.10), and this is also true for $\mathfrak{g} = sl(n)$ [30] (Prop. 6.1). Yet if $\mathfrak{g}$ is simple and different from $sl(n)$, then $\mathfrak{g}^*$ admits a single non-trivial orbit $\mathcal{O}_0$ of minimal dimension in $\mathfrak{g}^*$ and this is not polarizable. (Other non-regular orbits may not be polarizable. For example the short root eigenvector in G_2 generates a non-minimal, non-polarizable orbit.)

In a natural fashion our previous construction [20] associates with $\mathcal{O}_0$ a unique completely prime two-sided ideal J_0 in U ($\mathfrak{g}$). More specifically we show (Sect. 4) that U ($\mathfrak{g}$)/J_0 admits a unique embedding in the enveloping field of the tangent space to $\mathcal{O}_0$ identified with a subalgebra of $\mathfrak{g}$. A simple explicit formula for this embedding in given in Section 5. It turns out that J_0 is a primitive ideal and in Section 6, we determine its central character.

* Work supported by the C.N.R.S.

In Section 7, we show that J_0 is always a maximal ideal and in Section 8 that it is never an induced ideal. In Section 9 we show that the Weyl group acts through the automorphism group of the embedding field, a fact which gives an alternative and more elegant proof of the existence of the embedding. Finally in Section 10, we show that J_0 is the unique completely prime two-sided ideal whose characteristic variety is $\mathcal{O}_0 \cup \{0\}$. This suggest a generalization of quantization for non-polarizable orbits.

INDEX OF NOTATION. — Symbols frequently used in the text are given below in order of appearance.

Section 1 : $\mathbf{C}$, $\mathfrak{g}$, G, $\mathfrak{g}^*$, $\mathcal{O}$, U ($\mathfrak{g}$), $\mathcal{O}_0$, J_0.

Section 2 : $\mathfrak{h}$, Δ, π, $\mathfrak{g}^\alpha$, $\mathfrak{n}$, $|.|$, β, Δ_λ^+, π_λ, π_λ^c, $\mathfrak{g}_\lambda$, $\mathfrak{p}_\lambda$, Γ, Γ_0, $\mathfrak{g}^\Gamma$, W, $k(\alpha)$, Γ_1, Γ_2, $\mathfrak{n}_\beta$, Γ_3, B.

Section 3 : $\mathcal{O}_X$, $\mathcal{S}$, $l(\mathfrak{g})$, $k(\mathfrak{g})$, E_β.

Section 4 : $\mathfrak{r}$, $\mathfrak{s}$, $\mathcal{A}_n$, $\mathcal{A}'_n$, Φ.

Section 5 : $N_{\alpha,\beta}$, F_α, $\mathcal{F}$, σ, D, θ, ρ.

Section 6 : $\mathcal{O}_0^W$, M_λ, M_s.

Section 7 : $\mathcal{H}$, Z ($\mathfrak{g}$).

Section 8 : $\mathfrak{a}$, $\mathfrak{m}$, $\mathcal{A}^\wedge(\mathfrak{m})$, μ, $\mathcal{B}$, P, B_f^P, χ_E.

Section 9 : $\mathcal{R}$.

Section 10 : $\mathcal{V}$ (J).

2. The Highest Root

Let $\mathfrak{g}$ be a simple Lie algebra with fixed Cartan subalgebra $\mathfrak{h}$. Relative to $\mathfrak{h}$, let Δ (resp. Δ^+, Δ^-) denote the set of all non-zero (resp. positive, negative) roots with π a simple system corresponding to Δ^+. Let $\mathfrak{g}^\alpha$ be the root subspace for the root α and set $\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^\alpha$, $\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}^\alpha$. Given $\alpha \in \Delta$, let $|\alpha|$ denote the sum of its coefficients with respect to π. Recall [6] (pp. 198-199), that Δ admits a unique highest root β (i. e. $|\beta| \geqq |\alpha| : \alpha\in\Delta^+$).

Let n be a positive integer and recall that a Heisenberg Lie algebra $\mathfrak{a}_n$ is a Lie algebra with generators X_i, Y_i, Z and relations $[X_i, Y_i] = Z : i = 1, 2, \ldots, n$, and where all other brackets vanish. In [19] we identified an important Heisenberg subalgebra of $\mathfrak{n}$ associated with β. For *sl* (n) this had previously been noticed by Dixmier and in the general case had also been discovered by Kostant [28] and independently by Tits [33]. Here we develop some further properties of this subalgebra which we require later on.

Let $\mathfrak{h}_{\mathbf{R}}^*$ denote the real dual of $\mathfrak{h}$ in which Δ is defined and set

$$\mathcal{D} = \{\lambda\in\mathfrak{h}_{\mathbf{R}}^* : (\lambda, \alpha_i) \geqq 0, \text{ for all } \alpha_i\in\pi\}.$$

Note that $\beta\in\mathcal{D}$. Given $\lambda\in\mathcal{D}$, set $\Delta_\lambda^\pm = \{\alpha\in\Delta^\pm : (\lambda, \alpha) = 0\}$, $\pi_\lambda = \pi\cap\Delta_\lambda^+$ and π_λ^c the complement of π_λ in π. Let $\mathbf{N}$ denote the natural numbers.

LEMMA 2.1. — π_λ *generates* Δ_λ^+ *over* **N**.

Proof. — Immediate from the definition of $\mathscr{D}$.

The conclusion of the lemma can be expressed by saying that Δ_λ^+ is a positive root system for the semisimple subalgebra $\mathfrak{g}_\lambda$ obtained from the Dynkin diagram for $\mathfrak{g}$ by suppressing the simple roots not contained in π_λ. Note also that $\mathfrak{p}_\lambda = \mathfrak{g}_\lambda + \mathfrak{h} + \mathfrak{n}^-$ is a parabolic subalgebra of $\mathfrak{g}$ with reductive part $\mathfrak{g}_\lambda + \mathfrak{h}$. For the highest root β, the corresponding $\mathfrak{g}_\beta$ can be recovered from the extended Dynkin diagram for $\mathfrak{g}$ [6] (p. 198).

Set $\Gamma = \{ \gamma \in \Delta : (\gamma, \beta) > 0 \}$. Obviously $\Gamma \subset \Delta^+$ and is the complement of Δ_β^+ in Δ^+. Set $\Gamma_0 = \{ \gamma \in \Gamma : \gamma \neq \beta \}$.

LEMMA 2.2. — *For all* $\gamma \in \Gamma_0$, *we have* $(\gamma, \beta) = 1/2\,(\beta, \beta)$.

Proof. — Choose $\gamma \in \Gamma_0$. Since β is the highest root, $\gamma+\beta$, $\gamma - 2\beta$ are not roots. Yet $(\gamma, \beta) > 0$, so $\gamma - \beta$ is a root and the assertion follows from [16] [(18), p. 116].

Set $\mathfrak{g}^\Gamma = \text{lin span} \{ \mathfrak{g}^\gamma : \gamma \in \Gamma \}$.

COROLLARY 2.3. — $\mathfrak{g}^\Gamma$ *is a Heisenberg Lie algebra with centre* $\mathfrak{g}^\beta$.

Proof. — Given $\gamma \in \Gamma_0$, $\beta - \gamma$ is a root and by Lemma 2.2, $\beta - \gamma \in \Gamma_0$. Again given γ, $\delta \in \Gamma_0$ for which $\gamma + \delta$ is a root, then from the definition of Γ, we have $\gamma + \delta \in \Gamma$. Yet by Lemma 2.2, $(\beta, \gamma+\delta) = 1/2\,(\beta, \beta) + 1/2\,(\beta, \beta)$, so $\gamma + \delta = \beta$. This proves the assertion.

The semisimple Lie algebra $\mathfrak{g}_\beta$ need not be simple. Let $\Delta_\beta^+ = \bigcup_i \Delta_{\beta i}^+$ be a decomposition of Δ_β^+ into simple components. Let β' be a highest root for $\Delta_{\beta 1}^+$ and set $\Gamma' = \{ \gamma \in \Delta_{\beta_1}^+ : (\gamma, \beta') > 0 \}$.

LEMMA 2.4. — *For each* $\delta \in \Gamma'$, *there exists* $\gamma \in \Gamma$, *such that* $(\gamma, \delta) > 0$.

Proof. — For each $\alpha \in \pi_\beta^c$, $\delta - \alpha$ is not a root, so $(\alpha, \delta) \leqq 0$. If $(\alpha, \delta) \neq 0$, take $\gamma = \alpha$. Otherwise, note that for some $\alpha \in \pi_\beta^c$ we have $(\alpha, \beta') < 0$. Now again, $\beta' - \delta$ is a root, so let n be the largest positive integer such that $\delta_n = \beta' - n\,\delta$ is a root. Then $(\delta, \delta_n) < 0$ and $(\delta_n, \alpha) = (\beta', \alpha) < 0$. Hence $\gamma = \alpha + \delta_n$ is a root and satisfies the requirements of the lemma.

LEMMA 2.5. — *There exists* $\gamma \in \Gamma$ *such that* $(\gamma, \beta') > 0$.

Proof. — Recall that $(\alpha, \beta') < 0$ for some $\alpha \in \pi_\beta^c$, and let n be the largest positive integer such that $\gamma = \alpha + n\,\beta'$ is a root. Then $\gamma \in \Gamma$ and $(\gamma, \beta') > 0$, as required.

PROPOSITION 2.6. — *Let* ω *be a weight for a finite dimensional module* M *of* $\mathfrak{g}$. *Suppose for all* $\gamma \in \Gamma$, *that* $\omega + \gamma$ *is not a weight for* M. *Then* $\omega = 0$, *or* $\omega - \beta$ *is a weight for* M.

Proof. — If $\omega - \beta$ is not a weight, then $(\omega, \beta) = 0$. Suppose $\omega - \gamma$ is a weight for some $\gamma \in \Gamma_0$. Then $(\beta, (\omega - \gamma)) = -(\beta, \gamma) < 0$, so $\omega - \gamma + \beta$ is a weight. Yet $\beta - \gamma \in \Gamma_0$, so this contradicts the hypothesis of the proposition. Hence $(\omega, \gamma) = 0$, for all $\gamma \in \Gamma$.

Suppose $\omega-\beta'$ is a weight. By Lemma 2.5, there exists $\gamma \in \Gamma$, such that $(\gamma, \beta') > 0$. Then $((\omega-\beta'), \gamma) = -(\beta', \gamma) < 0$. Hence $\omega+(\gamma-\beta')$ is a weight, which contradicts the fact that $\gamma-\beta' \in \Gamma$.

Suppose $\omega+\delta$ is a weight for some $\delta \in \Gamma'$. By Lemma 2.4, there exists $\gamma \in \Gamma$ such that $(\gamma, \delta) < 0$. Then $(\gamma, \omega+\delta) = (\gamma, \delta) < 0$, so $\omega+(\gamma+\delta)$ is a weight, which contradicts the fact that $\gamma+\delta \in \Gamma$.

The first part of the proof now establishes that $(\omega, \delta) = 0$, for all $\delta \in \Gamma'$. Hence by the obvious induction we obtain $(\omega, \delta) = 0$, for all $\delta \in \Delta^+$. That is $\omega = 0$.

COROLLARY 2.7. — *Let α be a root. If $\alpha+\gamma$ is not a root for all $\gamma \in \Gamma$, then $\alpha = \beta$.*

Proof. — Obviously $\alpha \neq 0$, so by Proposition 2.6, it follows that $\alpha-\beta$ is a root. Since β is the highest root, it follows that $\alpha \in \Delta^+$. If $\alpha \neq \beta$, then $\alpha \in \Gamma_0$, and by Lemma 2.2, $\gamma = \beta-\alpha \in \Gamma_0$. Then $\alpha+\gamma$ is a root which contradicts the hypothesis.

Call a root $\alpha \in \Delta$ long if $(\alpha, \alpha) \geqq (\alpha', \alpha')$, for all $\alpha' \in \Delta$. Recall that β is always a long root (in fact this follows from Lemma 2.2) and that any two long roots are conjugate under the Weyl group W. Again recall that for each simple Lie algebra and for any $\alpha \in \Delta$, the quantity $(\beta, \beta)/(\alpha, \alpha)$ is a positive integer which can have at most two values. In particular, call $\mathfrak{g}$ simply-laced, if $(\beta, \beta) = (\alpha, \alpha)$ for all $\alpha \in \Delta$. If $\mathfrak{g}$ is not simply-laced, call a root $\alpha \in \Delta$ short if $(\alpha, \alpha) < (\beta, \beta)$. Recall that any two shorts roots are conjugate under the Weyl group.

PROPOSITION 2.8. — *If $\mathfrak{g}$ is not simply-laced, then Γ_0 admits a short root.*

Proof. — Let $\alpha \in \Delta^+$ be a short root. By Corollary 2.7, there exists $\gamma \in \Gamma_0$ such that $\alpha+\gamma$ is a root. If γ, $\gamma+\alpha$ are long, then $(\alpha, \alpha)+2(\alpha, \gamma) = 0$, so

$$4(\alpha, \gamma)^2/(\alpha, \alpha)(\gamma, \gamma) < 1$$

which contradicts [16] [(18), p. 116]. So $\alpha+\gamma$ is short and $\alpha+\gamma \in \Gamma_0$, since β is long.

A subset $\beta_1, \beta_2, \ldots, \beta_r \in \Delta$ is said to be a strongly orthogonal set of roots if $\beta_i \pm \beta_j$ is not a root for all pairs i, j. For example, the sequence of roots obtained by taking the highest root of Δ, the corresponding highest roots of Δ_{β_i} and so on, is a strongly orthogonal set. Kostant points out to me that any orthogonal set of roots determines a strongly orthogonal set (by taking sums and differences) and any maximal strongly orthogonal set is unique up to W. This can be proved as follows.

LEMMA 2.9. — *Let $\beta_1, \beta_2, \ldots, \beta_r \in \Delta$ be a maximal strongly orthogonal set of roots. Then at least one β_i is long.*

Proof. — Obviously we can assume that $\mathfrak{g}$ is not simply-laced. Since the assertion can be verified for G_2 by inspection, it remains to consider B_n, C_n, F_4. Now in each such case, if $\alpha \in \Delta$ is long and if the β_i are all short, then $(\alpha, \beta_i)/(\beta_i, \beta_i) = 0, \pm 1$. Furthermore from the orthogonality of the β_i, it is evident that

$$\beta_{r+1} = \alpha - \sum_{i=1}^{r} k_i \beta_i : \quad k_i = \frac{(\alpha, \beta_i)}{(\beta_i, \beta_i)}$$

is a root. Now $(\beta_i, \beta_{r+1}) = 0$ for all i, so by maximality of $\{ \beta_i \}$, it follows that $\beta_{r+1} = 0$. Then $\alpha = \sum k_i \beta_i$ with $k_i \neq 0$ for at least one i, say $i = 1, 2, \ldots, s$. Then $(\beta_s, \alpha) = k_s \neq 0$, so $\alpha - k_s \beta_s$ is a root and by induction $k_1 \beta_1 + k_2 \beta_2$ is a root which contradicts strong orthogonality.

