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THE MINIMUM JEANS MASS OR WHEN FRAGMENTATION MUST STOP

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SUMMARY

By considering the Jeans mass fragmentation of a gas cloud we derive a minimum fragment mass of $0{\cdot}025(m_{\rm H}/m)^{16/7}(\kappa_0/\kappa_{\rm f})^{1/7}\,M_\odot$, where $\kappa_0/\kappa_{\rm f}$ is the ratio of the electron scattering opacity to the gas opacity at the final fragmentation, and $m/m_{\rm H}$ is the ratio of the mass of a gas particle to that of the hydrogen atom. For a dark molecular cloud this would give a $0{\cdot}007\,M_\odot$ fragment. This result is very insensitive to changes in the opacity of the gas. We have also studied the effects of isotropic blackbody radiation on the minimum mass, with particular reference to primeval star formation, and find it important for epochs with redshift $z \sim 2{\cdot}7$. A number of different opacities were considered and we conclude that the most important source of opacity, both for present and past star formation, is dust grains.

1. INTRODUCTION

Jeans's criterion for gravitational instability (Jeans 1928) has proved to be exactly correct when rigorously derived for perturbations of isotropic collapses (see for example Lynden-Bell 1973; Hunter 1964; Peebles 1971). There are lingering doubts still hanging over fragmentation theory when the more realistic inhomogeneous collapses are considered, but an unpublished investigation by Mills indicates that fragmentation does proceed for perturbations of Penston's pressure-free inhomogeneous similarity collapse (Penston 1969). Thus when pressure forces are sufficiently weak fragmentation will occur, and in what follows we assume that Jeans's criterion is the correct one.

This paper neglects many of the complications of the real world, such as rotation and magnetic fields, since these make progress difficult and furthermore they always act so as to increase the Jeans mass. Thus minimum masses may be calculated in their absence. This paper is therefore devoted to a discussion of the minimum masses at which fragmentation must cease. Our main drive is to calculate the masses of the non-fragmenting fragments and in this we follow the tradition set by Hoyle (1953). For discussion of the full problem with the realistic complications of magnetic fields and rotation the reader is referred to review papers by Mestel (1965).

Jeans's criterion is that a medium of density ρ will be unstable to fragmentation at all wavelengths greater than $\lambda_{\rm J} = 2\pi/k_{\rm J}$ where $k_{\rm J}{}^2 c_{\rm s}{}^2 = 4\pi G\rho$ and $c_{\rm s}{}^2$ is the ratio of pressure to density perturbations. Thus we may expect the medium to break up into masses of about

$$M_{\rm J} = \rho\lambda_{\rm J}{}^3 = \pi^{3/2} c_{\rm s}{}^3 (G^3\rho)^{-1/2}. \tag{1}$$

Larger masses would themselves break up into masses of this size. However, during the continued collapse $c_s{}^3\rho^{-1/2}$ continues to diminish, thus we expect the fragments to fragment into masses that themselves fragment and this goes on until $c_s{}^3\rho^{-1/2}$ begins to increase. At this point the final fragments form and presumably remain to make stars. Subsequent accretion can increase their masses but will not decrease them. It is important to note that these are very much minimum masses since for a given mass to split it must be at least twice the minimum Jeans mass. In addition a Jeans mass perturbation has a zero growth rate and we would expect a real growing perturbation to be somewhat larger than a Jeans mass.

The importance of heat loss for continued fragmentation is readily illustrated, for if we take a gas with pressure $p \propto \rho^\gamma$ then $c_s{}^2 = dp/d\rho \propto \gamma\rho^{\gamma-1}$ and so $c_s{}^3\rho^{-1/2} \propto \rho^{3\gamma-4/2}$. Only when the effective $\gamma < 4/3$ as for an isothermal or cooled collapse does the Jeans mass continue to decrease as the density increases. In particular, once the collapse becomes adiabatic with $\gamma = 5/3$ as for a monatomic gas $c_s{}^3\rho^{-1/2}$ increases as ρ increases. The fragments no longer fragment.

The heating, cooling and opacity of the interstellar gas are picturesquely considered in a log T – log ρ plot (Fig. 1). For each point in the diagram we have a Jeans mass given by

$$M_J = \left(\frac{\pi k}{Gm}\right)^{3/2} T^{3/2}\rho^{-1/2} \tag{2}$$

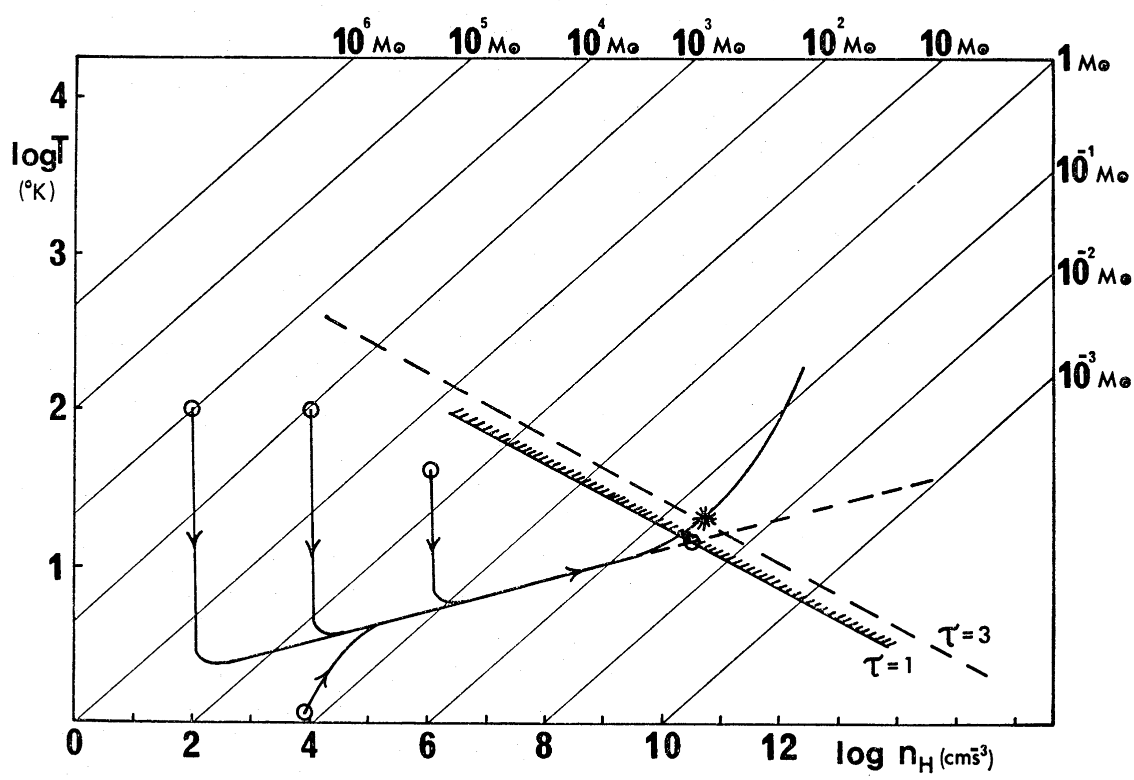

FIG. 1. *The log T – log ρ diagram. The lines of constant Jeans mass are indicated. To the right of the hatched line Jeans mass fragments are optically thick with $\tau \geq 1$. The other curve is the cooling curve for the idealized opacity law $\kappa = KT^2$, applicable to dust in equilibrium with gas. Notice the strong convergence of out-of-equilibrium solutions to the $C = 0$ solution (see equation (20)). The last fragmentation is at * where the trajectory bends away from the optically thin trajectory and becomes parallel to the lines of constant Jeans mass. This occurs at the optical depth $\tau \sim 3$. Notice that the point * corresponds very closely to the same mass as o, the intersection of the $\tau = 1$ curve with the cooling curve calculated in the absence of opacity. It is the conditions at o that were calculated in the Introduction (equations (6)–(11)). $m = m_H$.*

where we have taken the isothermal sound speed $c_s^2 = kT/m$, k is Boltzmann's constant and m is the mass of the constituent atoms or molecules (e.g. hydrogen atoms or molecules). The lines of constant Jeans mass have a slope of one-third in the diagram. In order to cool, it is important that the material in the middle of each fragment should be able to see the outside world. Thus considering blackbody radiation at temperature T we ask: 'What is the optical depth to this radiation through a length $\lambda_J/2$ of this material?'. The dark line on the diagram corresponds to an optical depth through a Jeans mass fragment of $\tau = 1$. On the shaded side of this line the material is fragmented into opaque Jeans masses, while significantly to the left of the line opacity is unimportant and the cloud temperature adjusts itself towards the dynamical balance of heating and cooling. In fact, on this transparent side of the diagram we find that the time scale for adjustment of the temperature is considerably faster than the $(16\pi G\rho)^{-1/2}$ time scale of the dynamic collapse, so that there is a rapid convergence in temperature for clouds starting away from thermal equilibrium towards a line which gives approximate balance between heating and cooling during the dynamical collapse.

Thus clouds starting from diverse initial conditions approach the region of optical thickness almost as though they all came from a balanced initial state. Once opaque the fragmentation ceases, so diverse initial conditions all yield non-fragmenting fragments of the same minimum size.

The simplest way to obtain a rough estimate of the conditions at the last fragmentation is to say that as the fragment becomes optically thick the radiative cooling balances the $-p\,dV$ rate of working. Thus when

$$\kappa_P \rho \frac{\lambda_J}{2} = \tau = 1 \qquad (3)$$

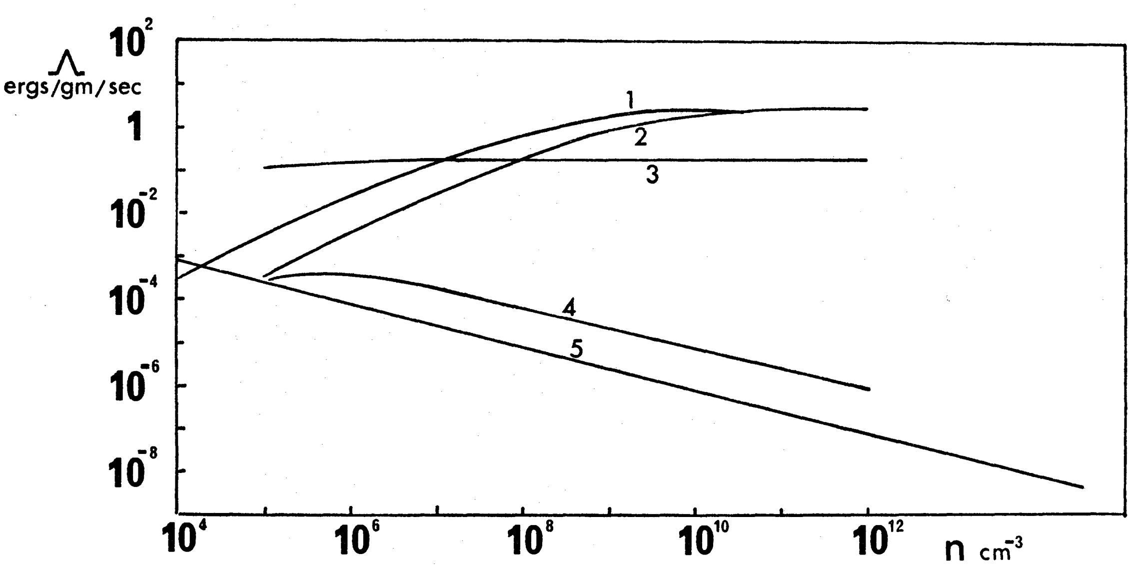

FIG. 2. *Cooling per unit mass as a function of density, at a temperature of* 15 *K, of dust, CO and C* I. *This demonstrates the cooling efficiency of dust at high densities compared with other coolants. The curves are:* (1) *Dust cooling; energy transfer parameter* $\Theta = 1$ *(very 'inelastic' dust).* (2) *Dust cooling; energy transfer parameter* $\Theta = 0{\cdot}1$ *('elastic' dust).* (3) *CO cooling, neglecting optical depth, assuming all carbon in this form.* (4) *CO cooling, including effect of optical thickness through a Jeans mass, assuming all carbon in this form.* (5) *C* I *cooling, including effect of optical thickness through a Jeans mass.*

