



# The Mixed Scalar Curvature of Almost-Product Metric-Affine Manifolds

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**Abstract.** We study two Einstein–Hilbert type actions on an almost-product metric-affine manifold, considered as functionals of the contorsion tensor. The first one is the total mixed scalar curvature of the linear connection, and the second one is based on a new type of curvature, recently introduced by B. Opozda for statistical structures. We deduce Euler–Lagrange equations of the actions and examine critical contorsion tensors associated with general and distinguished classes of connections, e.g. metric, statistical and adapted. The existence of such critical tensors depends on simple geometric properties of the almost-product structure, expressed only in terms of the Levi-Civita connection.

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## 1. Introduction

Distributions on manifolds appear in various situations, e.g. as tangent bundles of foliations or kernels of differential forms. An important role in understanding geometry of distributions and foliations play linear connections and the mixed sectional curvature, i.e., sectional curvature of planes that non-trivially intersect the distribution and its orthogonal complement, see [2, 7]. The mixed scalar curvature, i.e., an averaged mixed sectional curvature, is one of the simplest curvature invariants of an almost product manifold. The Euler–Lagrange equations for the total mixed scalar curvature, as functional on the space of metrics, has been studied in [1] as analog of Einstein–Hilbert action, and then in [8, 9] for distributions of any dimension.

The Metric-Affine Geometry (founded by E. Cartan) generalizes Riemannian Geometry: it uses an asymmetric connection with torsion,  $\bar{\nabla}$ , instead of the Levi-Civita connection  $\nabla$  of  $g$ , and appears in such context as homogeneous and almost Hermitian manifolds, Finsler geometry and gauge theory of gravity. The important distinguished cases are: Riemann–Cartan manifolds, where *metric* connections, i.e.,  $\bar{\nabla}g = 0$ , are used, e.g. [4], and *statistical* manifolds [3, 5], where the torsion is zero and the tensor  $\bar{\nabla}g$  is symmetric in all its entries. The main notion of Information Geometry is that of statistical manifold, and the theory of affine hypersurfaces in  $\mathbb{R}^{n+1}$  is a natural source of such manifolds. Riemann–Cartan spaces are central in gauge theory of gravity, where the torsion is represented by the spin tensor of matter.

The difference  $\mathfrak{T} := \bar{\nabla} - \nabla$  is called the *contorsion tensor*. For the curvature tensor  $\bar{R}_{X,Y} = [\bar{\nabla}_Y, \bar{\nabla}_X] + \bar{\nabla}_{[X,Y]}$  of  $\bar{\nabla}$ , using similar formula for the curvature tensor  $R$  of  $\nabla$ , we get

$$\bar{R}_{X,Y} - R_{X,Y} = (\nabla_Y \mathfrak{T})_X - (\nabla_X \mathfrak{T})_Y + [\mathfrak{T}_Y, \mathfrak{T}_X]. \tag{1}$$

Let  $M^{n+p}$  be a connected manifold with a pseudo-Riemannian metric  $g$  of index  $q$  and complementary orthogonal non-degenerate distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$  (subbundles of the tangent bundle  $TM$  of ranks  $\dim_{\mathbb{R}} \mathcal{D}_x = n$  and  $\dim_{\mathbb{R}} \mathcal{D}_x^\perp = p$  for every  $x \in M$ ) called an *almost-product structure* on  $M$ , see [2]. When  $q = 0$ ,  $g$  is a Riemannian metric, resp. a Lorentz metric when  $q = 1$ . Let  $^\top$  and  $^\perp$  denote  $g$ -orthogonal projections onto  $\mathcal{D}$  and  $\mathcal{D}^\perp$ , respectively. The following convention is adopted for the range of indices:

$$a, b, c \dots \in \{1 \dots n\}, \quad i, j, k \dots \in \{1 \dots p\}. \tag{2}$$

The function on  $(M, g, \bar{\nabla})$  endowed with orthogonal complementary distributions  $(\mathcal{D}, \mathcal{D}^\perp)$ ,

$$\bar{S}_{\text{mix}} = \frac{1}{2} \sum_{a,i} \epsilon_a \epsilon_i (g(\bar{R}_{E_a, \mathcal{E}_i} E_a, \mathcal{E}_i) + g(\bar{R}_{\mathcal{E}_i, E_a} \mathcal{E}_i, E_a)) \tag{3}$$

is called the *mixed scalar curvature w.r.t.*  $\bar{\nabla}$ . Here (and in further parts of the paper),  $\{E_a, \mathcal{E}_i\}$  is a local orthonormal frame on  $M$  adapted to  $\mathcal{D}$  and  $\mathcal{D}^\perp$ , and  $\epsilon_i = g(\mathcal{E}_i, \mathcal{E}_i) \in \{1, -1\}$ ,  $\epsilon_a = g(E_a, E_a) \in \{1, -1\}$ . We will also use the notation  $e_\mu$  and  $\epsilon_\mu = g(e_\mu, e_\mu)$  when we consider elements of the orthonormal adapted frame without distinction to which distribution they belong. In particular,

$$S_{\text{mix}} = \sum_{a,i} \epsilon_a \epsilon_i g(R_{E_a, \mathcal{E}_i} E_a, \mathcal{E}_i), \tag{4}$$

is the *mixed scalar curvature*, see [10]. Definitions (3) and (4) do not depend on the order of distributions and on the choice of a local frame.

In [5], the  $\mathcal{K}$ -sectional curvature of a symmetric  $(1, 2)$ -tensor  $\mathcal{K}$  (on any subspace of a vector space endowed with a scalar product and a cubic form) was introduced and applied to statistical connections. It is defined for any  $X, Y \in \mathfrak{X}_M$  by the following formula:  $K(X, Y) = g([\mathcal{K}_X, \mathcal{K}_Y] Y, X)$ . This way, for any  $(1, 2)$ -tensor  $\mathcal{K}$  on a pseudo-Riemannian manifold  $(M, g)$  endowed with

a pair  $(\mathcal{D}, \mathcal{D}^\perp)$ , we introduce the following invariant, called the *mixed scalar  $\mathcal{K}$ -curvature*:

$$S_{\text{mix}, \mathcal{K}} := \frac{1}{2} \sum_{a,i} \epsilon_a \epsilon_i (g([\mathcal{K}_i, \mathcal{K}_a] E_a, \mathcal{E}_i) + g([\mathcal{K}_a, \mathcal{K}_i] \mathcal{E}_i, E_a)). \tag{5}$$

If  $\mathcal{K}_X$  ( $X \in TM$ ) is either symmetric or anti-symmetric then (5) reads

$$S_{\text{mix}, \mathcal{K}} = \sum_{a,i} \epsilon_a \epsilon_i g([\mathcal{K}_i, \mathcal{K}_a] E_a, \mathcal{E}_i).$$

Observe that the *mixed scalar  $\mathfrak{T}$ -curvature* (associated with contorsion tensor) can be recognized as a part of  $\bar{S}_{\text{mix}}$ , see (3). Indeed, using (1), one can decompose  $\bar{S}_{\text{mix}}$  of (3) into the sum:

$$\bar{S}_{\text{mix}} = S_{\text{mix}} + S_{\text{mix}, \mathfrak{T}} + \frac{1}{2} Q, \tag{6}$$

where

$$Q = \sum_{a,i} \epsilon_a \epsilon_i (g((\nabla_i \mathfrak{T})_a E_a, \mathcal{E}_i) - g((\nabla_a \mathfrak{T})_i E_a, \mathcal{E}_i) + g((\nabla_a \mathfrak{T})_i \mathcal{E}_i, E_a) - g((\nabla_i \mathfrak{T})_a \mathcal{E}_i, E_a)). \tag{7}$$

Due to (6) and (7), we will consider  $\bar{S}_{\text{mix}}$  as a function of a  $(1, 2)$ -tensor  $\mathfrak{T}$ .

We study  $(1, 2)$ -tensors  $\mathfrak{T}$  on  $(M, g)$ , which are critical for the functionals

$$\bar{J}_{\text{mix}, \Omega} : \mathfrak{T} \mapsto \int_{\Omega} \bar{S}_{\text{mix}}(\mathfrak{T}) \, d \text{vol}_g, \quad I_{\text{mix}, \Omega} : \mathfrak{T} \mapsto \int_{\Omega} S_{\text{mix}, \mathfrak{T}} \, d \text{vol}_g. \tag{8}$$

The integral is taken over  $M$  if it converges; otherwise, one integrates over arbitrarily large, relatively compact domain  $\Omega \subset M$ . We consider arbitrary variations  $\bar{\nabla}^t = \nabla + \mathfrak{T}^t$ ,

$$\mathfrak{T}^t, \quad \mathfrak{T}^0 = \mathfrak{T}, \quad |t| < \epsilon, \tag{9}$$

and variations corresponding to distinguished classes of connections (e.g. metric and statistical), while  $\Omega$  contains supports of infinitesimal variations  $\partial_t \mathfrak{T}^t$ . In such cases, the Divergence Theorem states that  $\int_M (\text{div } \xi) \, d \text{vol}_g = 0$  when  $\xi \in \mathfrak{X}_M$  is supported in  $\Omega$ .