COROLLARY 2.10. — *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then any two maximal strongly orthogonal sets are conjugate under* W.

Proof. — It is enough to prove this for the simple components of $\mathfrak{g}$. The proof is by induction on rank $\mathfrak{g}$, the case rank $\mathfrak{g} = 1$ being trivial. Let $\{ \beta_i \}_{i=1}^r$ be a maximal strongly orthogonal set for $\mathfrak{g}$. By Lemma 2.9, we can assume that β_1 (say) is long, so $\beta = \beta_1$ to within W. Then $\beta_i \in \Delta_\beta^+ \cup \Delta_\beta^-$, for all $i > 1$, so $\{ \beta_i \}_{i=2}^r$ is a strongly orthogonal set for $\mathfrak{g}_\beta$ which is evidently maximal. Since rank $\mathfrak{g} >$ rank $\mathfrak{g}_\beta$, the proof is completed.

Obviously r of the lemma satisfies $r \leqq$ rank $\mathfrak{g}$ with equality only if -1 is in the Weyl group (actually if and only if [28]). These numbers are listed in [19] (Sect. 6, Table 1). They coincide with number of independent generators of cent U (n) which, incidentally, is a polynomial algebra. The latter was proved for *sl* (n) by Dixmier [9] (Thm. 1, 4) and in the general case in an unpublished result of Kostant [28] and [31]. A proof is given in [19] (Thm. 6.6).

For each $\alpha \in \Delta$, let $k(\alpha)$ denote the sum of the coefficients of the $\alpha_i \in \pi_\beta^c$. By Lemma 2.1, $k(\delta) = 0$, for all $\delta \in \Delta_\beta^+ \cup \Delta_\beta^-$.

LEMMA 2.11. — $k(\beta) = 1$, *if and only if* $|\beta| = 1$. *Otherwise* $k(\beta) = 2$ *and* $k(\gamma) = 1$, *for all* $\gamma \in \Gamma_0$.

Proof. — $|\beta| = 1$ implies $k(\beta) = 1$ trivially. We show $k(\beta) = 2$ otherwise. By Corollary 2.3, this will also prove the second part.

Take any $\alpha \in \pi_\beta^c$. Since $|\beta| > 1$, we have $\beta \neq \alpha$ and so by Lemma 2.2, $\gamma_1 = \beta - \alpha \in \Gamma_0$. Now there exists $\alpha' \in \pi$ such that $\gamma_2 = \gamma - \alpha'$ is a non-negative root. If $\alpha' \in \pi_\beta^c$, then $\gamma_2 \in \{ \Delta_\beta^+, 0 \}$ by Lemma 2.2, and so $k(\beta) = 2$. Otherwise $\gamma_2 \in \Gamma_0$ and reapplying the argument to γ_2 eventually proves $k(\beta) = 2$.

COROLLARY 2.12. — Card $\pi_\beta^c = 1$ *or* 2.

When card $\pi_\beta^c = 2$, we have a natural decomposition of Γ_0 into two disjoint sets Γ_1, Γ_2. That is if we write $\gamma = \sum_{i=1}^{n} k_i \alpha_i : \alpha_1, \alpha_n \in \pi_\beta$. Then $\Gamma_1 = \{ \gamma \in \Gamma_0 : k_1 = 1 \}$, $\Gamma_2 = \{ \gamma \in \Gamma_0 : k_n = 1 \}$. Actually card $\pi_\beta^c = 2$, only for *sl* $(n+1) : n \geqq 2$, and this case is very special. It is an empirical fact that when card $\pi_\beta^c = 1$, then $|\beta|$ is an odd integer. In this case, we set

$$\Gamma_1 = \left\{ \gamma \in \Gamma_0 : |\gamma| < \frac{1}{2} |\beta| \right\}, \qquad \Gamma_2 = \left\{ \gamma \in \Gamma_0 : |\gamma| > \frac{1}{2} |\beta| \right\}.$$

LEMMA 2.13. — *For all* $\alpha \in \Delta^+$, $\alpha \notin \Gamma_1 \cup \beta$, *there exists* $\gamma \in \Gamma_1$, *such that* $\alpha+\gamma$ *is a root.*

Proof. — Assume card $\pi_\beta^c = 1$. Since $\alpha \neq \beta$, by assumption, these exists by Corollary 2.7, a root $\delta \in \Gamma_1 \cup \Gamma_2$, such that $\alpha+\delta$ is a root. If $\delta \notin \Gamma_1$, then $\delta \in \Gamma_2$ and $\delta+\alpha \in \Gamma_2$, since $|\alpha| > 0$ and $\alpha \notin \Gamma_1$. Then by Corollary 2.3, there exists $\gamma_1, \gamma_2 \in \Gamma_1$ such that $\gamma_1+\delta = \beta$, $\gamma_2+\delta+\alpha = \beta$. Hence $\gamma_1 = \gamma_2+\alpha$, which proves the assertion.

Assume card $\pi_\beta^c = 2$. Then it is easily verified that for all $\alpha \in \Delta$, $\alpha \notin \Gamma_1 \cup \beta$, there exists $\gamma \in \Gamma_1$, such that $\alpha+\gamma$ is a root.

COROLLARY 2.14. — $\beta \cup \Gamma_1$ *spans* h^*.

Proof. — Choose $\lambda \in \mathfrak{h}^*$ non-zero and suppose that $(\lambda, \gamma) = 0$, for all $\gamma \in \beta \cup \Gamma_1$. Since π spans $\mathfrak{h}^*$, there exists $\alpha \in \pi$, such that $(\alpha, \lambda) \neq 0$. By Lemma 2.13, there exists $\gamma \in \Gamma_1$ such that $\alpha+\gamma$ is a root and it follows that $\alpha+\gamma \in \Gamma_2$. Then by Corollary 2.3, there exists $\gamma' \in \Gamma_1$, such that $\alpha+\gamma+\gamma' = \beta$, which leads to an easy contradiction.

Set $\mathfrak{g}^{\Gamma_i} = \text{lin span}\{\mathfrak{g}^\gamma : \gamma \in \Gamma_i\}$, for $i = 0, 1, 2$, and $\mathfrak{g}^{\Gamma_1 \cup \beta} = \mathfrak{g}^{\Gamma_1} \oplus \mathfrak{g}^\beta$. Obviously $\mathfrak{g}^{\Gamma_1 \cup \beta}$ is commutative. When card $\pi_\beta^c = 2$, $\mathfrak{g}^{\Gamma_1 \cup \beta}$ is complemented in $\mathfrak{g}$ by a subalgebra $\mathfrak{p}$ (which is in fact a maximal parabolic subalgebra of $\mathfrak{g}$). However from the theory of parabolics, it follows from Lemma 2.11, that this must fail if card $\pi_\beta^c = 1$. An eventual consequence of this is the fact that J_0 (*see* Sects. 1,8) is not an induced ideal for $\mathfrak{g} \neq sl(n+1)$. Yet we do have the following weaker fact. Set $\mathfrak{n}_3 = \text{lin span}\{\mathfrak{g}^\alpha : \alpha \in \Delta_\beta^+\}$.

LEMMA 2.15. — *Suppose* card $\pi_\beta^c = 1$ *and let* $\alpha \in \pi_\beta^c$. *Then* $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{g}^{-\alpha}$ *is a subalgebra of* $\mathfrak{g}$. *Furthermore* $\mathfrak{g}^{\Gamma_1 \cup \beta}$ *is a complemented in* $\mathfrak{k}$ *by the subalgebra*

$$\mathfrak{l} = \mathfrak{g}^{\Gamma_2} \oplus \mathfrak{n}_\beta \oplus \mathfrak{h} \oplus \mathfrak{g}^{-\alpha}.$$

Proof. — Apply Lemma 2.11.

LEMMA 2.16. — (card $\pi_\beta^c = 1$). *For all* $\gamma \in \Gamma_0 : \gamma \notin \pi_\beta^c$, *there exists* $\alpha_i \in \pi_\beta$ *such that* $\gamma-\alpha_i \in \Gamma_0$. *In particular* $\mathfrak{g}^{\Gamma_0}$ *is a simple* $\mathfrak{g}_\beta$ *module.*

Remark. — This is just special case of a general well-known fact about parabolics. We give the proof for completion.

Proof. — Suppose that $\gamma-\alpha_i$ is not a root for all $\alpha_i \in \pi_\beta$. Then $\gamma-\alpha : \alpha \in \pi_\beta^c$ must be a root and since (Lemma 2.11) $k(\gamma) = 1$, we have

$$\alpha = \gamma - \sum_{\alpha_i \in \pi_\beta^c} k_i \alpha_i : \quad k_i \geqq 0, \text{ and integer.}$$

Since $\gamma \neq \alpha$, at least one $k_i > 0$. This contradicts [12] (Thm. 5.1), which implies that$(\gamma, \alpha_1, \alpha_2, \ldots, \alpha_r) : r = \text{card}\,\pi_\beta$, form a simple system of roots for some regular semisimple subalgebra of $\mathfrak{g}$.

Set $\Gamma_3 = \{\gamma \in \Gamma_2 : |\gamma| = 1/2(|\beta|+1)\}$.

COROLLARY 2.17. — Γ_3 *is non-empty.*

Remark. — Card $\Gamma_3 = 1$, for G_2, $C_n : n \geqq 2$, card $\Gamma_3 = 2$ for B_n, $D_{n+1} : n \geqq 3$, card $\Gamma_3 = 3$ for E_8.

LEMMA 2.18. — *Set* $B = \Gamma_2 \cup -\Gamma_1 \cup \Delta_\beta^+ \cup -\beta$. *Then there exists a unique* $\omega \in W$ *such that* $\omega^{-1}(B) = \Delta^+$.

Proof. — Observe that $B \cup -B = \Delta$. Hence by [16] (Thm. 2, p. 242), it is enough to show that B admits a hyperplane of support though the origin. Now if no such hyperplane exists, then by Caratheodory's theorem and the rationality of the Cartan matrix, there exist positive integers k_i such that $\sum_{i \in I} k_i \alpha_i = 0 : \alpha_i \in B$. From Lemma 2.11 and the definition of Γ_1, Γ_2, it is easily checked that $B+B \subset B$. Now for any $i \in I$, we have $(\alpha_i, \alpha_j) < 0$ for some $j \in I$, so $\alpha_i + \alpha_j \in B$ and by induction we obtain the contradiction $0 \in B$.

LEMMA 2.19. — *Let* B_0 *be the subset of* B *generated additively by* $\Gamma_2 \cup -\beta$. *Then* $B_0 \supset -\Gamma_1$.

Proof. — Apply Corollary 2.3.

3. Orbits of Minimal Dimension in $\mathfrak{g}^*$

Assume $\mathfrak{g}$ simple. Here we characterize the non-trivial orbits of minimal dimension in $\mathfrak{g}^*$. The results are fairly well-known; but we give proofs for completion.

Recall that $\mathfrak{g}^*$ identifies with $\mathfrak{g}$ through the Killing form B. Furthermore for each $X \in \mathfrak{g}$, it follows by [11] (1.11.11), that codim $\mathfrak{g}^X = \dim \mathcal{O}_X$. Hence to characterize minimal orbits, it suffices to determine the conjugacy classes of the set

$$\mathcal{S} = \{X \in \mathfrak{g} - \{0\} : \dim \mathfrak{g}^X \geqq \dim \mathfrak{g}^Y, \text{ for all } Y \in \mathfrak{g} - \{0\}\}.$$

Define for each simple Lie algebra the numbers

$$k(\mathfrak{g}) = \frac{1}{2}(\operatorname{card} \Gamma + 1),$$

$$l(\mathfrak{g}) = \operatorname{Inf}\{\operatorname{codim} \mathfrak{p} : \mathfrak{p} \text{ a proper parabolic subalgebra of } \mathfrak{g}\}.$$

The numbers $k(\mathfrak{g})$, $l(\mathfrak{g})$ are listed in [20], Table (1), inspection of this and the root tables, [6] (pp. 250-275), gives.

LEMMA 3.1. — (1) $k(\mathfrak{g}) \leqq l(\mathfrak{g})$ *with equality if and only if* $\mathfrak{g} = sl(n+2) : n \in \mathbf{N}$; (2) *Suppose that,* $1 + \operatorname{codim} \mathfrak{p} \leqq 2k(\mathfrak{g})$, *with* $\mathfrak{p}$ *parabolic. Then* $\mathfrak{p}$ *is maximal, or* $\mathfrak{g} = sl(n+3) : n \in \mathbf{N}$, *and equality holds. If* $\mathfrak{p}$ *is maximal and the simple root defining* $\mathfrak{p}$ *has coefficient* $k > 1$ *in* β, *then equality holds.*

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(1) As W. Borho has pointed out to me, the entries for E_7, E_8 were incorrectly given and should read :

	$k(\mathfrak{g})$	$l(\mathfrak{g})$
E_7.....	17	27
E_8.....	29	57

Restrict B to $\mathfrak{h}$ and given $\lambda \in \mathfrak{h}^*$ non-zero, define $H_\lambda \in \mathfrak{h}$ uniquely, through

$$B(H, H_\lambda) = \langle \lambda, H \rangle,$$

for all $H \in \mathfrak{h}$.

LEMMA 3.2. — *Let* $H \in \mathfrak{g} - \{0\}$ *be semisimple. Then* $\dim \mathcal{O}_H \geqq 2\, l(\mathfrak{g})$.

Remark. — Equality can hold.

Proof. — Recall that $\mathfrak{g}^H$ contains a Cartan subalgebra, so $H \in \mathfrak{h}$, up to conjugacy. Write $H = H_\lambda : \lambda \in \mathfrak{h}^*$ and set $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in \mathfrak{h}^*_{\mathbf{R}}$. Clearly $\mathfrak{g}^{H_\lambda} = \mathfrak{g}^{H_{\lambda_1}} \cap \mathfrak{g}^{H_{\lambda_2}}$, with $\mathfrak{g}^{H_{\lambda_1}} = \mathfrak{g}^{H_{\lambda_2}}$ only if λ_1, λ_2 are proportional, so it is enough to consider λ real. Then up to conjugacy we may write $H = H_\lambda$ for some unique $\lambda \in \mathcal{D}$. By Lemma 2.1, $\dim \mathfrak{g}^{H_\lambda} = \operatorname{rank} \mathfrak{g} + 2 \operatorname{card} \Delta_\lambda^+$, which gives, $\dim \mathcal{O}_{H_\lambda} = \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g} - 2 \operatorname{card} \Delta_\lambda^+$. Yet $\operatorname{card} \Delta_\lambda^+ = \dim \mathfrak{p}_\lambda - 1/2 (\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g})$, so $\dim \mathcal{O}_H = 2 (\dim g - \dim \mathfrak{p}_\lambda)$. Recalling that $\mathfrak{p}_\lambda$ is a parabolic subalgebra, this gives the assertion of the lemma.

