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Research Article

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The mixed metric dimension of flower snarks and wheels

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Abstract: New graph invariant, which is called a mixed metric dimension, has been recently introduced. In this paper, exact results of the mixed metric dimension on two special classes of graphs are found: flower snarks J_n and wheels W_n. It is proved that the mixed metric dimension for J_5 is equal to 5, while for higher dimensions it is constant and equal to 4. For W_n, the mixed metric dimension is not constant, but it is equal to n when $n \geq 4$, while it is equal to 4, for $n = 3$.

Keywords: wheel graphs, mixed metric dimension, flower snarks, graph theory, discrete mathematics

MSC 2020: 05C12

1 Introduction

Let $G = (V, E)$ be a connected graph, where V represent a set whose elements are called vertices and E represent a set whose elements are called edges. The mixed metric dimension of graphs was introduced by Kelenc et al. (2017) in [1]. This dimension of graph G is the mixed version of the metric dimension and edge metric dimension.

In the connected graph G, the distance between two vertices u and v is the length of a shortest $u - v$ path in G. The vertex w resolves a pair $u, v \in V$ if $d(u, w) \neq d(v, w)$. The metric coordinates $r(v, S)$ of vertex v with respect to an ordered set of vertices $S = \{w_1, w_2, \ldots, w_k\}$ are defined as $r(v, S) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set S is a resolving set if every two vertices u and v from G are resolved by at least one vertex from the set S. The metric basis of graph G is the resolving set of the minimum cardinality. The metric dimension of graph G is the cardinality of the metric basis for graph G and is denoted by $\beta(G)$. Slater in [2] and Harary and Melter in [3] independently of one another introduced resolving sets for graphs. Also, several works are published about applications and some theoretical properties of this invariant. For instance, applications to the direction of robots in networks are analyzed in [4] and applications to chemistry in [5,6], among others. In general, there are several other variations of metric dimension in the literature: resolving dominating sets [7], strong metric dimension [8], local metric dimension [9], k-metric dimension [10], k-metric antidimension [11,12], etc.

The distance between a vertex w and an edge uv of graph G is defined as $d(uv, w) = \min\{d(u, w), d(v, w)\}$. The vertex w resolves a pair $e, f \in E$ if $d(w, e) \neq d(w, f)$. The metric coordinates $r(e, S)$ of edge e with respect to an ordered set of vertices $S = \{w_1, w_2, \ldots, w_k\}$ are defined as $r(e, S) = (d(e, w_1), d(e, w_2), \ldots, d(e, w_k))$. The set S is an edge resolving set if every two edges e and f from G are resolved by at least one vertex from the set S. The edge metric basis of the graph G is the edge resolving set of the minimum cardinality. The edge metric dimension of G is the cardinality of the edge metric basis for graph G and is denoted by $\beta_E(G)$. The concept of the edge metric dimension of graph G was introduced by Kelenc et al. in [13].

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For a given graph G, since every vertex of the graph is uniquely determined by resolving set and every edge of the graph is uniquely determined by the edge resolving set, the logical question is: whether every edge resolving set of the graph G is also the resolving set and *vice versa*? In [13], authors proved that there are several graph families for which the edge resolving set is also a resolving set, but in general case, it is not valid for every graph G. Similarly, for every graph G resolving set is not necessarily the edge resolving set for G.

Let $V \cup E$ be a set of items, where each item is either a vertex or an edge. The vertex v resolves a pair of items if $d(v, a) \neq d(v, b)$. The metric coordinates $r(a, S)$ of item a with respect to an ordered set of vertices $S = \{w_1, w_2, \ldots, w_k\}$ are defined as $r(a, S) = (d(a, w_1), d(a, w_2), \ldots, d(a, w_k))$. The set S is mixed resolving set if every two items from G are resolved by at least one vertex from the set S. The mixed metric basis of graph G is the mixed resolving set of the minimum cardinality. The mixed metric dimension of G is the cardinality of the mixed metric basis for graph G and is denoted by $\beta_M(G)$.

Since concept of mixed metric dimension was introduced in [1], in this paper the mixed metric dimension for several well-known classes of graphs are founded: path, cycle, tree, grid, and complete bipartite graph. Exact values of the mixed metric dimension of unicyclic graphs and tight upper bounds for graphs which have unicyclic subgraph are given in [14]. Extremal difference of mixed metric dimension and some graph invariants (cyclomatic number, edge, and strong metric dimension) are given in [15] and [16], respectively.

In this paper, this dimension will be studied for two special classes of graphs: flower snarks and wheels.

A flower snark is connected, bridgeless 3-regular graph. This graph is denoted by J_n and have $4n$ vertices and $6n$ edges, where vertex-set is $V(J_n) = \{a_i, b_i, c_i, d_i | i = 0, \ldots, n-1\}$. Vertices $\{a_i | 0 \leq i \leq n-1\}$ are called central vertices, $\{b_i | 0 \leq i \leq n-1\}$ is called a set of inner vertices and they induce the inner cycle, while set of vertices $\{c_i | 0 \leq i \leq n-1\}$ and $\{d_i | 0 \leq i \leq n-1\}$ are outer vertices and they induce the outer cycle. Let the edge set be $E(J_n) = \{a_ib_i, a_ic_i, a_id_i, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} | i = 0, 1, \ldots, n-1\}$, where the edges $a_{n-1}a_0$ and $b_{n-1}b_0$ are replaced by the edges $a_{n-1}b_0$ and $b_{n-1}a_0$. The indices are taken modulo n.