κ_P being the Planck mean opacity per unit mass at temperature T, we have for the energy balance

$$4\kappa_P \sigma T^4 = -p\,\frac{d(1/\rho)}{dt} = \frac{kT}{m}\,(16\pi G\rho)^{1/2}. \tag{4}$$

At the last equality we have assumed that the system is in a slightly pressure-resisted free fall so that

$$\frac{d\log\rho}{dt} = (16\pi G\rho)^{1/2}. \tag{5}$$

This value is taken from the behaviour of Penston's characteristic density $\rho_c = [4\pi G(t_0-t)^2]^{-1}$ in his isothermal similarity solution (Penston 1969). In general κ_P will be a function of ρ and T but when we solve equations (3) and (4) for the conditions at the final fragmentation T_f and ρ_f it is only the opacity at that moment that comes in. Calling this κ_{Pf} and solving (3) and (4) we have

$$\left.\begin{aligned} T_f &= \left(\frac{4G^2k}{m\sigma^2\kappa_{Pf}^4}\right)^{1/7} = \left[\frac{3^6.5^2}{2^4.\pi^6}\left(\frac{m_e}{m}\right)\left(\frac{Gm_H^2}{hc}\right)^2\left(\frac{e^2}{\hbar c}\right)^{-8}\right]^{1/7}\frac{m_e c^2}{k}\left(\frac{\kappa_0}{\kappa_{Pf}}\right)^{4/7}\\ &= 4{\cdot}1\left(\frac{\kappa_0}{\kappa_{Pf}}\right)^{4/7}\,\mathrm{K}\\ \rho_f &= \left(\frac{2^{12}m^8G^5\sigma^2}{\pi^7k^8\kappa_{Pf}^{10}}\right)^{1/7} = \left[\frac{3^8}{5^2.2^{17}.\pi^8}\left(\frac{Gm_H^2}{hc}\right)^5\left(\frac{e^2}{\hbar c}\right)\left(\frac{m}{m_H}\right)^8\left(\frac{m_e}{m_H}\right)^{-1}\right]^{1/7}\\ &\qquad\times\frac{m_H}{(e^2/m_e c^2)^3}\left(\frac{\kappa_0}{\kappa_{Pf}}\right)^{10/7}\\ &= 1{\cdot}5\times10^{-15}\left(\frac{\kappa_0}{\kappa_{Pf}}\right)^{10/7}\,\mathrm{g\,cm^{-3}}. \end{aligned}\right\} \tag{6}$$

In these numerical evaluations we have set $m = m_H$.

In the above we have expressed κ_{Pf} as a fraction of κ_0, the opacity due to free electron scattering in ionized hydrogen; $\kappa_0 = (8\pi/3)(e^2/m_e c^2)^2\,(1/m_H)$, where m_H is the mass of the hydrogen atom and $e^2/m_e c^2$ is the classical radius of the electron. We now express the Jeans mass at this final fragmentation by using equation (2), and we obtain

$$\begin{aligned} M_{Jf} &= \left[\frac{\pi^{14}}{2^3}\left(\frac{k}{m}\right)^{16}\sigma^{-4}G^{-10}\kappa_{Pf}^{-1}\right]^{1/7}\\ &= \left[\frac{3^5.5^4}{2^8.\pi^5}\left(\frac{e^2}{\hbar c}\right)^{-2}\left(\frac{Gm_H^2}{hc}\right)^{-10}\left(\frac{m_e}{m_H}\right)^2\left(\frac{m_H}{m}\right)^{16}\right]^{1/7} m_H\left(\frac{\kappa_0}{\kappa_{pf}}\right)^{1/7}\\ &= 0{\cdot}025\left(\frac{m_H}{m}\right)^{16/7}\left(\frac{\kappa_0}{\kappa_{Pf}}\right)^{1/7}M_\odot = 5.10^{31}\left(\frac{m_H}{m}\right)^{16/7}\left(\frac{\kappa_0}{\kappa_{Pf}}\right)^{1/7}\,\mathrm{g}. \end{aligned} \tag{7}$$

Notice how weakly the mass at the final fragmentation depends on the opacity. Thus a change in final opacity by a factor of 10^7 only gives a change of mass by a factor 10. However, the density and temperature at the final fragmentation are much more opacity dependent. The average distance between fragments at last fragmentation is

$$\left(\frac{\rho_f}{M_{Jf}}\right)^{-1/3} = 3.10^{16}\left(\frac{m}{m_H}\right)^{-8/7}\left(\frac{\kappa_{Pf}}{\kappa_0}\right)^{3/7} \text{cm} = 0{\cdot}01\left(\frac{m}{m_H}\right)^{-8/7}\left(\frac{\kappa_{Pf}}{\kappa_0}\right)^{3/7} \text{pc}$$

which should be compared with wide binary star separations.

When the opacity is a strongly varying function of ρ and T the above formulae are not terribly useful, for they express the density temperature and mass at the final fragmentation in terms of the unknown opacity then (albeit to the one-seventh power). If we take for example an opacity that varies with density and temperature

$$\kappa_P = K\rho^\alpha T^\beta \tag{8}$$

we need expressions for the final fragmentation variables in terms of K rather than κ_{Pf}. These are readily determined by writing κ_f in the form given by equation (8) and solving equations (6) for T and ρ:
Putting $E = (7+10\alpha+4\beta)^{-1}$

$$T_f = \left[2^{(2-4\alpha)}\pi^{4\alpha}G^2\left(\frac{k}{m}\right)^{(6\alpha+1)}\sigma^{-(4\alpha+2)}K^{-4}\right]^E \tag{9}$$

$$\rho_f = \left[2^{4\beta+12}\pi^{-(4\beta+7)}G^5\left(\frac{k}{m}\right)^{-(6\beta+8)}\sigma^{(4\beta+2)}K^{-10}\right]^E \tag{10}$$

$$M_{Jf} = \left[\frac{\pi^{14+21\alpha+8\beta}}{2^{3+6\alpha+2\beta}}G^{-(10+15\alpha+6\beta)}\left(\frac{k}{m}\right)^{16+24\alpha+9\beta}\sigma^{-2(3\alpha+\beta+2)}K^{-1}\right]^E. \tag{11}$$

It is significant that in equation (11) the final fragment mass depends only on the opacity, and that to the $-E$th power and $E \sim \frac{1}{10}$ in most cases. While we will examine sources of opacity of widely varying nature, the final fragment mass will not vary from 0·025 $M_\odot$ by more than a factor of 5 either way for realistic opacities. Notice also that G comes into the final mass with a power that is normally negative. Thus in variable G cosmologies larger G will lead to smaller masses and vice versa. Effects of magnetic fields, rotation, etc., may very crudely be thought of as offsetting some of the gravity, lowering the effective G and thus increasing the mass.

2. EFFECTS OF A HOT ENVIRONMENT

When background blackbody radiation is important we can obtain a lower limit to the fragment masses by assuming that the gas is at the background temperature T_b as it becomes optically thick. The gas cannot cool below this temperature. This is of obvious importance when considering star formation at an early epoch, when $T_b = 2{\cdot}7(1+z)$ K where z is the cosmological redshift factor.

Equation (3) gives

$$(K\rho_f{}^\alpha T_b{}^\beta)\,\rho_f\left[\left(\frac{\pi k}{Gm}\right)^{1/2}\frac{T_b{}^{1/2}\rho_f{}^{-1/2}}{2}\right] = \tau = 1 \tag{12}$$

and we have

$$\rho_f = \left[2\left(\frac{\pi k}{Gm}\right)^{-1/2}\frac{1}{K}\right]^{(2/2\alpha+1)} T_b{}^{-(2\beta+1)/(2\alpha+1)} \tag{13}$$

and

$$M_{\mathrm{Jf}} = \left(\frac{\pi k}{Gm}\right)^2 \frac{\kappa_0}{2} T_{\mathrm{b}}^2 \left(\frac{\kappa_{\mathrm{Pf}}}{\kappa_0}\right) \tag{14}$$

$$= 1{\cdot}54 . 10^{-3} \left(\frac{m_{\mathrm{H}}}{m}\right)^2 T_{\mathrm{b}}^2 \left(\frac{\kappa_{\mathrm{Pf}}}{\kappa_0}\right) M_{\odot}. \tag{15}$$

The dependence on opacity is now greatly increased and the minimum mass depends sensitively on the background temperature.

In the following sections we will examine all these processes in greater detail and justify some of our assertions.

3. EQUATIONS GOVERNING COOLING AND COLLAPSE

While detailed collapse calculations have been performed numerically—for references see review by Larson (1973), the more complex problem of fragmentation has not been solved. It is a characteristic of such solutions that in an isothermal collapse the ratio J of gravitational potential energy to gas thermal energy increases as the collapse proceeds. Since

$$J \propto \left(\frac{M}{M_{\mathrm{J}}}\right)^{2/3} \tag{16}$$

our hypothesis is that when J increases above a certain value the mass M is unstable to fragmentation into fragments of mass M_{J} whose number can be estimated from (16), and that the density contrast between the fragments and the mean density of the cloud increases with time. We will ignore the details of the dynamics and concentrate on the temperature of the gas at given density since this determines the Jeans mass. If we denote by Λ the cooling per g s^{-1} and by Γ the heating per g s^{-1} then taking a perfect gas we have from the conservation of energy

$$\frac{3}{2}\frac{k}{m}\frac{dT}{dt} = \Gamma - \Lambda. \tag{17}$$

We denote by Γ' any heating not due to the $p\, dV$ work of the collapsing gas. The $p\, dV$ contribution, as in (4) is $(kT/m)(16\pi G\rho)^{1/2}$ erg g^{-1} s^{-1} and from (5)

$$\frac{dT}{dt} = \frac{dT}{d(\log \rho)} (16\pi G\rho)^{1/2}$$

thus (17) may be put in the form

$$\frac{d \log T}{d \log \rho} = \frac{2}{3} - \frac{2}{3}\frac{m}{k}\frac{(\Lambda - \Gamma')}{(16\pi G\rho)^{1/2}\, T}. \tag{18}$$

From any given initial conditions in the $\log T - \log \rho$ diagram equation (18) enables us to compute a trajectory and hence a minimum mass for that trajectory. As remarked earlier, for most opacities this equation gives rapid convergence to the solution in which at any point heating $\simeq$ cooling. This is easily shown for opacities of the form $\kappa_{\mathrm{P}} = K\rho^{\alpha}T^{\beta}$, where we have $\Lambda = 4K\rho^{\alpha}T^{\beta}\sigma T^4$ and $\Gamma' = 0$. Equation (18) takes the form

$$\frac{d \log T}{d \log \rho} = \frac{2}{3} - \frac{2}{3}\frac{m}{k}\frac{4\sigma K\rho^{\alpha-1/2}T^{3+\beta}}{(16\pi G)^{1/2}}, \tag{19}$$

i.e.