LEMMA 3.3. — *Let* $E \in \mathfrak{g} - \{0\}$ *be nilpotent. Then* $\dim \mathcal{O}_E \geqq 2\, k(\mathfrak{g})$ *with equality if and only if* E *is conjugate to a fixed non-zero vector in* $\mathfrak{g}^\beta$.

Proof. — By the Jacobson-Morosov theorem [25] (Thm. 3.4), there exist $H, F \in \mathfrak{g}$ such that (E, H, F) span an $sl(2)$ subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Let $\mathfrak{p}$ be the parabolic subalgera of $\mathfrak{g}$ with reductive part $\mathfrak{g}^H$. Up to conjugacy $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}^-$ (*cf.* [11], Prop. 1.10.20). Decompose $\mathfrak{g}$ as a direct sum of simple $\mathfrak{t}$-modules and let t be the number of $\mathfrak{g}_i$ having even dimension. Then $\dim \mathfrak{g}^E = \dim \mathfrak{g}^H + t$. Now $\mathfrak{t} = \mathfrak{g}_j$, for some j, and so the relation $\dim \mathfrak{g}^H + 2 \operatorname{codim} \mathfrak{p} = \dim \mathfrak{g}$ established in Lemma 3.2 implies that $t \leqq \operatorname{codim} \mathfrak{p} - 1$. Hence $\dim \mathcal{O}_E \geqq 1 + \operatorname{codim} \mathfrak{p}$. Suppose $\mathfrak{g} \neq sl(n)$. By Lemma 3.1 (2), the relation $\dim \mathcal{O}_E \leqq 2\, k(\mathfrak{g})$ implies that $\mathfrak{p}$ is maximal and so defined by some $\alpha \in \pi$. If the highest root has coefficient $k = 1$ in α, then $t = 0$ and $\dim \mathcal{O}_E = 2 \operatorname{codim} \mathfrak{p} \geqq 2\, l(\mathfrak{g}) > 2\, k(\mathfrak{g})$, by Lemma 3.1 (1). Hence $k > 1$ and so by Lemma 3.1 (2), $t = \operatorname{codim} \mathfrak{p} - 1$. Hence $\mathfrak{g}^E \supset \mathfrak{n}$, and so $E = E_\beta$. Suppose $\mathfrak{g} = sl(n+3) : n \in \mathbf{N}$. If $\mathfrak{p}$ is maximal, then $t = 0$ and $\dim \mathcal{O}_E = 2 \operatorname{codim} \mathfrak{p} \geqq 2\, l(\mathfrak{g}) = 2\, k(\mathfrak{g})$. Furthermore equality determines $\mathfrak{p}$ up to outer conjugation and then it is easy to check that H cannot be of the required form. Hence $E = E_\beta$ as before. Finally application of $\exp \operatorname{ad} H : (\beta, H) \neq 0$, to a non-zero vector E_β in $\mathfrak{g}^\beta$ shows that $\mathfrak{g}^\beta - \{0\}$ is contained in a single G-orbit.

Remark. — Of course all conjugacy classes of $sl(2)$ subalgebras (and hence all nilpotent orbits) were classified by Dynkin [12] (Chap. III).

LEMMA 3.4. — *Given* $X \in \mathcal{S}$, *then* X *is either semisimple or nilpotent.*

Proof. — Recall that each $X \in \mathfrak{g}$ can be written uniquely as the sum $X = E + H$ of its nilpotent and semisimple components which lie in $\mathfrak{g}$. Furthermore $\mathfrak{g}^X = \mathfrak{g}^E \cap \mathfrak{g}^H$. If $E, H \neq 0$, expressing $\operatorname{ad}_{\mathfrak{g}} X$ in Jordan canonical form shows that $\mathfrak{g}^E \neq \mathfrak{g}^H$ and so then $X \notin \mathcal{S}$.

PROPOSITION 3.5. — *Suppose* $\mathfrak{g}$ *is simple and different from* $sl(n+1): n = 1, 2, \ldots$ *Then* $\mathscr{S}$ *consists of a single orbit containing* $\mathfrak{g}^{\beta}-\{0\}$, *which furthermore does not admit a polarization.*

Proof. — The first part follows from Lemmas 3.1.-3.4. For the second part, note that $\mathfrak{g}^{\Gamma} \oplus \mathbf{C}\,H_{\beta}$ identifies with tangent space to the point $E_{-\beta} \in \mathscr{S}$ and apply Lemma 2.16.

The situation for $sl(n+1): n = 2, 3, \ldots$, is rather different. It may be described as follows. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \pi$ be chosen such that $\alpha_1, \alpha_n \in \pi_{\beta}^c$, and let $\alpha^1, \alpha^2, \ldots, \alpha^n \in \mathscr{D}$ denote the corresponding fundamental weights. Then the parabolics with the minimal codimension $l(\mathfrak{g})$ are precisely $\mathfrak{g}_{\alpha^1}, \mathfrak{g}_{\alpha^n}$. By Lemma 3.2, the semisimple orbits of minimal dimension form a two parameter family corresponding to $c_1 H_{\alpha^1}, c_n H_{\alpha^n} : c_1, c_n \in \mathbf{C}^+$, where $\mathbf{C}^+ = \{z \in \mathbf{C}-0 : \operatorname{Re} z \geqq 0\}$. (Here we note that H_{α^1} is equivalent to $-H_{\alpha^n}$ under W). By Lemma 3.1, the minimal nilpotent orbit has the same dimension and in fact is a limit point for both families of semisimple orbits. Moreover the minimal orbits admit a polarization (indeed so do all orbits in $sl(n)$ [30], Prop. 6.1) and the inducing construction associates with them a family (parametrized by $\mathbf{C}$) of completely prime, primitive (but not necessarily maximal) two-sided ideals in $U(\mathfrak{g})$. For $sl(3)$, Dixmier [10] has further shown that the orbits in $\mathfrak{g}^*$ are in one to one correspondence with the class of completely prime, primitive two-sided ideals in $U(\mathfrak{g})$; but this bijection cannot be made continuous. In the sequel we shall generally ignore $sl(n)$.

4. The Embedding Theorem

Let $\mathfrak{g}$ be a simple Lie algebra and assume that card $\pi_{\beta}^c = 1$. Set $\mathfrak{r} = \mathfrak{g}^{\Gamma} \oplus \mathbf{C}H_{\beta}$, $\mathfrak{s} = \mathfrak{g}_{\beta}+\mathfrak{r}$. Observe that $\mathfrak{r}$ identifies with the tangent space to the point $E_{-\beta}$ on the minimal orbit $\mathscr{O}_0$. Hence to associate a two-sided ideal of $U(\mathfrak{g})$ with $\mathscr{O}_0$ it is natural to consider an embedding of $U(\mathfrak{g})$ in $U(\mathfrak{r})$. Actually some localization is required. Thus we set $E = E_{\beta}$ and $U(\mathfrak{r})_E = \{E^{-s} a : a \in U(\mathfrak{r}) : s = 0, 1, 2, \ldots\}$.

LEMMA 4.1. — $U(\mathfrak{r})_E$ *is isomorphic to a Weyl algebra* $\mathscr{A}_{n-1} \times \mathscr{A}'_1$ *of order* $n = 1/2\,(\text{card}\,\Gamma+1)$, *localized at one generator.*

Proof. — Recall [11] (Sect. 4.6.), and apply Corollary 2.3. Under this isomorphism we have $U(\mathfrak{r}) \subset \mathscr{A}_n$ and for short we write $\mathscr{A}'_n = \mathscr{A}_{n-1} \times \mathscr{A}'_1$.

LEMMA 4.2. — (Card $\pi_{\beta}^c = 1$). *There exists a unique embedding of* $U(\mathfrak{s})$ *in* $U(\mathfrak{r})_E$ *for which* $\varphi \,|\, \mathfrak{r} = \text{Id}$.

Proof. — Uniqueness. By Lemma 4.1, we obtain Cent $U(\mathfrak{r})_E = \mathbf{C}\,1$. Consequently the relations $[\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{r}$, $[H_{\beta}, \mathfrak{s}] = \{0\}$ determine $\varphi(\mathfrak{s})$ up to scalars. For each $\delta \in \Delta_{\beta}$, the relations $[H_{\delta}, E_{\delta}] = (\delta, \delta)\,E_{\delta}$ determine these scalars on $\varphi(E_{\delta})$ and the relations $[E_{\delta}, E_{-\delta}] = (\delta, \delta)\,H_{\delta}$ determine these scalars on $\varphi(H_{\delta})$.

Existence. To each $X \in \mathfrak{g}_{\beta}$, assign an element $\varphi(X) \in U(\mathfrak{g}^{\Gamma})_E$ of the form

$$\varphi(X) = \sum_{\gamma, \gamma' \in \Gamma_0} c_{\gamma\gamma'}(X)\,E^{-1}\,E_{\gamma}\,E_{\gamma'}\,; \quad c_{\gamma\gamma'}(X) \in \mathbf{C} \text{ and symmetric.}$$

Recalling that $[\mathfrak{g}_\beta, \mathfrak{g}^\Gamma] \subset \mathfrak{g}^{\Gamma_0}$, $[\mathfrak{g}_\beta, E] = 0$ and that $\mathfrak{g}^\Gamma$ is a Heisenberg Lie algebra, it is easy to check that there is a unique choice of the scalars for which $[X-\varphi(X), \mathfrak{r}] = 0$. φ is clearly linear and for all $X, Y \in \mathfrak{g}_\beta$, we have $[X-\varphi(X), Y-\varphi(Y)] = [X-\varphi(X), Y]$. Through the Jacobi identity the left-hand side commutes with $\mathfrak{g}^\Gamma$. Since ad Y leaves the $c_{\gamma\gamma'}(X)$ symmetric, their uniqueness gives $[X, \varphi(Y)] = \varphi([X, Y])$. Then resubstitution gives $[\varphi(X), \varphi(Y)] = \varphi([X, Y])$. Setting $\varphi = \mathrm{Id}$ on $\mathfrak{r}$ defines the required embedding.

Remark 1. — Existence can also be established through [11] (10.1.4).

Remark 2. — When card $\pi^c_\beta = 2$, uniqueness fails because of the relation rank $\mathfrak{g}$ − rank $\mathfrak{g}_\beta = 2$. This allows the inclusion of an additional scalar, in agreement with the conclusions given in the latter part of Section 3.

THEOREM 4.3 [$\mathfrak{g}$ *simple and different from* $sl(n+1) : n = 1, 2, \ldots$]. — *There exists a unique embedding* Φ *of* $U(\mathfrak{g})/J_0$ *in* $U(\mathfrak{r})_E$ *for which* $\Phi|\mathfrak{r} = \mathrm{Id}$.

Proof. — Uniqueness. By Lemma 4.2, Φ is uniquely determined on $U(\mathfrak{s})$. From the relations $[\mathfrak{g}^\Gamma, \mathfrak{g}^{-\Gamma_0}] \subset \mathfrak{g}_\beta \oplus \mathfrak{g}^{\Gamma_0} \oplus \mathbf{C}\, H_\beta$, it follows that Φ is determined to within an element of $\mathbf{C}[E, E^{-1}]$ on $\mathfrak{g}^{-\Gamma_0}$. Yet for all $\gamma \in \Gamma_0$, we have $[H_\beta, E_{-\gamma}] = -(\beta, \gamma) E_{-\gamma}$, where $(\beta, \gamma) = 1/2 (\beta, \beta)$ by Lemma 2.2. Hence Φ is uniquely determined on $\mathfrak{g}^{-\Gamma_0}$ and hence on $\mathfrak{g}$.

Existence. Take φ in the conclusion of Lemma 4.2. By (say) 5.3 φ extends uniquely to a linear map $\varphi : \mathfrak{s} \oplus \mathfrak{g}^{-\Gamma_0} \to U(\mathfrak{r})_E$ satisfying

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)], \quad X \in \mathfrak{r}, \quad Y \in \mathfrak{g}^{-\Gamma_0}. \tag{4.1}$$

Extend φ linearly to $\mathfrak{g}$ by setting

$$\varphi([E_{-\alpha}, E_{-\beta+\alpha}]) = [\varphi(E_{-\alpha}), \varphi(E_{-\beta+\alpha})],$$

where α is the unique simple root in π^c_β.

Set $\eta(X, Y) = [\varphi(X), \varphi(Y)] - \varphi([X, Y]) : X, Y \in \mathfrak{g}$, considered as an element of $U(\mathfrak{r})_E$. It remains to show that η vanishes. This will follow from the Jacobi identity and successive application of ad $\varphi(X) : X \in \mathfrak{r}$.

Take $X \in \mathfrak{s}$, $Y \in \mathfrak{g}^{-\Gamma_0}$, $Z \in \mathfrak{g}^\Gamma$. Through (4.1), Lemma 4.2 and the Jacobi identity, it follows that $[\varphi(Z), \eta(X, Y)] = 0$. Hence $\eta(X, Y) \in \mathbf{C}[E, E^{-1}]$. Yet by Corollary 2.3 :

$$[\varphi(H_\beta), \eta(X, Y)] = \frac{1}{2}(\beta, \beta)\eta(X, Y),$$

so

$$\eta(X, Y) = 0: \quad X \in \mathfrak{s}, \quad Y \in \mathfrak{g}^{-\Gamma_0}.$$

Now take $X \in \mathfrak{g}^{-\beta}$, $Y \in \mathfrak{r}$. From the above definition of $\varphi(E_{-\beta})$, the Jacobi identity and the established properties of φ, we obtain $\eta(E_{-\beta}, Y) = 0 : Y \in \mathfrak{r}$.

Now consider $\eta(E_{-\gamma_1}, E_{-\gamma_2}) : \gamma_1, \gamma_2 \in \Gamma_0, \quad \gamma_1+\gamma_2 \neq \beta$. Commutation with $\varphi(X) : X \in \mathfrak{r}$, shows that $\eta(E_{-\gamma_1}, E_{-\gamma_2}) \in \mathbf{C}\, E^{-1}$. Then taking $\lambda = \gamma_1+\gamma_2-\beta$, commutation with $\varphi(H_\lambda)$ shows that $\eta(E_{-\gamma_1}, E_{-\gamma_2}) = 0$. Applying ad $\varphi(E_\delta) : \delta \in \Delta_\beta$ to this last expression gives $\eta(E_\delta, E_{-\beta}) = 0$, and so we have established $\eta(X, E_{-\beta}) = 0 : X \in \mathfrak{s}$.

Now consider $\eta(E_{-\alpha}, E_{-\beta}) : \alpha \in \pi^c_\beta$. Commutation with $\varphi(Z) : Z \in \mathfrak{g}^\Gamma$ shows that $\eta(E_{-\alpha}, E_{-\beta}) \in \mathbf{C}\,[E, E^{-1})$, and then commutation with $\varphi(H_\beta)$ shows that $\eta(E_{-\alpha}, E_{-\beta}) = 0$. Application of ad $\varphi(E_\delta) : \delta \in \Delta_\beta$, to this relation, implies through Lemma 2.16 that $\eta(E_{-\alpha}, E_{-\beta}) = 0$, for all $\gamma \in \Gamma_0$.

