The Minimum-Area Spanning Tree Problem

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Abstract. Motivated by optimization problems in sensor coverage, we formulate and study the Minimum-Area Spanning Tree (MAST) problem: Given a set $\mathcal{P}$ of n points in the plane, find a spanning tree of $\mathcal{P}$ of minimum "area," where the area of a spanning tree $\mathcal{T}$ is the area of the union of the $n-1$ disks whose diameters are the edges in $\mathcal{T}$. We prove that the Euclidean minimum spanning tree of $\mathcal{P}$ is a constant-factor approximation for MAST. We then apply this result to obtain constant-factor approximations for the Minimum-Area Range Assignment (MARA) problem, for the Minimum-Area Connected Disk Graph (MACDG) problem, and for the Minimum-Area Tour (MAT) problem. The first problem is a variant of the power assignment problem in radio networks, the second problem is a related natural problem, and the third problem is a variant of the traveling salesman problem.

1 Introduction

We introduce and study the Minimum-Area Spanning Tree (MAST) problem. Given a set $\mathcal{P}$ of n points in the plane, find a spanning tree of $\mathcal{P}$ of minimum area, where the area of a spanning tree $\mathcal{T}$ of $\mathcal{P}$ is defined as follows. For each edge e in $\mathcal{T}$ draw the disk whose diameter is e. The *area* of $\mathcal{T}$ is then the area of the union of these $n-1$ disks. Although this problem seems natural (see also applications below), we are not aware of any previous work on it.

One of the main results of this paper (presented in Section 2) is that the minimum spanning tree of $\mathcal{P}$ is a constant-factor approximation for MAST. This is an important property of the minimum spanning tree as is shown below. (See, e.g., [7, 9] for background on the minimum spanning tree.)

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We apply the result above to three problems from a class of problems that has received considerable attention. The first problem is a variant of the power assignment problem (also called the range assignment problem). Let $\mathcal{P}$ be a set of n points in the plane, representing n transmitters-receivers (or transmitters for short). In the standard version of the power assignment problem one needs to assign transmission ranges to the transmitters in $\mathcal{P}$, so that (i) the resulting communication graph is strongly connected (that is, the graph in which there exists a directed edge from $p_i \in \mathcal{P}$ to $p_j \in \mathcal{P}$ if and only if p_j lies in the disk D_{p_i} is strongly connected, where the radius of D_{p_i} is the transmission range, r_i, assigned to p_i), and (ii) the total power consumption (i.e., the cost of the assignment of ranges) is minimal, where the total power consumption is $\sum_{p_i \in \mathcal{P}} \text{area}(D_{p_i})$.

The power assignment problem is known to be NP-hard (see Kirousis et al. [10] and Clementi et al. [6]). Kirousis et al. [10] also obtain a 2-approximation for this problem, based on the minimum spanning tree of $\mathcal{P}$, and this is the best approximation known.

Consider now the variant of the power assignment problem in which the second requirement above is replaced by (ii') the area of the union of the disks $D_{p_1}, \ldots, D_{p_n}$ is minimum. We refer to this problem as the Minimum-Area Range Assignment (MARA) problem. In general, the presence of a foreign receiver (whether friendly or hostile) in the region $D_{p_1} \cup \cdots \cup D_{p_n}$ is undesirable, and the smaller the area of this region, the lower the probability that such a foreign receiver is present. In Section 3 we prove that the range assignment of Kirousis et al. (that is based on the minimum spanning tree) is also a constant-factor approximation for MARA.

Another related and natural problem for which we obtain a constant-factor approximation (in Section 4) is the following. Let $\mathcal{P}$ be a set of n points in the plane. For each point $p \in \mathcal{P}$, draw a disk D_{p_i} of radius 0 or more, such that (i) the resulting disk graph is connected (that is, the graph in which there exists an edge between $p_i \in \mathcal{P}$ and $p_j \in \mathcal{P}$ if and only if $D_{p_i} \cap D_{p_j} \neq \emptyset$ is connected), and (ii) the area of the union of the disks $D_{p_1}, \ldots, D_{p_n}$ is minimized. We refer to this problem as the Minimum-Area Connected Disk Graph (MACDG) problem. (See, e.g., [8, 11] for background on intersection graphs and on disk graphs in particular.)