$$\frac{d}{d\rho}(\rho^{2/3(3+\beta)}T^{-(3+\beta)}) = A(3+\beta)\,\rho^{(\alpha+1/2+2/3\beta)}$$

where

$$A = \frac{2}{3}\frac{m}{k}\frac{4\sigma K}{(16\pi G)^{1/2}}.$$

This has the solution

$$T = \rho^{-((\alpha-1/2)/(3+\beta))}\left(\frac{A(3+\beta)}{(\alpha+\frac{3}{2}+\frac{2}{3}\beta)}+C\rho^{-(\alpha+2/3\beta+3/2)}\right)^{-1/(\beta+3)} \quad (20)$$

where C is a constant determined from the initial conditions. If $(\alpha+\frac{2}{3}\beta+\frac{3}{2})>0$ then the term containing the initial conditions becomes less and less important with increasing density and the solution converges to

$$T = \rho^{-((\alpha-1/2)/(3+\beta))}\Big/\left\{\frac{A(3+\beta)}{(\alpha+\frac{3}{2}+\frac{2}{3}\beta)}\right\}^{1/(\beta+3)} \quad (21)$$

which compares with the ' heating equals cooling ' line

$$T = \rho^{-((\alpha-1/2)/(3+\beta))}/\{\tfrac{3}{2}A\}^{1/(\beta+3)}. \quad (22)$$

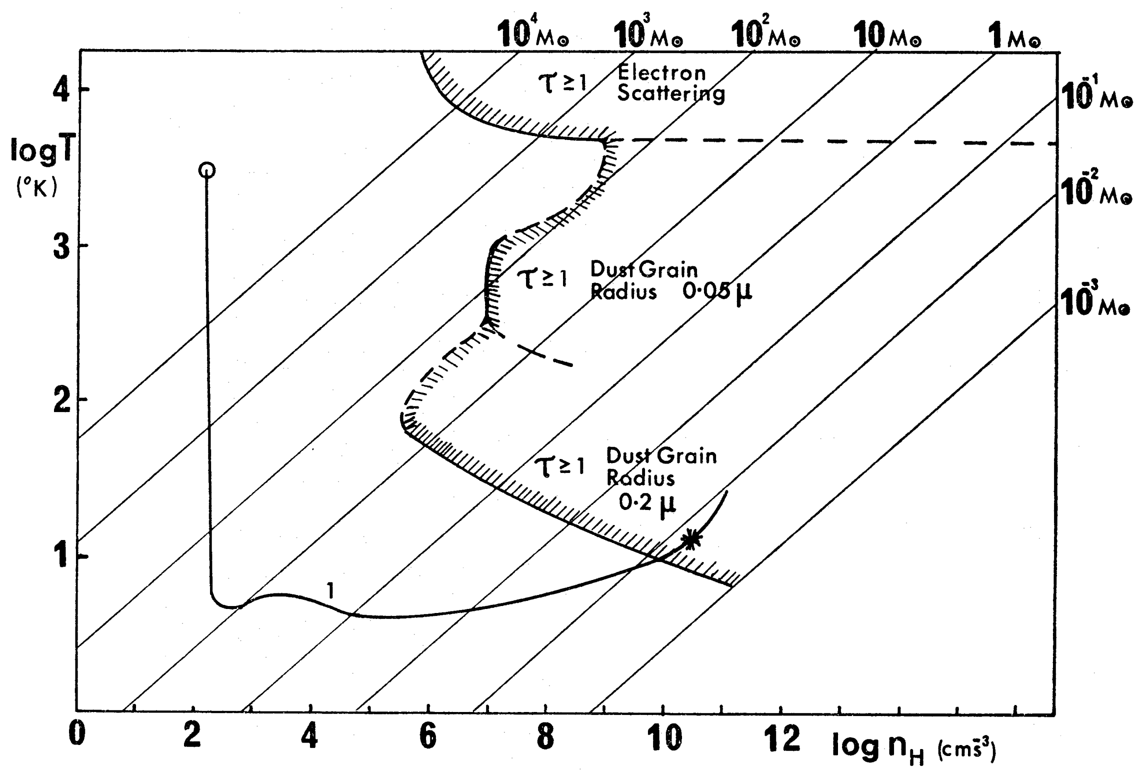

FIG. 3. *Cooling curve for a realistic dust opacity, including cooling by the lines of O I and C I. The minimum Jeans mass is shown by *. This minimum mass is that obtained in present-day regions of star formation in the Galaxy. Note that cosmic-ray heating at low densities $\sim 10^2$–10^3 cm^{-3} has not been included. This will not affect the high density portion of the curve $> 10^6$ cm^{-3}. The hatched region separates the optically thin region of the log T–log ρ diagram from the optically thick. The sources of opacity are dust and electron scattering. A rough indication has been made of the effect of the evaporation of grain mantles ~ 100 K and destruction of grain cores ~ 1000 K. Since grains can certainly survive for considerable periods at much higher temperatures this part of the ' opacity blanket ' is speculative. Molecular hydrogen $m = 2m_H$.*

When the temperature change is slow then the amount of $-p\,dV$ work that goes into heating the gas per unit time as compared with that radiated is small, and the two solutions are almost identical. The convergence to (21) displayed by (20) is extremely rapid when appropriate values are substituted for the opacity (Fig. 1). Numerical solutions of (18) display the same characteristic (Figs 3–7).

The other case of interest is when $(\alpha+\frac{2}{3}\beta+\frac{3}{2})<0$ and we have a thermal instability. The solution tends to $T \propto \rho^{2/3}$ which is just adiabatic compression. This instability proved to be important when considering cooling by atomic line radiation in the infrared where the excitation temperature of the levels differed by a large

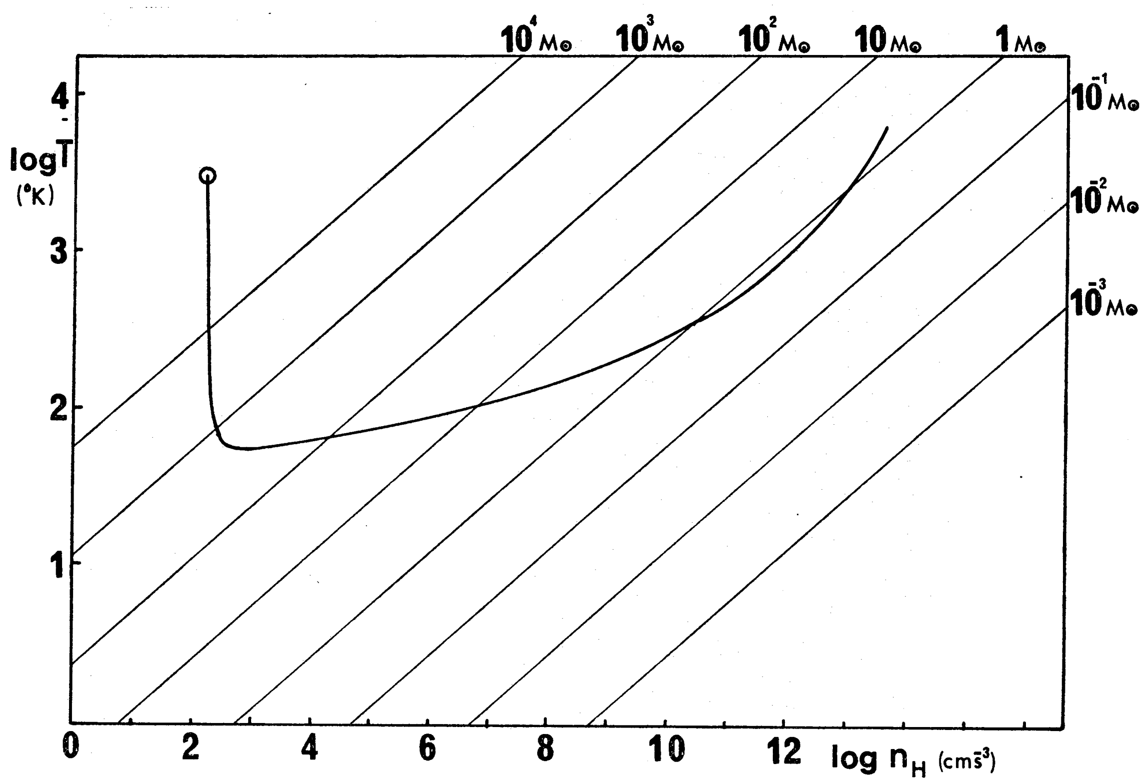

FIG. 4. *Trajectory with cooling due to molecular hydrogen with* $n_{H_2}/n_H = 1$. *Cooling is by the rotational line spectrum emission only.* $m = 1{\cdot}5\, m_H$.

amount. When a level is populated thermally the cooling $\Lambda \propto \exp(-T_0/T)$ while the $-p\,dV$ work $\propto \rho^{1/2}T$. For the equilibrium temperature $T < T_0$ the cooling is a sensitive function of temperature and the temperature increases slowly due to the $\rho^{1/2}$ factor in the heating. When the temperature reaches $T \geqslant T_0$, $\exp(-T_0/T) \rightarrow 1$ and the gas rapidly heats up with $T \propto \rho^{2/3}$ until another level at a higher excitation temperature begins to contribute.

The solution (20) does not have a minimum mass. We have not considered the effect of optical depth and we will discuss this in the next section. The following discussion may be omitted at a first reading. Equation (19) gives the trajectory in the log T – log ρ plane of an optically thin gas, and this is modified in Section 4, to give equation (41), which is the same equation taking into account optical thickness through a fragment. Some solutions to equation (41) are shown in Fig. 1.

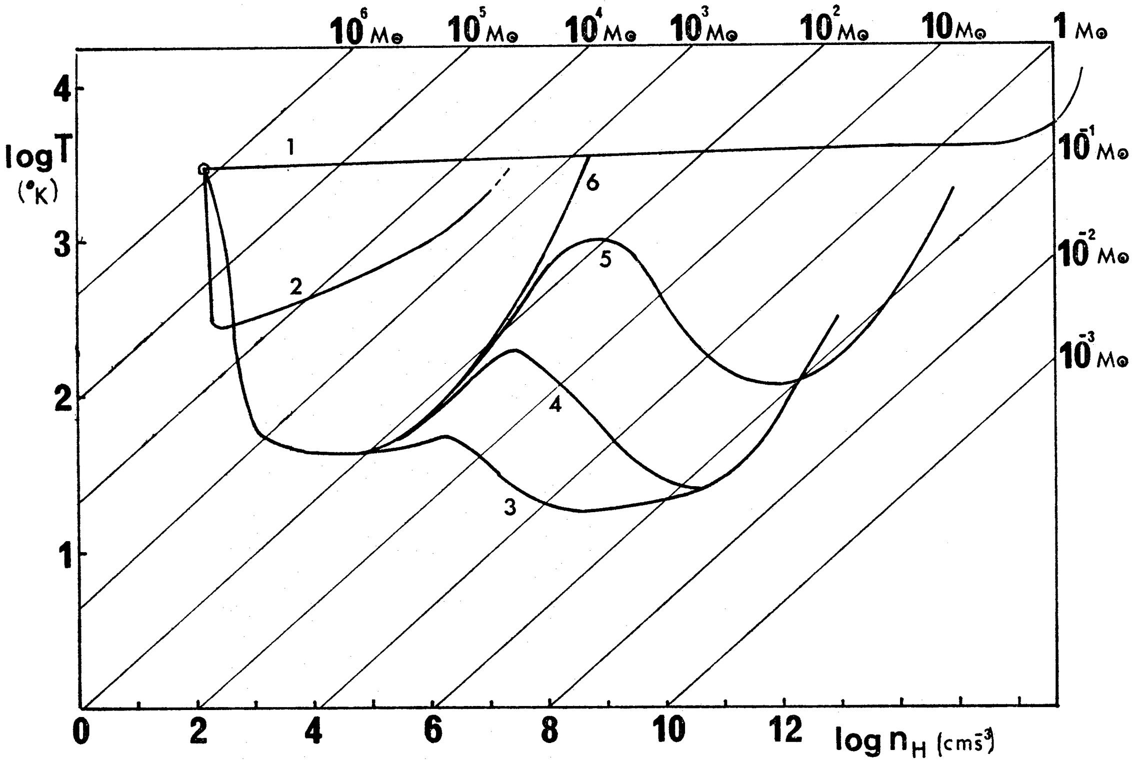

FIG. 5. *Curve* 1. *Trajectory with cooling due to bound–free emission from atomic hydrogen only. Curve* 2. *Cooling due to molecular hydrogen only,* $n_{H_2}/n_H = 10^{-3}$. *Curves* 3–6. *Cooling by* O I, C I *and dust. The abundance of* O I *and* C I *is reduced from solar in all curves by a factor of* 100. *The dust parameters are:*

