THE MIXING SET WITH FLOWS*

MICHELE CONFORTI†, MARCO DI SUMMA†, AND LAURENCE A. WOLSEY‡

Abstract. We consider the mixing set with flows:

$$s + x_t \geq b_t,\ x_t \leq y_t \text{ for } 1 \leq t \leq n;\ \ s \in \mathbb{R}^1_+,\ x \in \mathbb{R}^n_+,\ y \in \mathbb{Z}^n_+.$$

It models a "flow version" of the basic mixing set introduced and studied by Günlük and Pochet [5], as well as the most simple stochastic lot-sizing problem with recourse. More generally it is a relaxation of certain mixed integer sets that arise in the study of production planning problems.

We study the polyhedron defined as the convex hull of the above set. Specifically we provide an inequality description, and we also characterize its vertices and rays.

Key words. mixed integer programming, mixing, lot-sizing

AMS subject classifications. 90C11, 90C57

1. Introduction.

We give an inequality (external), and extreme point and extreme ray (internal) description for the convex hull of the *mixing set with flows* X^{FM}:

$$s + x_t \geq b_t \text{ for } 1 \leq t \leq n \tag{1.1}$$

$$x_t \leq y_t \text{ for } 1 \leq t \leq n \tag{1.2}$$

$$s \in \mathbb{R}^1_+,\ x \in \mathbb{R}^n_+,\ y \in \mathbb{Z}^n_+ \tag{1.3}$$

where $0 \leq b_1 \leq \ldots \leq b_n,\ b \in \mathbb{R}^n$.

This set is a relative of the *mixing set* X^{MIX}:

$$s + y_t \geq b_t \text{ for } 1 \leq t \leq n \tag{1.4}$$

$$s \in \mathbb{R}^1_+,\ y \in \mathbb{Z}^n_+ \tag{1.5}$$

with $b \in \mathbb{R}^n$ introduced formally by Günlük and Pochet [5] and studied by Pochet and Wolsey [7] and Miller and Wolsey [6]. Internal and external descriptions of the convex hull of X^{MIX} are given in [5].

The original motivation for studying X^{FM} was to generalize X^{MIX} by introducing the continuous (flow) variables x, noting that $\text{conv}(X^{MIX})$ is a face of $\text{conv}(X^{FM})$. However X^{FM} is also closely related to two lot-sizing models that we now present.

The constant capacity lot-sizing model can be formulated as

$$s_0 + \sum_{u=1}^{t} w_u \geq \sum_{u=1}^{t} d_u \text{ for } 1 \leq t \leq n \tag{1.6}$$

$$w_u \leq z_u \text{ for } 1 \leq u \leq n \tag{1.7}$$

$$s_0 \in \mathbb{R}^1_+,\ w \in \mathbb{R}^n_+,\ z \in \{0,1\}^n \tag{1.8}$$

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*This work was partly carried out within the framework of ADONET, a European network in Algorithmic Discrete Optimization, contract no. MRTN-CT-2003-504438. This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

†Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova. Via Trieste 63, 35121 Padova, Italy (`conforti@math.unipd.it, mdsumma@math.unipd.it`).

‡Center of Operations Research and Econometrics (CORE), Université catholique de Louvain. 34, Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium (`wolsey@core.ucl.ac.be`).

where d_t is the demand in period t, s_0 is the initial stock variable, w_t is the amount produced in t bounded above by the capacity C (we take $C = 1$ throughout wlog), and z_t is a 0-1 set-up variable with $z_t = 1$ if $x_t > 0$. Summing the constraints (1.7) over $1 \le u \le t$ (for each $t = 1, \dots, n$) leads to the relaxation

$$s_0 + \textstyle\sum_{u=1}^t w_u \ge \sum_{u=1}^t d_u \text{ for } 1 \le t \le n$$
$$\textstyle\sum_{u=1}^t w_u \le \sum_{u=1}^t z_u \text{ for } 1 \le t \le n$$
$$s_0 \in \mathbb{R}^1_+,\ \textstyle\sum_{u=1}^t w_u \in \mathbb{R}^1_+,\ \sum_{u=1}^t z_u \in \mathbb{Z}^1_+ \text{ for } 1 \le t \le n.$$

With $s := s_0$, $x_t := \sum_{u=1}^t w_u$ and $y_t := \sum_{u=1}^t z_u$, this is precisely the set X^{FM}.

The second link is to the two period stochastic lot-sizing model with constant capacities. Specifically, at time 0 one must choose to produce a quantity s at a per unit cost of h. Then in period 1, n different outcomes are possible. For $1 \le t \le n$, the probability of event t is ϕ_t, the demand is b_t and the unit production cost is p_t, with production in batches of size up to $C = 1$. There are also a fixed cost of q_t per batch and a possible bound k_t on the number of batches. If we want to minimize the total expected cost, the resulting problem is

$$\min\{hs + \sum_{t=1}^n \phi_t(p_t x_t + q_t y_t) : (s, x, y) \in X^{FM};\ y_t \le k_t,\ 1 \le t \le n\}. \tag{1.9}$$

Note that when $k_t = 1,\ 1 \le t \le n$, this is the standard lot-sizing variant. Also the uncapacitated case when $b_t \le 1,\ 1 \le t \le n$ has been treated in Guan et al. [4].

It is also interesting to view X^{MIX} and X^{FM} as simple mixed integer sets with special structure. One observation is that the associated constraint matrices are totally unimodular, but the right-hand sides are typically non-integer as $b \in \mathbb{Q}^n$. Miller and Wolsey [6] and Van Vyve [9] have introduced and studied a different extension, called a *continuous mixing set*, again having a totally unimodular system of constraints.

We now describe the contents of this paper. We terminate the introduction with some notation. In §2 we develop a polyhedral result used later to establish that a given polyhedron is "integral" (i.e. its vertices are points of the mixed integer set under consideration). In §3 we find an external description of $\mathrm{conv}(X^{FM})$ and two closely related sets, and in §4 we give an internal description that leads to a simple polynomial time algorithm for optimization over the set X^{FM}. We conclude in §5 with a brief indication of related work on other generalizations of mixing sets.