It remains to show that $\eta(E_{-\gamma}, E_{-\beta+\gamma}) = 0 : \gamma \in \Gamma_0$. This follows from the relation

$$0 = [\varphi(E_\gamma), [\varphi(E_{-\gamma}), \varphi(E_{-\beta})]] = [\varphi([E_\gamma, E_{-\gamma}]), \varphi(E_{-\beta})] + [\varphi(E_{-\gamma}), \varphi([E_\gamma, E_{-\beta}])].$$

Remark. — Uniqueness for $sl(n+1) : n \geqq 2$ fails through Remark 2 above. Uniqueness for $sl(2)$ fails because Γ_0 is empty.

COROLLARY 4.4. — *Given a linear map* $\Psi : \mathfrak{g} \to U(\mathfrak{r})_E$ *satisfying*

$$\Psi|\mathfrak{r} = \mathrm{Id}, \tag{1}$$

$$[\Psi(X), \Psi(Y)] = \Psi([X, Y]) : \quad X \in \mathfrak{r}, \quad Y \in \mathfrak{g}, \tag{2}$$

$$[\Psi(H_\delta), \Psi(E_\delta)] = \Psi([H_\delta, E_\delta]) : \quad \delta \in \Delta_\beta, \tag{3}$$

$$[\Psi(E_\delta), \Psi(E_{-\delta})] = \Psi([E_\delta, E_{-\delta}]) : \quad \delta \in \Delta_\beta, \tag{4}$$

$$[\Psi(E_{-\alpha}), \Psi(E_{-\beta+\alpha})] = \Psi([E_{-\alpha}, E_{-\beta+\alpha}]) : \quad \alpha \in \pi^c_\beta. \tag{5}$$

Then $\Psi = \Phi$. *In particular* Ψ *extends to an embedding of* $U(\mathfrak{g})$ *in* $U(\mathfrak{r})_E$.

This result shows how many relations one must check to confirm that a given candidate Ψ is indeed an embedding. Based on this we derive an explicit formula for Φ in the next section.

5. The Embedding Construction

For all $\gamma \in \Gamma$, choose a non-zero vector $E_\gamma \in \mathfrak{g}^\gamma$ and define non-zero scalars $N_{\gamma, \beta-\gamma}$ through $[E_\gamma, E_{\beta-\gamma}] = N_{\gamma, \beta-\gamma} E_\beta$. In particular we write $E_0 = H_\beta$, so that $N_{\beta, 0} = -(\beta, \beta)$. Set $E = E_\beta$ and $F_\gamma = N^{-1}_{\gamma, \beta-\gamma} E^{-1} E_{\beta-\gamma} : \gamma \in \Gamma$. Then for all $\gamma_1, \gamma_2 \in \Gamma$:

$$[E_{\gamma_1}, F_{\gamma_2}] = \delta_{\gamma_1 \gamma_2} + \frac{1}{2} \delta_{\gamma_2 \beta}(1-\delta_{\gamma_1 \beta}) N_{\beta-\gamma_1, \gamma_1} F_{\beta-\gamma_1} \tag{5.1}$$

where $\delta_{\gamma_1 \gamma_2}$ is the Kronecker delta. Set $\mathscr{F} = \text{lin span}\,\{F_\gamma : \gamma \in \Gamma\}$, let $S(\mathscr{F})$ denote the symmetric algebra over $\mathscr{F}$ and $\sigma : S(\mathscr{F}) \to U(\mathfrak{r})_E$ the symmetrization with respect to the given basis of $\mathscr{F}$. Note that σ is independent of choice of basis and is not onto. Define $D' : \mathfrak{g} \otimes S(\mathscr{F}) \to \mathfrak{g} \otimes S(\mathscr{F})$, through

$$D' = \sum_{\gamma \in \Gamma} \mathrm{ad}\, E_\gamma \otimes F_\gamma \qquad \text{and} \qquad D : \quad \mathfrak{g} \otimes \sigma(S(\mathscr{F})) \to \mathfrak{g} \otimes \sigma(S(\mathscr{F}))$$

through D' and transport under σ, that is $D(1 \otimes \sigma) = (1 \otimes \sigma) D'$. Define $e_- \in \mathfrak{g}^*$, through $\langle e_-, E \rangle = 1, \langle e_-, E_\gamma \rangle = 0 : \gamma \in \Delta, \ \gamma \notin \beta, \langle e_-, h \rangle = 0$. That is e_- identifies with $E_{-\beta}$ under the Killing form. Similarly define $e \in \mathfrak{g}^*$ through identification with E_β under the Killing form. Define $\theta : \mathfrak{g}_{\mathbb{A}} \otimes_{\mathbb{E}} U_{\mathbb{A}}(\mathfrak{r})_E \to U(\mathfrak{r})_E$ through

$$\theta(X \otimes y) = \langle e_-, X \rangle E y$$

(left multiplication).

LEMMA 5.1. — *For all* $\gamma \in \Gamma$, *we have*

$$(1) \qquad [(1 \otimes \operatorname{ad} E_\gamma, D] = \operatorname{ad} E_\gamma \otimes 1 - \frac{1}{2}[(\operatorname{ad} E_\gamma \otimes 1), D],$$

$$(2) \qquad [[(\operatorname{ad} E_\gamma \otimes 1), D], D] = 0.$$

Proof. — It suffices to derive these formulae for D′ and these result from (5.1) by direct computation. For the reader's convenience we note that

$$[(\operatorname{ad} E_\gamma \otimes 1), D'] = -(\operatorname{ad} E_\beta \otimes E_\beta^{-1} E_\gamma)(1 - \delta_{\gamma\beta}).$$

Note that D is nilpotent, so exp D is well-defined.

LEMMA 5.2. — *For all* $\gamma \in \Gamma$:

$$[(1 \otimes \operatorname{ad} E_\gamma), \exp D] = \exp D (\operatorname{ad} E_\gamma \otimes 1).$$

Proof. — From (1) of Lemma 5.1, we obtain

$$\begin{aligned} &[(1 \otimes \operatorname{ad} E_\gamma + 1/2 \operatorname{ad} E_\gamma \otimes 1), \exp D] \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \frac{1}{n!} D^j (\operatorname{ad} E_\gamma \otimes 1) D^{n-j-1} \\ &= \exp D \frac{1}{\operatorname{ad} D}(1 - \exp(-\operatorname{ad} D))(\operatorname{ad} E_\gamma \otimes 1). \end{aligned}$$

Rearrangement using (2) of Lemma 5.1 gives the required result.

Remark. — Lemma 5.2 fails if D is replaced by $c D : c \in \mathbf{C}; c \neq 1$.

Direct computation further establishes :

$$(5.2) \qquad [(\operatorname{ad} H_\beta \otimes 1 + 1 \otimes \operatorname{ad} H_\beta), D] = 0,$$

$$(5.3) \qquad (\operatorname{ad} E_\gamma)\theta = \theta(1 \otimes \operatorname{ad} E_\gamma) : \quad \gamma \in \Gamma,$$

$$(5.4) \qquad (\operatorname{ad} H_\beta)\theta = \theta(\operatorname{ad} H_\beta \otimes 1 + 1 \otimes \operatorname{ad} H_\beta).$$

THEOREM 5.3. — [$\mathfrak{g}$ *simple and different from* $sl(n+1)$: $n = 1, 2, \ldots$]. *Let* Φ *be in the conclusion of Theorem* 4.3. *Then*

$$\Phi(X) = \theta(\exp D(X \otimes 1)) + c(\mathfrak{g})(\beta, \beta)^2 E^{-1} \langle e, X \rangle : \quad X \in \mathfrak{g}$$

where $c(\mathfrak{g})$ *is a rational number dependent on* $\mathfrak{g}$.

Remark. — The second term affects only $\Phi(E_{-\beta})$. There seems to be no easy way of absorbing it by say adjusting the θ map.

Proof. — The given Φ is obviously linear and so it suffices to establish (1)-(5) of Corollary 4.4. (1) is verified by an easy computation. Then for all $\gamma \in \Gamma$:

$$\begin{aligned}[\Phi(E_\gamma), \Phi(X)] &= (\operatorname{ad} E_\gamma)\theta(\exp D(X \otimes 1)),\\ &= \theta((1 \otimes \operatorname{ad} E_\gamma)\exp D(X \otimes 1)), \quad \text{by (5.3)},\\ &= \theta(\exp D(\operatorname{ad} E_\gamma \otimes 1)(X \otimes 1)), \quad \text{by Lemma 5.2},\\ &= \Phi([E_\gamma, X]), \quad \text{as required.}\end{aligned}$$

A similar argument using (5.2), (5.4) establishes (2). Parts (3), (4) derive from symmetrization and the contruction of Lemma 4.2. Finally the argument of Theorem 4.3 shows that (5) can only fail by an element of $\mathbf{C}\,E^{-1}$ and this determines $c(\mathfrak{g})$ which is easily seen to be rational.

Set $J_0 = \ker \Phi$. Then J_0 is a two-sided ideal in $U(\mathfrak{g})$.

LEMMA 5.4. — J_0 *is completely prime and is primitive.*

Proof. — The first part follows from Lemma 4.1 and the fact that $\mathscr{A}'_n$ has no zero divisors. Again we further obtain that Cent $(U(g)/J_0) \subset$ Cent $\mathscr{A}'_n = \mathbf{C}\,1$, so J_0 is primitive by [11] (Sect. 8.5.7).

Set $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$. Given $\omega \in W$ define $\omega_\rho \in \operatorname{End} \mathfrak{h}^*$, through $\omega_\rho \lambda = \omega(\lambda + \rho) - \rho$, for all $\lambda \in \mathfrak{h}^*$. Set $W^T = \{\omega_\rho : \omega \in W\}$. It is well-known that the maximal ideals of Cent $U(\mathfrak{g})$ are in one to one correspondence with the orbits of $\mathfrak{h}^*$ under W^T. Now in particular $J_0 \cap$ Cent $U(\mathfrak{g})$ is a maximal ideal in Cent $U(\mathfrak{g})$ and in the next section we determine the corresponding W^T orbit.

6. The Central Character of J_0

Let $\mathcal{O}_0^W$ denote the W^T orbit defining $J_0 \cap$ Cent $U(\mathfrak{g})$. To determine a $\lambda \in \mathcal{O}_0^W$, we construct a $U(\mathfrak{g})$ module M_λ with highest weight vector v_λ such that $\varphi_\lambda : U(\mathfrak{g}) \to \operatorname{End} M_\lambda$ satisfies $\ker \varphi_\lambda = J_0$.

Let s be a positive integer and set

$$M_s = \mathbf{C}[E_{\gamma_1}, E_{\gamma_2}, \ldots, E_{\gamma_m}, E^{1/s}, E^{-1/s}],$$

where γ_i runs over Γ_1. Define M_s as an $\mathfrak{r}$-module by letting $E_\gamma : \gamma \in \Gamma_1 \cup \beta$ act through multiplication and $E_{\gamma'} : \gamma' \in \{\Gamma_2, 0\}$ $(E_0 = H_\beta)$ through adjoint action. Then M_s extends to a $U(\mathfrak{r})_E$ module and through Φ to a $U(\mathfrak{g})$ module.

Now let t be an integer and set $v_{s,t} = E_\beta^{t/s}$. By Theorem 5.3 and the definition of Γ_2.

LEMMA 6.1. — $\Phi(E_\gamma)\,v_{s,t} = 0$, *for all* $\gamma \in \Gamma_2 \cup \Delta_\beta^+$, *and*

$$\Phi(H_\beta)\,v_{s,t} = t/s(\beta, \beta)\,v_{s,t}.$$

Let B be as defined in Lemma 2.18. Set

$$\mathfrak{n}_B = \operatorname{lin\,span}\{\mathfrak{g}^\gamma : \gamma \in B\}, \qquad \mathfrak{n}_B^- = \operatorname{lin\,span}\{\mathfrak{g}^{-\gamma} : \gamma \in B\}.$$

Set $D'_1 = \sum_{\gamma \in \Gamma_0} \operatorname{ad} E_\gamma \otimes F_\gamma : D'_2 = D' - D'_1$ and define D_1, D_2 from D'_1, D'_2 by transport under σ. Then $[D_1, D_2] = 0$, so from Theorem 5.3 we obtain

$$(6.1) \qquad \Phi(E_{-\gamma}) = \theta(D_1 D_2 (E_{-\gamma} \otimes 1)) + \frac{1}{3!}\theta(D_1^3(E_{-\gamma} \otimes 1)) : \quad \gamma \in \Gamma_0.$$

LEMMA 6.2. — *Fix* $\gamma \in \Gamma_3$. *Then* $\Phi(E_{-\beta+\gamma}) v_{s,t} = 0$, *and for a suitable choice of* $u = s/t : \Phi(E_{-\gamma}) v_{s,t} = 0$.

Proof. — Both parts are similar and we prove just the second. Consider

$$\theta(D_1^3(E_{-\gamma} \otimes 1)) v_{s,t}.$$

The θ map gives a non-zero contribution, only if $\gamma_1 + \gamma_2 + \gamma_3 - \gamma = \beta$, where $\gamma_1, \gamma_2, \gamma_3 \in \Gamma_0$ from the summations in D_1^3. Thus $|\gamma_1| + |\gamma_2| + |\gamma_3| = (3/2)|\beta| + 1/2$. Hence we must have $\gamma_i \in \Gamma_1$ for at least one value of $i \in \{1, 2, 3\}$.

Suppose $i = 1$. To annihilate $v_{s,t}$ we move the corresponding F_{γ_1} to the left and this gives a non-zero contribution only if $\gamma_1 + \gamma_j = \beta$, for some $j \in \{2, 3\}$. Suppose $j = 2$, then $\gamma_3 = \gamma$ and we obtain a term proportional to $E^{-1} E_{\beta-\gamma} v_{s,t}$. Obviously

$$\theta(D_1^3(E_{-\gamma} \otimes 1)) v_{s,t}$$

consists of only terms having this form. A similar computation for

$$\theta(D_1 D_2 (E_{-\gamma} \otimes 1)) v_{s,t}$$

shows that besides such terms we obtain a non-zero contribution proportional to $E^{-1} E_{\beta-\gamma} H_\beta v_{s,t}$. Applying Lemma 6.1, we can cancel these terms for a suitable choice of s/t, which is easily verified to be rational.

Let s, t be in the conclusion of Lemma 6.2 and set $v_{\lambda'} = v_{s,t} : \lambda' \in \mathfrak{h}^*$.

COROLLARY 6.3. — *For all* $X \in \mathfrak{n}_B$, $\Phi(X) v_{\lambda'} = 0$.