In the paper, we deduce the Euler–Lagrange equations of (8)<sub>1</sub> and (8)<sub>2</sub> and examine critical contorsion tensors  $\mathfrak{T}$  (and their connections) in general and in distinguished classes. In Sect. 2, we prove (Theorems 1, 2) that  $\mathfrak{T}$  is critical for (8)<sub>1</sub> if and only if both distributions are totally umbilical with respect to the Levi-Civita connection and  $\mathfrak{T}$  obeys certain linear system; for statistical connections the geometrical condition is integrability of distributions instead of their umbilicity (Theorem 3). In Sect. 3, we prove (Theorem 4) that a tensor  $\mathfrak{T}$  is critical for (8)<sub>2</sub> if and only if it obeys certain linear system, and for adapted connections the necessary geometric conditions are integrability and minimality of distributions. These results show how the Riemannian geometry of an almost-product manifold restricts existence of linear connections critical for (8)<sub>1</sub> and (8)<sub>2</sub>. As an example, in Sect. 4 we discuss double-twisted product of metric-affine manifolds, where above conditions can be realized. Throughout

the paper everything (manifolds, distributions, etc.) is assumed to be smooth and oriented.

## 2. The Mixed Scalar Curvature of Connection

### 2.1. Preliminaries

We will define several geometric objects for the almost-product structure  $(M, \mathcal{D}, \mathcal{D}^\perp, g)$ . Let  $\mathfrak{X}_M$  (resp.,  $\mathfrak{X}_{\mathcal{D}}$ ) be the module over  $C^\infty(M)$  of all vector fields on  $M$  (resp. all vector fields with values in  $\mathcal{D}$ ). A *metric-affine space* is a manifold  $M$  endowed with a metric  $g$  of certain signature and a linear connection  $\bar{\nabla}$ . A connection  $\bar{\nabla} : \mathfrak{X}_M \times \mathfrak{X}_M \rightarrow \mathfrak{X}_M$  on  $TM$  has the properties:

$$\bar{\nabla}_{fX_1+X_2}Y = f\bar{\nabla}_{X_1}Y + \bar{\nabla}_{X_2}Y, \quad \bar{\nabla}_X(fY + Z) = X(f)Y + f\bar{\nabla}_X Y + \bar{\nabla}_X Z.$$

A unique metric and torsion free connection on  $(M, g)$  is the Levi-Civita connection  $\nabla$ , it is given by

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X).$$

Let  $T^\perp, h^\perp : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}^\perp$  be the integrability tensor and the second fundamental form (w.r.t.  $\nabla$ ) of  $\mathcal{D}$ , respectively,

$$T^\perp(X, Y) = (1/2)[X, Y]^\perp, \quad h^\perp(X, Y) = (1/2)(\nabla_X Y + \nabla_Y X)^\perp.$$

The mean curvature vector of  $\mathcal{D}$  is  $H^\perp = \text{Tr}_g h^\perp = \sum_a \epsilon_a h^\perp(E_a, E_a)$ . Let  $g^\top = g|_{\mathcal{D} \times \mathcal{D}}$ . A distribution  $\mathcal{D}$  is called *totally umbilical*, *harmonic*, or *totally geodesic*, if  $h^\perp = \frac{1}{n}H^\perp g^\top$ ,  $H^\perp = 0$ , or  $h^\perp = 0$ , respectively. Similarly, we define  $T, h : \mathcal{D}^\perp \times \mathcal{D}^\perp \rightarrow \mathcal{D}$  by

$$T(X, Y) = (1/2)[X, Y]^\top, \quad h(X, Y) = (1/2)(\nabla_X Y + \nabla_Y X)^\top,$$

and  $H = \text{Tr}_g h = \sum_i \epsilon_i h(\mathcal{E}_i, \mathcal{E}_i)$ .

For  $Z \in \mathcal{D}^\perp, X, Y \in \mathcal{D}$ , the shape operator  $A_Z$  (of  $\mathcal{D}$  w.r.t.  $Z$ ) and the operator  $T_Z^\sharp$  are defined by

$$g(A_Z(X), Y) = g(h^\perp(X, Y), Z), \quad g(T_Z^\sharp(X), Y) = g(T^\perp(X, Y), Z).$$

Similarly, for  $X \in \mathcal{D}$  and  $W, Z \in \mathcal{D}^\perp$  we have  $g(A_X(W), Z) = g(h(W, Z), X)$  and  $g(T_X^\sharp(W), Z) = g(T(W, Z), X)$ . We will use the following convention for various tensors:  $T_i^\sharp := T_{\mathcal{E}_i}^\sharp, A_i := A_{\mathcal{E}_i}, \mathfrak{I}_i = \mathfrak{I}_{\mathcal{E}_i}$  etc. Using the adapted orthonormal frame, we calculate

$$\begin{aligned} \langle h^\perp, h^\perp \rangle &= \sum_{a,b} \epsilon_a \epsilon_b g(h^\perp(E_a, E_b), h^\perp(E_a, E_b)), \\ \langle T^\perp, T^\perp \rangle &= \sum_{a,b} \epsilon_a \epsilon_b g(T^\perp(E_a, E_b), T^\perp(E_a, E_b)). \end{aligned}$$

The following formula, see [8, 10]:

$$S_{\text{mix}} = g(H, H) - \langle h, h \rangle + \langle T, T \rangle + g(H^\perp, H^\perp) - \langle h^\perp, h^\perp \rangle + \text{div}(H + H^\perp), \quad (10)$$

has found many applications.

Given  $\mathfrak{T}$ , define the (1,2)-tensor  $\mathfrak{T}^*$  by

$$g(\mathfrak{T}_X^* Y, Z) = g(\mathfrak{T}_X Z, Y), \quad X, Y, Z \in \mathfrak{X}_M.$$

Two partial traces of  $\mathfrak{T}$  (and similarly, of  $\mathfrak{T}^*$ ) are defined by

$$\text{Tr}^\top \mathfrak{T} := \sum_a \epsilon_a \mathfrak{T}_a E_a, \quad \text{Tr}^\perp \mathfrak{T} := \sum_i \epsilon_i \mathfrak{T}_i \mathcal{E}_i.$$

Define  $\mathfrak{T}_i^\top : \mathcal{D} \rightarrow \mathcal{D}$  by  $\mathfrak{T}_i^\top X := (\mathfrak{T}_i(X^\top))^\top$ . Similarly, put  $\mathfrak{T}_a^\perp X = (\mathfrak{T}_a(X^\perp))^\perp$  for  $\mathfrak{T}_a^\perp : \mathcal{D}^\perp \rightarrow \mathcal{D}^\perp$ . We have the following decomposition into symmetric and antisymmetric parts:

$$\mathfrak{T}_i^\top = (\mathfrak{T}_i^\top + \mathfrak{T}_i^{\top*})/2 + (\mathfrak{T}_i^\top - \mathfrak{T}_i^{\top*})/2.$$

Set  $\text{Id}_\mathcal{D}(X) = X^\top$  for all  $X \in \mathfrak{X}_M$ .

### 2.2. Arbitrary Variations

**Theorem 1.** *Let  $(M, g, \bar{\nabla})$  be a metric-affine manifold endowed with a non-degenerate distribution  $\mathcal{D}$ . A tensor  $\mathfrak{T}$  is critical for action (8)<sub>1</sub> if and only if  $\mathcal{D}$  and its orthogonal distribution  $\mathcal{D}^\perp$  are both **totally umbilical** and  $\mathfrak{T}$  satisfies the linear system:*

$$(\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i)^\top = -2T(\mathcal{E}_i, \mathcal{E}_j), \tag{11a}$$

$$(\text{Tr}^\perp \mathfrak{T}^*)^\top = H = -(\text{Tr}^\perp \mathfrak{T})^\top, \quad \text{for } n > 1, \tag{11b}$$

$$\mathfrak{T}_i^\top - (\mathfrak{T}_i^\top)^* = 2T_i^\sharp, \tag{11c}$$

$$\mathfrak{T}_i^\top + (\mathfrak{T}_i^\top)^* - g(\text{Tr}^\perp \mathfrak{T} + \text{Tr}^\perp \mathfrak{T}^*, \mathcal{E}_i) \text{Id}_\mathcal{D} = 0, \tag{11d}$$

$$(\text{Tr}^\perp \mathfrak{T} - \text{Tr}^\perp \mathfrak{T}^*)^\perp = \frac{2(n-1)}{n} H^\perp, \tag{11e}$$

$$(\mathfrak{T}_a E_b + \mathfrak{T}_b^* E_a)^\perp = -2T^\perp(E_a, E_b), \tag{11f}$$

$$(\text{Tr}^\top \mathfrak{T}^*)^\perp = H^\perp = -(\text{Tr}^\top \mathfrak{T})^\perp, \quad \text{for } p > 1, \tag{11g}$$

$$\mathfrak{T}_a^\perp - (\mathfrak{T}_a^\perp)^* = 2T_a^\sharp, \tag{11h}$$

$$\mathfrak{T}_a^\perp + (\mathfrak{T}_a^\perp)^* - g(\text{Tr}^\top \mathfrak{T} + \text{Tr}^\top \mathfrak{T}^*, E_a) \text{Id}_{\mathcal{D}^\perp} = 0, \tag{11i}$$

$$(\text{Tr}^\top \mathfrak{T} - \text{Tr}^\top \mathfrak{T}^*)^\top = \frac{2(p-1)}{p} H. \tag{11j}$$