Notation. Throughout we will use the following notation: $N := \{1, \dots, n\}$, e_S for the characteristic vector of a subset $S \subseteq N$, $e_i := e_{\{i\}}$ for the ith unit vector, and $\underline{0} := e_\emptyset$ and $\underline{1} := e_N$ for the n-vectors of 0s and 1s respectively.

2. Some Equivalences of Polyhedra. In the next section we will relate the polyhedra $\mathrm{conv}(X^{FM})$ and $\mathrm{conv}(X^{MIX})$. To do this, we will need some polyhedral equivalences that we introduce here.

For a nonempty polyhedron P in $\mathbb{R}^n$ and a vector $\alpha \in \mathbb{R}^n$, define $\mu_P(\alpha) := \min\{\alpha x : x \in P\}$ and let $M_P(\alpha)$ be the face $\{x \in P : \alpha x = \mu_P(\alpha)\}$, where $M_P(\alpha) = \emptyset$ whenever $\mu_P(\alpha) = -\infty$.

LEMMA 2.1. *Let $P \subseteq Q$ be two nonempty polyhedra in $\mathbb{R}^n$ and let α be a nonzero vector in $\mathbb{R}^n$. Then the following conditions are equivalent:*

1. $\mu_P(\alpha) = \mu_Q(\alpha)$*;*

2. $M_P(\alpha) \subseteq M_Q(\alpha)$.

Proof. Suppose $\mu_P(\alpha) = \mu_Q(\alpha)$. Since $P \subseteq Q$, every point in $M_P(\alpha)$ belongs to $M_Q(\alpha)$. So if 1 holds, then 2 holds as well. The converse is obvious. □

LEMMA 2.2. *Let $P \subseteq Q$ be two nonempty polyhedra in $\mathbb{R}^n$, where P is not an affine variety. Suppose that for every inequality $\alpha x \geq \beta$ that is facet-inducing for P, at least one of the following holds:*

1. *$\mu_P(\alpha) = \mu_Q(\alpha)$;*
2. *$M_P(\alpha) \subseteq M_Q(\alpha)$.*

Then $P = Q$.

Proof. We prove that if $M_P(\alpha) \subseteq M_Q(\alpha)$ for every inequality $\alpha x \geq \beta$ that is facet-inducing for P, then every facet-inducing inequality for P is a valid inequality for Q and every hyperplane containing P also contains Q. This shows $Q \subseteq P$ and therefore $P = Q$. By Lemma 2.1, the conditions $\mu_P(\alpha) = \mu_Q(\alpha)$ and $M_P(\alpha) \subseteq M_Q(\alpha)$ are equivalent and we are done.

Let $\alpha x \geq \beta$ be a facet-inducing inequality for P. Since $M_P(\alpha) \subseteq M_Q(\alpha)$, then $\beta = \mu_P(\alpha) = \mu_Q(\alpha)$ and $\alpha x \geq \beta$ is an inequality which is valid for Q. Now let $\gamma x = \delta$ be a hyperplane containing P. If $Q \not\subseteq \{x : \gamma x = \delta\}$, then there exists $\bar{x} \in Q$ such that $\gamma \bar{x} \neq \delta$. We assume wlog $\sigma = \gamma \bar{x} - \delta > 0$. Since P is not an affine variety, there exists an inequality $\alpha x \geq \beta$ which is facet-inducing for P (and so it is valid for Q). Then, for $\lambda > 0$ the inequality $(\lambda\alpha - \gamma)x \geq \lambda\beta - \delta$ is also facet-inducing for P, so it is valid for Q. Choosing $\lambda > 0$ such that $\lambda(\alpha\bar{x} - \beta) < \sigma$ gives a contradiction, as $(\lambda\alpha - \gamma)\bar{x} = \lambda\alpha\bar{x} - \gamma\bar{x} < \lambda\beta + \sigma - \gamma\bar{x} = \lambda\beta - \delta$. □

If P is not full-dimensional, for each facet F of P there are infinitely many distinct inequalities that define F (two inequalities are distinct if their associated half-spaces are distinct: that is, if one is not the positive multiple of the other). Observe that the hypotheses of the lemma must be verified for *all* distinct facet-defining inequalities (not just one facet-defining inequality for each facet), otherwise the result is false. For instance, consider the polyhedra $P = \{(x, y) : 0 \leq x \leq 1,\ y = 0\} \subset Q = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$. The hypotheses of Lemma 2.1 are satisfied for the inequalities $x \geq 0$ and $x \leq 1$, which define all the facets of P.

Also note that the assumption that P is not an affine variety cannot be removed: indeed, in such a case P does not have proper faces, so the hypotheses of the lemma are trivially satisfied, even if $P \neq Q$.

COROLLARY 2.3. *Let $P \subseteq Q$ be two pointed polyhedra in $\mathbb{R}^n$, with the property that every vertex of Q belongs to P. Let $Cx \geq d$ be a system of inequalities that are valid for P such that for every inequality $\gamma x \geq \delta$ of the system, $P \not\subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$.*

If for every $\alpha \in \mathbb{R}^n$ such that $\mu_P(\alpha)$ is finite but $\mu_Q(\alpha) = -\infty$, $Cx \geq d$ contains an inequality $\gamma x \geq \delta$ such that $M_P(\alpha) \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$, then $P = Q \cap \{x \in \mathbb{R}^n : Cx \geq d\}$.

Proof. We first show that $\dim(P) = \dim(Q)$. If not, there exists a hyperplane $\alpha x = \beta$ containing P but not Q. Wlog we can assume that $\mu_Q(\alpha) < \beta = \mu_P(\alpha)$. So $\mu_Q(\alpha) = -\infty$, otherwise there would exist an α-optimal vertex $\bar{x}$ of Q such that $\alpha\bar{x} < \beta$, contradicting the fact that $\bar{x} \in P$. Now the system $Cx \geq d$ must contain an inequality $\gamma x \geq \delta$ such that $P = M_P(\alpha) \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$, a contradiction.