The last problem for which we obtain a constant-factor approximation (in Section 5) is a variant of the well-known traveling salesman problem. Given a set $\mathcal{P}$ of n points in the plane, find a tour of $\mathcal{P}$ of minimum area, where the area of a tour T is the area of the n disks whose diameters are the edges of the tour. We refer to this problem as the Minimum-Area Tour (MAT) problem. The constant-factor approximation that we obtain for this problem is also based on results concerning the traveling salesman problem with a parameterized triangle inequality.

A potentially interesting property concerning the area of the minimum spanning tree that is obtained as an intermediate result in Section 2 is that the depth of the arrangement of the disks corresponding to the edges of the minimum spanning tree is bounded by some constant. Notice that this property does not follow immediately from the fact that the degree of the minimum spanning tree is at most 6, as is shown in Figure 2.

Finally, all the above results hold in any fixed dimension d (with small modifications).

2 MST is a Constant-Factor Approximation for MAST

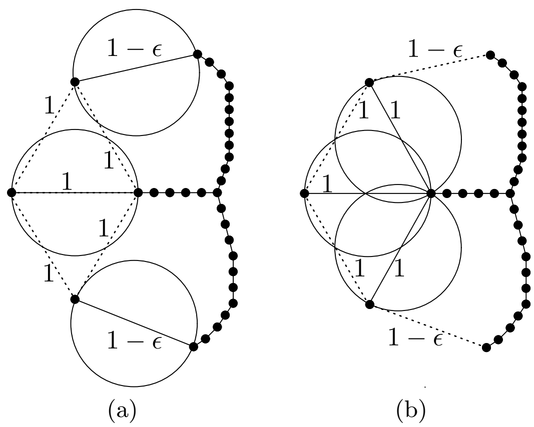

Fig. 1. A minimum spanning tree is not necessarily a minimum-area spanning tree. (a) The minimum spanning tree. (b) A minimum-area spanning tree.

Let $\mathcal{T}$ be any spanning tree of $\mathcal{P}$. For an edge e in $\mathcal{T}$, let $D(e)$ denote the disk whose diameter is e. Put $D(\mathcal{T}) = \{D(e) \,|\, e \text{ is an edge in } \mathcal{T}\}$, $\bigcup_{\mathcal{T}} = \bigcup_{e \in \mathcal{T}} D(e)$, and $\sigma_{\mathcal{T}} = \sum_{e \in \mathcal{T}} \text{area}(D(e))$. Let MST be a minimum spanning tree of $\mathcal{P}$. MST is not necessarily a solution for the Minimum-Area Spanning Tree (MAST) problem; see Figure 1. In this section we prove that MST is a constant-factor approximation for MAST, that is, $\text{area}(\bigcup_{\text{MST}}) = O(\text{area}(\bigcup_{\text{OPT}}))$, where OPT is an optimal spanning tree, i.e., a solution to MAST.

We begin by showing another interesting property of MST, namely, that the depth of any point p in the interior of a cell of the arrangement of the disks in $D(\text{MST})$ is bounded by a small constant. This property does not follow directly from the fact that the degree of MST is bounded by 6; see Figure 2. Let MST_p be a minimum spanning tree for $\mathcal{P} \cup \{p\}$. We need the following known and easy claim.

Claim 1 *We may assume that there is no edge (a, b) in MST_p, such that (a, b) is not in MST and both a and b are points of $\mathcal{P}$.*

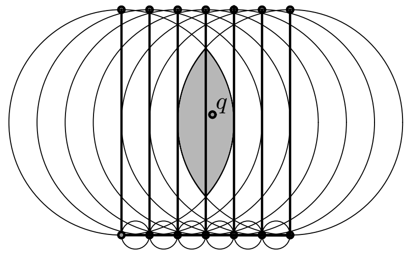

Fig. 2. A spanning tree $\mathcal{T}$ of degree 3, and a point q (in the interior of a cell of the arrangement of the disks in $D(\mathcal{T})$) of depth $O(n)$.