Property 1. [17] *Let J_n be a flower snark graph and $j \in \{0, 1, \ldots, n-1\}$ be an arbitrary number. Then the function $h_j : V(J_n) \to V(J_n)$ defined as:*

$$
\begin{aligned}
h_j(a_i) &= \begin{cases} a_{j-i}, & i \leq j \leq n-1, \\ a_{n+j-i}, & 0 \leq j < i, \end{cases} \\
h_j(b_i) &= \begin{cases} b_{j-i}, & i \leq j \leq n-1, \\ b_{n+j-i}, & 0 \leq j < i, \end{cases} \\
h_j(c_i) &= \begin{cases} c_{j-i}, & i \leq j \leq n-1, \\ d_{n+j-i}, & 0 \leq j < i, \end{cases} \\
h_j(d_i) &= \begin{cases} d_{j-i}, & i \leq j \leq n-1, \\ c_{n+j-i}, & 0 \leq j < i, \end{cases}
\end{aligned}
\tag{1}
$$

is an isomorphism of flower snark graph J_n.

A wheel graph is a cycle of length at least three, with a single vertex in the center connected to every vertex on the cycle. These graphs are denoted by W_n and have $n+1$ vertices and $2n$ edges, where the vertex-set is $V(W_n) = \{v_0, v_1, \ldots, v_n\}$ and the edge-set is $E(W_n) = \{v_0v_i | 1 \leq i \leq n\} \cup \{v_iv_{i+1} | 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. Vertex v_0 is called as an interior vertex of the graph, and all other vertices are called external vertices.

Figure 1 shows the flower snark graph J_9. Its mixed metric dimension is 4, which is obtained through total enumeration. The one mixed metric basis is $\{b_0, c_1, c_6, d_3\}$. In the below figure, the vertices that are elements of the basis are shown in larger circles, too.

Figure 2 shows the wheel graph W_8. Its mixed metric dimension is 8, which is obtained through total enumeration. The one mixed metric basis is $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. In the below figure, also the vertices that are elements of the basis are shown in larger circles.

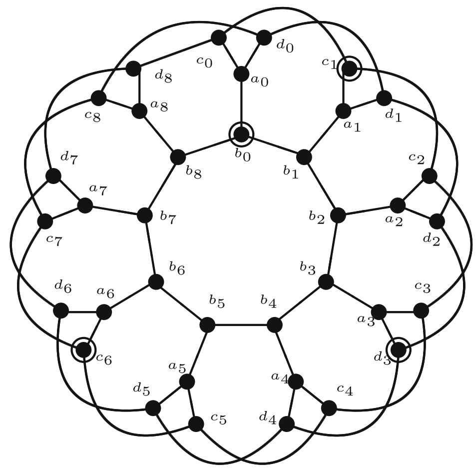

Figure 1: Graph J_9.

In the next part of the section, some theoretical properties of metric dimension, edge metric dimension, and mixed metric dimension will be presented following the literature.

Property 2. [1] *For any graph G, it holds that*

$$\beta_M(G) \geq \max\{\beta(G), \beta_E(G)\}.$$

In the next theorem [18], for the flower snarks, metric dimension J_n is given.

Theorem 1. [18] *Let J_n be the flower snark. Then for every odd positive integer $n \geq 5$, it holds that $\beta(J_n) = 3$.*

Proposition 1. [1] *Let v be an arbitrary vertex in a graph G and let $S = V(G)\setminus\{v\}$. If $(\forall w \in N(v))$ $(\exists x \in S)$ $d(vw, x) \neq d(w, x)$, then S is a mixed resolving set for the graph G.*

Proposition 2. [19] *Let G be a connected graph, then $\beta_E(G) \geq 1 + \lceil \log_2 \delta(G) \rceil$.*

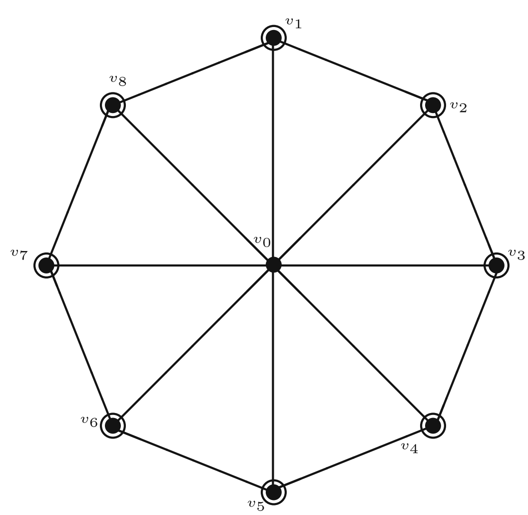

Figure 2: Graph W_8.

2 New theoretical results

In this section, the mixed metric dimension of two important classes of graphs is considered: flower snarks and wheel graph.

2.1 Mixed metric dimension of flower snarks

The metric dimension for flower snarks is given in [18]. So it was interesting to determine the value for mixed metric dimension for these graphs. In Theorem 2, the result about the mixed metric dimension is given and here it is shown that the dimension is different for $n = 5$ and odd $n \geq 7$. It should be noted that we will omit all cases which are equivalent with regard to isomorphism given in Property 1.

First, let set of vertices $V(J_n)$ be partitioned as follows: $V_1 = \{a_i | 0 \leq i \leq n-1\}$, $V_2 = \{b_i | 0 \leq i \leq n-1\}$, $V_3 = \{c_i | 0 \leq i \leq n-1\}$, and $V_4 = \{d_i | 0 \leq i \leq n-1\}$.

Next, several lemmas are proposed and proved, which help us to prove Theorem 2.