Let $Q' = Q \cap \{x \in \mathbb{R}^n : Cx \geq d\}$. Note that $P \subseteq Q' \subseteq Q$, thus $\dim(P) = \dim(Q') = \dim(Q)$. Let $\alpha x \geq \beta$ be a facet-inducing inequality for P. If $\mu_Q(\alpha)$ is finite, then Q contains an α-optimal vertex which is in P and therefore $\beta = \mu_P(\alpha) = \mu_{Q'}(\alpha) = \mu_Q(\alpha)$. If $\mu_Q(\alpha) = -\infty$, the system $Cx \geq d$ contains an inequality $\gamma x \geq \delta$ such that $M_P(\alpha) \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$ and $P \not\subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$. It follows that

$\gamma x \geq \delta$ is a facet-inducing inequality for P and that it defines the same facet of P as $\alpha x \geq \beta$ (that is, $M_P(\alpha) = M_P(\gamma)$). This means that there exist $\nu > 0$, a vector λ and a system $Ax = b$ which is valid for P such that $\gamma = \nu\alpha + \lambda A$ and $\delta = \nu\beta + \lambda b$. Since $\dim(P) = \dim(Q')$ and $P \subseteq Q'$, the system $Ax = b$ is valid for Q', as well. As $\gamma x \geq \delta$ is also valid for Q', it follows that $\alpha x \geq \beta$ is valid for Q' (because $\alpha = \frac{1}{\nu}\gamma - \frac{\lambda}{\nu}A$ and $\beta = \frac{1}{\nu}\delta - \frac{\lambda}{\nu}b$). Therefore $\beta = \mu_P(\alpha) = \mu_{Q'}(\alpha)$.

Now assume that P consists of a single point and $P \neq Q$. Then Q is a cone having P as apex. Given a ray α of Q, $\mu_P(\alpha)$ is finite while $\mu_Q(\alpha) = -\infty$, so the system $Cx \geq d$ contains an inequality $\gamma x \geq \delta$ such that $P \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$, a contradiction. So we can assume that P is not a single point and thus P is not an affine variety, as it is pointed. Now we can conclude by applying Lemma 2.2 to the polyhedra P and Q'. □

We remark that in the statement of Corollary 2.3 the condition that the two polyhedra are pointed is not necessary: if we replace the property "every vertex of Q belongs to P" with "every minimal face of Q belongs to P", the proof needs a very slight modification to remain valid. (However, in this case we should assume that P is not an affine variety, so that we can apply Lemma 2.2 in the proof.)

We also observe that the condition "for every inequality $\gamma x \geq \delta$ of the system, $P \not\subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$" is necessary. For instance, consider the polyhedra $P = \{(x, y) : 0 \leq x \leq 1,\ y = 0\} \subset Q = \{(x, y) : x \geq 0,\ y = 0\}$ and the system consisting of the single inequality $y \geq 0$.

3. An external description of X^{FM}. The approach taken to derive an inequality description of $\text{conv}(X^{FM})$ is first outlined briefly. We work with two intermediate mixed integer sets Z and X^{INT} for which we establish several properties. The first two link $\text{conv}(X^{FM})$ and $\text{conv}(Z)$, and the next two provide an external description of $\text{conv}(Z)$:

(i) First we observe that $X^{FM} = Z \cap \{(s, x, y) : \underline{0} \leq x \leq y\}$.

(ii) Using Corollary 2.3, we prove that $\text{conv}(X^{FM}) = \text{conv}(Z) \cap \{(s, x, y) : \underline{0} \leq x \leq y\}$.

(iii) We then show that the polyhedra $\text{conv}(Z)$ and $\text{conv}(X^{INT})$ are in 1-1 correspondence via an affine transformation.

(iv) Finally we note that X^{INT} is the intersection of mixing sets, and therefore external descriptions of $\text{conv}(X^{INT})$ and $\text{conv}(Z)$ are known.

3.1. A relaxation of X^{FM}. Consider the set Z:

$$s + y_t \geq b_t \text{ for } 1 \leq t \leq n \tag{3.1}$$

$$s + x_k + y_t \geq b_t \text{ for } 1 \leq k < t \leq n \tag{3.2}$$

$$s + x_t \geq b_t \text{ for } 1 \leq t \leq n \tag{3.3}$$

$$s \in \mathbb{R}^1_+,\ x \in \mathbb{R}^n,\ y \in \mathbb{Z}^n_+. \tag{3.4}$$

PROPOSITION 3.1. *Let X^{FM} and Z be defined on the the same vector b. Then $X^{FM} \subseteq Z$ and $X^{FM} = Z \cap \{(s, x, y) : \underline{0} \leq x \leq y\}$.*

Proof. To see that $X^{FM} \subseteq Z$, observe that for $(s, x, y) \in X^{FM}$, $s + y_t \geq s + x_t \geq b_t$, so $s + y_t \geq b_t$ is a valid inequality. Also $s + y_t \geq b_t$ and $x_k \geq 0$ imply that $s + x_k + y_t \geq b_t$ is a valid inequality. The only inequalities that define X^{FM} but do not appear in the definition of Z are the inequalities $\underline{0} \leq x \leq y$. □

Since the left-hand sides of inequalities (1.1)–(1.3) and (3.1)–(3.4) have integer coefficients, the recession cones of X^{FM} and Z coincide with the recession cones of their linear relaxations. Thus we have the following:

Observation 1. *The extreme rays of* $\text{conv}(X^{FM})$ *are the following* $2n+1$ *vectors:* $(1,\underline{0},\underline{0})$, $(0,\underline{0},e_k)$, $(0,e_k,e_k)$. *The* $2n+1$ *extreme rays of* $\text{conv}(Z)$ *are* $(0,\underline{0},e_k)$, $(0,e_k,\underline{0})$, $(1,-\underline{1},\underline{0})$. *Therefore both recession cones of* $\text{conv}(X^{FM})$ *and* $\text{conv}(Z)$ *are full-dimensional simplicial cones, thus showing that* $\text{conv}(X^{FM})$ *and* $\text{conv}(Z)$ *are both full-dimensional polyhedra.*

Observation 2. *Let* (s^*,x^*,y^*) *be a vertex of* $\text{conv}(Z)$. *Then*

$$s^* = \max\begin{cases} 0 \\ b_t - y_t^*,\ 1\le t\le n \\ b_t - x_t^*,\ 1\le t\le n \\ b_t - y_t^* - x_k^*,\ 1\le k<t\le n \end{cases}$$

$$x_k^* = \max\begin{cases} b_k - s^* \\ b_t - s^* - y_t^*,\ k<t\le n. \end{cases}$$

Lemma 3.2. *Let* (s^*,x^*,y^*) *be a vertex of* $\text{conv}(Z)$. *Then* $\underline{0}\le x^*\le y^*$.