3	$\Delta = 10^{-2}$	$a = 0{\cdot}2\ \mu$	$\Theta = 1{\cdot}0$	$\Gamma_g = 6\times 10^{-15}$
4	$\Delta = 10^{-2}$	$a = 0{\cdot}2\ \mu$	$\Theta = 0{\cdot}1$	$\Gamma_g = 6\times 10^{-15}$
5	$\Delta = 10^{-5}$	$a = 0{\cdot}02\ \mu$	$\Theta = 1{\cdot}0$	$\Gamma_g = 6\times 10^{-15}$
6	$\Delta = 10^{-5}$	$a = 0{\cdot}02\ \mu$	$\Theta = 0{\cdot}1$	$\Gamma_g = 6\times 10^{-15}$

where Δ *is the factor whereby the total mass in grains is reduced,* a *is the radius in* μ, Θ *is the energy transfer parameter (inelasticity) and* Γ_g *is the grain number ratio* n_g/n_H *where* n_g *is the grain number density.* $m = m_H$.

4. THE RADIATIVE TRANSFER PROBLEM WITHIN A JEANS MASS

Throughout the paper we will consider only the optical depth through one Jeans mass fragment at a given density and temperature to be important. This is obviously not correct since each fragment is surrounded by many others in the cluster and these may be important in preventing the escape of radiation. However, this may only be calculated from a detailed examination of the dynamics of the collapse, and besides, it can only increase the final minimum Jeans mass.

Let $I_\nu(r, \Theta)\, d\omega\, d\nu$ be the flow per cm² per sec of radiation travelling in directions within an infinitesimal solid angle $d\omega$ at an inclination Θ to the radial direction at a distance r from the centre of a spherical cloud, and in the frequency range ν to $\nu + d\nu$. In a length ds the radiation will lose by absorption $I_\nu\, d\omega\, d\nu \kappa_\nu \rho\, ds$ and gain by emission $\rho j_\nu (d\omega/4\pi)\, d\nu\, ds$. The emission j_ν is isotropic and stimulated emission has been included in the opacity κ_ν as a negative absorption. The energy actually lost per g s^{-1} is

$$\epsilon_\nu\, d\nu = \int_\omega \left(\frac{j_\nu}{4\pi} - \kappa_\nu I_\nu\right) d\nu\, d\omega \qquad (24)$$

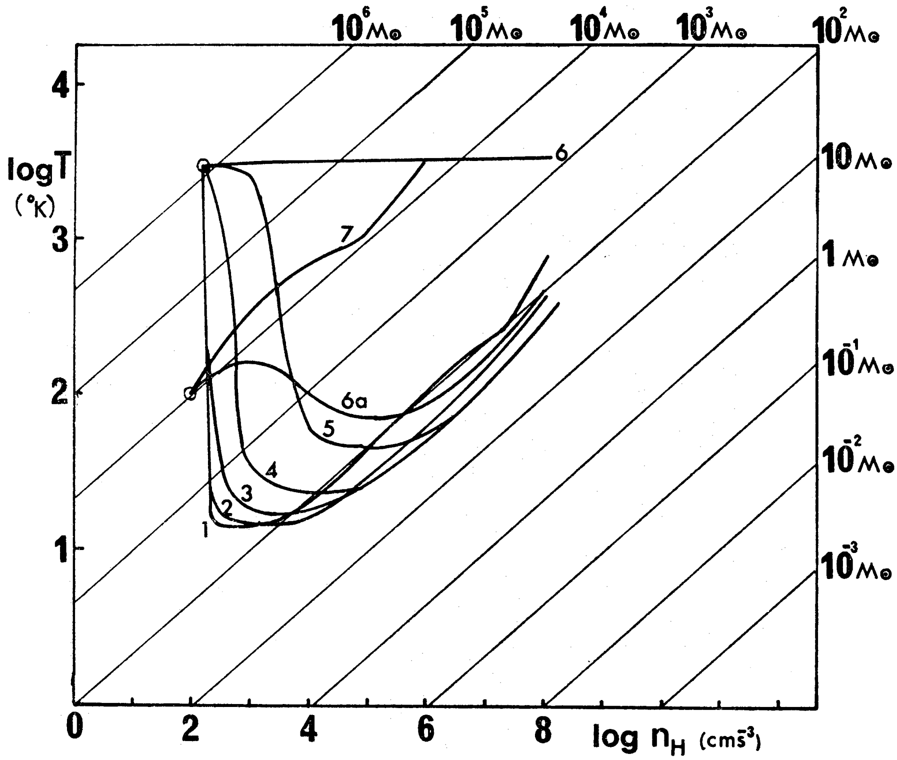

FIG. 6. *Cooling in an ionizing medium with no dust. We assume that there is sufficient ionizing radiation to enable Fe* II, *Si* II, *C* II *to exist and consider the cooling due to them with the addition of O* I. *The metal abundance ratio z/z_0 was varied from* 1·0 *by the following amounts:* 1, 0·3; 2, 0·03; 3, 0·01; 4, 0·003; 5, 0·001; 6, 0·0003; 6*a*, 0·0003; 7, 0·0001. $m = m_H$.

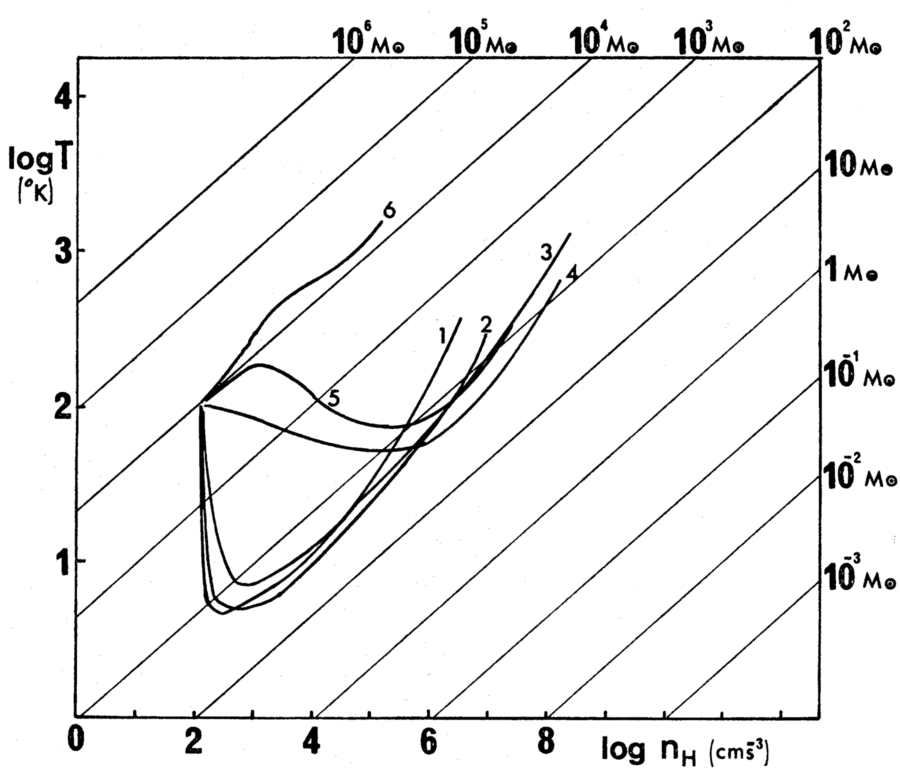

FIG. 7. *Cooling curves for a non-ionizing medium in the absence of dust. The only coolants are O* I *and C* I. *The metal abundance ratio z/z_0 was varied from* 1·0 *by the following amounts:* 1, 0·3; 2, 0·03; 3, 0·01; 4, 0·001; 5, 0·0003; 6, 0·0001. $m = m_H$.

and the flow will change according to the equation

$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho I_\nu + \frac{j_\nu \rho}{4\pi}. \tag{25}$$

Now from the geometry $dr = ds \cos\theta$ and $d\theta = -ds \sin\theta/r$ and

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds} = \cos\theta \frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I_\nu}{\partial \theta}. \tag{26}$$

Hence equation (25) becomes

$$\cos\theta \frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I_\nu}{\partial \theta} = -\kappa_\nu \rho I_\nu + \frac{j_\nu \rho}{4\pi}. \tag{27}$$

Multiply this first by $d\omega/4\pi$ and integrate over the sphere; then by $\cos\theta\, d\omega/4\pi$ and integrate over the sphere; we obtain

$$\frac{dH_\nu}{dr} - \frac{H_\nu^*}{r} = -\kappa_\nu \rho J_\nu + \frac{\rho j_\nu}{4\pi} = \frac{\rho \epsilon_\nu}{4\pi} \tag{28}$$

$$\frac{dK_\nu}{dr} - \frac{K_\nu^*}{r} = -\kappa_\nu \rho H_\nu \tag{29}$$

where

$$J_\nu = \frac{1}{4\pi}\int I_\nu(r,\theta)\, d\omega \qquad H_\nu = \frac{1}{4\pi}\int I_\nu(r,\theta)\cos\theta\, d\omega$$

$$H_\nu^* = \frac{1}{4\pi}\int \sin\theta \frac{\partial I_\nu}{\partial\theta}\, d\omega \qquad K_\nu = \frac{1}{4\pi}\int I_\nu(r,\theta)\cos^2\theta\, d\omega$$

$$K_\nu^* = \frac{1}{4\pi}\int \frac{\partial I_\nu}{\partial \theta}\sin\theta\cos\theta\, d\omega.$$

We now make the Eddington approximation at each r

$$I_\nu(\theta) = \begin{cases} \text{const.} = I_{\nu 1}(r) & \text{for} \quad \theta < \pi/2 \\ \text{const.} = I_{\nu 2}(r) & \text{for} \quad \theta > \pi/2. \end{cases}$$

Then

$$J_\nu = \tfrac{1}{2}(I_{\nu 1} + I_{\nu 2}) \qquad H_\nu = \tfrac{1}{4}(I_{\nu 1} - I_{\nu 2})$$

$$H_\nu^* = \tfrac{1}{2}(I_{\nu 2} - I_{\nu 1}) \qquad K_\nu = \tfrac{1}{3}J_\nu$$

$$K^* = 0$$

and

$$\frac{\partial H_\nu}{\partial r} + \frac{2H_\nu}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 H_\nu) = \frac{\rho\epsilon_\nu}{4\pi} \tag{30}$$

$$\frac{dJ_\nu}{dr} = -3\kappa_\nu \rho H_\nu. \tag{31}$$

In the centre of the cloud we must have the flux zero, i.e. $H = 0$ and at the outside we must have no incoming radiation, i.e. at $r = r_0$, $I_{\nu 2} = 0$ and $J_\nu = 2H_\nu$.