Proof. — By choice of s, t and Lemma 6.2, we have $\Phi(E_{-\beta}) v_{\lambda'} = 0$. The assertion then follows from Lemmas 2.19 and 6.1.

It is clear that $\Phi(H) v_{\lambda'} \in \mathbf{C} v_{\lambda'} : H \in h$, so we may write

$$\Phi(Y) v_{\lambda'} = \langle \lambda', Y \rangle v_{\lambda'} : \quad Y \in \mathfrak{n}_B \oplus \mathfrak{h}.$$

With respect to this choice of Borel subalgebra, $v_{\lambda'}$ is a highest weight vector for the infinite dimensional $U(g)$ module $M_{\lambda'} = \Phi(U(\mathfrak{n}_B^-)) v_{\lambda'}$. To determine λ' we note from Theorem 5.3, that for all $\delta \in \Delta_\beta$:

$$\Phi(H_\delta) = -\frac{1}{2} \sum_{\gamma_1 \in \Gamma_1} (\gamma_1, \delta) N_{\gamma_1, \beta-\gamma_1}^{-1} E^{-1} (E_{\gamma_1} E_{\beta-\gamma_1} + E_{\beta-\gamma_1} E_{\gamma_1}).$$

Hence

(6.2) $$\Phi(H_\delta)v_{\lambda'} = \frac{1}{2}\sum_{\gamma_1 \in \Gamma_1} (\gamma_1, \delta) v_{\lambda'}.$$

Now with respect to our previous choice of $\gamma \in \Gamma_3$, we obtain from Lemmas 6.1 and 6.2 that $(\gamma, \lambda') = 0$. This and (6.2) determines λ'. Of course λ' is then calculated relative to B and we must refer it back to Δ^+. To do this let ω be in the conclusion of Lemma 2.18. Then $\omega^{-1}(\lambda') \in \mathcal{O}_0^W$. Hence $\lambda = \omega(\omega^{-1}(\lambda')+\rho) - \rho \in \mathcal{O}_0^W$ and $\lambda = \lambda'+\omega\rho-\rho$. Summarizing.

TABLE

The U (*g*)*-module* M_λ *has highest weight* λ *and* $\varphi_\lambda : U(g) \to \text{End } M_\lambda$ *satisfies* $\ker \varphi_\lambda = J_0$

Cartan Label	Central Character $\lambda + \rho \in \mathcal{D}$
B_n	$\sum_{i=1}^{n-3} \alpha^i + \frac{1}{2}\alpha^{n-2} + \frac{1}{2}\alpha^{n-1} + \alpha^n$
C_n	$\sum_{i=1}^{n-1} \alpha^i + \frac{1}{2}\alpha^n$
D_n	$\sum_{i=1}^{n-3} \alpha^i + \alpha^{n-1} + \alpha^n$
E_6	$\alpha^1 + \alpha^2 + \alpha^3 + \alpha^5 + \alpha^6$
E_7	$\alpha^1 + \alpha^2 + \alpha^3 + \alpha^5 + \alpha^6 + \alpha^7$
E_8	$\alpha^1 + \alpha^2 + \alpha^3 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8$
F_4	$\frac{1}{2}\alpha^1 + \frac{1}{2}\alpha^2 + \alpha^3 + \alpha^4$
G_2	$\alpha^1 + \frac{1}{3}\alpha^2$

PROPOSITION 6.4. — *Set*

$$\lambda = \frac{-1}{2(\beta, \beta)}\Big(\sum_{\gamma_1 \in \Gamma_1} (\beta, \beta)\gamma_1 + 2(\gamma, \gamma_1)\beta\Big) - \beta,$$

where $\gamma \in \Gamma_3$. *Then* $\lambda \in \mathcal{O}_0^W$, *that is it defines the central character for* J_0.

Based on this formula, the above Table gives the unique representative of $\lambda+\rho : \lambda \in \mathcal{O}_0^W$ lying in the fundamental domain $\mathcal{D}$. The fundamental weights $\alpha^i : i = 1, 2, \ldots$, rank g, are defined through the relation $\alpha^i = \bar{\omega}_i$, where the $\bar{\omega}_i$ are taken from Bourbaki [6] (pp. 250-275). Define $\varphi_\lambda : U(g) \to \text{End } M_\lambda$ through $\varphi_\lambda(a)m = \Phi(a)m$: $a \in U(g)$, $m \in M_\lambda$. Then $\ker \varphi_\lambda \supset \ker \Phi = J_0$. Conversely given $a \in \ker \varphi_\lambda$, then $\Phi(a)m = 0$, for all $m \in M_\lambda$. Since M_λ contains, up to a displacement of $E^{t/s}$, the polynomial algebra on which the $\Phi(a)$ act as differential operators, we obtain $\Phi(a) = 0$ and so $a \in J_0$. Thus $\ker \varphi_\lambda = J_0$, as required.

7. The Maximality of J_0

We show that J_0 is maximal [for $\mathfrak{g}$ simple and different from $sl(n+1) : n = 1, 2, \ldots$]. The original proof was simplified by the following lemma [4] (Kor. 3.5) (whose full generality we do not require). We outline the proof. Define Dim as in Section 8.

LEMMA 7.1. — *That* I *be a prime ideal of* U ($\mathfrak{g}$) *and* J *a two-sided ideal of* U ($\mathfrak{g}$) *properly containing* I. *Then* Dim U ($\mathfrak{g}$)/J $\leqq$ Dim U ($\mathfrak{g}$)/I$-$1.

Proof. — Set V = U ($\mathfrak{g}$)/I, $\bar{J}$ = J/I. Let gr be the gradation functor for the filtration of V defined in Section 10. Since gr (V) is finitely generated, it follows by use of the Hilbert-Samuel polynomial that dim V^m is a polynomial in m, for all m sufficiently large. Say this polynomial is of degree l. Then [*cf.* (10.1)] Dim V = l. Now since I is prime and V is Noetherian, there exists (*see* [11], 3.5.10, 3.5.11) $a \in \bar{J}$ which is not a divisor of zero in V. Suppose $a \in V^k$. Then for all $m > k$, we have dim $(V^m \cap \bar{J}) \geqq$ dim V^{m-k}. Consequently Dim U ($\mathfrak{g}$)/J = Dim V/$\bar{J}$ $\leqq l-1$, as required.

LEMMA 7.2. — *Let* J *be a two-sided ideal of* U ($\mathfrak{g}$) *properly containing* K_0. *Then* dim U ($\mathfrak{g}$)/J $< \infty$.

Proof. — Since J_0 is completely prime (Lemma 5.4) it follows by Lemmas 8.8 and 7.1, that Dim U ($\mathfrak{g}$)/J $< 2\,k(\mathfrak{g})$ = dim $\mathcal{O}_0$. Yet $\mathcal{O}_0$ is the orbit of minimal non-zero dimension, so by Lemma 10.1. It follows that

$$\text{Dim}\, U(\mathfrak{g})/J = \dim\{0\} = 0.$$

Hence dim U ($\mathfrak{g}$)/J $< \infty$.

Remark. — We sketch an alternative proof. Let J be as above. From the given form of Φ one shows easily that $E_\beta^k \in J$, for some non-negative integer k. Now Borho [3] has shown that if the power of some root eigenvector lies in a two-sided ideal J of U ($\mathfrak{g}$), then $E_\delta^l \in J$, for l large, and all $\delta \in \Delta$. Indeed this is immediate if $\mathfrak{g}$ is simply-laced and also if the given root is a short one. Otherwise it suffices to show that $E_\gamma^l \in J$, for a short root γ. Here one can conveniently use Proposition 2.8, the only really delicate case being G_2. It follows that dim U ($\mathfrak{n}$)/U ($\mathfrak{n}$) $\cap$ J $< \infty$ and hence that dim $\mathcal{H}/\mathcal{H} \cap J < \infty$, where $\mathcal{H}$ denotes the set of harmonic elements of U ($\mathfrak{g}$) [16] (Sect. 0). After Kostant [26], we have U ($\mathfrak{g}$) = $\mathcal{H}$ Z ($\mathfrak{g}$), where Z ($\mathfrak{g}$) = Cent U ($\mathfrak{g}$). (Actually this is a tensor product; but we do not require this hard result). Some J_0 is primitive and J $\supset$ J_0, it follows that J $\cap$ Z ($\mathfrak{g}$) contains a maximal ideal of Z ($\mathfrak{g}$). Combined with the above observation, we have dim U ($\mathfrak{g}$)/J $< \infty$, as required. This argument was essentially my original proof. It also gives a special case of Borho's lemma [3], namely.

LEMMA 7.3. — ($\mathfrak{g}$ simple). *If* J *is a two-sided ideal of* U ($\mathfrak{g}$) *containing a power of some non-zero root eigenvector and an element of* Prim U ($\mathfrak{g}$), *then* dim U ($\mathfrak{g}$)/J $< \infty$.

THEOREM 7.4 [$\mathfrak{g}$ *simple and different from* $sl(n+1) : n = 1, 2, \ldots$]. — J_0 *is a maximal ideal.*

Proof. — Let J be a two-sided ideal of U (g) properly containing J_0. By Lemma 7.2, dim U (g)/J < ∞. Hence if J ≠ U (g), its central character coincides with that of J_0 given in the Table, namely λ. Yet, this is impossible since λ is never a dominant integral form, hence J = U (g) as required.

Remark. — In $sl(n+1) : n = 1, 2, \ldots$, there is a family (parametrized by **C**) of ideals corresponding to the minimal orbits (Sect. 3). These are maximal except (as usual) on the integers.

8. J_0 is not Induced

Let g be a finite dimensional Lie algebra over **C**. The theory of induced representations translated to an algebraic setting [11] (Chap. 5), leads to the following definition. A two-sided ideal J of U (g) is said to be induced from a subalgebra a of g, if there exists a two-sided ideal I of U (a) such that J is the largest two-sided ideal of U (g) contained in U (g) I. Our main result is that J_0 is not induced (except trivially from g itself). This can be expected since J_0 is associated with a non-polarizable orbit. Yet for the moment we are unable to apply this fact and we rely on the dimensionality estimate below.

Given an associative algebra $\mathscr{A}$ over **C**, we recall that its Gelfand-Kirillov dimension $\mathrm{Dim}_{\mathbf{C}} \mathscr{A}$ over **C** is defined follows [15] (Sect. 4). (In a non-associative algebra context *see* [23]). Let $a = (a_1, a_2, \ldots, a_n)$ be any finite subset of elements of $\mathscr{A}$, (a, m) the set of monomials of degree $\leq m$ and $d(a, m)$ its dimension over **C**. Then

$$\mathrm{Dim}_{\mathbf{C}} \mathscr{A} = \sup_{a} \overline{\lim_{m \to \infty}} \frac{\log d(a, m)}{\log m}.$$

We drop **C** in the sequel. For general information on Dim, *see* [21] (Chap. 2) and [4].

If $\mathscr{A}$ is commutative and integral [4] (2.1), then Dim $\mathscr{A}$ is just the maximal number of algebraically independent elements of $\mathscr{A}$. Now suppose $\mathscr{A}$ is a filtered algebra with the filtration $\{ \mathscr{A}^m \}_{m=-\infty}^{\infty}$ satisfying $\bigcap_{m=-\infty}^{\infty} \mathscr{A}^m = \{ 0 \}$, and such the associated graded algebra gr $(\mathscr{A})$ is commutative. Then from [21] (2.3) or [4] (5.1), we have

$$\mathrm{Dim}\, \mathscr{A} \geq \mathrm{Dim}\, \mathrm{gr}(\mathscr{A}). \tag{8.1}$$

Furthermore equality holds if $\mathscr{A}^{-n-1} = \{ 0 \}$ for some integer n and if gr $(\mathscr{A})$ is finitely generated [4] (5.5).

Let a be a subalgebra of g and m a complementary subspace for a in g. Let S (m) be the symmetric algebra over m and $\mathscr{A}^{\wedge}(\mathfrak{m})$ the algebra of infinite order differential operators over m with polynomial coefficients. Let ν be a representation for a and $\mathscr{E}$ the associated U (a) module. The representation μ induced to g by ν is defined to be the left regular representation in $U(\mathfrak{g}) \otimes_{U(\mathfrak{a})} \mathscr{E}$. Set $J = \ker \mu$, and $\mathscr{B} = \nu(U(\mathfrak{a})) \subset \mathrm{End}_{\mathbf{C}} \mathscr{E}$. We construct a representation μ^* of U (g) in $S(\mathfrak{m}) \otimes \mathscr{E}$ equivalent to μ [7]. Then [7] (Prop. 2.2), $U(\mathfrak{g})/J \cong \mu^*(U(\mathfrak{g})) \subset \mathscr{A}^{\wedge}(\mathfrak{m}) \otimes \mathscr{B}$.

Furthermore μ^* can be given explicitly. Let $\{X_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$ such that $\{X_r\}_{r=1}^m$, $\{X_s\}_{s=m+1}^n$ are bases for $\mathfrak{m}$, $\mathfrak{a}$ respectively. For $r \in \{1, 2, \ldots, m\}$ use x_r to denote X_r considered as an element of $S(\mathfrak{m})$ and set $y = (y_1, y_2, \ldots, y_m)$, where $y_r = \partial/\partial x_r$. Let g denote the function $g : x \to g(x) = x^{-1}(-1+x+\exp -x)$. Let $P : \mathfrak{g} \to \mathfrak{m}$ be the (linear) projection of $\mathfrak{g}$ onto $\mathfrak{m}$. Extend $\operatorname{ad} X_{\mathfrak{g}} : X \in \mathfrak{m}$, to a derivation ad X of $\mathbf{C}[[y]] \otimes \mathfrak{g}$ by $\mathbf{C}[[y]]$-linearity. Define (analytic) functions h_{ji} through identification of coefficients of X_i in the equation

$$(8.2) \qquad \sum_{j=1}^{n} X_j h_{ji}(y) = (1-g(D(y))P)^{-1}(\exp - D(y))X_i,$$

where

$$(8.3) \qquad D(y) = \sum_{r=1}^{m} y_r \operatorname{ad} X_r.$$

Then [22] (Equation 3.9), we have

$$(8.4) \qquad \mu^*(X_i) = \sum_{r=1}^{m} x_r h_{ri}(y) + \sum_{s=m+1}^{n} \nu(X_s) h_{si}(y).$$

Let μ_1^* denote the representation of g induced from the trivial representation of $\mathfrak{a}$. Set $\mathfrak{i} = \{X \in g : \mu^*(X) \in \mathbf{C}[[y]] \otimes \mathscr{B}\}$. Obviously $\mu^*[\mathfrak{g}, \mathfrak{i}] \subset \mathbf{C}[[y]] \otimes \mathscr{B}$, so $\mathfrak{i}$ is an ideal in $\mathfrak{g}$. By (8.2), $h_{ji}(0) = \delta_{ji}$, where δ_{ji} is the Kronecker delta, so $\mathfrak{i} \subset \mathfrak{a}$.