Proof. Assume $x_k^*<0$ for some index k. Then $s^*>0$, otherwise, if $s^*=0$, the constraints $s+x_k\ge b_k$, $b_k\ge 0$ imply $x_k^*\ge 0$.

We now claim that there is an index $t\in N$ such that $s^*=b_t-y_t^*$. If not, $s^*>b_t-y_t^*$, $1\le t\le n$, and there is an $\varepsilon\ne 0$ such that $(s^*,x^*,y^*)\pm\varepsilon(1,-\underline{1},\underline{0})$ belong to $\text{conv}(Z)$, a contradiction.

So there is an index $t\in N$ such that $s^*=b_t-y_t^*>0$. Since $b_t-y_t^*\ge b_t-y_t^*-x_k^*$, $1\le k<t$, this implies $x_k^*\ge 0$, $1\le k<t$. Observation 2 also implies $b_t-y_t^*\ge b_k-x_k^*$, $1\le k\le n$. Together with $y_t^*\ge 0$ and $b_t\le b_k$, $k\ge t$, this implies $x_k^*\ge y_t^*\ge 0$, $k\ge t$. This completes the proof that $x^*\ge\underline{0}$.

Assume $x_k^*>y_k^*$ for some index k. Then $y_k^*\ge 0$ implies $x_k^*>0$. Assume $x_k^*=b_k-s^*$. Then $y_k^*\ge b_k-s^*$ implies that $x_k^*\le y_k^*$, a contradiction. Therefore by Observation 2, $x_k^*=b_t-s^*-y_t^*$ for some $t>k$. Since $x_k^*>0$, then $b_t-s^*-y_t^*>0$, a contradiction to $s^*+y_t^*\ge b_t$. This shows $x^*\le y^*$. □

We now can state the main theorem of this section:

Theorem 3.3. *Let* X^{FM} *and* Z *be defined on the the same vector* b. *Then* $\text{conv}(X^{FM})=\text{conv}(Z)\cap\{(s,x,y):\ \underline{0}\le x\le y\}$.

Proof. By Proposition 3.1, $\text{conv}(X^{FM})\subseteq\text{conv}(Z)$. By Lemma 3.2 and Proposition 3.1, every vertex of $\text{conv}(Z)$ belongs to $\text{conv}(X^{FM})$.

Let $\alpha=(h,p,q)$, $h\in\mathbb{R}^1$, $p\in\mathbb{R}^n$, $q\in\mathbb{R}^n$ be such that $\mu_{\text{conv}(X^{FM})}(\alpha)$ is finite and $\mu_{\text{conv}(Z)}(\alpha)=-\infty$. Since by Observation 1, the extreme rays of $\text{conv}(Z)$ that are not rays of $\text{conv}(X^{FM})$ are $(0,e_k,\underline{0})$ and $(1,-\underline{1},\underline{0})$, then either $p_k<0$ for some index k or $h<\sum_{t=1}^n p_t$.

If $p_k<0$, then $M_{\text{conv}(X^{FM})}(\alpha)\subseteq\{(s,x,y):x_k=y_k\}$.

If $h<\sum_{t=1}^n p_t$, let $N^+=\{j\in N:p_j>0\}$ and $k=\min\{j:j\in N^+\}$. We show that $M_{\text{conv}(X^{FM})}(\alpha)\subseteq\{(s,x,y):x_k=0\}$. Suppose that $x_k>0$ in some optimal solution. As the solution is optimal and $p_k>0$, we cannot decrease only the variable x_k and remain feasible. Thus $s+x_k=b_k$, which implies that $s<b_k$. However this implies that for all $j\in N^+$, we have $x_j\ge b_j-s>b_j-b_k\ge 0$ as $j\ge k$. Now as $x_j>0$ for all $j\in N^+$, we can increase s by $\varepsilon>0$ and decrease x_j by ε for all $j\in N^+$. The new point is feasible in X^{FM} and has lower objective value, a contradiction.

To complete the proof, since $\text{conv}(X^{FM})$ is full-dimensional, the system $\underline{0}\le x\le y$ does not contain an improper face of $\text{conv}(X^{FM})$. So we can now apply Corollary 2.3 to $\text{conv}(X^{FM})$, $\text{conv}(Z)$ and the system $\underline{0}\le x\le y$. □

3.2. The intersection set. The following set is the *intersection set* X^{INT}:

$$\sigma_k + y_t \geq b_t - b_k \text{ for } 0 \leq k < t \leq n$$
$$\sigma \in \mathbb{R}_+^{n+1},\ y \in \mathbb{Z}_+^n.$$

where $0 = b_0 \leq b_1 \leq \ldots \leq b_n$.

Note that X^{INT} is the intersection of the following $n+1$ mixing sets X_k^{MIX}, each one associated with a single variable σ_k:

$$\sigma_k + y_t \geq b_t - b_k \text{ for } k < t \leq n$$
$$\sigma_k \in \mathbb{R}_+^1,\ y \in \mathbb{Z}_+^{n-k}.$$

THEOREM 3.4. *Let X^{INT} be an intersection set and let X^{FM} be defined on the same vector b. The affine transformation $\sigma_0 = s$ and $\sigma_t = s + x_t - b_t,\ 1 \leq t \leq n$, maps* $\mathrm{conv}(X^{FM})$ *into* $\mathrm{conv}(X^{INT}) \cap \{(\sigma, y) : 0 \leq \sigma_k - \sigma_0 + b_k \leq y_k,\ 1 \leq k \leq n\}$.