There are two solutions of (30) and (31) which are of interest. The first is when the temperature remains constant with r. For simplicity we have assumed $\rho(r) = $ const. This applies when the gas is being cooled by a number of different processes and one of these cooling processes, say an emission line, becomes optically thick. The gas is still optically thin to the other coolants so that the temperature will remain almost constant through the fragment and since by Kirchoff's Law $j_\nu = 4\pi\kappa_\nu B_\nu(T)$ we have j_ν constant with r. We can differentiate (30) with respect to r and substituting (31) we obtain

$$\tau_\nu^2 H_\nu'' + 2\tau_\nu H_\nu' - (2+3\tau_\nu^2)\, H_\nu = 0 \tag{32}$$

where the differentiation is with respect to $\tau_\nu = \kappa_\nu \rho r$. Notice this optical 'height' increases outwards from the centre. The solution of equation (32) that is finite at $\tau_\nu = 0$ is

$$H_\nu = A\left[-\frac{\sinh\sqrt{3}\tau_\nu}{3\tau_\nu^2} + \frac{\cosh\sqrt{3}\tau_\nu}{\sqrt{3}\tau_\nu}\right]$$

from which we deduce $\epsilon_\nu = 4\pi\kappa_\nu(A\sinh\sqrt{3}\tau_\nu)/\tau_\nu$ and the average ϵ_ν,

$$\langle\epsilon_\nu\rangle = \frac{\int \epsilon_\nu \tau_\nu^2\, d\tau}{\int \tau_\nu^2\, d\tau} = 4\pi\kappa_\nu A(\sqrt{3}\tau_{\nu_0}^{-2}\cosh\sqrt{3}\tau_{\nu_0} - \tau^{-3}\sinh\sqrt{3}\tau_{\nu_0}).$$

A is determined from the conditions that $J_\nu = 2H_\nu$ at the outside $\tau = \tau_{\nu_0}$, and $J_\nu = (1/4\pi\kappa_\nu)(j_\nu - \epsilon_\nu)$ by equation (28). Thus we have for the net average emission rate per unit mass in the range ν to $\nu + d\nu$

$$\langle\epsilon_\nu\rangle\, d\nu = j_\nu \beta(\tau_{\nu_0})\, d\nu = 4\pi\kappa_\nu B_\nu(T)\, \beta(\tau_{\nu_0})\, d\nu \tag{33}$$

where $\beta(\tau_{\nu_0})$ is the average 'escape probability' of a photon and in this case

$$\beta(\tau_{\nu_0}) = \frac{3}{2}\frac{1}{\tau_{\nu_0}}\left\{\frac{\dfrac{\cosh(\sqrt{3}\tau_{\nu_0})}{\tau_{\nu_0}} - \dfrac{1}{\sqrt{3}\tau_{\nu_0}^2}\sinh(\sqrt{3}\tau_{\nu_0})}{\dfrac{3}{2}\dfrac{\sinh(\sqrt{3}\tau_{\nu_0})}{\sqrt{3}\tau_{\nu_0}} + \dfrac{\cosh(\sqrt{3}\tau_{\nu_0})}{\tau_{\nu_0}} - \dfrac{\sinh(\sqrt{3}\tau_{\nu_0})}{\sqrt{3}\tau_{\nu_0}^2}}\right\} \tag{34}$$

and τ_{ν_0} is the optical depth to the centre of a fragment at a frequency ν. This has the simple asymptotic forms

$$\beta \to 1 \quad \text{as} \quad \tau_{\nu_0} \to 0 \tag{35}$$

$$\beta \to \frac{1}{\tau_{\nu_0}} \times \frac{3}{2+\sqrt{3}} \quad \text{as} \quad \tau_{\nu_0} \to \infty. \tag{36}$$

The form $\beta \to 1/\tau$ is similar to the result obtained by de Jong, Shih-I. Chu & Dalgarno (1975) with the difference that the length scale for τ_{ν_0} is the radius of a Jeans mass fragment. That β has the correct form for large optical depths is easily shown since the luminosity $L_\nu\, d\nu$ is $L_\nu\, d\nu = \frac{4}{3}\pi r_0^3 \rho \bar{\epsilon}_\nu\, d\nu$ and $\bar{\epsilon}_\nu = 4\pi\kappa_\nu B_\nu(T)\, \beta(\tau_{\nu_0})$, i.e. $L_\nu\, d\nu = \frac{4}{3}\pi r_0^3 \rho 4\pi\kappa_\nu B_\nu(T)\, \beta(\tau_{\nu_0})\, d\nu$.

The asymptotic form $\beta \to 3/(2+\sqrt{3})(1/\tau_{\nu_0})$ for $\tau_{\nu_0} = \kappa_\nu \rho r_0 \to \infty$ gives

$$L_\nu\, d\nu \to 4\pi r_0^2 \pi B_\nu(T)\, d\nu \frac{4}{2+\sqrt{3}}$$

or integrating over ν

$$L = 4\pi r_0^2 \sigma T^4 \left(\frac{4}{2+\sqrt{3}}\right).$$

The extra factor $4/(2+\sqrt{3})$ indicates that the Eddington approximation is not quite accurate at large τ. The form of β given in equation (34) was used for calculating the effect of opacity on emission lines. (Typically molecular rotation lines and magnetic dipole emission lines.)

The second solution of interest occurs when the fragment becomes optically thick to all coolants. The temperature will start to rise first in the centre of the cloud and the previous solution does not apply. Before opacity starts damming back the heat, the temperature and density are uniform and so is the rate of $-p\,dV$ working and the resultant emissions. When opacity effects set in the temperature of the inner parts will begin to rise. We shall now calculate this temperature change on the assumption that the energy generated by the work is still emitted uniformly throughout the mass so $\epsilon = \int \epsilon_\nu\, d\nu$ is constant with r. One trial numerical model using a time-dependent stellar evolution code shows this is a good approximation.

Equations (30) and (31) are now easily soluble but before we do this it is necessary to make some more approximations. As we will show the most important source of opacity will be dust which has a non-grey opacity. We can handle this approximately by integrating (30) and (31) over all frequencies to obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 H) = \frac{\rho}{4\pi}(j - 4\pi\kappa_{\rm P} J) = \frac{\epsilon\rho}{4\pi} = \text{const.} \tag{37}$$

$$\frac{\partial J}{\partial r} = -3\kappa_{\rm R}\rho H \tag{38}$$

$$j = 4\kappa_{\rm P}\sigma T^4(r) \tag{39}$$

where $\kappa_{\rm P}$ and $\kappa_{\rm R}$ are the Planck and Rosseland mean opacities, respectively. These opacity weightings in equations (37) and (38) will only be correct at large optical depths when $J_\nu \sim B_\nu(T)$, but since the solution has the correct limit at small optical depths the error should not be great. The solution that has $H(0) = 0$ is $H = \epsilon\rho r/12\pi$. Integrating equation (38) under the boundary condition $J = 2H$ at $r = r_1$ (outside boundary) gives

$$J = \frac{\epsilon\rho^2}{4\pi}\int_r^{r_1} \kappa_{\rm R} r\, dr + \frac{\epsilon\rho r_1}{6\pi}.$$

We define

$$\tau_{\rm R}(r) = \frac{2}{r_1}\int_r^{r_1} \rho\kappa_{\rm R} r\, dr.$$

Notice that $\tau_{\rm R}$ is like an optical depth but the r weighting is different. However, $\tau_{\rm R}(0) = \rho\kappa_{\rm R} r_1$, when $\kappa_{\rm R}$ is constant. Substituting this J into equation (37) we have

$$4\kappa_{\rm P}\sigma T^4 - \epsilon\rho r_1\kappa_{\rm P}(\tau_{\rm R}/2 + \tfrac{2}{3}) = \epsilon$$

and so defining $\tau_1(T) = \rho r_1 \kappa_{\rm P}(T)$

$$\epsilon = \frac{4\kappa_{\rm P}(T)\,\sigma T^4}{1+\tau_1(\tau_{\rm R}/2+\frac{2}{3})} = \frac{4\kappa_{\rm P}(T_0)\,\sigma T_0^4}{1+\tau_1(T_0)(\tau_{\rm R}(0)/2+\frac{2}{3})} \tag{40}$$

which equation tells us the temperature distribution and expresses ϵ in terms of the central temperature T_0. The luminosity is given by

$$L = \tfrac{4}{3}\pi r_1^3\rho\epsilon = 4\pi r_1^2\sigma T_0^4\left[\frac{\frac{4}{3}\tau_1(T_0)}{1+\tau_1(T_0)(\frac{1}{2}\tau_{\rm R}(0)+\frac{2}{3})}\right].$$

Notice that the final bracket behaves as $\frac{4}{3}\tau_1$, for small τ_1, and like $8/(3\tau_R(0)+4)$ for large τ_1. To lower the effective temperature by a factor 2 below the central temperature we would need $\tau_R(0) \simeq 43$. Now in equation (18) we are interested in the luminosity, or rate of heat loss, from a Jeans mass. Once the temperature has become non-uniform we shall replace it by $T_J = \langle T^{3/2}\rangle^{2/3}$ since that average gives the same Jeans mass. We therefore need to express L not in terms of T_0 but rather T_J. However, even for $\tau_R(0) = 4$ (which is further than we shall need) the effective temperature only falls below the central one by a factor $2^{1/4} = 1 \cdot 18$ so the temperature variation is not very great. It is therefore sense to approximate $\kappa_R(T)$ by $\overline{\kappa_R}$ and thus to have $\tau_R(r) = \rho\overline{\kappa_R}(r_1^2-r^2)/r_1$. If we then put $T = T_0+\delta T$ we have from equation (40) with $\kappa_P \propto T^\beta$ (defining $\tau_1 = \overline{\kappa_R}\rho r_1$)

$$\frac{\delta T(r)}{T_0} = -\zeta\frac{r^2}{r_1^2}$$

where

$$\zeta = \frac{\tau_1\bar{\tau}_1}{8+2\beta+4\tau_1(\bar{\tau}_1+\frac{4}{3})}.$$

For T_J we find $T_J = (1-\frac{3}{5}\zeta)\, T_0$ and so

$$\epsilon = \frac{4(1+\frac{3}{5}\zeta)^{4+\beta}\,\kappa_P(T_J)\,\sigma T_J^4}{1+(1+\frac{3}{5}\zeta)^\beta\,\tau_1(T_J)(\frac{1}{2}\tau_1+\frac{2}{3})}.$$

Thus

$$L = 4\pi r_1^2\sigma T_J^4\left[\frac{\frac{4}{3}(1+\frac{3}{5}\zeta)^{4+\beta}\,\tau_1}{1+(1+\frac{3}{5}\zeta)^\beta\,\tau_1(\frac{1}{2}\tau_1+\frac{2}{3})}\right].$$

Notice that for large τ_1 the effective temperature is proportional to $(\bar{\tau}_1)^{-1/4}\,T_J$ and $\tau \to \frac{1}{4}$. For small optical depths $\zeta \to 0$ and ϵ tends to its old value $\epsilon = 4\kappa_P\sigma T^4$ and $T \to T_0 \to T_J$. The energy loss rate as modified by optical depth may now be incorporated in equation (17) where we have used it in calculations involving dust opacity. For the idealized opacities $\kappa_P = K\rho^\alpha T^\beta$ equation (19) is modified to the form

$$\frac{d\log T_J}{d\log\rho} = \frac{2}{3} - \frac{\frac{2}{3}m/k\,4(1+\frac{3}{5}\zeta)^{4+\beta}\,\sigma K\rho^{\alpha-1/2}T_J^{3+\beta}}{[1+(1+\frac{3}{5}\zeta)^{4+\beta}\,\tau_1(T_J)(\frac{1}{2}\bar{\tau}_1+\frac{2}{3})](16\pi G)^{1/2}}. \quad (41)$$

For $\alpha = 0$, $\beta = 2$ the results of computing solutions to this equation starting from the convergent $C = 0$ solution of its optically thin limit (19) are shown in Fig. 1. The minimum Jeans mass is achieved at $\tau \sim 3$, soon after optical depth effects set in. Indeed this is always the case as expected. One may think of a black blanket of opacity on the right of the $\log T - \log\rho$ plots such that the clouds follow a track that is almost the optically thin track until they meet the black blanket and thereafter achieve their minimum Jeans masses almost at once. In practice no appreciable error in minimum mass is made if it is taken to be the Jeans mass at the point where track computed on the basis of the optically thin equations hits the $\tau = 1$ blanket line. Thus in practice the crude estimates of Section 1 are correct provided the right final opacities are used. The rest of this paper is devoted to looking at the opacities and coolants in the low temperature region. In practice the opacities are smaller than κ_0 in the far infrared so the right final temperature is above 4·1 K.