Define a filtration in $\mathscr{A}^\wedge(\mathfrak{m}) \otimes \mathscr{B}$ through the degree of an element considered as a polynomial in $x = (x_1, x_2, \ldots, x_n)$ and let gr denote the associated gradation functor. Set $J_1 = \ker \mu_1^*$.

LEMMA 8.1:

$$(1) \qquad \operatorname{Dim} U(\mathfrak{g})/J = \operatorname{Dim} \operatorname{gr}(U(\mathfrak{g})/J),$$

$$(2) \qquad \operatorname{Dim} \operatorname{gr}(U(\mathfrak{g})/J) \geqq \operatorname{Dim} \operatorname{gr}(U(\mathfrak{g})/J_1).$$

Proof. — Observe that $\deg \mu^*(X) = 1 : X \notin \mathfrak{i}$ and $\deg \mu^*(X) = 0 : X \subset \mathfrak{i}$. Hence gr is induced by the canonical filtration of $U(\mathfrak{g}/\mathfrak{i})$, and (1) follows from [4] (5.5).

Since $\operatorname{gr} \mathscr{A}^\wedge(\mathfrak{m})$ is commutative and integral [7], Lemme 1.4 (i), $\operatorname{Dim} \operatorname{gr}(U(g)/J_1) = r$, where r is the transcendence degree of Fract $\operatorname{gr}(U(\mathfrak{g})/J_1)$. Since $\operatorname{gr} \mu_1^*(X) : X \in \mathfrak{g}$ generates $\operatorname{gr}(U(\mathfrak{g})/J_1)$, there exist $Y_1, Y_2, \ldots, Y_r \in \mathfrak{g}$ such that the $\operatorname{gr} \mu_1^*(Y_i)$ are algebraically independent. Now $\mu_1^*(X) = 0 : X \in \mathfrak{i}$, so $Y_i \notin \mathfrak{i}$ and hence

$$\operatorname{gr} \mu^*(Y_i) = \operatorname{gr} \mu_1^*(Y_i),$$

from (8.4). This gives (2).

Below we estimate Dim gr $(\mathrm{U}(\mathfrak{g})/\mathrm{J}_1)$. In this we may take $\nu = 0$ in (8.4) and treat x, y as independent variables. Set $z = (x, y)$ and

$$w_i = \sum_{r=1}^{m} x_r h_{ri}(y): \quad i = 1, 2, \ldots, n. \tag{8.5}$$

Then it is elementary that

LEMMA 8.2. — Dim gr $(\mathrm{U}(\mathfrak{g})/\mathrm{J}_1) \geqq$ rank $(\partial w_i/\partial z_j)$, *with equality if the* h_{ij} *are rational.*

Set

$$h_{ji,k} = \partial h_{ji}/\partial y_k: \quad k = 1, 2, \ldots, m.$$

LEMMA 8.3. — Rank $(\partial w_i/\partial z_j) \geqq$ codim $\mathfrak{a}$, *with equality if and only if* $\mathfrak{a}$ *is an ideal in* $\mathfrak{g}$.

Proof. — Observe that

$$dw_i = \sum_{r=1}^{m} (h_{ri}(y)\, dx_r + x_r h_{ri,s}(y)\, dy_s). \tag{8.6}$$

Now from (8.2), we obtain $h_{ji}(0) = \delta_{ji}$, where δ_{ji} is the Kronecker delta. This gives rank $h_{ri}(y) = m$ in some neighbourhood N_0 of the origin. Since $m =$ codim $\mathfrak{a}$, we obtain the asserted inequality. For equality to hold, we require $h_{ri,s}(y) = 0 : y \in \mathrm{N}_0$; $r, s = 1, 2, \ldots, m$, $i = m+1, \ldots, n$. Evaluation at $y = 0$, using (8.2) gives $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$, as required.

Given $f \in \mathfrak{m}^*$, define the two-form $\mathrm{B}_f^{\mathrm{P}}$ on $\mathfrak{g}$ through $\mathrm{B}_f^{\mathrm{P}}(\mathrm{X}, \mathrm{Y}) = \langle f, \mathrm{P}[\mathrm{X}, \mathrm{Y}] \rangle$. Given $\mathrm{E} \in \mathfrak{m}$, define the map $\chi_{\mathrm{E}} : \mathfrak{g} \to \mathfrak{m}$, through

$$\chi_{\mathrm{E}} = \mathrm{P}(1 - g(\operatorname{ad} \mathrm{E})\mathrm{P})^{-1}(\exp - \operatorname{ad} \mathrm{E}).$$

LEMMA 8.4. — *Fix* $f \in \mathfrak{m}^*$, $\mathrm{E} \in \mathfrak{m}$. *Let* $\mathfrak{u}$, $\mathfrak{v}$ *be subspaces of* $\mathfrak{g}$ *with trivial intersection. Then*

$$\operatorname{rank}\left(\frac{\partial w_i}{\partial z_j}\right) \geqq \operatorname{rank}(\mathrm{B}_f^{\mathrm{P}}|_{\mathfrak{u}' \times \mathfrak{m}}) + \dim \chi_{\mathrm{E}}\, \mathfrak{v},$$

where

$$\mathfrak{u}' = \left\{\frac{1}{2}\mathrm{Y} + \mathrm{Z} : \mathrm{Y} + \mathrm{Z} \in \mathfrak{u};\ \mathrm{Y} \in \mathfrak{m},\ \mathrm{Z} \in \mathfrak{a}\right\}.$$

Proof. — Set $\langle f, \mathrm{X}_r \rangle = x_r : r = 1, 2, \ldots, m$. From (8.2) we obtain for all $s = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n$, that

$$\sum_{r=1}^{m} x_r h_{ri,s}(0) = c_i \mathrm{B}_f^{\mathrm{P}}(\mathrm{X}_s, \mathrm{X}_i): \quad c_i = \begin{cases} \frac{1}{2}: & i \in \{1, 2, \ldots, m\}, \\ 1: & \text{otherwise.} \end{cases} \tag{8.7}$$

Yet dim $\chi_{bE}\,\mathfrak{v} \geqq \dim \chi_E\,\mathfrak{v} : b \in C_0 : C_0$ some non-empty Zariski open set in $\mathbf{C}$. Hence in the y-space, there exists in each neighbourhood of the origin a point y' such that $\dim \chi_{E'}\,\mathfrak{v} \geqq \dim \chi_E\,\mathfrak{v}$, where $E' = \sum_{r=1}^{m} y'_i X_i$. $\mathfrak{u} \cap \mathfrak{v} = \{0\}$; (8.6) and (8.7) imply the required assertion.

Remarks. — If either $\mathfrak{m} \subset \mathfrak{u}$, or $\mathfrak{m} \cap \mathfrak{u} = \{0\}$, then $\mathfrak{u} = \mathfrak{u}'$. Suppose $[\mathfrak{m}, \mathfrak{u}] \subset \mathfrak{m}$, then B_f^P can be replaced by the two-form $B_f : (X, Y) \to \langle f, [X, Y] \rangle$. Finally if $\mathfrak{m}$ is a subalgebra, then $(1-P)\,\Delta(y)\,P = 0$, so $\chi_E = (1-g\,(\text{ad}\, E))^{-1}\,P \exp(-\text{ad}\, E)$. Hence in the lemma, we can replace χ_E by $\chi'_E = P \exp(-\text{ad}\, E)$.

From now on we assume that $\mathfrak{g}$ is simple. In the notation of Section 2, we set $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$, $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$. Given $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$, we can assume that $\mathfrak{p} \supset \mathfrak{b}^-$. We write $\mathfrak{p} = \mathfrak{r}_0 \oplus \mathfrak{n}_0^-$, where $\mathfrak{r}_0$, $\mathfrak{n}_0^-$ are respectively the reductive part and nilradical of $\mathfrak{p}$. Let $\mathfrak{n}_0$ be the unique subalgebra of $\mathfrak{n}$ complementing $\mathfrak{p}$ in $\mathfrak{g}$, satisfying $[\mathfrak{b}, \mathfrak{n}_0] \subset \mathfrak{n}_0$.

LEMMA 8.5. — *These exists* $f \in \mathfrak{n}_0^*$ *such that* rank $(B_f |_{\mathfrak{b} \times \mathfrak{n}_0}) = \dim \mathfrak{n}_0$.

Proof. — It suffices to prove the corresponding assertion for $\mathfrak{n}$. This follows form [19] (Lemma 5.7).

Remark. — This incidentally proves the statement given in [31].

Recall [25] that an S-triple (E, H, F) is a three dimensional subalgebra of $\mathfrak{g}$ satisfying the relations $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = 2H$. An S-triple parabolic $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{r}_0 = \mathfrak{g}^H$, where $H = H_\lambda : \lambda \in \mathscr{D}$ and is the semisimple element of an S-triple (E, H, F). For example, $\mathfrak{b}^-$ is an S-triple parabolic with respect to the principle S-triple [25] (Sect. 5). Again if β is the highest root, $\mathfrak{p}_\beta$ is an S-triple parabolic with respect to $(E_\beta, H_\beta, E_{-\beta})$. Unfortunately not all parabolics are of this form. For example, the parabolic of minimal (non-zero) codimension in $D_n : n \geqq 4$.

LEMMA 8.6. — *Suppose* $\mathfrak{p}$ *is an* S-*triple parabolic with respect to the* S-*triple* (E, H, F). *Then* $P\,(\exp -\text{ad}\, E)\,\mathfrak{n}_0 = \mathfrak{n}_0$, *where* $P : \mathfrak{g} \to \mathfrak{n}_0$ *is the projection onto* $\mathfrak{n}_0$.

Proof. — Let $g = \oplus\, \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into simple S-modules. Since $H = H_\lambda : \lambda \in \mathscr{D}$, it follows that $\mathfrak{n}_0$ (resp. $\mathfrak{n}_0^-$) is the linear span of positive (resp. negative) root subspaces of ad H. Hence $\mathfrak{n}_0 = \oplus\,(\mathfrak{n}_0 \cap \mathfrak{g}_i)$, $\mathfrak{n}_0^- = \oplus\,(\mathfrak{n}_0^- \cap \mathfrak{g}_i)$. Thus it suffices to prove that $\chi_E^i\,(\mathfrak{n}_0^- \cap \mathfrak{g}_i) = \mathfrak{n}_0 \cap \mathfrak{g}_i$, for each i, where $\chi_E^i = P\,(\exp -\text{ad}\, E)|_{\mathfrak{g}_i}$. Now since $\mathfrak{r}_0 = \mathfrak{g}^H$, we obtain $\dim(\mathfrak{n}_0 \cap \mathfrak{g}_i) = \dim(\mathfrak{n}_0^- \cap \mathfrak{g}_i) = m_i$, for suitable non-negative integers m_i. Then χ_E^i is an $m_i \times m_i$ matrix which in a suitable basis has entries $(\chi_E^i)_{rs} = 1/(n_i+s-r)!$, where $n_i = \dim \mathfrak{g}_i - m_i$. Then

$$\det \chi_E^i = \frac{1\,!\,2\,!\ldots\,!\,(m_i-1)\,!}{n_i\,!\,(n_i+1)\,!\ldots\,!\,(n_i+m_i-1)\,!} \neq 0,$$

as required.

PROPOSITION 8.7. — *Let* $\mathfrak{p}$ *a parabolic subalgebra of* $\mathfrak{g}$. *Suppose thet either,*

(1) *The nilradical of* $\mathfrak{p}$ *is commutative, or;*

(2) $\mathfrak{p}$ *is an* S-*triple parabolic.*

Then rank $(\partial w_i/\partial z_j) = 2$ codim $\mathfrak{p}$, *for a representation induced from* $\mathfrak{p}$.

Proof (1). — Let $\mathfrak{m}$, $\mathfrak{u}$, $\mathfrak{v}$ be in the hypothesis of Lemma 8.4. Set $E = 0$, $\mathfrak{m} = \mathfrak{v} = \mathfrak{n}_0$ and let $\mathfrak{u}$ be the complementary subalgebra of $\mathfrak{n}_0$ in $\mathfrak{b}$. Then $\dim \chi_E \mathfrak{v} =$ codim $\mathfrak{p}$, trivially. Again $\mathfrak{n}_0^-$ is commutative by hypothesis and hence so is $\mathfrak{m}$. Then applying Lemma 8.5 to the conclusion of Lemma 8.4, we obtain the required assertion.

(2) Set $\mathfrak{m} = \mathfrak{n}_0$, $\mathfrak{u} = \mathfrak{b}$, $\mathfrak{v} = \mathfrak{n}_0$ and E the nilpositive element of the defining S-triple. From Lemmas 8.4-8.6 we obtain the required result.

Let J_0 be the two-sided ideal of U ($\mathfrak{g}$) defined in Section 5, and $k(\mathfrak{g})$ the numbers defined in Section 3. Set $E = E_\beta$.

LEMMA 8.8 [$\mathfrak{g}$ *simple and different from* $sl(n+1)$ $n = 1, 2, \ldots$]. — *Let* J *be a completely prime, proper two-sided ideal of* U ($\mathfrak{g}$), *which is not the augmentation ideal. Then* Dim U ($\mathfrak{g}$)/J $\geqq 2k(\mathfrak{g})$, *with equality if and only if* $J = J_0$.

Proof. — We have $E \notin J$, otherwise J is the augmentation ideal. Hence since J is completely prime and ad E is locally nilpotent on U ($\mathfrak{g}$), it follows that $\{E^r\}_{r=0}^{\infty}$ is an Ore set for U ($\mathfrak{g}$)/J and so we can localize U ($\mathfrak{g}$)/I at E. Then by Lemma 4.1 it follows that $(U(\mathfrak{g})/J)_E$ contains the Weyl algebra $\mathscr{A}_n : n = 1/2$ (card $\Gamma+1) = k(g)$, defined in its conclusion. Hence $J \cap U(\mathfrak{r}) = \{0\}$ and so by (8.1):

$$\operatorname{Dim} U(\mathfrak{g})/J \geqq \operatorname{Dim} U(\mathfrak{r}) = \dim \mathfrak{r} = 2n,$$

with equality if $J = J_0$. Suppose $J \not\supset J_0$. Then there exists $a \in J_0$, $a \notin J$ which we can choose to be highest weight vector under the adjoint action of $\mathfrak{g}$. Suppose

$$\operatorname{Dim} U(\mathfrak{g})/J = 2k(\mathfrak{g}).$$

Then a is left algebraic over U ($\mathfrak{r}$) and commutation with $E_\gamma : \gamma \in \Gamma$ implies that a is algebraic over $\mathbf{C}[E]$. This gives the relation $u_n a^n + \ldots + u_0 = 0$, with $n \geqq 1$, $u_0 \neq 0$ and $u_i \in \mathbf{C}[E]$ for $i = 0, 1, 2, \ldots, n$. Since $a \in J_0$, this implies $u_0 \in J_0$, which contradicts the fact that $\mathbf{C}[E] \cap J_0 = 0$. Hence $J \supset J_0$, so $J = J_0$ by Lemma 7.2.