Proof. Assume there is such an edge (a, b) in MST_p. Consider the path in MST between a and b. At least one of the edges along this path is not in MST_p. Let e be such an edge. $|e| \leq |(a, b)|$, since otherwise (a, b) would have been chosen by the algorithm that computed MST (e.g., Kruskal's minimum spanning tree algorithm [5]). Therefore, we may replace the edge (a, b) in MST_p by e, without increasing the total weight of the tree.

An immediate corollary of this claim is that we may assume that if e is an edge in MST_p but not in MST, then one of e's endpoints is p.

Lemma 1. $\sigma_{\text{MST}} \leq 5\,\text{area}(\bigcup_{\text{MST}})$.

Proof. We prove that p belongs to at most 5 of the disks in $D(\text{MST})$. Let $D(q_1, q_2)$ be a disk in $D(\text{MST})$, such that $p \in D(q_1, q_2)$. (Notice that p is not on the boundary of $D(q_1, q_2)$, since p is in the interior of a cell of the arrangement of the disks in $D(\text{MST})$.) We show that the edge (q_1, q_2) is not in MST_p. If it is, then either the path from q_1 to p or the path from q_2 to p includes the edge (q_1, q_2) (but not both). Assume, e.g., that the path from q_1 to p includes the edge (q_1, q_2). Then, since (q_1, p) is shorter than (q_1, q_2), we can decrease the total weight of MST_p by replacing (q_1, q_2) in MST_p by (q_1, p). We conclude that (q_1, q_2) is not in MST_p.

Thus, by the corollary immediately preceding the lemma, each disk $D \in D(\text{MST})$ such that $p \in D$, induces a distinct edge in MST_p that is connected to p. But the degree of p is at most 6 (this is true for any vertex of any Euclidean minimum spanning tree), so there can be at most 5 disks covering p, since one of the edges connected to p is present due to the increase in the number of points (i.e., p was added to $\mathcal{P}$).

Remark. Ábrego et al. [1] have shown that the constant 5 can be improved to a constant 3, with a significantly more delicate argument. Their result appeared in an earlier (unpublished) draft of their manuscript.

Let OPT be an optimal spanning tree of $\mathcal{P}$, i.e., a solution to MAST. We use OPT to construct another spanning tree, ST, of $\mathcal{P}$. Initially ST is empty. Let e_1

be the longest edge in OPT. Draw two concentric disks C_1 and C_1^3 around the mid point of e_1 of diameters $|e_1|$ and $3|e_1|$, respectively. Compute a minimum spanning tree of the points of $\mathcal{P}$ lying in C_1^3, using Kruskal's algorithm [5]. Whenever an edge is chosen by Kruskal's algorithm, it is immediately added to ST. See Figure 3. Let S_1 denote the set of edges that have been added to ST in this (first) iteration.

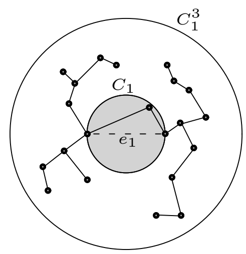

Fig. 3. ST after choosing e_1.

Next, let e_2 be the longest edge in OPT, such that at least one of its endpoints lies outside C_1^3. As for e_1, draw two concentric disks C_2 and C_2^3 around the mid point of e_2 of diameters $|e_2|$ and $3|e_2|$, respectively. Apply Kruskal's minimum spanning algorithm to the points of $\mathcal{P}$ lying in C_2^3 with the following modification. The next edge in the sorted list of potential edges is chosen by the algorithm if and only if it is not already in ST and its addition to ST does not create a cycle in ST. Moreover, when an edge is chosen by the algorithm it is immediately added to ST; see Figure 4 (a) and (b). Let S_2 denote the set of edges that have been added to ST in this iteration.