5. STAR FORMATION WITH REALISTIC OPACITIES

We will now consider star formation in two different environments and examine the sources of opacity available. In the first case we will consider star formation occurring under conditions presently existing in our Galaxy; in the second case we will consider primeval star formation under conditions which are largely unknown but which will probably be characterized by a marked depletion of the heavier elements and non-negligible thermal blackbody radiation.

There is considerable evidence that star formation at present is associated with the large molecular dust clouds present in the Galaxy, such as those in Orion and the ρ Oph dust cloud. One observes molecular emission and absorption lines, thermal infrared radiation, knots of infrared emission, H_2O and OH masers and high dust extinction, and the molecular observations have indicated densities 10^3–10^5 cm and temperatures 6–70 K. For reviews of dark clouds see Lequeux (1972) and Heiles (1971). Possible sources of gas cooling in such regions include the neutral oxygen atom, neutral and ionized carbon atoms, molecular hydrogen and HD rotational transitions, other molecules such as carbon monoxide, and dust (Dalgarno & McCray 1972). Werner (1970) has shown that for densities $> 10^4$ cm^{-3} the extinction of starlight by dust is sufficient that C II ions recombine and so C II ions may be neglected. The low temperatures obtainable with C I atoms considered alone means that we can neglect the transitions of O I, H_2 and HD since they require higher temperatures to excite them. This statement is not true when the gas temperature rises to > 50 K.

Carbon monoxide has received some attention as a cooling agent (see de Jong *et al.* 1975 for references) and the observed flux in the $J = 1$–0 rotational transition of $^{12}C^{16}O$ and $^{13}C^{16}O$ in the ρ Oph cloud is sufficient to maintain its gravitational collapse (Berger & Simon 1973). It is observed in high abundance with possibly 20 per cent of carbon in this form (Tucker, Kutner & Thaddeus 1973).

As cooling agents we will consider C I atoms, carbon monoxide and dust.

The most important gas heating mechanisms appear to be cosmic rays, X-rays, radiation absorption by grains at long wavelengths, and the compressional heating produced by the collapse. We will assume that the large molecular line widths seen in dark clouds (Heiles 1971) are not due to supersonic turbulence and that this mechanism does not contribute to the heating.

Glassgold & Langer (1973) have estimated that a 2 MeV cosmic ray transfers 15 eV of energy into heat in each collision with an H_2 molecule. If we take a collisional rate of 4×10^{16} cm^{-3} s^{-1} this gives a heating rate of $5{\cdot}8 \times 10^{-3}$ erg g^{-1} s^{-1}. Formula (4) gives the compressional heating as $(kT/m)(16\pi G\rho)^{1/2} = 1{\cdot}52 \times 10^5 \rho^{1/2} T$ erg g^{-1} s^{-1}. Even assuming no attenuation cosmic ray heating will become unimportant at densities greater than 10^{-18} g cm^{-3} ($\sim 10^6$ cm^{-3}). While this is a high density for a molecular cloud it is a low density for a star. We will show that the collapse can continue to a density of $\sim 10^{-14}$ g cm^{-3}. In addition, a $10^3 M_\odot$ cloud at a density of 10^4 cm^{-3} has a radius of just over 1 pc, whereas the stopping length of a 2 MeV proton under these conditions is 0·057 pc (Dalgarno & McCray 1972). X-rays are even less effective than cosmic rays.

As heating mechanisms we will consider compressional heating and dust.

Since we are concerned with low gas temperatures it is necessary to know the opacity of dust at these temperatures. Unfortunately the opacity of dust in the far-infrared is most uncertain. A large number of grain models exist and grains have been constructed from silicates, graphite, ices, hydrocarbons and mixtures of all of them (Aannestad & Purcell 1973), with parameters that vary from model to model. For this reason we will simply choose what seems a 'reasonable' model of grains in which there is a refractory core probably $\sim 0{\cdot}05\ \mu$ and a dirty ice mantle to a total of $0{\cdot}2\ \mu$, with the ratio of the grain number density n_g to hydrogen atom number density n_H, which we will call Γ_g, equal to 6×10^{-13}. This would make the grains about 1 per cent of the interstellar medium by mass. In fact we will show that when the grain temperature is coupled to the gas temperature the cooling is not dependent on the grain size distribution.

The radiation (s^{-1}) emitted from a grain of radius a is

$$4\pi a^2 \pi \int_0^\infty B(\lambda, T_d)\, Q(\lambda, a)\, d\lambda \tag{42}$$

where $B(\lambda, T_d)$ is the Planck distribution function at the grain temperature T_d and $Q(\lambda, a)$ is the radiation absorption efficiency factor for a spherical body of radius a at a wavelength λ. If $a \ll \lambda$ then

$$Q(\lambda, a) = \frac{8\pi a}{\lambda} \operatorname{Im}\left\{\frac{m^2-1}{m^2+2}\right\} \tag{43}$$

where m is the complex refractive index of the grain material (Van de Hulst 1957). For m frequency independent we would have $Q(\lambda, a) \propto \lambda^{-1}$ as in the optical interstellar reddening. However, there is no reason to believe that m will be constant, and using the Helmholtz relation for absorption bands we would expect in the very long wavelength limit past the last band $Q(\lambda, a) \propto \lambda^{-2}$. We will define the Planck mean absorption efficiency by

$$Q_P = \frac{\int_0^\infty Q(\lambda, a)\, B(T, \lambda)\, d\lambda}{\int_0^\infty B(T, \lambda)\, d\lambda}$$

and the energy balance for a grain can now be written

$$\sigma T_d{}^4 Q_P(T_d, a) = \frac{\frac{3}{2} nkT}{(2\pi mkT)^{1/2}}\, k(T-T_d)\, \Theta + \gamma Q_P(T_b, a)\, \sigma T_b{}^4 \tag{44}$$

where the first term is the heat loss due to radiation of a dust grain at temperature T_d. The second is the collisional energy gain from a gas of particles of mass m, number density n, temperature T and Θ is the energy transfer parameter, 'inelasticity'. The last term represents heating from an isotropic blackbody radiation of temperature T_b and dilution factor γ. If we let $Q_P(a, T) = aQT^\mu$ then we can solve (44) for the grain temperature given by the gas temperature. Kellmann & Gaustad (1969) have calculated $Q_P(T)$ for water ice, graphite and silicate grains in the temperature range 10–1000 K. While ice is not conspicuously present in infrared emission spectra (Knacke, Cudaback & Gaustad 1969) we feel that the low dependence on opacity in equation (11) justifies using one likely mixture rather than investigating the behaviour of hypothetical dirty-ice mixtures, and so from now on we will assume the value $Q_P(T) = 4{\cdot}1 \times 10^{-7}\, T^{2{\cdot}67}$ for $0{\cdot}2$-μ grains.

In Fig. 2 we have calculated the radiation loss (g^{-1}) as a function of density for dust, neutral carbon atoms and carbon monoxide molecules, all at a temperature of 15 K. The radiation from the latter two is calculated by the method given in the

Appendix taking into account optical thickness in the lines through a Jeans mass. It can be seen that at high densities the only coolant of importance is dust. Knowing this we can calculate a minimum mass for dust directly from equation (11). At 10 K, Q_P for dust $\sim 10^{-4}$ which gives the grain an effective cross-section of $\sim 10^{-13}$ cm^2. Combined with a relative abundance of 6×10^{-13} the opacity is $\sim 0{\cdot}04$ cm^2 g^{-1} and $(\kappa_0/\kappa_{Pf})^{1/7} = 1{\cdot}4$. For molecular hydrogen $m/m_H = 2$ and so the minimum mass for dust in a molecular hydrogen cloud is $7 \times 10^{-3} M_\odot$. A trajectory in the log T – log ρ diagram for normal abundances is shown in Fig. 3. As a minimum protostellar mass this is not an unreasonable number, since there are many processes that would increase this mass substantially, e.g. magnetic fields, angular momentum and grain heating by external radiation. If we extrapolate the Salpeter mass function backwards towards stars of small mass then this minimum enables us to estimate the proportion of stellar mass that could conceivably be locked up in such small objects. If $N(M)\,dM$ is the number of stars of mass between M and $M + dM$ then $N(M)\,dM \propto M^{-(2+\delta)}\,dM$. If the mass of the smallest observed stars is M_0 then the ratio of the mass of unobserved stars to the total mass of observed stars is

$$\int_{M_{\min}}^{M_0} MN(M)\,dM \Big/ \int_{M_0}^{\infty} MN(M)\,dM \sim \left(\frac{M_0}{M_{\min}}\right)^{\delta}$$

$\delta \sim 0{\cdot}35$ and $M_0 \sim 0{\cdot}1$ which implies that even with so small a minimum mass there would only be $2{\cdot}5$ times as much unobservable matter as observable.

It is interesting to ask 'How little dust is required to make stars?'. The answer is that remarkably little dust is required to dominate the cooling in the later stages of a collapse. The relevant parameters are the grain radius a, the grain number ratio Γ_g (i.e. if the number of grains cm^{-2} is n_g then $n_g = \Gamma_g n_H$) and the energy transfer parameter Θ. To solve for the minimum mass we must recognize that the temperature of the dust T_d is not always identical with gas temperature and so in additions to equations (3) and (4) for the opacity and heating–cooling we must add equation (44) for the thermal balance of the dust. The complete equations for the final fragment density and temperature when dust dominates the cooling are

$$\Gamma_g \pi a^2 Q_P \frac{\lambda_J}{2} n_H = \Gamma_g \pi a^2 a Q T_d{}^{\mu} \frac{n_H}{2} \left(\frac{\pi k}{Gm}\right)^{1/2} \frac{T^{1/2}}{\rho^{1/2}} = \tau = 1 \quad (45)$$

$$p\frac{dV}{dt} = \frac{kT}{m}(16\pi G\rho)^{1/2} = \frac{\Gamma_g}{m_H} 4\pi a^2 . aQT_d{}^{\mu}\sigma T^4 \quad (46)$$

$$QaT_d{}^{\mu}\sigma T_d{}^4 = \frac{\frac{3}{2}nkT}{(2\pi mkT)^{1/2}} \Theta k(T - T_d) + \gamma Q a \sigma T_b{}^{4+\mu} \quad (47)$$

$QaT_d{}^{\mu}$ is the Planck mean efficiency defined earlier. However, at the high densities involved in the final fragmentation and for normal dust abundances, the dust temperature is closely coupled to the gas temperature and $T_d \sim T$, which can be deduced from equation (47). In this circumstance the grain parameters mentioned above only occur in the combination $\Gamma_g a^3$. Since the total mass of grains/cm^3 of radius $a = (\Gamma_g a^3)(4\pi/3)\,\rho_g n_H$ where ρ_g is the density of a single grain, this implies that the final fragment mass does not depend on the grain size distribution but only on the total mass of material in grains regardless of its state of subdivision (within the limits of equation (43)).