Remark. — The only if part of the lemma was pointed out to be by W. Borho, who observes that the conclusion fails in general for primitive ideals (of infinite codimension). A primitive ideal is prime; but not necessarily completely prime [11] (3.1.6 and Thm. 3.7.2).

PROPOSITION 8.9. — [$\mathfrak{g}$ *simple and different from* $sl(n+1) : n = 1, 2, \ldots$]. J_0 *is not induced by any proper parabolic subalgebra of* $\mathfrak{g}$.

Proof. — By Lemmas 8.1-8.3 and 8.8, it suffices to consider parabolics for which codim $\mathfrak{p} < 2k(\mathfrak{g})$, that is codim $\mathfrak{p} \leqq$ codim $\mathfrak{p}_\beta$. It is easy to verify that all such parabolics are maximal and furthermore satisfy either (1) or (2) of Proposition 8.7. [In fact only B_n, D_{n+1} : $n \geqq 3$, E_6, E_7 admit a parabolic of strictly smaller codimension and this satisfies (1). Apart from the S-triple parabolic $\mathfrak{p}_\beta$, only F_4, G_2 admit a parabolic of equal codimension and this satisfies (2) because there exists a root proportional to the corresponding fundamental weight.] Then the assertion of the proposition obtains from Lemma 3.1 and the conclusion of Proposition 8.7.

Consider now induction from an arbitrary subalgebra. By stepwise induction it suffices to consider only maximal subalgebras. From [24], [12] (Sect. 7.23), and (Thms. 7.3 and 5.5), a maximal subalgebra $\mathfrak{p}$ is either parabolic or semisimple. Furthermore :

LEMMA 8.10 [$\mathfrak{g}$ *simple and different from* $sl(n+1)$: $n = 1, 2, \ldots$]. — *Let* $\mathfrak{g}_0$ *be a maximal semisimple subalgebra of* $\mathfrak{g}$ *satisfying* $\dim \mathfrak{g} - \dim \mathfrak{g}_0 \leqq 2k(\mathfrak{g}) - 1$. *Then either:*

(1) $\mathfrak{g} = \mathrm{so}(m+1)$, $\mathfrak{g}_0 = \mathrm{so}(m)$: $m \geqq 6$, *or;*

(2) $\mathfrak{g} = \mathrm{so}(7)$, $\mathfrak{g}_0$ *of type* G_2 *embedded in* $\mathrm{so}(7)$ *through its seven dimensional representation.*

Proof. — If $\mathfrak{g}_0$ is maximal and semisimple, then in the terminology of Dynkin ([12], [13], [29]), $\mathfrak{g}_0$ is either a regular or an S-subalgebra. Furthermore the maximal regular subalgebras of $\mathfrak{g}$ are determined by suppressing a simple root in the extended Dynkin diagram [12] (Chap. II, Sect. 5); [4] (pp. 250-275), of $\mathfrak{g}$. Computation then shows $\mathfrak{g}_0$ cannot be regular. Again if $\mathfrak{g}$ is an exceptional Lie algebra, then from [12] (Thm. 14.1 and Table 39), it is easy to verify that $\mathfrak{g}_0$ cannot be an S-subalgebra. Finally assume that $\mathfrak{g}$ is a classical Lie algebra. If $\mathfrak{g}_0$ is a direct sum of classical simple Lie algebras, then the requirement rank $\mathfrak{g} \geqq$ rank $\mathfrak{g}_0$ is sufficient to give (1) as the only possible choice. If $\mathfrak{g}_0$ contains an exceptional Lie algebra, then by [13] (Chap. 1 and Thm. 1.5), $\mathfrak{g}_0$ is a maximal S-subalgebra of so (m), or sp (m) (m even) only if it admits a representation τ of dimension m. Given Ω the highest weight vector for τ, let $\Omega(\pi)$ denote the sum of the coefficients of the simple roots in Ω. Then $\dim \tau \geqq 2\,\Omega(\pi)$ and this estimate suffices to give (2) as the only possible choice.

Given $\mathfrak{g}_0$ a semisimple subalgebra of $\mathfrak{g}$, let $\mathfrak{m}$ be the complementary invariant subspace for the adjoint action of $\mathfrak{g}_0$ in $\mathfrak{g}$. Set $l(\mathfrak{g}_0, \mathfrak{m}) = \sup \{ f \in \mathfrak{m}^* : \mathrm{rank}\,(B_f|_{\mathfrak{g}_0 \times \mathfrak{m}}) \}$. For $\mathfrak{g}$ simple and $\mathfrak{m}$ a simple $\mathfrak{g}$ module, a complete listing of these numbers derives from [14] (Table 1). Furthermore

LEMMA 8.11. — *Let* J *be a two-sided ideal of* U$(\mathfrak{g})$. *If* J *is induced from* $\mathfrak{g}_0$, *then*

$$\mathrm{Dim}\, U(\mathfrak{g})/J \geqq \dim \mathfrak{m} + l(\mathfrak{g}_0, \mathfrak{m}).$$

Proof. — By Lemmas 8.1-8.3, it suffices to prove that

$$\mathrm{rank}\left(\frac{\partial w_i}{\partial z_j}\right) \geqq \dim \mathfrak{m} + l(\mathfrak{g}_0, \mathfrak{m}).$$

In Lemma 8.4, set $E = 0$, $\mathfrak{u} = \mathfrak{g}_0$, $\mathfrak{v} = \mathfrak{m}$. Its conclusion gives the required result.

THEOREM 8.12. — [$\mathfrak{g}$ *simple and different from* $sl(n+1)$: $n = 1, 2, \ldots$] J_0 *is not induced by any proper subalgebra of* $\mathfrak{g}$.

Proof. — By Proposition 8.9 and the above discussion, it remains to examine cases (1), (2) of Lemma 8.10. Take $\mathfrak{g} = \mathrm{so}\,(m+1) : m \geqq 6$. Then $\mathrm{Dim}\,(\mathrm{U}\,(\mathfrak{g})/\mathrm{J}_0) = 2\,m-4$. Yet for an ideal J induced from its so (m) subalgebra, we have through Lemma 8.11 and [14] (Table 1), that $\mathrm{Dim}\,\mathrm{U}\,(\mathfrak{g})/\mathrm{J} \geqq 2\,m-1$. Again take $\mathfrak{g} = \mathrm{so}\,(7)$. Then $\mathrm{Dim}\,\mathrm{U}\,(\mathfrak{g})/\mathrm{J}_0 = 8$, whereas for an ideal induced from its G_2 subalgebra, we have [14] (Table 1), that $\mathrm{Dim}\,\mathrm{U}\,(g)/\mathrm{J} \geqq 13$. This completes the proof of the theorem.

9. Weyl Induction

Let $\mathfrak{g}$ be simple and different from $sl\,(n+1) : n = 1, 2, \ldots$ Set $\mathfrak{r} = \mathfrak{g}^{\Gamma} \oplus \mathbf{C}\,\mathrm{H}_{\beta}$, $\mathrm{E} = \mathrm{E}_{\beta}$. Since the embedding $\mathrm{U}\,(\mathfrak{g}) \subset \mathrm{U}\,(\mathfrak{r})_{\mathrm{E}}$ is rather asymmetric with respect to $\mathfrak{r}$, it is natural to consider the action of the Weyl group W, which permutes the possible choices of Γ. We show that W acts through Aut (Fract U $(\mathfrak{r})$). This provides an alternative proof of the existence of the ambedding.

Given $\alpha \in \Delta$, let $\omega_{\alpha} \in \mathrm{W}$ denote the reflection in the plane normal to α and $\omega_i = \omega_{\alpha_i}$ given $\alpha_i \in \pi$. Recall that the $\omega_i : i = 1, 2, \ldots$, rank $\mathfrak{g}$, generate W and that for all $\alpha \in \Delta^+ : \alpha \neq \alpha_i$, we have $\omega_i\,\alpha \in \Delta^+$. Again for all $\omega \in \mathrm{W}$, we have $\omega\, g^{\alpha} = g^{\omega\alpha}$ and we can choose $0 \neq \mathrm{E}_{\alpha} \in g^{\alpha}$ such that $\omega_{\alpha}\,(\mathrm{E}) = c\,(\omega, \alpha)\,\mathrm{E}_{\omega\alpha} : c\,(\omega, \alpha) = \pm 1$. (It will turn out that these ± 1 factors play absolutely no role in our analysis and could be ignored).

Recall the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and let $\mathscr{A}$ be an associative algebra. Given associative algebra homomorphisms $\varphi : \mathrm{U}\,(\mathfrak{n}) \to \mathscr{A}$, $\Phi : \mathrm{U}\,(\mathfrak{g}) \to \mathscr{A}$ and the group homomorphism $\psi : \mathrm{W} \to \mathrm{Aut}\,\mathscr{A}$, the pair (φ, ψ) [resp. (Φ, ψ)] will be called compatible on $\mathfrak{n}$ (resp. on $\mathfrak{g}$) if

$$(9.1) \qquad \psi\,(\omega_i)\,\varphi\,(\mathrm{E}_{\alpha}) = \varphi\,(\omega_i\,\mathrm{E}_{\alpha}) : \quad i = 1, 2, \ldots, \mathrm{rank}\,\mathfrak{g}, \quad \alpha \in \Delta^+, \quad \alpha \neq \alpha_i,$$

$$(9.2) \qquad [\text{resp. } \psi\,(\omega)\,\Phi\,(\mathrm{E}_{\alpha}) = \Phi\,(\omega\,\mathrm{E}_{\alpha}),\ \omega \in \mathrm{W},\ \alpha \in \Delta].$$

LEMMA 9.1 [$\mathfrak{g}$ *semisimple with* card $\Delta^+ >$ rank $\mathfrak{g}$]. — *A compatible pair* (φ, ψ) *on* $\mathfrak{n}$ *extends uniquely to a compatible pair* (Φ, ψ) *on* $\mathfrak{g}$.

Proof. — For each $\alpha \in \Delta^+$, set $\Phi\,(\mathrm{E}_{\alpha}) = \varphi\,(\mathrm{E}_{\alpha})$:

$$\Phi\,(\omega_{\alpha}\,\mathrm{E}_{\alpha}) = c\,(\omega_{\alpha}, \alpha)\,\Phi\,(\mathrm{E}_{-\alpha}) = \psi\,(\omega_{\alpha})\,\varphi\,(\mathrm{E}_{\alpha}).$$

To show that (9.2) holds, its suffices to take $-\alpha \in \Delta^-$ and $\alpha_i \in \pi$, $\alpha_i + \alpha \neq 0$. Ignoring the ± 1 factors we obtain

$$\begin{aligned} \psi\,(\omega_i)\,\Phi\,(\mathrm{E}_{-\alpha}) &= \psi\,(\omega_i)\,\psi\,(\omega_{\alpha})\,\varphi\,(\mathrm{E}_{\alpha}), \quad \text{by definition,} \\ &= \psi\,(\omega_i)\,\psi\,(\omega_{\alpha})\,\psi^{-1}\,(\omega_i)\,\varphi\,(\omega_i\,\mathrm{E}_{\alpha}), \quad \text{by (9.1)}, \\ &= \psi\,(\omega_{\omega_i\alpha})\,\varphi\,(\mathrm{E}_{\omega_i\alpha}), \quad \text{and by definition,} \\ &= \Phi\,(\mathrm{E}_{-\omega_i\alpha}), \quad \text{as required.} \end{aligned}$$

Again the ± 1 factors must come out right by the compatibility of the Weyl group action on $\mathfrak{g}$. Hence (9.2) holds. It remains to show that Φ extends to a homomorphism $U(\mathfrak{g}) \to \mathscr{A}$.

Given $\alpha, \beta \in \Delta$, with $\alpha+\beta \neq 0$, there exists by [16] (Thm. 2, p. 242), $\omega \in W$ such that $\omega\alpha, \omega\beta \in \Delta^+$. Ignoring ± 1 factors and recalling that $\Phi|_{\mathfrak{n}} = \varphi$ is a homomorphism, we obtain

$$\psi(\omega)[\Phi(E_\alpha), \Phi(E_\beta)] = [\Phi(E_{\omega\alpha}), \Phi(E_{\omega\beta})], \quad \text{by (9.2)},$$
$$= \Phi([E_{\omega\alpha}, E_{\omega\beta}]) = \psi(\omega)\Phi([E_\alpha, E_\beta]).$$

Then multiplication by $\psi^{-1}(\omega)$ gives the required identity.

Now with $(\alpha, \alpha) H_\alpha = [E_\alpha, E_{-\alpha}]$, set $(\alpha, \alpha)\Phi(H_\alpha) = [\Phi(E_\alpha), \Phi(E_{-\alpha})]$, for all $\alpha \in \Delta$ Through the Jacobi identity we obtain for all $\alpha, \beta \in \Delta : \alpha \neq \beta$, that

$$\Phi([H_\alpha, E_\beta]) = [\Phi(H_\alpha), \Phi(E_\beta)]$$

and hence $[\Phi(H_\alpha), \Phi(H_\beta)] = 0$, and $\Phi(H_{\alpha+\beta}) = \Phi(H_\alpha)+\Phi(H_\beta)$. Finally given card $\Delta^+ >$ rank $\mathfrak{g}$, this last relation implies that $\Phi([H_\alpha, E_\alpha]) = [\Phi(H_\alpha), \Phi(E_\alpha)]$, which concludes the proof.

Now let $\mathfrak{k}$, $\mathfrak{l}$ be the subalgebras of $\mathfrak{g}$ defined in Lemma 2.15. Its conclusion allows us to construct by induction from a character ν on $\mathfrak{l}$, a realization of $\mathfrak{k}$. In this (8.2)-(8.4) apply and because $\mathfrak{l}$ admits a commutative complement in $\mathfrak{k}$, the factor $(1-g(D(y))^{\cdot}P)^{-1}$ in (8.2) drops out. Furthermore by Corollary 2.3, we can identify y_r with $E^{-1} E_{\beta-\gamma_r}$ as γ_r runs over $\Gamma_1 \cup \beta$ [with $E_0 = H_\beta - 1/2 \sum_{\gamma \in \Gamma_1} N_{\gamma, \beta-\gamma}^{-1} (\beta, \beta) E^{-1} E_{\beta-\gamma} E_\gamma$]. Thus this construction gives an embedding φ_ν of $U(\mathfrak{k})$ in $U(\mathfrak{r})_E$. We remark that if Φ is defined by the conclusion of Theorem 5.3, then there exists a character ν_0 on $\mathfrak{l}$ such that $\Phi|_{\mathfrak{k}} = \varphi_{\nu_0}$. On the other hand using Lemma 9.1 we can reconstruct Φ from φ_ν. (More precisely we reconstruct Φ from $\varphi = \varphi_\nu|_{\mathfrak{k}'} : \mathfrak{k}' = \mathfrak{n} \oplus \mathbf{C} H_\alpha \oplus \mathbf{C} E_{-\alpha}$, which is independent of ν.) To do this it suffices to define $\psi : W \to \operatorname{Aut}(\operatorname{Fract} U(\mathfrak{r}))$ and show that (φ, ψ) is compatible on $\mathfrak{n}$. Now it is easy to verify that the uniqueness part of the proof of Theorem 4.3 implies that it is sufficient to assertain compatibility on $\mathfrak{r}$. However compatibility on $\mathfrak{r}$ exactly defines $\psi(\omega_i) : i = 1, 2, \ldots,$ rank $\mathfrak{g}$, if we can show that

$$\operatorname{Fract} \varphi(U(\omega_i \mathfrak{r})) = \operatorname{Fract} U(\mathfrak{r}).$$

For all $\alpha_i \in \pi_\beta$, it follows that $\omega_i \mathfrak{r} = \mathfrak{r}$, so there is nothing to prove. For $\alpha_1 = \alpha \in \pi_\beta^c$, we have

LEMMA 9.2. — $\operatorname{Fract} \varphi(U(\omega_\alpha \mathfrak{r})) = \operatorname{Fract} U(\mathfrak{r})$.