In the i'th iteration, let e_i be the longest edge in OPT, such that there is no path yet in ST between its endpoints. Draw two concentric circles C_i and C_i^3 around the mid point of e_i, and apply Kruskal's minimum spanning tree algorithm with the modification above to the points of $\mathcal{P}$ lying in C_i^3. Let S_i denote the set of edges that have been added to ST in this iteration. The process ends when for each edge e in OPT there already exists a path in ST between the endpoints of e.

Claim 2 *For each i, S_i is a subset of the edge set of the minimum spanning tree* MST$_i$ *that is obtained by applying Kruskal's algorithm,* without *the modification above, to the points in C_i^3.*

Proof. Let e be an edge that was added to ST during the i'th iteration. If e is not chosen by Kruskal's algorithm (without the modification above), it is only because, when considering e, a path between its two endpoints already existed in MST$_i$. But this implies that e could not have been added to ST, since, any edge already in MST$_i$ was either also added to ST or was not added since there already

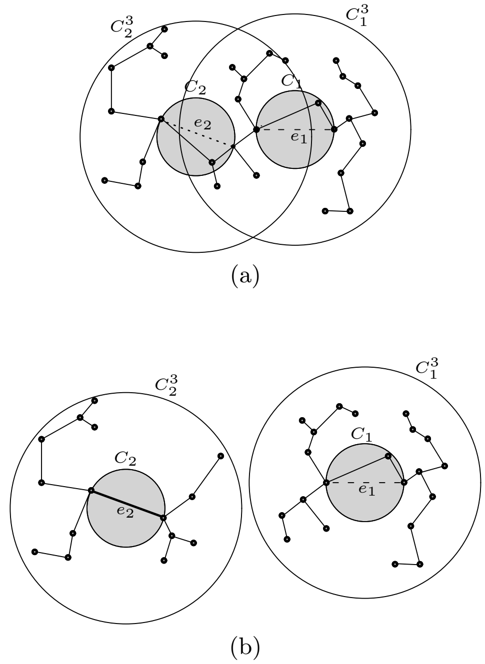

Fig. 4. ST after choosing e_1 and e_2. (a) One of the end points of e_2 is in C_1^3. (b) Both endpoints of e_2 are not in C_1^3.

existed a path in ST between its two endpoints. Thus, when e was considered by the modified algorithm it should have been rejected. We conclude that e must be in MST_i.

Claim 3 ST *is a spanning tree of* $\mathcal{P}$.

Proof. Since only edges that do not create a cycle in ST were added to ST, there are no cycles in ST. Also, ST is connected, since otherwise there still exists an edge in OPT that forces another iteration of the construction algorithm.

Let $\mathcal{C}$ denote the set of the disks $C_1, \ldots, C_k$, and let $\mathcal{C}^3$ denote the set of the disks $C_1^3, \ldots, C_k^3$, where k is the number of iterations in the construction of ST.

Claim 4 *For any pair of disks* C_i, C_j *in* $\mathcal{C}$, $i \neq j$, *it holds that* $C_i \cap C_j = \emptyset$.

Proof. Let C_i be any disk in $\mathcal{C}$. We show that for any disk $C_j \in \mathcal{C}$ such that $j > i$, $C_i \cap C_j = \emptyset$. From the construction of ST it follows that $|e_j|$, the diameter of C_j, is smaller or equal to $|e_i|$, the diameter of C_i. Moreover, at least one of the endpoints of e_j lies outside C_i^3 (since if both endpoints of e_j lie in C_i^3, then, by the end of the i'th iteration, a path connecting between these endpoints must already exist in ST). Therefore, C_j whose center coincides with the mid point of e_j, cannot intersect C_i.

Claim 5 $\sigma_{\mathrm{ST}} = O(\mathrm{area}(\bigcup_{\mathrm{OPT}}))$.