Proof. Let Z be defined on the same vector b. It is straightforward to check that the affine transformation $\sigma_0 = s$ and $\sigma_t = s + x_t - b_t,\ 1 \leq t \leq n$, maps $\mathrm{conv}(Z)$ into $\mathrm{conv}(X^{INT})$. By Theorem 3.3, $\mathrm{conv}(X^{FM}) = \mathrm{conv}(Z) \cap \{(s, x, y) : \underline{0} \leq x \leq y\}$ and the result follows. ☐

The above theorem shows that an external description of $\mathrm{conv}(X^{FM})$ can be obtained from an external description of $\mathrm{conv}(X^{INT})$. Such a description is already known:

PROPOSITION 3.5 (Günlük and Pochet [5]). *Consider the mixing set X^{MIX} defined in* (1.4)–(1.5). *For $t = 1, \ldots, n$ we define $f_t := b_t - \lfloor b_t \rfloor$. Let $T \subseteq N$ and suppose that $i_1, \ldots, i_{|T|}$ is an ordering of T such that $f_{i_{|T|}} \geq \cdots \geq f_{i_1} \geq f_{i_0} := 0$. Then the* mixing inequalities

$$s \geq \sum_{t=1}^{|T|} (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - y_{i_t}),$$

$$s \geq \sum_{t=1}^{|T|} (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - y_{i_t}) + (1 - f_{i_{|T|}})(\lfloor b_{i_1} \rfloor - y_{i_1})$$

are valid for X^{MIX}. Moreover, adding all mixing inequalities to the linear constraints defining X^{MIX} gives the convex hull of X^{MIX}.

PROPOSITION 3.6 (Miller and Wolsey [6]). *Let $X_k^{MIX}(n^k, s^k, y^k, b^k)$ for $1 \leq k \leq m$ be m mixing sets with some or all y variables in common. Let $X^* = \cap_{k=1}^m X_k^{MIX}$. Then*

$$\mathrm{conv}(X^*) = \bigcap_{k=1}^{m} \mathrm{conv}(X_k^{MIX}). \tag{3.5}$$

OBSERVATION 3. *Günlük and Pochet* [5] *have shown that there is a compact formulation of the polyhedron* $\mathrm{conv}(X^{MIX})$, *see also* [2]. *Therefore it follows from Theorem* 3.4 *and Proposition* 3.6 *that a compact formulation of* $\mathrm{conv}(X^{FM})$ *can be obtained by writing the compact formulations of all the mixing polyhedra* $\mathrm{conv}(X_k^{MIX})$, *together with the inequalities $0 \leq \sigma_t - \sigma_0 + b_t \leq y_t,\ 1 \leq t \leq n$ and then applying the transformation $s = \sigma_0$ and $x_t = -s + \sigma_t + b_t,\ 1 \leq t \leq n$.*

3.3. Variants of X^{FM}. Here for the purpose of comparison we examine the convex hulls of two sets closely related to X^{FM}.

The first is the relaxation obtained by dropping the non-negativity constraint on the flow variables x. The *unrestricted mixing set with flows* X^{UFM} is the set:

$$s + x_t \geq b_t \text{ for } 1 \leq t \leq n$$
$$x_t \leq y_t \text{ for } 1 \leq t \leq n$$
$$s \in \mathbb{R}^1_+,\ x \in \mathbb{R}^n,\ y \in \mathbb{Z}^n_+$$

where $0 < b_1 \leq \ldots \leq b_n$, $b \in \mathbb{Q}^n$. Its convex hull turns out to be much simpler and in fact the unrestricted mixing set with flows and the mixing set are closely related.

Proposition 3.7. *For an unrestricted mixing set with flows X^{UFM} and the mixing set X^{MIX} defined on the same vector b,*

$$\text{conv}(X^{UFM}) = \{(s,x,y):\ (s,y) \in \text{conv}(X^{MIX});\ b_t - s \leq x_t \leq y_t,\ 1 \leq t \leq n\}.$$

Proof. Let $P = \{(s,x,y):\ (s,y) \in \text{conv}(X^{MIX});\ b_t - s \leq x_t \leq y_t,\ 1 \leq t \leq n\}$. The inclusion $\text{conv}(X^{UFM}) \subseteq P$ is obvious. In order to show that $P \subseteq \text{conv}(X^{UFM})$, we prove that the extreme rays (resp. vertices) of P are rays (resp. feasible points) of $\text{conv}(X^{UFM})$.

The cone $\{(s,x,y) \in \mathbb{R}^1_+ \times \mathbb{R}^n \times \mathbb{R}^n_+ : -s \leq x_t \leq y_t,\ 1 \leq t \leq n\}$ is the recession cone of both P and $\text{conv}(X^{UFM})$, thus P and $\text{conv}(X^{UFM})$ have the same rays.

We now prove that if (s^*,x^*,y^*) is a vertex of P, then (s^*,x^*,y^*) belongs to $\text{conv}(X^{UFM})$. It is sufficient to show that y^* is integer. We do so by proving that (s^*,y^*) is a vertex of $\text{conv}(X^{MIX})$. If not, there exists a nonzero vector $(u,w) \in \mathbb{R} \times \mathbb{R}^n$ such that $(s^*,y^*) \pm (u,w) \in \text{conv}(X^{MIX})$ and $w_t = -u$ whenever $y^*_t = b_t - s^*$. Define a vector $v \in \mathbb{R}^n$ as follows: If $x^*_t = b_t - s^*$, set $v_t = -u$ and if $x^*_t = y^*_t$, set $v_t = w_t$. (Since x^*_t satisfies at least one of these two equations, this assignment is indeed possible). It is now easy to check that, for $\varepsilon > 0$ sufficiently small, $(s^*,x^*,y^*) \pm \varepsilon(u,v,w) \in P$, a contradiction. Therefore (s^*,y^*) is a vertex of $\text{conv}(X^{MIX})$ and thus $(s^*,y^*) \in X^{MIX}$. Then $(s^*,x^*,y^*) \in X^{UFM}$ and the result is proved. ∎

The second set we consider is a restriction of the set X^{FM} in which we add simple bounds and network dual constraints on the integer variables y. Specifically, consider the following inequalities:

$$l_i \leq y_i \leq u_i, \qquad 1 \leq i \leq n \tag{3.6}$$
$$\alpha_{ij} \leq y_i - y_j \leq \beta_{ij},\ 1 \leq i,j \leq n \tag{3.7}$$

where $l_i, u_i, \alpha_{ij}, \beta_{ij} \in \mathbb{Z} \cup \{+\infty, -\infty\}$ and define the following set:

$$W = \{(s,x,y) \in \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{Z}^n : y \text{ satisfies (3.6)–(3.7)}\}.$$

We assume that for every index i, W contains a vector with $y_i > 0$.