Lemma 1. *If S is an arbitrary mixed resolving set of J_n, then*:
(*a*) $S \bigcap (V_1 \bigcup V_2) \neq \varnothing$;
(*b*) $S \bigcap (V_3 \bigcup V_4) \neq \varnothing$;
(*c*) $(\forall i \in \{0, 1, \ldots, n-1\})\ S \bigcap \{a_j, b_j, c_j, d_j | i \leq j \leq i+k-1\} \neq \varnothing$.

Proof.
(a) Suppose the opposite, i.e., $S \bigcap (V_1 \bigcup V_2) = \varnothing$, which means that $S \subseteq (V_3 \bigcup V_4)$. For each i and j, such that $0 \leq i, j \leq n-1$, it holds that $d(b_j, c_i) = d(b_j, d_i) = d(a_j, c_i) + 1$ so $d(a_jb_j, c_i) = d(a_j, c_i) = d(a_jb_j, d_i) = d(a_j, d_i)$, which means that edge a_jb_j has the same metric coordinates as vertex a_j with respect to $V_3 \bigcup V_4$, i.e., $r(a_jb_j, V_3 \bigcup V_4) = r(a_j, V_3 \bigcup V_4)$. Since $S \subseteq (V_3 \bigcup V_4)$, then $r(a_jb_j, S) = r(a_j, S)$ holds. Therefore, the set S is not the mixed resolving set of J_n, which is a contradiction with a starting assumption.
(b) Suppose the opposite, i.e., $S \bigcap (V_3 \bigcup V_4) = \varnothing$, which means that $S \subseteq (V_1 \bigcup V_2)$. For each i and j, such that $0 \leq i, j \leq n-1$, and $i \neq j$, it holds that $d(c_j, a_i) = d(d_j, a_i) = d(a_j, a_i) - 1$ and $d(c_j, b_i) = d(d_j, b_i) = d(a_j, b_i) + 1$. For $i = j$, similarly $d(c_i, a_i) = d(d_i, a_i) = 1$ and $d(c_i, b_i) = d(d_i, b_i) = 2 = d(a_i, b_i) + 1$ hold. Next, for each j, such that $0 \leq j \leq n-1$ it holds that vertex c_j has the same metric coordinates as vertex d_j with respect to $V_1 \bigcup V_2$, i.e., $r(c_j, V_1 \bigcup V_2) = r(d_j, V_1 \bigcup V_2)$. Since $S \subseteq (V_1 \bigcup V_2)$, then $r(c_j, S) = r(d_j, S)$ holds. Therefore, the set S is not the mixed resolving set of J_n, which is a contradiction with a starting assumption.
(c) Suppose the opposite, i.e., $S \bigcap \{a_j, b_j, c_j, d_j | i \leq j \leq i+k-1\} = \varnothing$, which means that $S \subseteq \{a_j, b_j, c_j, d_j | i-k-1 \leq j \leq i-1\}$. For $i-k \leq j \leq i-1$, $d(b_{i+1}, a_j) = d(b_i, a_j) + 1$, $d(b_{i+1}, b_j) = d(b_i, b_j) + 1$, $d(b_{i+1}, c_j) = d(b_i, c_j) + 1$, and $d(b_{i+1}, d_j) = d(b_i, d_j) + 1$ hold, so $d(b_ib_{i+1}, a_j) = d(b_i, a_j)$, $d(b_ib_{i+1}, b_j) = d(b_i, b_j)$, $d(b_ib_{i+1}, c_j) = d(b_i, c_j)$ and $d(b_ib_{i+1}, d_j) = d(b_i, d_j)$, which means that edge b_ib_{i+1} has the same metric coordinates as vertex b_i with respect to $\{a_j, b_j, c_j, d_j | i-k \leq j \leq i-1\}$. For only remained case, when $j = i-k-1$, $d(b_{i+1}, a_{i-k-1}) = d(b_i, a_{i-k-1}) - 1$, $d(b_{i+1}, b_{i-k-1}) = d(b_i, b_{i-k-1}) - 1$, $d(b_{i+1}, c_{i-k-1}) = d(b_i, c_{i-k-1}) - 1$ and $d(b_{i+1}, d_{i-k-1}) = d(b_i, d_{i-k-1}) - 1$ hold, so $d(b_ib_{i+1}, a_{i-k-1}) = d(b_{i+1}, a_{i-k-1})$, $d(b_ib_{i+1}, b_j) = d(b_{i+1}, b_{i-k-1})$, $d(b_ib_{i+1}, c_{i-k-1}) = d(b_{i+1}, c_{i-k-1})$, and $d(b_ib_{i+1}, d_{i-k-1}) = d(b_{i+1}, d_{i-k-1})$, which means that edge b_ib_{i+1} has the same metric coordinates as vertex b_{i+1} with respect to $\{a_{i-k-1}, b_{i-k-1}, c_{i-k-1}, d_{i-k-1}\}$. Therefore, $r(b_ib_{i+1}, \{a_j, b_j, c_j, d_j | i-k-1 \leq j \leq i-1\}) = r(b_{i+1}, \{a_j, b_j, c_j, d_j | i-k-1 \leq j \leq i-1\})$. Having in mind that $S \subseteq \{a_j, b_j, c_j, d_j | i-k-1 \leq j \leq i-1\}$, we have $r(b_ib_{i+1}, S) = r(b_{i+1}, S)$, so that S is not the mixed resolving set of J_n, which is a contradiction with a starting assumption. □

It should be noted that, as is mentioned earlier, all indices in part (c) take modulo n. Moreover, without loss of generality, one vertex from $S \bigcap (V_3 \bigcup V_4)$, from Lemma 1 part (b), should be transformed into the vertex c_0 by isomorphism from Property 1.