If we define 'standard' values of Γ_g and a such that $\Gamma_{g_0} = 6\times10^{-13}$ and $a_0 = 2\times10^{-5}$ cm then we can define a new parameter $\Delta = (a^3\Gamma_g)/(a_0{}^3\Gamma_{g_0})$ which measures the mass cm^{-3} in grains relative to the value observed in the Galaxy near the Sun. Equations (45), (46) and (47) may be solved simultaneously for the final temperature and density T_f, ρ_f of the last fragment in terms of Θ, a, Γ_g and m/m_H being 2 for molecular hydrogen and 1 for atomic hydrogen. The results are shown in Table 1.

TABLE I

Δ	a (μ)	Γ_g	m/m_H	Θ	T_f (K)	T_{df} (K)	ρ_f (g cm^{-3})	M_{Jf} ($M_\odot$)
1	0·2	6×10^{-13}	2	1	9·4	9·0	$4·78\times10^{-14}$	0·006
10^{-2}	0·2	6×10^{-15}	2	1	27·8	26·7	$9·36\times10^{-13}$	0·006
$1·56\times10^{-4}$	0·05	6×10^{-15}	1	1	73·0	71	$1·95\times10^{-12}$	0·05
$1·56\times10^{-5}$	0·05	6×10^{-16}	1	1	134	120	$6·46\times10^{-12}$	0·075
10^{-5}	0·02	6×10^{-15}	1	1	140	133	$8·9\times10^{-12}$	0·068

It is in accord with equation (11) that despite large decreases in the amount of material in grains the Jeans mass changes slowly. This may be of some importance when considering the formation of the Galaxy, when, according to observations of old stars, heavy elements may have been less abundant than at present by a factor of 100. The value $\Delta = 10^{-5}$ is a generous depletion of material in grains. It is probably also the maximum depletion possible before grains become unimportant, as the numerical solutions to equation (18) (curve 1 on Fig. 3 and curves 3, 4, 5 and 6 on Fig. 5) indicate. These solutions show the path taken in the $\log T - \log \rho$ diagram by the gas where the cooling is due to magnetic dipole emission from O I and C I (see Appendix) and varying types of dust. The hump at $n = 10^7 - 10^8$ in Fig. 5, curves 3, 4, 5 and 6, is due to the inefficiency of O I and C I as coolants at this density (see Fig. 7) which causes the temperature to rise until the dust becomes important. As the amount of material in grains is decreased the hump becomes larger until in curve 6 the hydrogen ionizes and the grains will eventually be destroyed. Curve 6 is in fact an unphysical situation since the collapse becomes adiabatic, and dynamical collapse stops long before this temperature is reached. Even curve 5 is unrealistic since the temperature increases by a factor of 6 from the onset of adiabaticity to the top of the hump.

6. EFFECTS OF RADIATION ON MINIMUM MASS

To conclude the section on dust we will consider the effect of background radiation, either in the form of big-bang thermal background or radiation from nearby hot objects like newly formed stars. In the first case the radiation temperature is an absolute minimum while in the second the radiation would heat the grains above the temperature of the gas and the grains would heat the gas up to their own temperature. The degree to which this occurs depends on the presence of other coolants in the gas but above a density of 10^6 cm^{-3} it should be a reasonable approximation to say that the gas collapses isothermally at the grain temperature. Equation (45) in conjunction with equation (2) gives the following when $\Delta = 1$.

$T = T_d$ (K)	n_{Hf} (cm^{-3})	M_J ($M_\odot$)
10	$1 \cdot 1 \times 10^{10}$	$7 \cdot 3 \times 10^{-3}$
20	$1 \cdot 33 \times 10^{8}$	0·19
30	$1 \cdot 02 \times 10^{7}$	1·2
40	$1 \cdot 6 \times 10^{6}$	4·7
50	$4 \cdot 0 \times 10^{5}$	13·4

The grain temperature is a sensitive means of controlling the minimum Jeans mass. The range of temperatures observed in molecular clouds (5–70 K) might permit the formation of a large range of protostellar masses.

If we wish to know at what value of the redshift z the cosmic 2·7 K blackbody radiation becomes hot enough to influence star formation we must first decide on how much dust there is. For 'normal' dust abundance and a typical radius $a = 0{\cdot}2\ \mu$ blackbody radiation will be important for $z \geqslant 2{\cdot}7$. For a more likely situation in which the grain number is reduced by 100 and the radius $a = 0{\cdot}05\ \mu$ we would have to go to $z \geqslant 26$ before radiation is important, still a modest redshift. Background radiation would seem to have an important part in the study of the formation and early evolution of galaxies.

7. STAR FORMATION WITH UNUSUAL OPACITIES

An understanding of star formation is of importance in discussing galaxy formation and evolution and we will now repeat the previous arguments for an environment similar to that in the early Universe.

In the standard big-bang model we would expect the primeval gas to become unstable to gravitational clumping after recombination at a temperature of 4000 K and a density of between 10^{-20} and 10^{-22} g cm^{-3} depending on what one takes to be the present matter density. In the absence of any other cooling agent this gas can collapse almost isothermally due to the cooling of free–bound collisions between protons and electrons, at a temperature between 3000 and 4000 K. Hirasawa (1969) has investigated the formation of molecular hydrogen through the reaction $H^- + H \rightarrow H_2 + e^-$ and concludes that sufficient molecular hydrogen can be formed to have a significant effect on the cooling of the gas. This mechanism can only operate when the universal blackbody radiation is below 300 K. Since the free-fall collapse time for a gas of density $1{\cdot}66 \times 10^{-22}$ g cm^{-3} (5×10^6 yr) is the same as that for the Universe to cool to 300 K it seems that unless the collapse is significantly retarded the first generation of objects will be formed at a temperature of 3000–4000 K. For later generations of objects the heavy element abundance should have increased to the point at which this also must be considered. We have accordingly calculated the path in the log $T - \log \rho$ diagram followed by a mass of collapsing gas which is cooled by free–bound emission (Fig. 5, curve 1); molecular hydrogen transitions (Fig. 4 and Fig. 5, curve 2); magnetic dipole transitions in Fe II, Si II, C II, C I and O I (Fig. 6, curves 1–7, and Fig. 7, curves 1–6). The calculations on dust, which also fall in this section, have already been discussed. The details concerning these coolants are given in an Appendix. The heavy element ('metal') abundance ratio z/z_0 was varied from 1 to 10^{-4} and the molecular hydrogen abundance was varied from $n_{H_2}/n_H = 10^{-6}$ to $n_{H_2}/n_H = 1$. Equation (18) was solved numerically and the results may be summarized as follows:

(a) The solutions displayed the very rapid convergence to the heating ~ cooling trajectory as predicted in Section 3. Initial conditions did not influence the final fragment mass.

(b) molecular hydrogen was not found to be important in determining the final fragment mass. Even at the lowest observed metal abundance ratios (10^{-2}) the metals were more important at low densities $10^2 \sim 10^5$ cm^{-2} and at higher densities dust was more important. In the absence of metals and dust H_2 was important only in high abundance $n_{H_2}/n_H \sim 1$. This does not contradict Hirasawa's findings for he considered a different problem.

(c) Free–bound emission by hydrogen could maintain an isothermal collapse at an ionization of 10^{-5} up to densities of 10^{-8} g cm^{-3}. Substituting into equation (7) gives immediately $M_J = 0{\cdot}13\, M_\odot$, in agreement with the numerical work. This may not be the end of the story however. The fragment becomes optically thick at a temperature of 4500 K. If the gas has sufficient energy in the macroscopic motions of the collapse to increase the temperature to 10 000 K before equilibrium is attained, then the hydrogen will start to ionize and it is likely that the fragmentation will continue until ionization is complete (even though the gas is optically thick), the $p\,dV$ work going into ionization. We can obtain a rough estimate of the final density by integrating the $P.dV$ work at constant temperature and putting this equal to the energy required to ionize the hydrogen. This gives

$$\ln\left(\frac{\rho}{\rho_f}\right) = \frac{1{\cdot}57\times10^{-5}}{\langle T\rangle} \qquad (48)$$

where ρ_f is the density at which the fragment becomes optically thick, ρ the density where the hydrogen is all ionized and $\langle T\rangle$ the mean temperature at which all this occurs. Putting $\langle T\rangle = 12\,000$ K gives $\rho/\rho_f = 5\times10^5$ and a final Jeans mass of $10^{-3}\,M_\odot$.

(d) Magnetic dipole line cooling gave very large final fragment masses in the absence of any other cooling agents, $>30\,M_\odot$. At normal abundances of the coolants this might be expected from the high opacities in the lines. At lower abundances the opacity is correspondingly decreased and one might expect the collapse to continue longer. However, the thermalization of levels at the density n_{crit} (see Appendix) changes the density dependence of cooling from being $\propto\rho^2$ to being $\propto\rho$ and the gas starts to heat up sufficiently for the Jeans mass to increase. The final density attained is low, varying from 10^4 to 10^7 cm^3 and the temperature below the excitation temperature of 228 K of the principle coolant O I. In fact it seems unreasonable to expect that this situation should occur since we have already shown that dust is an effective coolant even at very low abundances and where we have metals we can surely expect to have some dust.

7. CONCLUSION

While a fragmentation theory of star formation does not lead immediately to stellar mass objects it is encouraging that the minimum masses derived are smaller than the smallest observed stars. The minimum mass in a typical dark molecular hydrogen cloud was found to be $7\times10^{-3}\,M_\odot$, a result which is extremely insensitive to variations in the amount of dust, size of grains or cosmic ray flux. This minimum mass can be increased by any process which tends to hold the grains at a temperature above that which they would normally assume. This could have important consequences for primeval star formation. Dust was found to maintain its cooling efficiency down to a reduction by mass of 10^{-5} of the normal value.

Cooling in the absence of dust was also investigated. Cooling by C II, C I, O I, Si II and Fe II resulted in minimum masses $> 30\ M_\odot$ and these species became unimportant for abundances relative to solar of $z/z_0 \sim 10^{-4}$. Molecular hydrogen by itself was only important in high abundance $n_{H_2}/n_H \sim 1$ and since it is unlikely to occur by itself it can be neglected in most situations.

Free–bound emission in hydrogen was found to be a very efficient coolant, producing a minimum mass of $0.13\ M_\odot$. However, in order to cool by this means the gas must reach a temperature of ~ 4000 K and besides, pure hydrogen is not a very common situation.

All the masses have been derived on the basis of the Jeans mass at the last fragmentation—this could be a severe oversimplification for the cloud as a whole will still contract and some of these minimum mass fragments will collide and recombine. It could be that a statistical theory of this recombination would lead to a prediction for the mass spectrum (Arny & Weissman 1973).