Theorem 3.8.

$$\text{conv}(X^{FM} \cap W) = \text{conv}(X^{FM}) \cap W.$$

Proof. The proof uses the same technique as in §3.1–3.2, where Z (resp. X^{FM}) has to be replaced with $Z \cap W$ (resp. $X^{FM} \cap W$). We only point out the main differences.

To see that the proof of Theorem 3.3 is still valid, note that the extreme rays of $\text{conv}(Z \cap W)$ are of the following types:

(i) $(1, \underline{0}, \underline{0})$ and $(0, e_k, \underline{0})$;
(ii) $(0, \underline{0}, y)$ for suitable vectors $y \in \mathbb{Z}^n$.

However, the rays of type (ii) are also rays of $\mathrm{conv}(X^{FM} \cap W)$. Also, the condition that for every index i, W contains a vector with $y_i > 0$, shows that none of the inequalities $0 \le x_i \le y_i$ defines an improper face of $\mathrm{conv}(X^{FM} \cap W)$ and Corollary 2.3 can still be applied. Thus the proof of Theorem 3.3 is still valid.

Finally, the following extension of equation (3.5) (due to Miller and Wolsey [6]) is needed: $\mathrm{conv}(X^* \cap W) = \cap_{k=1}^m \mathrm{conv}(X_k^{MIX}) \cap W$. □

Note that since the feasible region of problem (1.9) is of the type $X^{FM} \cap W$, Theorem 3.8 yields a linear inequality description of the feasible region of the two period stochastic lot-sizing model with constant capacities.

4. An internal description of X^{FM}. Since the extreme rays of $\mathrm{conv}(X^{FM})$ are described in Observation 1, in order to give a complete internal description of $\mathrm{conv}(X^{FM})$ we only have to characterize its vertices. These will then be used to describe a simple polynomial algorithm for optimizing over X^{FM}.

First we state a result concerning the vertices of any mixed integer set.

LEMMA 4.1. *Let $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{Z}^p : Ax + By \le c\}$. If (x^*, y^*) is a vertex of* $\mathrm{conv}(P)$*, then x^* is a vertex of the polyhedron $P(y^*) = \{x \in \mathbb{R}^n : Ax \le c - By^*\}$.*

Proof. If x^* is not a vertex of $P(y^*)$, there exists a nonzero vector $\varepsilon \in \mathbb{R}^n$, $\varepsilon \ne \underline{0}$, such that $A(x^* \pm \varepsilon) \le c - By^*$. But then $(x^*, y^*) \pm (\varepsilon, \underline{0})$ is in P and thus (x^*, y^*) is not a vertex of $\mathrm{conv}(P)$. □

In the following, given a point $p = (\bar{s}, \bar{x}, \bar{y})$ in $\mathrm{conv}(X^{FM})$, we denote by $f_{\bar{s}}$ the fractional part of $\bar{s}$.

CLAIM 4.2. *Let $v = (s^*, x^*, y^*)$ be a vertex of* $\mathrm{conv}(X^{FM})$*. If $s^* > 0$, there exists $j \in N$ such that $s^* + x_j^* = b_j$, $f_{s^*} = f_j$ and $s^* \le b_j$.*

Proof. By Lemma 4.1, (s^*, x^*) is a vertex of the polyhedron $P(y^*)$ defined by

$$s + x_t \ge b_t \text{ for } 1 \le t \le n \tag{4.1}$$

$$x_t \le y_t^* \text{ for } 1 \le t \le n \tag{4.2}$$

$$s \in \mathbb{R}_+^1, x \in \mathbb{R}_+^n. \tag{4.3}$$

Then among the constraints defining $P(y^*)$ there exist $n+1$ inequalities which are tight for (s^*, x^*) and whose left-hand sides form a nonsingular $(n+1) \times (n+1)$ matrix. Therefore, if $s^* > 0$ there exists an index j such that $s^* + x_j^* = b_j$ and either $x_j^* = y_j^*$ or $x_j^* = 0$. Thus $x_j^* \in \mathbb{Z}$ and thus $f_{s^*} = f_j$. Also $x_j^* \ge 0$ implies $s^* \le b_j$. □

CLAIM 4.3. *Let $v = (s^*, x^*, y^*)$ be a vertex of* $\mathrm{conv}(X^{FM})$*. Then for $1 \le t \le n$*

$$y_t^* = \max\{0, \lceil b_t - s^* \rceil\}. \tag{4.4}$$

Proof. Suppose $b_t - s^* < 0$. Then either $x_t^* = 0$ or $x_t^* = y_t^*$. Now if $y_t^* \ge 1$, in the first case both points $v \pm (0, \underline{0}, e_t)$ are in X^{FM}, in the second case both points $v \pm (0, e_t, e_t)$ are in X^{FM}, a contradiction.

Suppose $b_t - s^* \ge 0$. If $y_t^* \ge \lceil b_t - s^* \rceil + 1$ then, setting $\varepsilon = \min\{x_t^* - (b_t - s^*), 1\}$, both points $v \pm (0, \varepsilon e_t, e_t)$ are in X^{FM}, a contradiction. □

CLAIM 4.4. *Let $v = (s^*, x^*, y^*)$ be a vertex of* $\mathrm{conv}(X^{FM})$*. Then for $1 \le t \le n$*

$$x_t^* = \begin{cases} 0 & \text{if } \; b_t - s^* < 0, \\ b_t - s^* \quad \text{or} \quad \lceil b_t - s^* \rceil & \text{if } \; b_t - s^* \ge 0. \end{cases} \tag{4.5}$$