*Proof.* For the terms of  $Q$ , see (7), we have

$$\begin{aligned} & \sum_{a,i} \epsilon_a \epsilon_i g((\nabla_i \mathfrak{T})_a E_a, \mathcal{E}_i) = \text{div}((\text{Tr}^\top \mathfrak{T})^\perp) + g(\text{Tr}^\top \mathfrak{T}, H^\perp - H) \\ & \quad + \sum_{a,i,j} \epsilon_a \epsilon_i \epsilon_j [g(\mathfrak{T}_j E_a, \mathcal{E}_i) + g(\mathfrak{T}_a \mathcal{E}_j, \mathcal{E}_i)] g((A_a + T_a^\sharp) \mathcal{E}_i, \mathcal{E}_j), \\ & \sum_{a,i} \epsilon_a \epsilon_i g((\nabla_a \mathfrak{T})_i E_a, \mathcal{E}_i) = \text{div}((\text{Tr}^\perp \mathfrak{T}^*)^\top) + g(\text{Tr}^\perp \mathfrak{T}^*, H - H^\perp) \\ & \quad + \sum_{a,b,i} \epsilon_a \epsilon_b \epsilon_i [g(\mathfrak{T}_i E_a, E_b) + g(\mathfrak{T}_b E_a, \mathcal{E}_i)] g((A_i + T_i^\sharp) E_a, E_b). \end{aligned}$$

From the above formula, its dual (with respect to interchanging distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$ ) and (7) we obtain

$$\begin{aligned}
 Q &= \operatorname{div} \left( (\operatorname{Tr}^\top \mathfrak{T})^\perp - (\operatorname{Tr}^\perp \mathfrak{T}^*)^\top + (\operatorname{Tr}^\perp \mathfrak{T})^\top - (\operatorname{Tr}^\top \mathfrak{T}^*)^\perp \right) + Q_1, \quad \text{where} \\
 Q_1 &= \sum_{a,i} \epsilon_a \epsilon_i \left[ g(\mathfrak{T}_i E_a + \mathfrak{T}_i^* E_a + \mathfrak{T}_a \mathcal{E}_i + \mathfrak{T}_a^* \mathcal{E}_i, (A_a - T_a^\sharp) \mathcal{E}_i + (A_i - T_i^\sharp) E_a) \right] \\
 &\quad + g(\operatorname{Tr}^\top \mathfrak{T} - \operatorname{Tr}^\perp \mathfrak{T} + \operatorname{Tr}^\perp \mathfrak{T}^* - \operatorname{Tr}^\top \mathfrak{T}^*, H^\perp - H). \tag{12}
 \end{aligned}$$

Since  $S_{\text{mix}}$  does not depend on  $\mathfrak{T}$ ,

$$\frac{d}{dt} \bar{J}_{\text{mix},\Omega}(\mathfrak{T}^t) = \frac{d}{dt} I_{\text{mix},\Omega}(\mathfrak{T}^t) + \frac{1}{2} \frac{d}{dt} \int_\Omega Q_1(\mathfrak{T}^t) \, d \operatorname{vol}_g.$$

Set  $S = \partial_t \mathfrak{T}^t|_{t=0}$  for one-parameter family (9) of (1, 2)-tensors. We have

$$\begin{aligned}
 \partial_t g(\mathfrak{T}_i \mathfrak{T}_a E_a, \mathcal{E}_i) &= \sum_b \epsilon_b g(S_i E_b, \mathcal{E}_i) g(\mathfrak{T}_a E_a, E_b) \\
 &\quad + \sum_j \epsilon_j g(S_i \mathcal{E}_j, \mathcal{E}_i) g(\mathfrak{T}_a E_a, \mathcal{E}_j) \\
 &\quad + \sum_b \epsilon_b g(S_a E_a, E_b) g(E_b, \mathfrak{T}_i^* \mathcal{E}_i) \\
 &\quad + \sum_j \epsilon_j g(S_a E_a, \mathcal{E}_j) g(\mathcal{E}_j, \mathfrak{T}_i^* \mathcal{E}_i), \tag{13a}
 \end{aligned}$$

$$\begin{aligned}
 -\partial_t g(\mathfrak{T}_a \mathfrak{T}_i E_a, \mathcal{E}_i) &= -\sum_b \epsilon_b g(S_a E_b, \mathcal{E}_i) g(\mathfrak{T}_i E_a, E_b) \\
 &\quad - \sum_j \epsilon_j g(S_a \mathcal{E}_j, \mathcal{E}_i) g(\mathfrak{T}_i E_a, \mathcal{E}_j) \\
 &\quad - \sum_b \epsilon_b g(S_i E_a, E_b) g(\mathfrak{T}_a E_b, \mathcal{E}_i) \\
 &\quad - \sum_j \epsilon_j g(S_i E_a, \mathcal{E}_j) g(\mathfrak{T}_a \mathcal{E}_j, \mathcal{E}_i). \tag{13b}
 \end{aligned}$$

Since  $Q_1$  is linear in  $\mathfrak{T}$ , to get its  $t$ -derivatives one should replace  $\mathfrak{T}$  by  $S$  in (12)<sub>2</sub>. Using this together with (13a)–(13b) and their dual equations, and removing integrals of divergences of compactly supported vector fields, we get

$$\begin{aligned}
 &\left( \frac{d}{dt} \int_\Omega \bar{S}_{\text{mix}}(\mathfrak{T}^t) \right) \Big|_{t=0} \\
 &= \frac{1}{2} \int_\Omega \left\{ \sum g(S_a E_b, E_c) (g(\operatorname{Tr}^\perp \mathfrak{T}^* - H, E_c) \delta_{ab} + g(H + \operatorname{Tr}^\perp \mathfrak{T}, E_b) \delta_{ac}) \right. \\
 &\quad + \sum g(S_a E_b, \mathcal{E}_i) (g(\operatorname{Tr}^\perp \mathfrak{T}^* + H^\perp, \mathcal{E}_i) \delta_{ab} - g((A_i - T_i^\sharp) E_a, E_b) - g(\mathfrak{T}_i E_a, E_b)) \\
 &\quad + \sum g(S_a \mathcal{E}_i, E_b) (g(\operatorname{Tr}^\perp \mathfrak{T} - H^\perp, \mathcal{E}_i) \delta_{ab} + g((A_i + T_i^\sharp) E_b, E_a) - g(\mathfrak{T}_i E_b, E_a)) \\
 &\quad + \sum g(S_a \mathcal{E}_i, \mathcal{E}_j) (g((A_a - T_a^\sharp) \mathcal{E}_i, \mathcal{E}_j) - g((A_a + T_a^\sharp) \mathcal{E}_i, \mathcal{E}_j) - g(\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i, E_a)) \\
 &\quad \left. + \sum g(S_i \mathcal{E}_j, \mathcal{E}_k) (g(\operatorname{Tr}^\top \mathfrak{T}^* - H^\perp, \mathcal{E}_k) \delta_{ij} + g(H^\perp + \operatorname{Tr}^\top \mathfrak{T}, \mathcal{E}_j)) \delta_{ik} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum g(S_i \mathcal{E}_j, E_a) (g(\text{Tr}^\top \mathfrak{T}^* + H, E_a) \delta_{ij} - g((A_a + T_a^\sharp) \mathcal{E}_j, \mathcal{E}_i) - g(\mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_j)) \\
 & + \sum g(S_i E_a, \mathcal{E}_j) (g(\text{Tr}^\top \mathfrak{T} - H, E_a) \delta_{ij} + g((A_a + T_a^\sharp) \mathcal{E}_j, \mathcal{E}_i) - g(\mathfrak{T}_a \mathcal{E}_j, \mathcal{E}_i)) \\
 & + \sum g(S_i E_a, E_b) (g((A_i - T_i^\sharp) E_a, E_b) \\
 & - g((A_i + T_i^\sharp) E_a, E_b) - g(\mathfrak{T}_a E_b + \mathfrak{T}_b^* E_a, \mathcal{E}_i)) \}, \tag{14}
 \end{aligned}$$

where both integrals are with respect to the volume form  $\text{dvol}_g$ , all sums are taken over repeated indices and factors  $\epsilon_\mu$  are omitted. Since no further assumptions are made about  $S$  or  $\mathfrak{T}$ , all the components  $g(S_\mu e_\lambda, e_\rho)$  are independent and the above formula gives rise to the following Euler–Lagrange equations:

$$g(\text{Tr}^\perp \mathfrak{T}^* - H, E_c) \delta_{ab} + g(H + \text{Tr}^\perp \mathfrak{T}, E_b) \delta_{ac} = 0, \tag{15a}$$

$$g(\text{Tr}^\top \mathfrak{T}^* - H^\perp, \mathcal{E}_k) \delta_{ij} + g(H^\perp + \text{Tr}^\top \mathfrak{T}, \mathcal{E}_j) \delta_{ik} = 0, \tag{15b}$$

$$g(\text{Tr}^\perp \mathfrak{T}^* + H^\perp, \mathcal{E}_i) \delta_{ab} - g((A_i - T_i^\sharp) E_a, E_b) - g(\mathfrak{T}_i E_a, E_b) = 0, \tag{15c}$$

$$g(\text{Tr}^\top \mathfrak{T}^* + H, E_a) \delta_{ij} - g((A_a - T_a^\sharp) \mathcal{E}_i, \mathcal{E}_j) - g(\mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_j) = 0, \tag{15d}$$

$$g(\text{Tr}^\perp \mathfrak{T} - H^\perp, \mathcal{E}_i) \delta_{ab} + g((A_i + T_i^\sharp) E_b, E_a) - g(\mathfrak{T}_i E_b, E_a) = 0, \tag{15e}$$

$$g(\text{Tr}^\top \mathfrak{T} - H, E_a) \delta_{ij} + g((A_a + T_a^\sharp) \mathcal{E}_j, \mathcal{E}_i) - g(\mathfrak{T}_a \mathcal{E}_j, \mathcal{E}_i) = 0, \tag{15f}$$

$$2g(T_a^\sharp \mathcal{E}_i, \mathcal{E}_j) + g(\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i, E_a) = 0, \tag{15g}$$

$$2g(T_i^\sharp E_a, E_b) + g(\mathfrak{T}_a E_b + \mathfrak{T}_b^* E_a, \mathcal{E}_i) = 0. \tag{15h}$$

To simplify (15a)–(15h), first we consider (15a). It may yield three equations, in the following cases:

1.  $a = b = c$ :

$$g(\text{Tr}^\perp \mathfrak{T} + \text{Tr}^\perp \mathfrak{T}^*, E_a) = 0. \tag{16}$$

From here it follows that  $(\text{Tr}^\perp \mathfrak{T} + \text{Tr}^\perp \mathfrak{T}^*)^\top = 0$ .