Theorem 2. *For odd $n \geq 5$, it holds that* $\beta_M(J_n) = \begin{cases} 5, & n = 5, \\ 4, & n \geq 7. \end{cases}$

Proof. Step 1 : Exact value for $n \in \{5, 7, 9\}$

By using total enumeration technique, it can be shown that:

- $\beta_M(J_5) = 5$, with mixed metric basis $\{a_3, b_0, b_1, c_2, d_3\}$;
- $\beta_M(J_7) = 4$, with mixed metric basis $S = \{b_0, c_1, c_5, d_3\}$;
- $\beta_M(J_9) = 4$, with mixed metric basis $S = \{b_0, c_1, c_6, d_3\}$.

Step 2 : Upper bound equals 4 for $n \geq 11$.

Let $S = \{b_0, c_1, c_{k+2}, d_3\}$. It will be proved that S is the mixed resolving set. The representations of coordinates of each vertex and each edge, with respect to S, are shown in Tables 1 and 2.

As it can be seen from Tables 1 and 2, all items have mutually different metric coordinates, so S is the mixed resolving set. Therefore, $\beta_M(J_n) \leq 4$.

Step 3 : Lower bound equals 4 for $n \geq 11$.

Suppose the opposite, i.e., $\beta_M(J_n) \leq 3$. By Property 2 and Theorem 1, we have $\beta_M(J_n) = 3$. Let S be the mixed resolving set of J_n. Then, from Lemma 1, parts (a) and (b), two members of set S could be derived, i.e., $c_0 \in S$ and $(\exists i)(a_i \in S \vee b_i \in S)$. Let the remaining third member of set S has index j, i.e., $(\exists j)(a_j \in S \vee b_j \in S \vee c_j \in S \vee d_j \in S)$.

From Lemma 1, part (c), it holds that indices i and j have not arbitrary values:

(I) $i \neq 0 \ \wedge \ j \neq 0$;
(II) $1 \leq i \leq k \ \Rightarrow \ k + 1 \leq j \leq 2k$;
(III) $k + 1 \leq i \leq 2k \ \Rightarrow \ 1 \leq j \leq k$.

Table 1: Metric coordinates of vertices of J_{2k+1}

Vertex	Cond.	$r(v, S)$
a_i	$0 \leq i \leq 1$	$(i + 1, 2 - i, k + i, 4 - i)$
a_2		$(3, 2, k + 1, 2)$
a_i	$3 \leq i \leq k$	$(i + 1, i, k + 3 - i, i - 2)$
a_{k+1}		$(k + 1, k + 1, 2, k - 1)$
a_i	$k + 2 \leq i \leq k + 3$	$(2k + 2 - i, 2k + 3 - i, i - 1 - k, i - 2)$
	$k + 4 \leq i \leq 2k$	$(2k + 2 - i, 2k + 3 - i, i - 1 - k, 2k + 5 - i)$
b_i	$0 \leq i \leq 1$	$(i, 3 - i, k + i + 1, 5 - i)$
b_2		$(2, 3, k + 2, 3)$
b_i	$3 \leq i \leq k$	$(i, i + 1, k + 4 - i, i - 1)$
b_{k+1}		$(k, k + 2, 3, k)$
b_i	$k + 2 \leq i \leq k + 3$	$(2k + 1 - i, 2k + 4 - i, i - k, i - 1)$
	$k + 4 \leq i \leq 2k$	$(2k + 1 - i, 2k + 4 - i, i - k, 2k + 6 - i)$
c_0		$(2, 1, k + 1, 5)$
c_i	$1 \leq i \leq 2$	$(i + 2, i - 1, k + 2 - i, 5 - i)$
	$3 \leq i \leq k$	$(i + 2, i - 1, k + 2 - i, i - 1)$
	$k + 1 \leq i \leq k + 2$	$(2k + 3 - i, i - 1, k + 2 - i, i - 1)$
	$k + 3 \leq i \leq 2k$	$(2k + 3 - i, 2k + 4 - i, i - 2 - k, 2k + 4 - i)$
d_0		$(2, 3, k - 1, 3)$
d_i	$1 \leq i \leq 2$	$(i + 2, i + 1, k - 1 + i, 3 - i)$
	$3 \leq i \leq k$	$(i + 2, i + 1, k + 4 - i, i - 3)$
	$k + 1 \leq i \leq k + 2$	$(2k + 3 - i, 2k + 2 - i, k + 4 - i, i - 3)$
	$k + 3 \leq i \leq k + 4$	$(2k + 3 - i, 2k + 2 - i, i - k, i - 3)$
	$k + 5 \leq i \leq 2k$	$(2k + 3 - i, 2k + 2 - i, i - k, 2k + 6 - i)$