Proof. As (s^*, x^*) is a vertex of the polyhedron $P(y^*)$ defined by (4.1)–(4.3), it is easy to verify as in the proof of Claim 4.2 that for each t one of the following holds: either $s^* + x^*_t = b_t$ or $x^*_t = 0$ or $x^*_t = y^*_t = \max\{0, \lceil b_t - s^* \rceil\}$ (where the last equality follows from Claim 4.3). It follows that if $b_t - s^* < 0$ then $x^*_t = 0$ (otherwise inequality $x^*_t \geq 0$ would be violated) and that if $b_t - s^* \geq 0$ then $x^*_t \in \{b_t - s^*, \lceil b_t - s^* \rceil\}$ (otherwise inequality $s^* + x^* \geq b_t$ would be violated). □

Given a point $p = (\bar{s}, \bar{x}, \bar{y})$ in $\text{conv}(X^{FM})$, we define the following subsets of N:

$$N_p = \{t \in N : -1 < b_t - \bar{s} \leq 0\},$$
$$P_p = \{t \in N : 0 < b_t - \bar{s} < 1\}.$$

CLAIM 4.5. *Let $v = (s^*, x^*, y^*)$ be a vertex of* $\text{conv}(X^{FM})$*. If $s^* \geq 1$ then $N_v \cup P_v \neq \emptyset$. Moreover, if $s^* \geq 1$ and $N_v = \emptyset$ then there exists $t \in P_v$ such that $0 < x^*_t < 1$.*

Proof. Suppose $s^* \geq 1$ and $N_v \cup P_v = \emptyset$. Then $|b_t - s^*| \geq 1$, $1 \leq t \leq n$. Let $I \subseteq N$ be the set of indices t such that $b_t - s^* \geq 1$. Note that if $t \in I$, then $x^*_t \geq 1$ by Claim 4.4, and that if $t \notin I$, then $s^* + x^*_t \geq b_t + 1$. It follows that both points $v \pm (1, -e_I, -e_I)$ are in X^{FM}, a contradiction as v is a vertex of $\text{conv}(X^{FM})$.

Now suppose $s^* \geq 1$ and $N_v = \emptyset$ and assume that for every $t \in P_v$ either $x^*_t = 0$ or $x^*_t \geq 1$. Then Claim 4.4 implies that $x^*_t = 1$ for every $t \in P_v$. If $t \notin P_v$ then either $b_t - s^* \leq -1$ or $b_t - s^* \geq 1$, as $N_v = \emptyset$. Let I be the set of indices t such that $b_t - s^* \geq 1$. Note that if $t \in I$, then $x^*_t \geq 1$, and that if $t \notin P_v \cup I$, then $s^* + x^*_t \geq b_t + 1$. Thus it follows that both points $v \pm (1, -e_{P_v \cup I}, -e_{P_v \cup I})$ are in X^{FM}, again a contradiction. □

We need the following Lemma.

LEMMA 4.6. *Let $p = (\bar{s}, \bar{x}, \bar{y}) \in$* $\text{conv}(X^{FM})$*. Suppose that the components of p satisfy both conditions* (4.4) *and* (4.5)*. If for every convex combination of points in X^{FM} giving p, all the points appearing with nonzero coefficient have s-component equal to $\bar{s}$, then p is a vertex of* $\text{conv}(X^{FM})$*.*

Proof. Consider any convex combination of points in X^{FM} giving p and let C be the set of points in X^{FM} appearing with nonzero coefficient in such combination. Given $t \in N$, either $\bar{y}_t = 0$ or $\bar{y}_t = \lceil b_t - \bar{s} \rceil$. If $\bar{y}_t = 0$ then, since all points in C satisfy $y_t \geq 0$, they all satisfy $y_t = 0$. If $\bar{y}_t = \lceil b_t - \bar{s} \rceil$ then, since all points in C satisfy $y_t \geq \lceil b_t - \bar{s} \rceil$, they all satisfy $y_t = \lceil b_t - \bar{s} \rceil$. Thus all points in C have the same y-components. As to the x-components, either $\bar{x}_t = 0$ or $\bar{x}_t = b_t - \bar{s}$ or $\bar{x}_t = \lceil b_t - \bar{s} \rceil$. If $\bar{x}_t = 0$ then, since all points in C satisfy $x_t \geq 0$, they all satisfy $x_t = 0$. If $\bar{x}_t = b_t - \bar{s}$ then, since all points in C satisfy $x_t \geq b_t - \bar{s}$, they all satisfy $x_t = b_t - \bar{s}$. If $\bar{x}_t = \lceil b_t - \bar{s} \rceil$ then $\bar{x}_t = \bar{y}_t$ and so, since all points in C satisfy $x_t \leq y_t$, they all satisfy $x_t = y_t$. Thus all points in C have the same x-components. Therefore all points in C are identical. This shows that p cannot be expressed as a convex combination of points in X^{FM} distinct from p, thus p is a vertex of $\text{conv}(X^{FM})$. □

CLAIM 4.7. *Let $p = (\bar{s}, \bar{x}, \bar{y}) \in$* $\text{conv}(X^{FM})$*. Suppose that the components of p satisfy both conditions* (4.4) *and* (4.5)*. If $\bar{s} = 0$, or $\bar{s} = f_j$ for some $j \in N$, or $\bar{s} = b_j$ for some $j \in N$, then p is a vertex of* $\text{conv}(X^{FM})$*.*

Proof. Consider an arbitrary convex combination of points in X^{FM} giving p and let C be the set of points appearing with nonzero coefficient in such combination. Suppose $\bar{s} = 0$. Then all points in C satisfy $s = 0$. Thus, by Lemma 4.6, p is a vertex of $\text{conv}(X^{FM})$. Suppose $\bar{s} = f_j$ for some j. Condition (4.4) implies that $\bar{s} + \bar{y}_j = b_j$. Then all points in C satisfy $s + y_j = b_j$ and thus they all have $f_s = f_j$, in particular $s \geq f_j$. It follows that they all satisfy $s = f_j$. The conclusion now follows from

Lemma 4.6. Suppose $\bar{s} = b_j$ for some j. Then $\bar{x}_j = 0$, thus all points in C satisfy $x_j = 0$ and so they satisfy $s \geq b_j$. It follows that they all satisfy $s = b_j$. Again the conclusion follows from Lemma 4.6. □