2.  $a = b \neq c$  (note that this requires  $n > 1$ ):

$$g(\text{Tr}^\perp \mathfrak{T}^* - H, E_c) = 0.$$

From this we obtain (11b)<sub>1</sub>.

3.  $a = c \neq b$  (note that this requires  $n > 1$ ):

$$g(H + \text{Tr}^\perp \mathfrak{T}, E_b) = 0.$$

From this we obtain (11b)<sub>2</sub>. Note that case  $a \neq b \neq c \neq a$  gives no new conditions.

Next, we consider (15c), which can be presented as:

$$g(H^\perp + \text{Tr}^\perp \mathfrak{T}^*, \mathcal{E}_i) \text{Id}_{\mathcal{D}} - A_i + T_i^\sharp - \mathfrak{T}_i^\top = 0. \tag{17}$$

Then we examine (15e), which can be written as

$$g(\text{Tr}^\perp \mathfrak{T} - H^\perp, \mathcal{E}_i) \text{Id}_{\mathcal{D}} + A_i + T_i^\sharp - \mathfrak{T}_i^\top = 0. \tag{18}$$

Finally, we consider (15g):

$$2g(T_a^\sharp \mathcal{E}_i, \mathcal{E}_j) + g(\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i, E_a) = 0,$$

which can be presented as (11a) and implies (16). Equations (15b), (15d), (15f) and (15h) are dual to the ones considered above.

The antisymmetric part of (17), which is the same as antisymmetric part of (18), yields (11c), while the symmetric part of (17) is the following:

$$g(H^\perp + \text{Tr}^\perp \mathfrak{T}^*, \mathcal{E}_i) \text{Id}_{\mathcal{D}} - A_i - (\mathfrak{T}_i^\top + \mathfrak{T}_i^{\top*})/2 = 0. \tag{19}$$

On the other hand, the symmetric part of (18) reads as

$$g(\text{Tr}^\perp \mathfrak{T} - H^\perp, \mathcal{E}_i) \text{Id}_{\mathcal{D}} + A_i - (\mathfrak{T}_i^\top + \mathfrak{T}_i^{\top*})/2 = 0. \tag{20}$$

Taking the sum of (19) and (20), we obtain (11d), while taking the difference of those equations yields

$$2(h^\perp - H^\perp g^\top) + (\text{Tr}^\perp \mathfrak{T} - \text{Tr}^\perp \mathfrak{T}^*)^\perp g^\top = 0. \tag{21}$$

Equation (21) yields that  $h^\perp$  is proportional to  $g^\top$ , and so  $h^\perp = \frac{1}{n} H^\perp g^\top$  follows. From the trace of (21) we obtain (11e). In this way, we obtain the first half, (11a)–(11e), of the Euler–Lagrange equations. The second half, (11f)–(11j), follows by interchanging the roles of  $\mathcal{D}$  and  $\mathcal{D}^\perp$ .  $\square$

*Remark 1.* Solutions of (11a)–(11j) form an affine subspace in the linear space of all tensors  $\mathfrak{T}$ . Among all solutions there exists one with minimal norm, whose properties might be interesting.

**Corollary 1.** *Let  $n + p > 2$ , then  $\mathfrak{T} = 0$  (corresponding to the Levi-Civita connection) is critical for action (8)<sub>1</sub> if and only if both distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$  are totally geodesic and integrable.*

*Proof.* Let  $\mathfrak{T} = 0$  in (11a)–(11j) and use (11c) to get  $T^\perp = 0$ , then either (11e) or (11g) to get  $H^\perp = 0$ , which together with  $h^\perp = \frac{1}{n} H^\perp g^\top$  yields  $h^\perp = 0$ . Then use the dual equations to get  $T = 0 = h^\perp$ .  $\square$

**Proposition 1.** *All tensors critical for action (8)<sub>1</sub> are parameterized by  $p^3 + p^2n + n^3 + n^2p - 2n + 2n\delta_{n1} - 2p + 2p\delta_{p1}$  functions on  $M$ , that correspond to independent components of  $\mathfrak{T}$ .*

*Proof.* First observe that  $\mathfrak{T}_i$  and  $\mathfrak{T}_a$  are independent, so we can consider two halves of the Euler–Lagrange equations separately. Components  $\mathfrak{T}_i^\perp$  fully determine components  $(\mathfrak{T}_i^\perp)^*$  and are restricted only for  $p > 1$ , by  $p$  scalar equations (11e) (for  $p = 1$  (11e) yields  $H^\perp = 0$  and no restrictions on  $\mathfrak{T}_i$ ). All components  $\mathfrak{T}_i^\top$  are fully determined by  $\mathfrak{T}_i^\perp$  according to (11c) and (11d) (antisymmetric and symmetric part of  $\mathfrak{T}_i^\top$ , respectively). Components  $(\mathfrak{T}_i \mathcal{E}_j)^\top$  and  $(\mathfrak{T}_i^* \mathcal{E}_j)^\top$  are restricted by  $p^2n$  scalar equations (11a) and for  $n > 1$ , additionally by  $n$  equations (11b)<sub>1</sub>. Note that from (11a) it follows that  $(\mathfrak{T}_i \mathcal{E}_i)^\top = (\mathfrak{T}_i^* \mathcal{E}_i)^\top$ ; hence, (11b)<sub>2</sub> yields no new restrictions.

In total, we have  $p^3$  components of  $\mathfrak{T}_i^\perp$  restricted for  $p > 1$  by  $p$  scalar, linear equations;  $2p^2n$  components  $(\mathfrak{T}_i \mathcal{E}_j)^\top$  and  $(\mathfrak{T}_i^* \mathcal{E}_j)^\top$ , which are restricted by  $p^2n$  scalar, linear equations and for  $n > 1$  by  $p^2n + n$  scalar, linear equations.



For the second half of the Euler–Lagrange equations we obtain the dual result (with  $n$  and  $p$  interchanged).  $\square$

**Proposition 2.** *Let  $n + p > 2$ , then a critical point of the action  $(8)_1$  is not an extremal point (also for variations in the subspaces of tensors  $\mathfrak{T}$  corresponding to metric connections and corresponding to statistical connections).*

*Proof.* Let  $\mathfrak{T}^t = \mathfrak{T} + t \cdot S$ . Then the only part of  $\bar{S}_{\text{mix}}$  that is quadratic in  $t$  comes from the difference  $g(\mathfrak{T}_i^t \mathfrak{T}_a^t E_a, \mathcal{E}_i) - g(\mathfrak{T}_a^t \mathfrak{T}_i^t E_a, \mathcal{E}_i)$  and its dual w.r.t. interchanging the distributions. Hence,  $\bar{S}_{\text{mix}}(\mathfrak{T}^t) = O(t) + t^2 \sigma$ , where

$$\sigma = \sum_{a,i} \epsilon_a \epsilon_i (g([S_i, S_a] E_a, \mathcal{E}_i) + g([S_a, S_i] \mathcal{E}_i, E_a)). \tag{22}$$

Let  $p \geq 2$ , and assume that we have  $S_{E_1} E_1 = \mathcal{E}_2$ ,  $S_{E_1} \mathcal{E}_1 = 0$ ,  $S_{E_1} \mathcal{E}_2 = -E_1$ , and if  $n > 1$  then for  $b > 1$  we have  $S_{E_b} = 0$ . Assume further that for  $i = 1$  we have  $S_{\mathcal{E}_1} E_1 = 0$ ,  $S_{\mathcal{E}_1} \mathcal{E}_1 = \pm \mathcal{E}_2$ ,  $S_{\mathcal{E}_1} \mathcal{E}_2 = \mp \mathcal{E}_1$ , and for  $j > 1$  we have  $S_{\mathcal{E}_j} = 0$ . Then we have

$$\begin{aligned} \sigma &= 2 \sum_{a,i} \epsilon_a \epsilon_i (g(S_i E_a, S_a \mathcal{E}_i) - g(S_a E_a, S_i \mathcal{E}_i)) \\ &= 2 g(E_1, E_1) g(\mathcal{E}_1, \mathcal{E}_1) g(\mathcal{E}_2, \pm \mathcal{E}_2). \end{aligned}$$

Hence,  $\sigma$  is neither positive definite, nor negative definite. This proof can be used also for variation among metric connections, because  $S$  defined above has all necessary symmetries.