Table 2: Metric coordinates of edges of J_{2k+1}

Edge	Cond.	$r(e, S)$
a_0b_0		$(0, 2, k, 4)$
a_ib_i	$1 \le i \le 2$	$(i, i, k+1, 4-i)$
	$3 \le i \le k$	$(i, i, k+3-i, i-2)$
$a_{k+1}b_{k+1}$		$(k, k+1, 2, k-1)$
a_ib_i	$k+2 \le i \le k+3$	$(2k+1-i, 2k+3-i, i-1-k, i-2)$
	$k+4 \le i \le 2k$	$(2k+1-i, 2k+3-i, i-1-k, 2k+5-i)$
a_0c_0		$(1, 1, k, 4)$
a_ic_i	$1 \le i \le 2$	$(i+1, i-1, k+2-i, 4-i)$
	$3 \le i \le k$	$(i+1, i-1, k+2-i, i-2)$
	$k+1 \le i \le k+2$	$(2k+2-i, i-1, k+2-i, i-2)$
	$k+3 \le i \le 2k$	$(2k+2-i, 2k+3-i, i-2-k, 2k+4-i)$
a_0d_0		$(1, 2, k-1, 3)$
a_id_i	$1 \le i \le 2$	$(i+1, i, k-1+i, 3-i)$
	$3 \le i \le k$	$(i+1, i, k+3-i, i-3)$
$a_{k+1}d_{k+1}$		$(k+1, k+1, 2, k-2)$
a_id_i	$k+2 \le i \le k+3$	$(2k+2-i, 2k+2-i, i-1-k, i-3)$
	$k+4 \le i \le 2k$	$(2k+2-i, 2k+2-i, i-1-k, 2k+5-i)$
b_0b_1		$(0, 2, k+1, 4)$
b_ib_{i+1}	$1 \le i \le 2$	$(i, i+1, k+3-i, 4-i)$
	$3 \le i \le k$	$(i, i+1, k+3-i, i-1)$
	$k+1 \le i \le k+2$	$(2k-i, 2k+3-i, 2, i-1)$
	$k+3 \le i \le 2k$	$(2k-i, 2k+3-i, i-k, 2k+5-i)$
c_0c_1		$(2, 0, k+1, 4)$
c_ic_{i+1}	$1 \le i \le 2$	$(i+2, i-1, k+1-i, 4-i)$
	$3 \le i \le k$	$(i+2, i-1, k+1-i, i-1)$
$c_{k+1}c_{k+2}$		$(k+1, k, 0, k)$
c_ic_{i+1}	$k+2 \le i \le 2k-1$	$(2k+2-i, 2k+3-i, i-2-k, 2k+3-i)$
$c_{2k}d_0$		$(2, 3, k-2, 3)$
c_0d_{2k}		$(2, 1, k, 5)$
d_0d_1		$(2, 2, k-1, 2)$
d_id_{i+1}	$1 \le i \le 2$	$(i+2, i+1, k-1+i, 2-i)$
	$3 \le i \le k$	$(i+2, i+1, k+3-i, i-3)$
$d_{k+1}d_{k+2}$		$(k+1, k, 2, k-2)$
d_id_{i+1}	$k+2 \le i \le k+3$	$(2k+2-i, 2k+1-i, i-k, i-3)$
	$k+4 \le i \le 2k-1$	$(2k+2-i, 2k+1-i, i-k, 2k+5-i)$

Part (I) holds, because if $(i = 0 \ \vee \ j = 0) \Rightarrow$

$$(S \bigcap \{a_p, b_p, c_p, d_p | 1 \le p \le k\} = \varnothing \ \vee \ S \bigcap \{a_p, b_p, c_p, d_p | k+1 \le p \le 2k\} = \varnothing),$$

right hand side of implication is in direct contradiction with Lemma 1, part (c).

If $1 \le i \le k$, then again by Lemma 1(c), $S \bigcap \{a_p, b_p, c_p, d_p | k+1 \le p \le 2k\} \ne 0$ holds. Since $i \notin \{p | k+1 \le p \le 2k\}$, then it must be $j \in \{p | k+1 \le p \le 2k\}$, so part (II) holds.

Part (III) also follows from Lemma 1, in a similar way to part (II).

We have eight possible cases for mixed resolving set S.

Case 1. $S = \{c_0, a_i, a_j\}$.

Since $i \neq 0 \ \wedge\ j \neq 0$, then

$$d(a_0c_0, c_0) = d(c_0, c_0) = 0,\ d(a_0c_0, a_i) = d(a_0, a_i) - 1 = d(c_0, a_i), \text{ and } d(a_0c_0, a_j) = d(a_0, a_j) - 1 = d(c_0, a_j).$$

Therefore, $r(a_0c_0, S) = r(c_0, S)$, which means that S is not the mixed resolving set.

Case 2. $S = \{c_0, a_i, b_j\}$

Subcase 1. $1 \le i \le k$

From part (II), it holds that $k + 1 \le j \le 2k$. Then

$$d(a_0c_0, c_0) = d(c_0d_{2k}, c_0) = 0 \quad \text{and} \quad d(a_0c_0, a_i) = d(c_0, a_i) = d(c_0d_{2k}, a_i).$$

Also, since

$$d(a_0c_0, b_j) = d(a_0, b_j) = d(b_0, b_j) + 1,\ d(c_0d_{2k}, b_j) = d(d_{2k}, b_j) = d(b_{2k}, b_j) + 2, \text{ and } d(b_0, b_j) = d(b_{2k}, b_j) + 1,$$

it follows that $d(a_0c_0, b_j) = d(c_0d_{2k}, b_j)$.
Therefore, $r(a_0c_0, S) = r(c_0d_{2k}, S)$, which means that S is not the mixed resolving set.

Subcase 2. $k + 1 \le i \le 2k$

From part (III), it holds that $1 \le j \le k$. Then

$$d(a_0c_0, c_0) = d(c_0c_1, c_0) = 0 \quad \text{and} \quad d(a_0c_0, a_i) = d(c_0, a_i) = d(c_0c_1, a_i).$$

Also, since

$$d(a_0c_0, b_j) = d(a_0, b_j) = d(b_0b_j) + 1, \quad d(c_0c_1, b_j) = d(c_1, b_j) = d(b_1, b_j) + 2, \quad \text{and} \quad d(b_0, b_j) = d(b_1, b_j) + 1,$$

it follows that $d(a_0c_0, b_j) = d(c_0c_1, b_j)$.