Proof. Recall that $\sigma_{\mathrm{ST}} = \Sigma_i \sigma_{S_i}$, where $\sigma_{S_i} = \Sigma_{e \in S_i} \mathrm{area}(D(e))$. We first show by the sequence of inequalities below that $\sigma_{S_i} = O(\mathrm{area}(C_i))$.

$$\sigma_{S_i} \leq^1 \sigma_{\mathrm{MST}_i} \leq^2 5\,\mathrm{area}(\bigcup_{\mathrm{MST}_i}) =^3 O(\mathrm{area}(C_i^3)) =^4 O(\mathrm{area}(C_i))\,.$$

The first inequality follows immediately from Claim 2. The second inequality is true by Lemma 1. Consider Equality 3. Since all edges in MST_i are contained in C_i^3, it holds that $\bigcup_{\mathrm{MST}_i}$ is contained in a disk that is obtained by expanding C_i^3 by some constant factor. It follows that $\mathrm{area}(\bigcup_{\mathrm{MST}_i}) = O(\mathrm{area}(C_i^3)) = O(\mathrm{area}(C_i))$.

Therefore,

$$\sigma_{\mathrm{ST}} = \Sigma_i \sigma_{S_i} = \Sigma_i O(\mathrm{area}(C_i))\,.$$

But according to Claim 4, the latter expression is equal to $O(\mathrm{area}(\bigcup_{\mathcal{C}}))$, and, since $\mathcal{C}$ is a subset of $D(\mathrm{OPT})$, we conclude that $\sigma_{\mathrm{ST}} = O(\mathrm{area}(\bigcup_{\mathrm{OPT}}))$.

We are now ready to prove the main result of this section.

Theorem 1. MST *is a constant-factor approximation for* MAST, *i.e.,* $\mathrm{area}(\bigcup_{\mathrm{MST}}) \leq c \cdot \mathrm{area}(\bigcup_{\mathrm{OPT}})$, *for some constant c.*

Proof.

$$\mathrm{area}(\bigcup_{\mathrm{MST}}) \leq^1 \sigma_{\mathrm{MST}} \leq^2 \sigma_{\mathrm{ST}} \leq^3 c \cdot \mathrm{area}(\bigcup_{\mathrm{OPT}})\,.$$

The first inequality is trivial. The second inequality holds for any spanning tree of $\mathcal{P}$; that is, for any spanning tree $\mathcal{T}$, $\sigma_{\mathrm{MST}} \leq \sigma_{\mathcal{T}}$. (Since if the lengths $|e|$ of the edges are replaced with weights $\pi|e|^2/2$, we remain with the same minimum spanning tree.) The third inequality is proven in Claim 5.

3 A Constant-Factor Approximation for MARA

MST induces an assignment of ranges to the points of $\mathcal{P}$. Let $p_i \in \mathcal{P}$ and let r_i be the length of the longest edge in MST that is connected to p_i, then the range that is assigned to p_i is r_i. Put $\mathrm{RA} = \{D_{p_1}, \ldots, D_{p_n}\}$, where D_{p_i} is the disk of radius r_i centered at p_i. In this section we apply the main result of the previous section (i.e., MST is a constant-factor approximation for MAST), in order to prove that the range assignment that is induced by MST is a constant-factor approximation for the Minimum-Area Range Assignment (MARA) problem. That is, (i) the corresponding (directed) communication graph is strongly connected, and (ii) the area of the union of the disks in RA is bounded by some constant times the area of the union of the transmission disks in an optimal range assignment, i.e., a solution to MARA.

The first requirement above was already proven by Kirousis et al. [10], who showed that the range assignment induced by MST is a 2-approximation for the standard range assignment problem. Let OPT^R denote an optimal range assignment, i.e., a solution to MARA. It remains to prove the second requirement above.

Claim 6 $\text{area}(\bigcup_{RA}) \leq 9\,\text{area}(\bigcup_{\text{MST}})$.

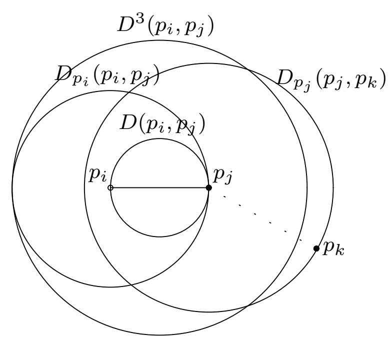

Fig. 5. $(p_i, p_j) \in$ MST; $D(p_i, p_j) \in D(\text{MST})$; $D_{p_i}(p_i, p_j), D_{p_j}(p_j, p_k) \in$ RA; $D^3(p_i, p_j) \in D^3(\text{MST})$.