For variation in the subspaces of tensors  $\mathfrak{T}$  corresponding to statistical connections the  $(1, 2)$ -tensor  $S$  is symmetric in all its indices; hence,

$$\begin{aligned} \sigma &= 2 \sum_{a,i} \epsilon_a \epsilon_i (g(S_a E_a, S_i \mathcal{E}_i) - g(S_i E_a, S_a \mathcal{E}_i)) \\ &= 2 \sum_{a,i} \epsilon_a \epsilon_i g(S_a E_a, S_i \mathcal{E}_i) - 2 \sum_{a,b,i} \epsilon_a \epsilon_b \epsilon_i g(S_a E_b, \mathcal{E}_i)^2 \\ &\quad - 2 \sum_{a,i,j} \epsilon_a \epsilon_i \epsilon_j g(S_a \mathcal{E}_i, \mathcal{E}_j)^2. \end{aligned}$$

Since for  $n + p > 2$  not every component of  $S$  is determined by  $\text{Tr}_{\mathcal{D}} S$  and  $\text{Tr}_{\mathcal{D}^\perp} S$ , we see that again  $\sigma$  is neither positive definite, nor negative definite.  $\square$

### 2.3. Variations Corresponding to Metric Connections

Let us examine the case when  $\mathfrak{T}$  corresponds to a *metric connection*, i.e.,  $\bar{\nabla} = \nabla + \mathfrak{T}$  preserves the metric:  $\bar{\nabla} g = 0$ . Then we have the following symmetry:

$$\mathfrak{T}_X^* = -\mathfrak{T}_X. \tag{23}$$

Using (23), we obtain

$$\begin{aligned} &g((\nabla_i \mathfrak{T})_a E_a, \mathcal{E}_i) - g((\nabla_a \mathfrak{T})_i E_a, \mathcal{E}_i) + g([\mathfrak{T}_i, \mathfrak{T}_a] E_a, \mathcal{E}_i) \\ &= g((\nabla_a \mathfrak{T})_i \mathcal{E}_i, E_a) - g((\nabla_i \mathfrak{T})_a \mathcal{E}_i, E_a) + g([\mathfrak{T}_a, \mathfrak{T}_i] \mathcal{E}_i, E_a). \end{aligned}$$

Considering a variation  $\mathfrak{T}^t$  with  $S = \partial_t \mathfrak{T}|_{t=0}$  and differentiating (23), while keeping the metric  $g$  fixed, leads to the following condition:

$$g(S_X Y, Z) + g(S_X Z, Y) = 0. \tag{24}$$

Also, the curvature tensor  $\bar{R}$  of a metric connection  $\bar{\nabla}$  has the same symmetries as  $R$ , its sectional curvature  $\bar{K}(X, Y)$  is well defined and we can interpret the mixed scalar curvature as the sum

$$\bar{S}_{\text{mix}} = \sum_{a,i} \bar{K}(E_a, \mathcal{E}_i).$$

**Theorem 2.** *Let  $(M, g)$  be a pseudo-Riemannian manifold endowed with orthogonal complementary non-degenerate distributions  $(\mathcal{D}, \mathcal{D}^\perp)$ . Then a tensor  $\mathfrak{T}$  obeying (23) is critical for  $(8)_1$  in the subspace of tensors obeying (23) if and only if  $(\mathcal{D}, \mathcal{D}^\perp)$  are both **totally umbilical** and  $\mathfrak{T}$  satisfies*

$$(\mathfrak{T}_j \mathcal{E}_i - \mathfrak{T}_i \mathcal{E}_j)^\top = 2T(\mathcal{E}_i, \mathcal{E}_j), \tag{25a}$$

$$\mathfrak{T}_i^\top = T_i^\sharp, \tag{25b}$$

$$(\text{Tr}^\perp \mathfrak{T})^\perp = \frac{n-1}{n} H^\perp, \tag{25c}$$

$$(\text{Tr}^\perp \mathfrak{T})^\top = -H \quad \text{for } n > 1, \tag{25d}$$

$$(\mathfrak{T}_b E_a - \mathfrak{T}_a E_b)^\perp = 2T^\perp(E_a, E_b), \tag{25e}$$

$$\mathfrak{T}_a^\perp = T_a^\sharp, \tag{25f}$$

$$(\text{Tr}^\top \mathfrak{T})^\top = \frac{p-1}{p} H, \tag{25g}$$

$$(\text{Tr}^\top \mathfrak{T})^\perp = -H^\perp \quad \text{for } p > 1. \tag{25h}$$

*Proof.* By (24) components of  $S$  in (14) are no longer independent and from (23) we obtain  $\mathfrak{T}^* = -\mathfrak{T}$ , which leads to the Euler–Lagrange equations (25a)–(25h).  $\square$

Assuming (23) in (11a)–(11e) yields equations equivalent to (25a)–(25h). This implies the following.

**Corollary 2.** *Any tensor  $\mathfrak{T}$  obeying (23) and critical for action  $(8)_1$  w.r.t. variations restricted only to the space of tensors obeying (23) is also critical w.r.t. arbitrary variations of  $\mathfrak{T}$ .*

*Remark 2.* Instead of using  $\nabla$ , the Euler–Lagrange equations (25a)–(25h) can be presented in terms of extrinsic geometry of the metric connection  $\bar{\nabla} = \nabla + \mathfrak{T}$ . For example, the second fundamental form  $\bar{h}^\perp$  of  $\mathcal{D}$  w.r.t.  $\bar{\nabla}$  is given by

$$\bar{h}^\perp(X, Y) = h^\perp(X, Y) + \frac{1}{2} (\mathfrak{T}_X Y + \mathfrak{T}_Y X)^\perp, \quad X, Y \in \mathfrak{X}_{\mathcal{D}},$$

and the mean curvature vector w.r.t.  $\bar{\nabla}$  is  $\bar{H}^\perp = H^\perp + (\text{Tr}^\top \mathfrak{T})^\perp$ . By the above and (25a)–(25h), the distributions are no longer totally umbilical w.r.t.  $\bar{\nabla}$  if  $n, p > 1$ , but are minimal, i.e.,  $\bar{H}^\perp = 0 = \bar{H}$ .

### 2.4. Variations Corresponding to Statistical Connections

Let us examine the case when  $\mathfrak{T}$  corresponds to a *statistical connection*, i.e.,  $\bar{\nabla} = \nabla + \mathfrak{T}$  is torsionless, with symmetric tensor  $\bar{\nabla}g$ . Then we have the following symmetries:

$$\mathfrak{T}_X Y = \mathfrak{T}_Y X, \quad \mathfrak{T}^* = \mathfrak{T}. \tag{26}$$

**Theorem 3.** *Let  $(M, g)$  be a pseudo-Riemannian manifold endowed with orthogonal complementary non-degenerate distributions  $(\mathcal{D}, \mathcal{D}^\perp)$ . Then a tensor  $\mathfrak{T}$  obeying (26) is critical for (8)<sub>1</sub> for variations in the subspace of tensors obeying (26) if and only if  $(\mathcal{D}, \mathcal{D}^\perp)$  are both **integrable** and  $\mathfrak{T}$  satisfies*

$$\begin{aligned} (\mathfrak{T}_i \mathcal{E}_j)^\top &= 0 = (\mathfrak{T}_a E_b)^\perp, & (27) \\ (\text{Tr}^\perp \mathfrak{T})^\perp &= 0 = (\text{Tr}^\top \mathfrak{T})^\top, \\ \text{if } n > 1 \text{ then } H &= 0, \quad \text{and} \quad \text{if } p > 1 \text{ then } H^\perp = 0. \end{aligned}$$

*Proof.* Note that  $\text{Tr}^\perp \mathfrak{T}^* = \text{Tr}^\perp \mathfrak{T}$  and  $\text{Tr}^\top \mathfrak{T}^* = \text{Tr}^\top \mathfrak{T}$ . Substituting (26) into (14), we find that the Euler–Lagrange equations consist of the system

$$g(\text{Tr}^\perp \mathfrak{T} - H, E_c) \delta_{ab} + g(H + \text{Tr}^\perp \mathfrak{T}, E_b) \delta_{ac} = 0, \tag{28a}$$

$$-g(T_a^\sharp \mathcal{E}_i, \mathcal{E}_j) - g(\mathfrak{T}_i \mathcal{E}_j, E_a) = 0, \tag{28b}$$

$$g(\text{Tr}^\perp \mathfrak{T}, \mathcal{E}_i) \delta_{ab} - g(T_i^\sharp E_a, E_b) - 2g(\mathfrak{T}_a E_b, \mathcal{E}_i) = 0, \tag{28c}$$

and the equations dual to the above (with interchanged roles of distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$ ), which we do not write here. From (28a) with  $a = b = c$  it follows that  $(\text{Tr}^\perp \mathfrak{T})^\top = 0$ .

For  $n > 1$ , (28a) with  $a = b \neq c$  yields additionally:  $H = 0 = (\text{Tr}^\perp \mathfrak{T})^\top$ . From (28b), we obtain

$$T(\mathcal{E}_j, \mathcal{E}_i) = (\mathfrak{T}_i \mathcal{E}_j)^\top, \tag{29}$$

but from the symmetry  $\mathfrak{T}_i \mathcal{E}_j = \mathfrak{T}_j \mathcal{E}_i$  it follows that  $T = 0$ ; hence,  $(\mathfrak{T}_i \mathcal{E}_j)^\top = 0$ .