Therefore, $r(a_0c_0, S) = r(c_0c_1, S)$, which means that S is not the mixed resolving set.

Case 3. $S = \{c_0, a_i, c_j\}$

Subcase 1. $1 \le i \le k$

From part (II), it holds that $k + 1 \le j \le 2k$. Then

$$d(a_0c_0, c_0) = d(c_0d_{2k}, c_0) = 0 \quad \text{and} \quad d(a_0c_0, a_i) = d(c_0, a_i) = d(c_0d_{2k}, a_i).$$

Also, since

$$d(a_0c_0, c_j) = d(a_0, c_j) = d(c_{2k}, c_j) + d(c_{2k}, a_0) = d(c_{2k}, c_j) + 2 \quad \text{and} \quad d(c_0d_{2k}, c_j) = d(d_{2k}, c_j) = d(c_{2k}, c_j) + 2,$$

it follows that $d(a_0c_0, c_j) = d(c_0d_{2k}, c_j)$.

Therefore, $r(a_0c_0, S) = r(c_0d_{2k}, S)$, which means that S is not the mixed resolving set.

Subcase 2. $k + 1 \le i \le 2k$

From part (III), it holds that $1 \le j \le k$. Then

$$d(a_0c_0, c_0) = d(c_0, c_0) = 0, \quad d(a_0c_0, a_i) = d(a_0, a_i) - 1 = d(c_0, a_i), \quad \text{and} \quad d(a_0c_0, c_j) = d(c_0, c_j).$$

Therefore, $r(a_0c_0, S) = r(c_0, S)$, which means that S is not the mixed metric resolving set.

Case 4. $S = \{c_0, a_i, d_j\}$

Subcase 1. $1 \le i \le k$

From part (II), it holds that $k + 1 \le j \le 2k$. Then

$$d(a_0c_0, c_0) = d(c_0, c_0) = 0, \quad d(a_0c_0, a_i) = d(a_0, a_i) - 1 = d(c_0, a_i), \quad \text{and}$$
$$d(a_0c_0, d_j) = d(a_0, d_j) - 1 = d(c_0, d_j).$$

Therefore, $r(a_0c_0, S) = r(c_0, S)$, which means that S is not the mixed resolving set.

Subcase 2. $k+1 \le i \le 2k$

From part (III), it holds that $1 \le j \le k$. Then

$$d(c_0a_0, c_0) = d(c_0c_1, c_0) = 0, \quad d(c_0a_0, a_i) = d(c_0c_1, a_i), \quad \text{and} \quad d(c_0a_0, d_j) = d(a_0, d_j) = d(c_1, d_j) = d(c_0c_1, d_j).$$

Therefore, $r(c_0a_0, S) = r(c_0c_1, S)$ which means that S is not the mixed resolving set.

Case 5. $S = \{c_0, b_i, a_j\}$ Reduce to Case 2. By substitution $j' = i$, $i' = j$.

Case 6. $S = \{c_0, b_i, b_j\}$

Subcase 1. $1 \le i \le k$

From part (II), it holds that $k+1 \le j \le 2k$. Then

$$d(a_0, c_0) = 1 = d(a_0d_0, c_0),$$
$$d(a_0, b_i) = d(b_0, b_i) + 1 = d(d_0, b_i) - 1 = d(a_0d_0, b_i), \text{ and}$$
$$d(a_0, b_j) = d(b_0, b_j) + 1 = d(d_0, b_j) - 1 = d(a_0d_0, b_j).$$

Therefore, $r(a_0, S) = r(a_0d_0, S)$, which means that S is not a mixed resolving set.

Subcase 2. $k+1 \le i \le 2k$

Reduce to Subcase 2. By substitution $j' = i$, $i' = j$.

Case 7. $S = \{c_0, b_i, c_j\}$

Subcase 1. $1 \le i \le k$

From part (II), it holds that $k+1 \le j \le 2k$. Then

$$d(b_0b_1, c_0) = d(b_0, c_0) = 2 = d(a_1, c_0) = d(a_1b_1, c_0),$$
$$d(b_0b_1, b_i) = d(b_1, b_i) = d(a_1, b_i) - 1 = d(a_1b_1, b_i), \text{ and}$$
$$d(b_0b_1, c_j) = d(b_0, c_j) = d(b_0, b_j) + 2 = d(c_0, c_j) = d(d_0, c_j) + d(d_0, c_0) = d(d_0, c_j) + d(d_0, a_1) =$$
$$d(a_1, c_j) = d(b_1, c_j) - 1 = d(a_1b_1, c_j), \text{ where } d(d_0, c_0) = d(d_0, a_1) = 2.$$

Therefore, $r(b_0b_1, S) = r(a_1b_1, S)$, which means that S is not a mixed resolving set.

Subcase 2. $k+1 \le i \le 2k$

From part (III), it holds that $1 \le j \le k$. Then

$$d(b_0, c_0) = 2 = d(d_{2k-1}, c_0),$$
$$d(b_0, b_i) = d(b_{2k-1}, b_i) + 2 = d(d_{2k-1}, b_i), \text{ and}$$
$$d(b_0, c_j) = d(b_0, b_j) + 2 = d(c_0, c_j) + 2 = d(c_0, c_j) + d(d_{2k-1}, c_0) = d(d_{2k-1}, c_j), \text{ where } d(d_{2k-1}, c_0) = 2.$$

Therefore, $r(b_0, S) = r(d_{2k-1}, S)$, which means that S is not a mixed resolving set.