Proof. We define an auxiliary set of disks. For each edge e in MST, draw a disk of diameter $|3e|$ centered at the mid point of e. Let $D^3(\text{MST})$ denote the set of these $n-1$ disks; see Figure 5. We now observe that $\text{area}(\bigcup_{RA}) \leq \text{area}(\bigcup_{D^3(\text{MST})})$. This is true since for each $p_i \in \mathcal{P}$, $D_{p_i} = D_{p_i}(p_i, p_j)$ for some point $p_j \in \mathcal{P}$ that is connected to p_i (in MST) by an edge, and $D_{p_i}(p_i, p_j)$ is contained in the disk of $D^3(\text{MST})$ corresponding to the edge (p_i, p_j). Finally, clearly $\text{area}(\bigcup_{D^3(\text{MST})}) \leq 9\,\text{area}(\bigcup_{\text{MST}})$.

Theorem 2. RA *is a constant-factor approximation for* MARA*, i.e.,* $\text{area}(\bigcup_{\text{RA}}) \leq c' \cdot \text{area}(\bigcup_{\text{OPT}^R})$*, for some constant* c'*.*

Proof. The proof is based on the observation that the (directed) communication graph corresponding to OPT^R contains a spanning tree, and on the main result of Section 2. Let p be any point in $\mathcal{P}$. We construct a spanning tree $\mathcal{T}$ of $\mathcal{P}$ as follows. For each point $q \in \mathcal{P}$, $q \neq p$, compute a shortest (in terms of number of hops) directed path from q to p, and add the edges in this path to $\mathcal{T}$. Now make all edges in $\mathcal{T}$ undirected. $\mathcal{T}$ is a spanning tree of $\mathcal{P}$. For each edge (p_i, p_j) in $\mathcal{T}$, the disk $D(p_i, p_j)$ is contained either in the transmission disk of p_i (in OPT^R), or in the transmission disk of p_j (in OPT^R). Hence, $\bigcup_{\mathcal{T}} \subseteq \bigcup_{\text{OPT}^R}$.

The following sequence of inequalities completes the proof. (OPT denotes a solution to MAST.)

$$\text{area}(\bigcup_{\text{RA}}) \leq^1 9\,\text{area}(\bigcup_{\text{MST}}) \leq^2 9c \cdot \text{area}(\bigcup_{\text{OPT}}) \leq^3 9c \cdot \text{area}(\bigcup_{\mathcal{T}}) \leq^4 9c \cdot \text{area}(\bigcup_{\text{OPT}^R}) .$$

The first inequality follows from Claim 6; the second inequality follows from Theorem 1; the third inequality follows from the definition of OPT; the fourth inequality was shown above.

4 A Constant-Factor Approximation for MACDG

MST induces an assignment of radii to the points of $\mathcal{P}$. Let $p_i \in \mathcal{P}$ and let r_i be the length of the longest edge in MST connected to p_i, then the radius that is assigned to p_i is $r_i/2$. Put $\text{DG} = \{D_{p_1}, \ldots, D_{p_n}\}$, where D_{p_i} is the disk of radius $r_i/2$ centered at p_i. In this section we apply the main result of Section 2, in order to prove that DG is a constant-factor approximation for the Minimum-Area Connected Disk Graph (MACDG) problem. That is, (i) viewing DG as an intersection graph, DG is connected, and (ii) the area of the union of the disks in DG is bounded by some constant times the area of the union of the disks in an optimal assignment of radii, i.e., a solution to MACDG.

The first requirement above clearly holds, since each edge in MST is also an edge in DG. Let OPT^D denote an optimal assignment of radii, i.e., a solution to MACDG. It remains to prove the second requirement above.