From the dual equation we obtain that also  $\mathcal{D}$  must be integrable. Finally, from (28c) we get

$$\delta_{ab} g(\text{Tr}^\perp \mathfrak{T}, \mathcal{E}_i) - 2g(\mathfrak{T}_i E_b, E_a) + g(T_i^\sharp E_b, E_a) = 0,$$

and furthermore,

$$\delta_{ab} (\text{Tr}^\perp \mathfrak{T})^\perp - 2(\mathfrak{T}_a E_b)^\perp + T^\perp(E_b, E_a) = 0;$$

hence,  $2(\mathfrak{T}_a E_b)^\perp = \delta_{ab} (\text{Tr}^\perp \mathfrak{T})^\perp$ . Equation dual to (29) yields  $(\mathfrak{T}_a E_b)^\perp = 0$ ; thus we obtain  $(\text{Tr}^\perp \mathfrak{T})^\perp = 0$ .  $\square$

*Remark 3.* By (26)<sub>1</sub> and (27)<sub>1</sub> we get  $g(\mathfrak{T}_i E_a, \mathcal{E}_j) = 0$ ; hence,  $\mathfrak{T}_i : \mathcal{D} \rightarrow \mathcal{D}$ . By (26)<sub>2</sub> and (27)<sub>2</sub> we get  $g(\mathfrak{T}_i E_a, E_b) = 0$ ; hence,  $\mathfrak{T}_i : \mathcal{D} \rightarrow \mathcal{D}^\perp$ . By the above,  $\mathfrak{T}_i|_{\mathcal{D}} = 0$ . Similarly,  $\mathfrak{T}_a|_{\mathcal{D}^\perp} = 0$ .

The Weyl–Cartan connections  $\bar{\nabla}$ , i.e.,  $\text{Tr}(\bar{\nabla}_X g) = 0$  ( $X \in \mathfrak{X}_M$ ), have been classified in [4].

**Corollary 3.** Any tensor  $\mathfrak{T}$  obeying (26) that is critical for  $(8)_1$  in the subspace of tensors obeying (26) corresponds to a Weyl–Cartan connection  $\bar{\nabla} = \nabla + \mathfrak{T}$ .

*Proof.* It follows from Theorem 3, that  $\text{Tr}^\top \mathfrak{T} = 0$  and  $\text{Tr}^\perp \mathfrak{T} = 0$ , which – given the symmetries of  $\mathfrak{T}$  – implies  $\text{Tr}(\bar{\nabla}_X g) = 0$  for every  $X \in \mathfrak{X}_M$ .  $\square$

As expected, varying  $\mathfrak{T}$  corresponding to statistical connection in the space of tensors obeying (26) gives weaker conditions for being a critical point of  $(8)_1$  than what we get from (11a)–(11j).

### 2.5. Critical Adapted Connections

Let us examine the case when  $\mathfrak{T}$  corresponds to an *adapted connection*  $\bar{\nabla} = \nabla + \mathfrak{T}$ , i.e., see [2]:

$$\bar{\nabla}_Z X \in \mathfrak{X}_{\mathcal{D}}, \quad \bar{\nabla}_Z Y \in \mathfrak{X}_{\mathcal{D}^\perp}, \quad X \in \mathfrak{X}_{\mathcal{D}}, \quad Y \in \mathfrak{X}_{\mathcal{D}^\perp}, \quad Z \in \mathfrak{X}_M.$$

An example is the contorsion tensor  $\mathfrak{T}$  of the Schouten–Van Kampen connection,

$$\mathfrak{T}_X Y = -(\nabla_{X^\top} Y^\top)^\perp - (\nabla_{X^\perp} Y^\perp)^\top + (A_{Y^\perp} + T_{Y^\perp}^\sharp) X^\top + (A_{Y^\top} + T_{Y^\top}^\sharp) X^\perp.$$

Recall that the *Schouten–Van Kampen connection*  $\hat{\nabla}$  is defined by

$$\hat{\nabla}_X Y = \nabla_X Y - (\nabla_{X^\top} Y^\perp)^\top - (\nabla_{X^\perp} Y^\top)^\perp - (\nabla_{X^\perp} Y^\perp)^\top - (\nabla_{X^\top} Y^\top)^\perp,$$

see [2], which can be also written as

$$\begin{aligned} \hat{\nabla}_X Y &= \nabla_X Y - (\nabla_{X^\top} Y^\top)^\perp - (\nabla_{X^\perp} Y^\perp)^\top \\ &\quad + (A_{Y^\perp} + T_{Y^\perp}^\sharp) X^\top + (A_{Y^\top} + T_{Y^\top}^\sharp) X^\perp. \end{aligned}$$

It follows that  $\hat{\nabla}$  is metric and

$$\begin{aligned} \mathfrak{T}_a E_b &= -(h^\perp + T^\perp)(E_a, E_b), \quad \mathfrak{T}_i \mathcal{E}_j = -(h + T)(\mathcal{E}_i, \mathcal{E}_j), \\ \mathfrak{T}_a \mathcal{E}_i &= (A_i + T_i^\sharp) E_a, \quad \mathfrak{T}_i E_a = (A_a + T_a^\sharp) \mathcal{E}_i, \end{aligned}$$

which yields  $(\mathfrak{T}_i \mathcal{E}_j)^\perp = 0 = (\mathfrak{T}_i E_a)^\top$ .

Using these formulas in (25b), (25c) and their dual equations, we obtain the following.

**Corollary 4.** Let distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$  be both totally umbilical w.r.t.  $\nabla$ .

- (i) If  $n, p > 1$  then the contorsion tensor  $\mathfrak{T}$  of Schouten–Van Kampen connection  $\hat{\nabla}$  satisfies (25a)–(25h) if and only if distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$  are both totally geodesic and integrable.
- (ii) If  $n = 1$  and  $p > 1$  then the contorsion tensor  $\mathfrak{T}$  of  $\hat{\nabla}$  satisfies (25a)–(25h) if and only if  $\mathcal{D}^\perp$  is totally geodesic and integrable.

We shall now examine the case when  $\mathfrak{T}$  corresponds to a metric adapted connection  $\bar{\nabla} = \nabla + \mathfrak{T}$ , which is given by the formula, see [2]:

$$\bar{\nabla}_X Y = \hat{\nabla}_X Y + (\mathfrak{T}_X(Y^\top))^\top + (\mathfrak{T}_X(Y^\perp))^\perp,$$

with Schouten–Van Kampen connection  $\hat{\nabla}$  and  $\mathfrak{T}_X Y = \bar{\nabla}_X Y - \nabla_X Y$ . Its contorsion tensor  $\mathfrak{T}$  satisfies

$$\begin{aligned} \mathfrak{T}_X Y &= -(\nabla_{X^\top} Y^\top)^\perp - (\nabla_{X^\perp} Y^\perp)^\top + (A_{Y^\perp} + T_{Y^\perp}^\sharp)X^\top + (A_{Y^\top} + T_{Y^\top}^\sharp)X^\perp \\ &\quad + (\mathfrak{T}_X(Y^\top))^\top + (\mathfrak{T}_X(Y^\perp))^\perp. \end{aligned}$$

Note that components  $(\mathfrak{T}_\mu E_a)^\top$  and  $(\mathfrak{T}_\mu \mathcal{E}_i)^\perp$  are not restricted by the definition of adapted connection.

**Corollary 5.** *Let distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$  be totally umbilical. A tensor  $\mathfrak{T}$  corresponding to an **adapted metric connection** obeys (25a)–(25h) if and only if it obeys (25b), (25c) and their dual (25f), (25g).*

Note that components  $(\mathfrak{T}_b E_a)^\top$  and  $(\mathfrak{T}_j \mathcal{E}_i)^\perp$  are restricted only by the symmetry (23), the scalar equations (25c) and (25g). Thus, on a manifold with two totally umbilical, orthogonal distributions there exist many adapted, metric connections critical for action (8)<sub>1</sub>.

Next, we consider  $\mathfrak{T}$  corresponding to arbitrary (not necessarily metric) adapted connections.

**Corollary 6.** *A tensor  $\mathfrak{T}$  corresponding to an **adapted connection** satisfies (11a)–(11e), respectively, (11f)–(11j), if and only if it satisfies (11c)–(11e), respectively, (11h)–(11j).*

*Proof.* In our case, we have

$$\begin{aligned} \mathfrak{T}_X Y &= \mathfrak{T}_X^\top Y + \mathfrak{T}_X^\perp Y - h(X, Y) - h^\perp(X, Y) \\ &\quad - T^\perp(X, Y) - T(X, Y) + (A_{Y^\perp} + T_{Y^\perp}^\sharp)X^\top + (A_{Y^\top} + T_{Y^\top}^\sharp)X^\perp, \end{aligned} \tag{30}$$

$$\begin{aligned} \mathfrak{T}_X^* Y &= (\mathfrak{T}^{\top*})_X Y + (\mathfrak{T}^{\perp*})_X Y + h(X, Y) + h^\perp(X, Y) \\ &\quad + T^\perp(X, Y) + T(X, Y) - (A_{Y^\perp} + T_{Y^\perp}^\sharp)X^\top - (A_{Y^\top} + T_{Y^\top}^\sharp)X^\perp, \end{aligned} \tag{31}$$

where  $\mathfrak{T}^\top$  and  $\mathfrak{T}^{\top*}$  are defined in Sect. 2.2. From the above we immediately see that (11b) and the dual ones are satisfied, regardless of the dimensions of distributions. Also we obtain

$$\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i = -T(\mathcal{E}_i, \mathcal{E}_j) + T(\mathcal{E}_j, \mathcal{E}_i) = -2T(\mathcal{E}_i, \mathcal{E}_j);$$

thus, (11a) is satisfied. Equations (11c), (11e) and (11d) are restrictions on (otherwise free) components:  $\mathfrak{T}_i^\top$ ,  $\text{Tr}^\perp \mathfrak{T} - \text{Tr}^\perp \mathfrak{T}^*$  and  $\text{Tr}^\perp \mathfrak{T} + \text{Tr}^\perp \mathfrak{T}^*$ . Finally,  $h^\perp = \frac{1}{n} H^\perp g^\top$  is a geometric condition in terms of  $\nabla$ .  $\square$

We do not consider variations of  $\bar{S}_{\text{mix}}$  on the space of tensors  $\mathfrak{T}$  corresponding to adapted connections because (8)<sub>1</sub> is identically zero on this subspace, since for every such tensor we have  $\bar{S}_{\text{mix}}(\mathfrak{T}) = 0$ .