Case 8. $S = \{c_0, b_i, d_j\}$

Subcase 1. $1 \le i \le k$

From part (II), it holds that $k+1 \le j \le 2k$. Then

$$d(b_0b_1, c_0) = d(b_0, c_0) = 2 = d(a_1, c_0) = d(a_1b_1, c_0), \quad d(b_0b_1, b_i) = d(b_1, b_i) = d(a_1, b_i) - 1 = d(a_1b_1, b_i)$$
$$d(b_0b_1, d_j) = d(b_0, d_j) = d(b_0, b_j) + 2 = d(d_0, d_j) = d(c_0, d_j) + d(c_0, a_1) = d(a_1, d_j) = d(b_1, d_j) - 1$$
$$= d(a_1b_1, d_j),$$

where $d(c_0, a_1) = d(c_0, c_1) + d(c_1, a_1) = 2$.

Therefore, $r(b_0b_1, S) = r(a_1b_1, S)$, which means that S is not a mixed resolving set.

Subcase 2. $k+1 \le i \le 2k$

From part (III), it holds that $1 \le j \le k$. Then

$$d(b_0, c_0) = 2 = d(a_{2k}, c_0), \quad d(b_0, b_i) = d(a_{2k}, b_i), \text{ and}$$
$$d(b_0, d_j) = d(b_0, b_j) + 2 = d(d_0, d_j) + 2 = d(d_0, d_j) + d(a_{2k}, d_0) = d(a_{2k}, d_j).$$

Therefore, $r(b_0, S) = r(a_{2k}, S)$, which means that S is not a mixed resolving set.

Since S is not the mixed resolving set in all eight cases, which is a contradiction with starting assumption, so $\beta_M(J_n) \geq 4$. Therefore, from the previous three steps, the proof of theorem is completed. □

It would be interesting to make comparison between mixed metric dimension and metric dimension for flower snarks. For $n = 5$ situation is easy since all three dimensions can be obtained by a total enumeration, so $\beta(J_n) = 3 < \beta_E(J_n) = 4 < \beta_M(J_n) = 5$. For odd $n \geq 7$, mixed metric dimension is larger than metric dimension, i.e., $\beta(J_n) = 3 < \beta_M(J_n) = 4$. From Theorem 2 it is easy to see that for $n \geq 7$, similar to metric dimension, mixed metric dimension for flower snarks is constant, i.e., it does not depend on n.

When we consider edge metric dimension, situation is not so clear. Theorem 2 has obvious corollary, that for $n \geq 7$ it is $\beta_E(J_n) \leq 4$. From Theorem 2 of [1], lower bound of edge metric dimension applied to flower snarks is $\beta_E(J_n) \geq 3$, so it holds $(\forall\ n \geq 7)\beta_E(J_n) \in \{3, 4\}$. It would be interesting to find the exact value.

2.2 Mixed metric dimension of wheel graphs

In the following, the mixed metric dimension of wheel graphs is obtained. In the next theorem, the mixed metric dimension of these graphs is determined.

Theorem 3.

$$\beta_M(W_n) = \begin{cases} 4, & n = 3, \\ n, & n \geq 4. \end{cases}$$

Proof. For $n = 3$ using total enumeration, we have $\beta_M(W_3) = 4 = |W_3|$ so that $S = V(W_3)$.

Therefore, the mixed metric dimension of wheels will be considered only for case when $n \geq 4$. In order to present it more clearly, the representations of the mixed metric coordinates of each vertex and each edge with respect to $V(W_n)$ are given in Table 3.

Two steps will be considered.

Step 1 : Upper bound for $n \geq 4$.

Let $S = \{v_i | 1 \leq i \leq n\}$. It will be proved that S is the mixed resolving set of W_n, for $n \geq 4$. Since $S = V \backslash \{v_0\}$, then for each item, vectors of metric coordinates with respect to S are presented in Table 4.

Table 3: Mixed metric representations for W_n with respect to V

Vertex	**Cond.**	$r(v, V)$
v_0		$(0, 1, \dots, 1)$
v_1		$(1, 0, 1, 2, \dots, 2, 1)$
v_2		$(1, 1, 0, 1, 2, \dots, 2)$
v_i	$3 \leq i \leq n-2$	$(1, 2, \dots, 2, 1, \underbrace{0}_{i}, 1, 2, \dots, 2)$
v_{n-1}		$(1, 2, \dots, 2, 1, 0, 1)$
v_n		$(1, 1, 2, \dots, 2, 1, 0)$
Edge	**Cond.**	$r(e, V)$
v_0v_i	$1 \leq i \leq n$	$(0, 1, \dots, 1, \underbrace{0}_{i}, 1, \dots, 1)$
v_1v_2		$(1, 0, 0, 1, 2, \dots, 2, 1)$
v_1v_n		$(1, 0, 1, 2, \dots, 2, 1, 0)$
v_2v_3		$(1, 1, 0, 0, 1, 2, \dots, 2)$
v_iv_{i+1}	$3 \leq i \leq n-3$	$(1, 2, \dots, 2, 1, \underbrace{0}_{i}, 0, 1, 2, \dots, 2)$
$v_{n-2}v_{n-1}$		$(1, 2, \dots, 2, 1, 0, 0, 1)$
$v_{n-1}v_n$		$(1, 1, 2, \dots, 2, 1, 0, 0)$

Table 4: Mixed metric representations for W_n with respect to $V\backslash\{v_0\}$