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NOTE

S. S. Kumar (1972) gives evidence for very low mass stars down to $\sim 0.01\ M_\odot$.

REFERENCES

Aannestad, P. A. & Purcell, E. M., 1973. *A. Rev. Astr. Astrophys.*, **11**, 309.
Allen, C. W., 1973. *Astrophysical quantities*, Athlone Press, London.
Arny, T. & Weissman, P., 1973. *Astr. J.*, **78**, 309.
Bahcall, J. N. & Wolf, R. A., 1968. *Astrophys. J.*, **152**, 701.
Berger, P. S. & Simon, M., 1973. *Astrophys. J.* (*Lett.*), **180**, L43.
Dalgarno, A. & McCray, R. A., 1972. *A. Rev. Astr. Astrophys.*, **10**, 375.
de Jong, T., Shih-I Chu & Dalgarno, A., 1975. *Astrophys. J.*, **199**, 69.
Field, G. B., Rather, J. D. G., Aannestad, P. A. & Orszag, S. A., 1968. *Astrophys. J.*, **151**, 953.
Gaustad, J. E., 1963. *Astrophys. J.*, **138**, 1050.
Glassgold, A. E. & Langer, W. D., 1973. *Astrophys. J.*, **186**, 859.
Goldreich, P. & Kwan, J., 1974. *Astrophys. J.*, **189**, 441.
Heiles, C., 1971. *A. Rev. Astr. Astrophys.*, **9**, 293.
Hirasawa, T., 1969. *Progress theor. Phys.*, **42**, 523.
Hoyle, F., 1953. *Astrophys. J.*, **118**, 512.
Hunter, C., 1964. *Astrophys. J.*, **139**, 570.
Jeans, Sir James, 1928. *Astronomy and cosmogony*, p. 337, Cambridge University Press.
Kellman, S. A. & Gaustad, J. E., 1969. *Astrophys. J.*, **157**, 1465.
Knacke, R. F., Cudaback, D. D. & Gaustad, J. E., 1969. *Astrophys. J.*, **158**, 151.
Kumar, S. S., 1972. *Astrophys. Space Sci.*, **17**, 219.
Larson, R. B., 1973. *A. Rev. Astr. Astrophys.*, **11**, 219.

Lequeux, J., 1972. *Symposium on the origin of the solar system (Nice)*, pp. 118, ed H. Reeves, Centre National de la Recherche Scientifique.
Lynden-Bell, D., 1973. *Dynamical structure and evolution of stellar systems*, Saas Fee, Geneva Observatory.
Mestel, L., 1965. *Q. Jl R. astr. Soc.*, **6**, 161 & 265.
Peebles, P. J. E., 1971. *Physical cosmology*, p. 215, Princeton University Press, New Jersey.
Penston, M. V., 1969. *Mon. Not. R. astr. Soc.*, **144**, 425.
Penston, M. V., 1970. *Astrophys. J.*, **162**, 771.
Seaton, M. J., 1955. *Ann. Astrophys.*, **18**, 188.
Spitzer, L., 1956. *Physics of fully ionized gases*, Interscience, New York.
Tucker, L. D., Kutner, M. L. & Thaddeus, P., 1973. *Astrophys. J., Lett.*, **186**, L13.
Van de Hulst, H. C., 1957. *Light scattering by small particles*, p. 270, Wiley, New York.
Werner, M. W., 1970. *Astrophys., Lett.*, **6**, 81.

APPENDIX

COOLING MECHANISMS

This will not be an exhaustive list of cooling agents, but a choice of those considered most important in the environments considered. For a review of heating and cooling in the interstellar medium see Dalgarno & McCray (1972).

Free–bound emission

Spitzer (1956) gives the cross-section for radiative capture into the nth quantum level of a hydrogenic atom as

$$\sigma_{\mathrm{cr}} = A_{\mathrm{r}} \frac{\nu_0}{\nu} \frac{h\nu_0}{\frac{1}{2}m_{\mathrm{e}}v^2} \frac{g}{n^3} \tag{A1}$$

where $-h\nu_0$ is the energy of the ground state, $\nu = (1/h)(\frac{1}{2}m_{\mathrm{e}}v^2 + h\nu_0)$ is the frequency of the emitted photon, m_{e} and v are the mass and velocity of the captured electron, g is the Gaunt factor and A_{r} is the recapture constant.

$$A_{\mathrm{r}} = \frac{2^4}{3^{3/2}} \frac{he^2}{m_{\mathrm{e}}^2 c^3} = 2{\cdot}11 \times 10^{-22}\ \mathrm{cm}^2. \tag{A2}$$

At low collision energies $g = 0{\cdot}797$.

Using this cross-section we can integrate over a Maxwellian electron velocity distribution and sum over all levels to obtain the energy radiated per unit volume per second

$$\Lambda_{\mathrm{b.f.}} = 4{\cdot}28 \times 10^{-22}\, n_{\mathrm{e}} n_{\mathrm{p}} T^{-1/2}\ \mathrm{erg\ cm^{-3}\ s^{-1}}. \tag{A3}$$

The opacity κ_{p} may be derived from Kirchoff's Law since

$$\Lambda_{\mathrm{b.f.}} = 4\rho\kappa_{\mathrm{p}}\sigma T^4.$$

In the pure hydrogen considered $n_{\mathrm{e}} = n_{\mathrm{p}}$ and Saha's equation (Allen 1973) gives

$$\log_{10}\left(\frac{n_{\mathrm{e}}}{n_{\mathrm{p}}}\right) = \frac{1}{2}\left(\frac{-6{\cdot}8544.10^4}{T} - \frac{3}{2}\log_{10}\left(\frac{5040}{r}\right) + 20{\cdot}9376 - \log_{10} n_{\mathrm{H}}\right). \tag{A4}$$

Magnetic dipole emission lines

The following data is taken from Bahcall & Wolf (1968); Field *et al.* (1968); Penston (1970); and Seaton (1955).

	Trans.	Wave (μ)	Excit. temp. (E/k K)	A (s^{-1})	N_{crit} (cm^{-3})
C II	$2p_{3/2}$–$2p_{1/2}$	156	92	$2 \cdot 36 \times 10^{-6}$	4×10^3
Si II	$2p_{3/2}$–$2p_{1/2}$	35	413	$2 \cdot 13 \times 10^{-4}$	4×10^5
Fe II	$6D_{7/2}$–$6D_{9/2}$	26	554	$5 \cdot 3 \times 10^{-3}$	4×10^6
	$6D_{5/2}$–$6D_{7/2}$	35	961	$1 \cdot 6 \times 10^{-3}$	10^6
O I	$3p_0$–$3p_1$	147	326	$1 \cdot 70 \times 10^{-5}$	10^4
	$3p_1$–$3p_2$	63	228	$8 \cdot 95 \times 10^{-5}$	5×10^4
C I	$3p_2$–$3p_1$	369	62·2	$2 \cdot 68 \times 10^{-7}$	10^3
	$3p_1$–$3p_0$	610	23·4	$7 \cdot 93 \times 10^{-8}$	5×10^2

A is the Einstein A-value for the transition. The energy radiated per unit volume per second in any transition between states X and Y is

$$n_x \Delta E_{xy} A_{xy} \tag{A5}$$

where n_x is the number of atoms cm^{-3} in state X, ΔE_{xy} is the energy difference between X and Y, and A_{xy} is the Einstein A-coefficient for the transition X to Y. Important to the discussion is the parameter C_{xy}/A_{xy} the ratio of collisional to radiative de-excitation. When this ratio is small level populations are calculated by equating the total number of collisions feeding a level to the depletion by spontaneous emission. When this ratio is large levels are populated according to the Maxwell–Boltzmann distribution. Approximate densities for this cross-over are designated N_{crit}.

To solve for the level populations the complete equations for collisional excitation and collisional and radiative de-excitation were used. This is simple for a two-state system like C II and Si II. Let C_{yx} be the rate at which collisions cause transitions between Y and X and let C_{xy} be the opposite. Then

$$n_y C_{yx} = n_x(A_{xy} + C_{xy}), \tag{A6}$$

i.e.

$$\frac{n_x}{n_y} = \frac{g_x/g_y \exp(-\Delta E_{xy}/kT)}{1 + A_{xy}/C_{xy}} \tag{A7}$$

since $C_{yx}/C_{xy} = g_x/g_y \exp(-\Delta E_{xy}/kT)$ by detailed balancing and g_x and g_y are the degeneracies of the states X and Y. For three-state systems like Fe II and O I the equations are similar but more lengthy. The collisional cross-sections were estimated by the formula given in Field *et al.* (1968).

The radiation from C I can be simply expressed when it is present in normal abundances, being optically thick through a Jeans mass for densities at which the obscuration is sufficient for the atom to exist. If n_0, n_1 and n_2 are the numbers of atoms cm^{-3} in the states 0, 1 and 2 then we have for the energy radiated

$$\Lambda_{CI} = (n_1 E_{10} A_{10} \beta_{10} + n_2 E_{21} A_{21} \beta_{21}) \tag{A8}$$

β is the photon escape probability defined in Section 3, and for large optical depths $\beta \to (3/(2+\sqrt{3}))\, 1/\tau$. The cross-section for reabsorption of line radiation in a line broadened by thermal broadening is, including stimulated emission,

$$\sigma_{xy} = \frac{c^3}{8\pi \nu_{xy}^3} \frac{g_x}{g_y} \left(\frac{m}{2\pi kT}\right)^{1/2} A_{xy}(1 - \exp(-\Delta E_{xy}/kT)) \tag{A9}$$

where c is the speed of light and ν_{xy} the radiation frequency, m being in this case the mass m_C of a carbon atom, and the optical depth through a Jeans length is

$$\tau_{xy} = \sigma_{xy} \frac{\lambda_J}{2} n_y. \qquad \text{(A10)}$$

Substituting (A9) and (A10) in (A8) gives

$$\Lambda_{CI} = \frac{2}{\lambda_J} \frac{3}{2+\sqrt{3}} \left(\frac{2\pi kT}{m_C}\right)^{1/2} \times \left\{ \frac{n_1}{n_0} E_{10} A_{10} \frac{8\pi\nu_{10}^3}{c^3} \frac{g_0}{g_1} \frac{1}{A_{10}} \frac{1}{(1-\exp(-E_{10}/kT))} + \frac{n_2}{n_1} E_{21} A_{21} \frac{8\pi\nu_{21}^3}{c^3} \frac{g_2}{g_1} \frac{1}{A_{21}} \frac{1}{(1-\exp(-E_{21}/kT))} \right\} \qquad \text{(A11)}$$

$$= \frac{2}{\lambda\nu} \frac{3}{2+\sqrt{3}} \left(\frac{2\pi kT}{m_C}\right)^{1/2} \frac{4\pi}{c} (B(T, \nu_{10})\, \nu_{10} + B(T, \nu_{21})\, \nu_{21})$$

since $n_x/n_y = g_x/g_y \exp(-\Delta E_{xy}/kT)$ and $B(T, \nu)$ is the Planck function.

The elemental abundances used are those given by Allen (1973) in the following proportions

H	C	O	Si	Fe
10^6	330	670	40	33

When depleted all elements were depleted proportionally.

Molecular cooling

The two most abundant molecules in the interstellar medium seem to be H_2 and CO. The cooling to be considered is by transitions among the rotational levels. The rotational ground state is at 5·5 K for CO and at 510 K for H_2. The transition for CO is electric dipole and so $\Delta J = \pm 1$ while for H_2 the transition is electric quadrupole and $\Delta J = \pm 2$. It was assumed that H_2 is an equilibrium mixture of ortho- and parahydrogen in the ratio 3:1. The levels for H_2 were taken to be in thermodynamic equilibrium since $A_{20} \sim 10^{-10}\,s^{-1}$, and the energy loss was summed over the first 12 levels, taking into account optical depth as in equation (33). The same was done for CO, except that the summation was discontinued at the first level where collisional de-excitation was less than the spontaneous transition rate. The data for CO was taken from Goldreich & Kwan (1974) and is as follows:

Energy of J level $E_J = hBJ(J+1)$ where h is Planck's constant and B is the molecules rotational constant, $B = 5{\cdot}77 \times 10^{10}\,s^{-1}$. The Einstein A-coefficient is

$$A_{J,J-1} = 7{\cdot}45 \times 10^{-8}\, J^4/(2J+1)\, s^{-1}.$$

The data for H_2 were taken from Field *et al.* (1968) and is

$$E_J = 1{\cdot}178 \times 10^{-14}\, J(J+1)\,\text{erg}$$

$$A_{J,J-2} = 7{\cdot}52 \times 10^{-13} \frac{J(J-1)(2J-1)}{(2J+1)}\, s^{-1}.$$