### 3. The Mixed Scalar Curvature of Contorsion Tensor

In this section we search for  $(1, 2)$ -tensors  $\mathfrak{T}$  on  $(M, g)$  endowed with orthogonal complementary distributions  $(\mathcal{D}, \mathcal{D}^\perp)$ , critical for the action  $(8)_2$ . We deduce Euler–Lagrange equations for  $(8)_2$  and find examples of critical connections in general and particular classes of almost product metric-affine manifolds. Since

$$S_{\text{mix}, \mathfrak{T}} = \frac{1}{2} \sum_{a,i} \epsilon_a \epsilon_i (g([\mathfrak{T}_i, \mathfrak{T}_a] E_a, \mathcal{E}_i) + g([\mathfrak{T}_a, \mathfrak{T}_i] \mathcal{E}_i, E_a))$$

does not contain covariant derivatives and is quadratic in  $\mathfrak{T}$ , the Euler–Lagrange equations for  $(8)_2$  will not contain any terms related to the geometry of distributions. Using the same approach as in Sects. 2.2, 2.3, 2.4, and 2.5, we obtain the following.

**Theorem 4.** *A  $(1, 2)$ -tensor  $\mathfrak{T}$  is critical for  $(8)_2$  among all  $(1, 2)$ -tensors if and only if*

$$(\mathfrak{T}_a^* E_b + \mathfrak{T}_b E_a)^\perp = 0, \tag{32a}$$

$$(\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i)^\top = 0, \tag{32b}$$

$$\delta_{ac} g(\text{Tr}^\perp \mathfrak{T}, E_b) + \delta_{ab} g(\text{Tr}^\perp \mathfrak{T}^*, E_c) = 0, \tag{32c}$$

$$\delta_{ij} g(\text{Tr}^\top \mathfrak{T}^*, \mathcal{E}_k) + \delta_{ik} g(\text{Tr}^\top \mathfrak{T}, \mathcal{E}_j) = 0, \tag{32d}$$

$$\mathfrak{T}_i^\top = g(\text{Tr}^\perp \mathfrak{T}, \mathcal{E}_i) \text{Id}_{\mathcal{D}}, \tag{32e}$$

$$\mathfrak{T}_a^\perp = g(\text{Tr}^\top \mathfrak{T}^*, E_a) \text{Id}_{\mathcal{D}^\perp}, \tag{32f}$$

$$(\text{Tr}^\perp \mathfrak{T} - \text{Tr}^\perp \mathfrak{T}^*)^\perp = 0, \tag{32g}$$

$$(\text{Tr}^\top \mathfrak{T} - \text{Tr}^\top \mathfrak{T}^*)^\top = 0 \tag{32h}$$

for all  $a, b, c, i, j, k$  of (2). Moreover, if  $n > 1$  and  $p > 1$  then (32c)–(32d) read as

$$(\text{Tr}^\perp \mathfrak{T})^\top = 0 = (\text{Tr}^\perp \mathfrak{T}^*)^\top, \quad (\text{Tr}^\top \mathfrak{T}^*)^\perp = 0 = (\text{Tr}^\top \mathfrak{T})^\perp.$$

*Proof.* We have  $S_{\text{mix}, \mathfrak{T}} = \frac{1}{2} (Q_{1, \mathfrak{T}} + Q_{2, \mathfrak{T}})$ , where

$$Q_{1, \mathfrak{T}} = \sum_{a,i} \epsilon_a \epsilon_i g([\mathfrak{T}_i, \mathfrak{T}_a] E_a, \mathcal{E}_i), \quad Q_{2, \mathfrak{T}} = \sum_{a,i} \epsilon_a \epsilon_i g([\mathfrak{T}_a, \mathfrak{T}_i] \mathcal{E}_i, E_a).$$

Consider one-parameter family of connections,  $\bar{\nabla}^t = \nabla + \mathfrak{T}^t$ . Let  $S = \partial_t \mathfrak{T}^t|_{t=0}$ . Since

$$Q_{1, \mathfrak{T}} = \sum_{a,i} \epsilon_a \epsilon_i [g((\mathfrak{T}_i \mathfrak{T}_a - \mathfrak{T}_a \mathfrak{T}_i) E_a, \mathcal{E}_i)]$$

and  $Q_{1, \mathfrak{T}}$  and  $Q_{2, \mathfrak{T}}$  are dual w.r.t. interchanging the distributions  $\mathcal{D}$  and  $\mathcal{D}^\perp$ , we can use equations (13a) and (13b) (and their dual equations) to obtain the following:

$$\begin{aligned} \partial_t (Q_{1, \mathfrak{T}} + Q_{2, \mathfrak{T}}) &= \int_\Omega \sum \left\{ g(S_a E_b, E_c) (\delta_{ac} g(\text{Tr}^\perp \mathfrak{T}, E_b) + \delta_{ab} g(\text{Tr}^\perp \mathfrak{T}^*, E_c)) \right. \\ &\quad \left. + g(S_i \mathcal{E}_j, \mathcal{E}_k) (\delta_{ij} g(\text{Tr}^\top \mathfrak{T}^*, \mathcal{E}_k) + \delta_{ik} g(\text{Tr}^\top \mathfrak{T}, \mathcal{E}_j)) \right\} \end{aligned}$$

$$\begin{aligned}
 & -g(S_i E_b, E_a)(g(\mathfrak{T}_a \mathcal{E}_i, E_b) + g(\mathfrak{T}_b E_a, \mathcal{E}_i)) \\
 & -g(S_a \mathcal{E}_i, \mathcal{E}_j)(g(\mathfrak{T}_i \mathcal{E}_j, E_a) + g(\mathfrak{T}_j E_a, \mathcal{E}_i)) \\
 & -g(S_i \mathcal{E}_j, E_a)(g(\mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_j) - \delta_{ij} g(\text{Tr}^\top \mathfrak{T}^*, E_a)) \\
 & -g(S_a \mathcal{E}_i, E_b)(g(\mathfrak{T}_i E_b, E_a) - \delta_{ab} g(\text{Tr}^\perp \mathfrak{T}, \mathcal{E}_i)) \\
 & -g(S_a E_b, \mathcal{E}_i)(g(\mathfrak{T}_i E_a, E_b) - \delta_{ab} g(\text{Tr}^\perp \mathfrak{T}^*, \mathcal{E}_i)) \\
 & -g(S_i E_a, \mathcal{E}_j)(g(\mathfrak{T}_a \mathcal{E}_j, \mathcal{E}_i) - \delta_{ij} g(\text{Tr}^\top \mathfrak{T}, E_a)) \Big\} d \text{vol}_g,
 \end{aligned}$$

where summing is over repeated indices and factors  $\epsilon_\mu$  are omitted. Since no further assumptions are made about  $S$  or  $\mathfrak{T}$ , all the components  $g(S_\mu e_\lambda, e_\rho)$  are independent and the above formula gives rise to the Euler–Lagrange equations stated above.  $\square$

Clearly, the Levi-Civita connection is a critical point for the total mixed scalar  $\mathfrak{T}$ -curvature.

For either metric or statistical connection  $\bar{\nabla} = \nabla + \mathfrak{T}$ , we have

$$S_{\text{mix}, \mathfrak{T}} = \sum_{a,i} \epsilon_a \epsilon_i g([\mathfrak{T}_i, \mathfrak{T}_a] \mathcal{E}_i, E_a).$$

**Corollary 7.** *A (1, 2)-tensor  $\mathfrak{T}$  corresponding to **statistical connection** is critical for (8)<sub>2</sub> among all tensors corresponding to statistical connections if and only if for all  $a, b, i, j$ :*

$$\begin{aligned}
 (\text{Tr}^\top \mathfrak{T})^\perp &= 0 = (\text{Tr}^\perp \mathfrak{T})^\top, \\
 (\mathfrak{T}_j \mathcal{E}_i)^\top &= (1/2) \delta_{ij} (\text{Tr}^\top \mathfrak{T})^\top \quad (\forall i, j), \\
 (\mathfrak{T}_a E_b)^\perp &= (1/2) \delta_{ab} (\text{Tr}^\perp \mathfrak{T})^\perp \quad (\forall a, b).
 \end{aligned}$$

If, in addition,  $\sum_a \epsilon_a \neq 0 \neq \sum_i \epsilon_i$  then the above Euler–Lagrange equations are reduced to

$$\begin{aligned}
 (\text{Tr}^\top \mathfrak{T})^\top &= 0 = (\text{Tr}^\perp \mathfrak{T})^\perp, \\
 (\mathfrak{T}_a E_b)^\perp &= 0 = (\mathfrak{T}_j \mathcal{E}_i)^\top \quad (\forall a, b, i, j).
 \end{aligned}$$

**Corollary 8.** *A (1, 2)-tensor  $\mathfrak{T}$  corresponding to a **metric connection** is critical for (8)<sub>2</sub> among all (1, 2)-tensors corresponding to metric connections if and only if*

$$(\mathfrak{T}_b E_a + \mathfrak{T}_a^* E_b)^\perp = 0 = (\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i)^\top, \tag{33a}$$

$$(\text{Tr}^\top \mathfrak{T})^\top = 0 = (\text{Tr}^\perp \mathfrak{T})^\perp, \tag{33b}$$

$$\mathfrak{T}_a^\perp = 0 = \mathfrak{T}_i^\top, \tag{33c}$$

and

- (i) if  $n > 1$  then  $(\text{Tr}^\perp \mathfrak{T})^\top = 0$ ;
- (ii) if  $p > 1$  then  $(\text{Tr}^\top \mathfrak{T})^\perp = 0$ .