Vertex	Cond.	$r(v, V\backslash\{v_0\})$
v_0		$(1, \ldots, 1)$
v_1		$(0, 1, 2 \ldots, 2, 1)$
v_2		$(1, 0, 1, 2 \ldots, 2)$
v_i	$3 \le i \le n-2$	$(2, \ldots, 2, 1, \underbrace{0}_{i}, 1, 2, \ldots, 2)$
v_{n-1}		$(2, \ldots, 2, 1, 0, 1)$
v_n		$(1, 2, \ldots, 2, 1, 0)$
Edge	**Cond.**	$r(e, V\backslash\{v_0\})$
v_0v_i	$1 \le i \le n$	$(1, \ldots, 1, \underbrace{0}_{i}, 1, \ldots, 1)$
v_1v_2		$(0, 0, 1, 2, \ldots, 2, 1)$
v_1v_n		$(0, 1, 2, \ldots, 2, 1, 0)$
v_2v_3		$(1, 0, 0, 1, 2, \ldots, 2)$
v_iv_{i+1}	$3 \le i \le n-3$	$(2, \ldots, 2, 1, \underbrace{0}_{i}, 0, 1, 2, \ldots, 2)$
$v_{n-2}v_{n-1}$		$(2, \ldots, 2, 1, 0, 0, 1)$
$v_{n-1}v_n$		$(1, 2, \ldots, 2, 1, 0, 0)$

As it can be seen from Table 4, edge v_iv_j, $1 \le i, j \le n$ has two zeros in the vector of metric coordinates ($d(v_iv_j, v_i) = d(v_iv_j, v_j) = 0$), while the vertex v_i and the edge v_0v_i ($1 \le i \le n$) have one zero in the vector of metric coordinates ($d(v_0v_i, v_i) = d(v_i, v_i) = 0$). Only vertex v_0 has the vector of metric coordinates equal to $(1, \ldots, 1)$ (without zeros). Since, for $l \ne i$ it is $d(v_0v_i, v_l) = 1$, but $d(v_i, v_l) = 2$, it follows that mixed metric coordinates of all items are mutually different. Therefore, S is the mixed resolving set, so it holds $\beta_M(W_n) \le n$.

Step 2 : Lower bound for $n \ge 4$.

Assume the opposite that it is $\beta_M(W_n) \le n-1$. Then mixed resolving set S exists, so that $|S| \le n-1$.

Case 1. $v_0 \notin S$.

Since $|V| = n+1 \wedge |S| \le n-1 \Rightarrow (\exists i)1 \le i \le n, v_i \notin S$, it holds that $r(v_0, S) = r(v_0v_i, S) = (1, \ldots, 1)$. It will be concluded that S is not the mixed resolving set, which is a contradiction with a starting assumption.

Case 2. $v_0 \in S$.

Since $|V| = n+1 \wedge |S| \le n-1 \Rightarrow (\exists i, j)1 \le i, j \le n, v_i, v_j \notin S$, it holds that $r(v_0, S) = r(v_0v_i, S) = (0, 1, \ldots, 1)$. It will be concluded that S is not the mixed resolving set, which is a contradiction with a starting assumption.

Since S is not the mixed resolving set in both cases, it follows that $\beta_M(W_n) \ge n$.

Therefore, from Steps 1 and 2, it follows that $\beta_M(W_n) = n$. □

Also, it should be noted that Step 1 could be indirectly proved using Proposition 1 from [1]. In order to give constructive proof, we have decided on the proof presented above.

All three previously mentioned metric invariants could be compared. It is already known from [20] that

$$\beta(W_n) = \begin{cases} 3, & n = 3, 6, \\ 2, & n = 4, 5, \\ \left\lfloor \dfrac{2n+2}{5} \right\rfloor, & n \ge 7, \end{cases}$$

and from [13], it follows that

$$\beta_E(W_n) = \begin{cases} n, & n = 3, 4, \\ n-1, & n \ge 5. \end{cases}$$

From the previous, it would be interesting to make comparison between mixed metric dimension, metric dimension and edge metric dimension, for wheel graphs. For $n = 3$ mixed metric dimension is larger than metric and edge metric dimension, i.e., $\beta(W_3) = \beta_E(W_3) = 3 < \beta_M(W_3) = 4$. For $n = 4$ edge metric dimension and mixed metric dimension are larger then metric dimension, i.e., $\beta(W_4) = 2 < \beta_E(W_4) = \beta_M(W_4) = 4$. Then, for $n = 5$ and $n = 6$ holds $\beta(W_5) = 2 < \beta_E(W_5) = 4 < \beta_M(W_5) = 5$ and $\beta(W_6) = 3 < \beta_E(W_6) = 5 < \beta_M(W_6) = 6$, respectively. And, for $n \geq 7$ edge metric dimension is smaller than mixed metric dimension, and larger than the metric dimension, i.e., $\beta(W_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor < \beta_E(W_n) = n - 1 < \beta_M(W_n) = n$. Unlike the flower snarks graphs described above, for wheel graphs, the mixed metric dimension depends on n.

3 Conclusions

In this paper, the mixed metric dimension for flower snarks and wheels graphs is considered. First, it is given lemma about some properties of the mixed resolving sets of flower snark graphs. Next, this lemma is used for obtaining exact value of the mixed metric dimension for flower snarks. It is proved that it is constant and equal to 4, for odd $n \geq 7$, while $\beta_M(J_5) = 5$. Last, it is present the mixed metric dimension for wheels, and it is proved to be equal to n, for $n \geq 4$, while $\beta_M(W_3) = 4$.

Further work can be directed in finding the mixed metric dimension of some other interesting classes of graphs. Other direction could be finding the exact value of the edge metric dimension of flower snarks.

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