Theorem 3. DG *is a constant-factor approximation for* MACDG*, i.e.,* $\text{area}(\bigcup_{\text{DG}}) \leq c'' \cdot \text{area}(\bigcup_{\text{OPT}^D})$*, for some constant* c''*.*

Proof. We only outline the proof, since it is very similar to the proof of the previous section. We first claim that $\text{area}(\bigcup_{\text{DG}}) \leq 9\,\text{area}(\bigcup_{\text{MST}})$. This follows immediately from Claim 6, since $\bigcup_{\text{DG}} \subseteq \bigcup_{\text{RA}}$. Next, we observe that if one doubles the radius of each of the disks in OPT^D, then the resulting set of disks contains the set of disks of some spanning tree $\mathcal{T}$ of $\mathcal{P}$. Thus, by Theorem 1, $\text{area}(\bigcup_{\text{MST}}) \leq c \cdot \text{area}(\bigcup_{\text{OPT}^D})$. We complete the proof by putting the two inequalities together.

5 A Constant-Factor Approximation for MAT

Consider the complete graph induced by $\mathcal{P}$. We assign weights to the edges of the graph, such that the weight $w(e)$ of an edge e is $|e|^2$. Let G^2 denote this graph. Define the weight $w(F)$ of a subset F of the edge-set of G^2 to be the sum of the weights of the edges in F.

Notice that the triangle inequality does not hold in G^2. However, the triangle inequality "almost" holds, in that $|uv|^2 \leq 2 \cdot (|uw|^2 + |wv|^2)$. For distance functions such that $d(u,v) \leq \tau \cdot (d(u,w) + d(w,v))$, constant-factor approximation algorithms for the TSP are known: Andreae and Bandelt [3] give a $(3\tau^2/2+\tau/2)$-approximation, which was refined by Andrea [2] to a $(\tau^2+\tau)$-approximation, and Bender and Chekuri [4] give a 4τ-approximation. For our case ($\tau = 2$), this implies that there is a 6-approximation.

Andreae and Bandelt actually compute a tour T in G^2, such that $w(T) \leq c \cdot w(\text{MST}_{G^2})$, where MST_{G^2} is the minimum spanning tree of G^2 and c is some constant. We show that T is a constant-factor approximation for the Minimum-Area Tour (MAT) problem.

For an edge e in T, let $D(e)$ denote the disk whose diameter is e. Put $D(T) = \{D(e) \mid e \text{ is an edge in } T\}$, $\bigcup_T = \bigcup_{e \in T} D(e)$, and $\sigma_T = \sum_{e \in T} \text{area}(D(e))$. Let OPT^T be an optimal tour, i.e., a solution to MAT. Clearly $\text{area}(\bigcup_{\text{OPT}^T}) \geq \text{area}(\bigcup_{\text{OPT}^S})$, where OPT^S is a solution to the Minimum Area Spanning Tree (MAST) problem. We need to show that $\text{area}(\bigcup_T) = O(\text{area}(\bigcup_{\text{OPT}^T}))$. Indeed

$$\text{area}(\bigcup_T) \leq \sigma_T \leq w(T) \leq c \cdot w(\text{MST}_{G^2}) \, .$$

But $w(\text{MST}_{G^2}) = \sum_{e \in \text{MST}} |e|^2$, where MST is the minimum spanning tree of $\mathcal{P}$ (since both trees are identical in terms of edges). So

$$\text{area}(\bigcup_T) = O(\sum_{e \in \text{MST}} |e|^2) = O(\sigma_{\text{MST}}) = O(\text{area}(\bigcup_{\text{MST}})) \, ,$$

where the latter equality follows from Lemma 1. And, by the main result of Section 2,

$$O(\text{area}(\bigcup_{\text{MST}})) = O(\text{area}(\bigcup_{\text{OPT}^S})) = O(\text{area}(\bigcup_{\text{OPT}^T})) \, .$$

The following theorem summarizes the result of this section.

Theorem 4. *T is a constant-factor approximation for* MAT*, i.e.,* $\text{area}(\bigcup_T) \leq \hat{c} \cdot \text{area}(\bigcup_{\text{OPT}^T})$*, for some constant $\hat{c}$.*

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