*Proof.* For variation among metric connections  $S = \partial_t \mathfrak{T}^t$  is antisymmetric, and for metric connection  $\mathfrak{T}$  is antisymmetric – using this we obtain the following Euler–Lagrange equations:

$$g(\mathfrak{T}_a^* E_b + \mathfrak{T}_b E_a, \mathcal{E}_i) = 0 = g(\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i, E_a), \tag{34a}$$

$$g(\text{Tr}^\perp \mathfrak{T}, \delta_{ac} E_b) + g(\text{Tr}^\perp \mathfrak{T}^*, \delta_{ab} E_c) = 0, \tag{34b}$$

$$g(\text{Tr}^\top \mathfrak{T}^*, \delta_{ij} \mathcal{E}_k) + g(\text{Tr}^\top \mathfrak{T}, \delta_{ik} \mathcal{E}_j) = 0, \tag{34c}$$

$$g(\mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_j) + g(\text{Tr}^\top \mathfrak{T}, \delta_{ij} E_a) = 0, \tag{34d}$$

$$g(\mathfrak{T}_i E_b, E_a) - g(\text{Tr}^\perp \mathfrak{T}, \delta_{ab} \mathcal{E}_i) = 0 \tag{34e}$$

for all  $a, b, c, i, j, k$  of (2). From (34a) we obtain (33a). Taking symmetric parts of (34d) and (34e) leads to (33b), while the antisymmetric parts of (34d) and (34e) yield (33c). Finally, setting  $a = b \neq c$  for  $n > 1$  in (34b) and  $i = j \neq k$  for  $p > 1$  in (34c) yields the remaining conditions.  $\square$

**Corollary 9.** *A (1, 2)-tensor  $\mathfrak{T}$  corresponding to an **adapted connection** is critical for (8)<sub>2</sub> among all (1, 2)-tensors  $\mathfrak{T}$  corresponding to adapted connections if and only if both  $\mathcal{D}$  and  $\mathcal{D}^\perp$  are **integrable** and*

*if  $n > 1$  then  $\mathcal{D}^\perp$  is **minimal** w.r.t.  $\nabla$ ;*

*if  $p > 1$  then  $\mathcal{D}$  is **minimal** w.r.t.  $\nabla$ .*

*Proof.* Let  $\mathfrak{T}$  be the contorsion tensor of an adapted connection. From (30) and (31) we obtain that for all  $X \in \mathfrak{X}_M$  all the components of  $\mathfrak{T}$  except  $\mathfrak{T}_X^\top$  and  $\mathfrak{T}_X^\perp$  are determined only by the Levi-Civita connection—hence they remain the same for all adapted connections. It follows that for  $S := \partial_t \mathfrak{T}^t$ , where  $\mathfrak{T}^t$  are contorsion tensors of adapted connections from some one-parameter family, we have  $g(S_\mu E_a, \mathcal{E}_i) = g(S_\mu \mathcal{E}_i, E_a) = 0$ . Hence, the only Euler–Lagrange equations that remain to be considered are

$$g(\mathfrak{T}_b E_a + \mathfrak{T}_a^* E_b, \mathcal{E}_i) = 0, \tag{35a}$$

$$g(\mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i, E_a) = 0, \tag{35b}$$

$$g(\text{Tr}^\perp \mathfrak{T}, \delta_{ac} E_b) + g(\text{Tr}^\perp \mathfrak{T}^*, \delta_{ab} E_c) = 0, \tag{35c}$$

$$g(\text{Tr}^\top \mathfrak{T}^*, \delta_{ij} \mathcal{E}_k) + g(\text{Tr}^\top \mathfrak{T}, \delta_{ik} \mathcal{E}_j) = 0 \tag{35d}$$

for all  $a, b, c, i, j, k$  of (2). Using (30) and (31) we can write (35a) as follows

$$\begin{aligned} 0 &= g((A_i + T_i^\sharp) E_a, E_b) + g(-h^\perp(E_b, E_a) - T^\perp(E_b, E_a), \mathcal{E}_i) \\ &= g(T_i^\sharp E_a, E_b) - g(T_i^\sharp E_b, E_a) = 2g(T_i^\sharp E_a, E_b), \end{aligned} \tag{36}$$

and (35c) as

$$0 = \delta_{ac} g(-H, E_b) + \delta_{ab} g(E_c, H) = g(\delta_{ab} E_c - \delta_{ac} E_b, H). \tag{37}$$

For  $n = 1$ , (37) is always satisfied, for  $n \geq 2$  and  $a = b \neq c$  we obtain  $H = 0$ . From (36) we obtain that  $T^\perp = 0$  and the claim follows from the fact that (35b), (35d) are dual to (35a), (35c), resp.  $\square$



**Proposition 3.** *Let  $n + p > 2$ , then a critical point of the action  $(8)_2$  is not an extremal point (also for variations in the subspaces of tensors  $\mathfrak{T}$  corresponding to metric connections and corresponding to statistical connections).*

*Proof.* The claim follows from the proof of Proposition 2 as for  $\mathfrak{T}^t = \mathfrak{T} + t \cdot S$  we have  $S_{\text{mix}, \mathfrak{T}}(\mathfrak{T}^t) = O(t) + t^2\sigma$ , with  $\sigma$  as in (22). □

### 4. Double-Twisted Metric-Affine Products

The *doubly-twisted product* of metric-affine manifolds  $(B, g_B, \mathfrak{T}_B)$  and  $(F, g_F, \mathfrak{T}_F)$  is a manifold  $M = B \times F$  with the metric  $g = g^\top + g^\perp$  and the contorsion tensor  $\mathfrak{T} = \mathfrak{T}^\top + \mathfrak{T}^\perp$ , where

$$g^\top(X, Y) = v^2 g_B(X^\top, Y^\top), \quad g^\perp(X, Y) = u^2 g_F(X^\perp, Y^\perp),$$

$$\mathfrak{T}_X^\top Y = u^2 (\mathfrak{T}_B)_{X^\top} Y^\top, \quad \mathfrak{T}_X^\perp Y = v^2 (\mathfrak{T}_F)_{X^\perp} Y^\perp,$$

and the warping functions  $u, v \in C^\infty(M)$  are positive. For  $v = 1$  we have the *twisted metric-affine product*; if, in addition,  $u \in C^\infty(B)$  then this is a *warped metric-affine product*, and for  $u = v = 1$  – the product. Denote this double-twisted metric-affine product by  $B \times_{(v,u)} F$ . Its second fundamental forms (w.r.t.  $\nabla$ ) are  $h = -\nabla^\top(\log u) g^\perp$  and  $h^\perp = -\nabla^\perp(\log v) g^\top$ , see [6], and the mean curvature vectors are  $H = -n \nabla^\top(\log u)$  and  $H^\perp = -p \nabla^\perp(\log v)$ . Hence, the *leaves*  $B \times \{y\}$  and the *fibers*  $\{x\} \times F$  of a RC doubly-twisted product  $B \times_{(v,u)} F$  are totally umbilical w.r.t.  $\bar{\nabla}$  and  $\nabla$ . By (7) and (10) we have  $S_{\text{mix}} = -n(\Delta^\top u)/u - p(\Delta^\perp v)/v$  and  $\bar{S}_{\text{mix}} = S_{\text{mix}} + n u(\text{Tr } \mathfrak{T}^\top)(u) + p v(\text{Tr } \mathfrak{T}^\perp)(v)$ , where  $\Delta^\top$  is the leafwise Laplacian and  $\Delta^\perp$  is the fiberwise Laplacian.

One may show that if given connections on  $B$  and  $F$  are either metric or statistical connections then the new connection  $\bar{\nabla} = \nabla + \mathfrak{T}$  on  $B \times_{(v,u)} F$  has the same property.

**Corollary 10 (of Theorem 1).** *A double-twisted product is critical for action  $(8)_1$  w.r.t. variations of  $\mathfrak{T}$  if and only if*

$$\nabla^\top u = 0 = \nabla^\perp v \quad (\text{hence, } h^\perp = 0 = h), \tag{38a}$$

$$\text{Tr } \mathfrak{T}^B = 0 = \text{Tr } \mathfrak{T}^F. \tag{38b}$$

**Corollary 11 (of Theorem 2).** *A double-twisted product of Riemann–Cartan manifolds is critical for action  $(8)_1$  w.r.t. variations of  $\mathfrak{T}$  obeying (23) if and only if (38a)–(38b) hold.*

**Corollary 12 (of Theorem 3).** *A double-twisted product of statistical manifolds is critical for action  $(8)_1$  w.r.t. variations of  $\mathfrak{T}$  obeying (26) if and only if (38a)–(38b) hold.*

For the mixed scalar  $\mathfrak{T}$ -curvature we obtain the following.

**Corollary 13 (of Theorem 4).** *A double-twisted product  $B \times_{(v,u)} F$  with  $\sum_a \epsilon_a \neq 0 \neq \sum_i \epsilon_i$  is critical for action  $(8)_2$  w.r.t. variations of  $\mathfrak{T}$  if and only if (38b) hold.*

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