Proof. — Set $\mathscr{R} = \operatorname{Fract} \varphi(U(\omega_\alpha \mathfrak{r}))$. The inclusion $\mathscr{R} \subset \operatorname{Fract} U(\mathfrak{r})$, holds by definition of φ. For the reverse inclusion, we may use the relation $\Phi|_{\mathfrak{k}'} = \varphi$ noted above, where Φ is defined by Theorem 5.3.

Prove E_β, $E_{\beta-\alpha} \in \mathscr{R}$. If $\beta-2\alpha$ is not root then ω_α interchanges β and $\beta-\alpha$ and the assertion is immediate. Otherwise we note from Lemma 2.11, that $\beta-3\alpha$ is not a root and so $((\beta-\alpha), \alpha) = 0$. Then $\omega_\alpha \beta = \beta-2\alpha$, $\omega_\alpha(\beta-\alpha) = \beta-\alpha$. From (5.3):

$$\varphi(E_{\beta-2\alpha}) = E_\beta^{-1} E_{\beta-\alpha}^2 \qquad \text{and} \qquad \varphi(E_{\beta-\alpha}) = E_{\beta-\alpha},$$

so E_β, $E_{\beta-\alpha} \in \mathscr{R}$ as required.

Let n be an integer > 1. Assume we have shown that $E_\gamma \in \mathscr{R}$, for all $\gamma \in \Gamma$ with $|\gamma| > n$. Then we show that $E_{\gamma_i} \in \mathscr{R}$, for all $\gamma_i \in \Gamma$ satisfying $|\gamma_i| = n$. In this we can assume that $n < |\beta|-1$. If $(\gamma_i, \alpha) = 0$, then $\omega_\alpha \gamma_i = \gamma_i$, so $E_{\gamma_i} \in \mathscr{R}$ trivially. If $(\gamma_i, \alpha) \neq 0$, then by Lemma 2.11, $\gamma_i+\alpha$, $\gamma_i-2\alpha$ are not roots, so $\omega_\alpha \gamma_i = \gamma_i-\alpha \in \Delta_\beta^+$. By Theorem 5.3:

$$\varphi(E_{\gamma_i-\alpha}) = \frac{1}{2} \sum_{\gamma \in \Gamma_0} N_{\gamma, \gamma_i-\alpha} N_{\gamma, \beta-\gamma}^{-1} E_\beta^{-1} E_{\beta-\gamma} E_{\gamma+\gamma_i-\alpha}. \tag{9.3}$$

Now since card $\pi_\beta^c = 1$, it follows that $|\gamma+\gamma_i-\alpha| \geqq \gamma_i$ with equality only if $\gamma = \alpha$. Again since $\gamma_i-\alpha \in \Delta_\beta^+$, we have $|\gamma+\gamma_i-\alpha| < |\beta|$, so $|\gamma+\gamma_i| \leqq \beta$. If equality holds, set $\beta-\gamma = \gamma_j$. In (9.3), we obtain a non-zero contribution only if $\beta-\gamma_j+\gamma_i-\alpha$ is a root. Yet $|\beta-\gamma_j+\gamma_i-\alpha| = |\beta-\alpha|$ and so by Corollary 2.3, it follows from the relation card $\pi_\beta^c = 1$, that $\gamma_i = \gamma_j$. Noting the identity $N_{\alpha,\gamma_i-\alpha} N_{\alpha,\beta-\alpha}^{-1} = N_{\beta-\gamma_i,\gamma_i-\alpha} N_{\beta-\gamma_i,\gamma_i}^{-1}$, we may rewrite (9.3) in the form

$$\varphi(E_{\gamma_i-\alpha}) = N_{\alpha,\gamma_i-\alpha} N_{\alpha,\beta-\alpha}^{-1} E_\beta^{-1} E_{\beta-\alpha} E_{\gamma_i}, \quad \operatorname{mod} \mathscr{R}.$$

Hence $E_{\gamma_i} \in \mathscr{R}$ as required.

It remains to show that E_α, $H_\beta \in \mathscr{R}$. Let $\mathscr{R}'$ denote the field generated by

$$\{E_\gamma : \gamma \in \Gamma, |\gamma| > 1\}.$$

From Theorem 5.3, it follows that $\omega_\alpha E_\alpha$ $(= E_{-\alpha})$ and $\omega_\alpha H_\beta$ are linear in E_α, H_β over $\mathscr{R}'$ Since they cannot be linearly dependent it follows that E_α, $H_\beta \in \mathscr{R}$, as required.

We may summarize our conclusions in the following manner:

PROPOSITION 9.3. — *Let* $\mathfrak{k}$, $\mathfrak{l}$ *be in the conclusion of Lemma* 2.15 *and set* $\mathscr{A}$ = Fract U ($\mathfrak{r}$). *Define a compatible pair* (φ, ψ) *on* $\mathfrak{n}$, *through the representation* μ^* (cf. *Sect.* 8) *on* $\mathfrak{k}$ *induced from the trivial representation of* $\mathfrak{l}$ *and through the conclusion of Lemma* 9.2. *Then* (φ, ψ) *extends to a compatible pair* (Φ, ψ) *on* $\mathfrak{g}$, *and* Φ *coincides with the homomorphism in the conclusion of Theorem* 5.3.

We can now verify the claims asserted in [20]. The proof of [20] (Lemma 2.1) is given in [21] (2.10). Given this, the results claimed in [20] (Thm. 4.1 follow from Lemma 4.1, Thm. 4.3, Lemma 5.4 and Prop. 9.3). Similar arguments give [20] (Thm. 5.1). The explicit formulae computed in [20] (Sections 9-11 coincide with that given by Theorem 5.3).

10. Quantization and Non-Polarizable Orbits

Here we establish a precise connection between $\mathcal{O}_0$ and J_0, and suggest a possible generalization of this relationship.

Define a filtration $\{U^m\}_{m=0}$ on U (g) through

$$U^m = \text{lin span}\{X^n : n = 0, 1, 2, \ldots, m, X \in \mathfrak{g}\},$$

and let gr denote the associated gradation functor. Recall that S (g) identifies with gr (U (g)) and let $\{\,,\}$ denote the Poisson bracket defined on S (g) [and hence on Fract S (g)] through gr [32] (Sect. 2). Given m an integer $\geqq 0$, set

$$U_m = \text{gr}_m(U^m), \qquad S^m = \bigoplus_{n=0}^{m} U_n, \qquad U = U(\mathfrak{g}), \qquad S = S(\mathfrak{g}).$$

Let J be a two-sided ideal in U (g). Define the characteristic variety $\mathcal{V}(J) \subset \mathfrak{g}^*$ of J to be the zero variety of gr (J). For each $\xi \in$ Fract U (g), set gr $(\xi) = (\text{gr}(a))^{-1}\,\text{gr}(b)$ and deg ξ = deg gr (b) − deg gr (a), given $\xi = a^{-1}b : a, b \in U(\mathfrak{g})$. Recall [15] (Lemma 4), that gr (ξ) and deg ξ are independent of the representatives a, b of ξ.

Define a filtration $\{(U/J)^m\}_{m=0}$ on U/J through $(U/J)^m = U^m/U^m \cap J$. Since gr (U) is commutative and Noetherian, it follows that gr (U/J) is commutative and finitely generated. Hence by the remark following (8.1) we have (*see* also [4], Kor., 5.4):

$$\text{(10.1)} \qquad \text{Dim}(U/J) = \text{Dim gr}(U/J) = \text{Dim S/gr}(J).$$

Now gr (J) is G-stable and hence by transposition so is $\mathcal{V}(J)$. Let $I_1, I_2, \ldots, I_n$ be prime ideals of S such that $I_1 \cap I_2 \cap \ldots \cap I_n = \sqrt{\text{gr}(J)}$, [11] (3.1.10). It is easy to check that each I_i is G-stable and hence so is each irreducible component $\mathcal{V}_i(J) = \mathcal{V}(I_i)$ of $\mathcal{V}(J)$. By [4] (3.1 *e*), Dim S/gr (J) = max $\{\text{Dim A/I} : i = i, 2, \ldots, n\}$. Now Dim A/I_i is the transcendence degree of Fract S/I_i and it is classical that this coincides with the dimension dim $\mathcal{V}(I_i)$ of the tangent space to a generic point of $\mathcal{V}(I_i)$. We obtain

LEMMA 10.1. — *Let* J *be a two-sided ideal in* (g). *Then* $\mathcal{V}(J)$ *is a union of* G-*orbits in* $\mathfrak{g}^*$ *and* Dim (U/J) = max $\{\dim \mathcal{V}(I_i) : i = 1, 2, \ldots, n\}$.

Remark. — This result is implicit in [4] (Sect. 7).

Assume g simple and different from $sl(n+1) : n = 1, 2, \ldots$ Let J_0 be the two-sided ideal of U (g) defined in Section 5 (following Thm. 5.3) and $\mathcal{V}_0$ the minimal non-zero orbit in $\mathfrak{g}^*$.

PROPOSITION 10.2. — *Let* J *be a completely prime two-sided ideal in* U (*g*). *Then* $\mathcal{V}(J) = \mathcal{V}_0 \cup \{0\}$, *if and only if* $J = J_0$.

Proof. — By Lemma 8.8, Dim (U/J) = $2k(\mathfrak{g})$, if and only if $J = J_0$. Excepting $\{0\}$, there is by Lemma 3.3 and by Proposition 3.5 only one orbit in $\mathfrak{g}^*$ of dimension $\leqq 2k(\mathfrak{g})$. Hence the required assertion follows from Lemma 10.1.

This result uniquely relates J_0 to $\mathscr{V}_0$. More generally, as R. Rentschler suggests, one may expect there to be a bijection (or very nearly one) between the family of orbits in $\mathfrak{g}^*$ and the class of primitive, completely prime, two-sided ideals in $U(\mathfrak{g})$. To discuss this the above procedure needs some refining since $\mathscr{V}(J)$ is always a cone (consisting of nilpotent orbits) and may also give too much. In particular one always gets the point $\{0\}$ corresponding to the augmentation ideal. Finally we should not expect the simple dimensionality arguments given above to be sufficient in general. Rather we should recall that $\deg \Phi(X) = 1$ and note that as a consequence we have

$$(10.4) \qquad \{\operatorname{gr}\Phi(X), \operatorname{gr}\Phi(Y)\} = \operatorname{gr}[\Phi(X), \Phi(Y)]: \quad X, Y \in \mathfrak{g}.$$

Thus $\varphi : X \to \varphi_X = \operatorname{gr}\Phi(X)$ is a Lie algebra homomorphism of $\mathfrak{g}$ into Fract $S(\mathfrak{r})$, sometimes called a classical realization of $\mathfrak{g}$. After Kostant [27], it is known that the set of zeros of $X - \varphi_X$ [which coincides with $\mathscr{V}(J_0)$] is a union of G-orbits in $\mathfrak{g}^*$, which in this case is just $\mathcal{O}_0 \cup \{0\}$. Conversely starting from a given orbit $\mathcal{O}$ in $\mathfrak{g}^*$, we choose local co-ordinates x_i, $y_i \in C^\infty(\mathcal{O})$ such that the Kirillov-Kostant symplectic form is given by

$$\sum_{i=1}^{m} dx_i \wedge dy_i : \quad 2m = \dim \mathcal{O}.$$

This gives rise to a linear map $\varphi : X \to \varphi_X$ of $\mathfrak{g}$ into $C^\infty(\mathcal{O})$ satisfying

$$\varphi_{[X, Y]} = \{\varphi_X, \varphi_Y\} = \sum_{i=1}^{m} \left(\frac{\partial \varphi_X}{\partial x_i} \frac{\partial \varphi_Y}{\partial y_i} - \frac{\partial \varphi_Y}{\partial x_i} \frac{\partial \varphi_X}{\partial y_i} \right).$$

Suppose that $\mathcal{O}$ admits a polarization $\mathfrak{a}$. Then by induction from the character $\mathfrak{a} \to \langle f, \mathfrak{a} \rangle : f \in \mathcal{O}$ on $\mathfrak{a}$, we may write φ_X as functions linear in homogeneous in x_1, $x_2, \ldots, x_m$ over $\mathbf{C}[[y_1, y_2, \ldots, y_m]]$, given by $\varphi_X = \mu^*(X)$ and (8.2)-(8.4). Moreover the linearity of φ_X enables one to replace y_r by $\partial/\partial x_r$ and so define a completely prime [7] (Cor. 3.2), two-sided ideal J in $U(\mathfrak{g})$. In fact $J = \ker \mu^*$ and is the ideal induced by the character $\mathfrak{a} \to \langle f, \mathfrak{a} \rangle$ on $\mathfrak{a}$. This process, though not obviously canonical, works rather well for $\mathfrak{g}$ solvable and $\mathfrak{g} = sl(3)$ ([1], [5], [10]) and represents the algebraic basis of Kostant's quantization [27].

When $\mathcal{O}$ is not polarizable, it is no longer possible to choose co-ordinates so that φ_X is linear in the x_i. Yet we might hope that at least φ_X can be chosen to be no more than polynomial over $x_1, x_2, \ldots, x_m$. Even then, $\Phi(X)$ must contain terms of lower order if it is to satisfy (10.4) with $\Phi(X) = \varphi_X$. Neither the existence or uniqueness of such terms is obvious. Indeed difficultes are known to arise when $\dim \mathfrak{g} = \infty$ [17], a fact responable for the failure of old-fashioned quantization. Nevertheless we do wish to point out that it is essentially the above process by which we associated J_0 with the minimal orbit $\mathcal{O}_0$.

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A. JOSEPH,

Permanent address:
Department of Mathematics,
Tel-Aviv University,
Ramat Aviv,
Israel;

Mailing address:
Département de Mathématiques,
Bât. 425,
91045 Orsay.