Claim 4.8. *Let $p = (\bar{s}, \bar{x}, \bar{y}) \in \mathrm{conv}(X^{FM})$. Let $\bar{s} = m + f_j$ for some $j \in N$, where $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$. Suppose that there exists an index h such that $-1 < b_h - \bar{s} < 0$. Suppose that the components of p satisfy both conditions (4.4) and (4.5). Then p is a vertex of $\mathrm{conv}(X^{FM})$.*

Proof. Consider an arbitrary convex combination of points in X^{FM} giving p and let C be the set of points appearing with nonzero coefficient in such a combination. Since $b_j - \bar{s} \geq 0$ by assumption, condition (4.4) implies that $\bar{s} + \bar{y}_j = b_j$; then all points in C satisfy $s + y_j = b_j$ and thus they all have $f_s = f_j = f_{\bar{s}}$. Since $b_h - \bar{s} < 0$, Claim 4.4 implies that $\bar{x}_h = 0$; then all points in C satisfy $x_h = 0$. Suppose that there exists a point in C satisfying $s \neq \bar{s}$. Then there exists a point in C satisfying $s < \bar{s}$, i.e. $s \leq \bar{s} - 1$. Therefore, for such point, $s + x_h = s \leq \bar{s} - 1 < b_h$, a contradiction. Thus all points in C satisfy $s = \bar{s}$. Lemma 4.6 concludes the proof. □

Claim 4.9. *Let $p = (\bar{s}, \bar{x}, \bar{y}) \in \mathrm{conv}(X^{FM})$. Let $\bar{s} = m + f_j$ for some $j \in N$, where $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$. Suppose that there exists an index h such that $0 < b_h - \bar{s} < 1$. Suppose that the components of p satisfy both conditions (4.4) and (4.5) and that $\bar{x}_h = b_h - \bar{s}$. Then p is a vertex of $\mathrm{conv}(X^{FM})$.*

Proof. Consider an arbitrary convex combination of points in X^{FM} giving p and let C be the set of points appearing with nonzero coefficient in such combination. Since by assumption $b_j - \bar{s} \geq 0$, condition (4.4) implies that $\bar{s} + \bar{y}_j = b_j$; then all points in C satisfy $s + y_j = b_j$ and thus they all have $f_s = f_j = f_{\bar{s}}$. Since $\bar{s} + \bar{x}_h = b_h$, all points in C satisfy $s + x_h = b_h$. Suppose that there exists a point in C satisfying $s \neq \bar{s}$. Then there exists a point in C satisfying $s > \bar{s}$, i.e. $s \geq \bar{s} + 1$ since $f_s = f_{\bar{s}}$. Therefore, for such point, $x_h = b_h - s \leq b_h - \bar{s} - 1 < 0$, a contradiction. Thus all points in C satisfy $s = \bar{s}$. Lemma 4.6 concludes the proof. □

Theorem 4.10. *The point $p = (s^*, x^*, y^*)$ is a vertex of $\mathrm{conv}(X^{FM})$ if and only if its components satisfy one of the following conditions:*

(i) $s^* = 0$
$x^*_t = b_t$ *or* $x^*_t = \lceil b_t \rceil$ *for* $1 \leq t \leq n$
$y^*_t = \lceil b_t \rceil$ *for* $1 \leq t \leq n$

(ii) $s^* = f_j$ *for some* $1 \leq j \leq n$
$$x^*_t = \begin{cases} 0 & \text{if } b_t - f_j < 0 \\ b_t - f_j \text{ or } \lceil b_t - f_j \rceil & \text{if } b_t - f_j \geq 0 \end{cases}$$
$y^*_t = \max\{0, \lceil b_t - f_j \rceil\}$ *for* $1 \leq t \leq n$

(iii) $s^* = b_j$ *for some* $1 \leq j \leq n$
$$x^*_t = \begin{cases} 0 & \text{if } b_t - b_j < 0 \\ b_t - b_j \text{ or } \lceil b_t - b_j \rceil & \text{if } b_t - b_j \geq 0 \end{cases}$$
$y^*_t = \max\{0, \lceil b_t - b_j \rceil\}$ *for* $1 \leq t \leq n$

(iv) $s^* = m + f_j$ *for some* $1 \leq j \leq n$*, where* $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$*, and* $-1 < b_h - s^* < 0$ *for some* $1 \leq h \leq n$
$$x^*_t = \begin{cases} 0 & \text{if } b_t - s^* < 0 \\ b_t - s^* \text{ or } \lceil b_t - s^* \rceil & \text{if } b_t - s^* \geq 0 \end{cases}$$
$y^*_t = \max\{0, \lceil b_t - s^* \rceil\}$ *for* $1 \leq t \leq n$

(v) $s^* = m + f_j$ *for some* $1 \le j \le n$*, where* $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$*, and* $0 < b_h - s^* < 1$ *for some* $1 \le h \le n$

$$x_t^* = \begin{cases} 0 & \text{if } b_t - s^* < 0 \\ b_t - s^* \text{ or } \lceil b_t - s^* \rceil & \text{if } b_t - s^* \ge 0 \text{ and } t \ne h \\ b_t - s^* & \text{if } t = h \end{cases}$$

$y_t^* = \max\{0, \lceil b_t - s^* \rceil\}$ *for* $1 \le t \le n$

Proof. Claim 4.7 shows that points of types (i), (ii) and (iii) are vertices of $\mathrm{conv}(X^{FM})$. Claim 4.8 and Claim 4.9 show that points of types (iv) and (v) are vertices of $\mathrm{conv}(X^{FM})$. It remains to prove that there are no other vertices. If $p = (s^*, x^*, y^*)$ is a vertex of $\mathrm{conv}(X^{FM})$ then its components satisfy conditions (4.4) and (4.5). By Claim 4.2, either $s^* = 0$ or $f_{s^*} \in \{f_1, \ldots, f_n\}$. If $s^* = 0$, p satisfies the conditions of case (i). If $s^* = f_j$ for some j, p satisfies the conditions of case (ii). If $s^* = b_j$ for some j, then p satisfies the conditions of case (iii). Otherwise, by Claim 4.2 there exists $j \in N$ such that $f_{s^*} = f_j$ and $1 \le s^* < b_j$. Then $s^* = m + f_j$, where $0 < m < \lfloor b_j \rfloor$, $m \in \mathbb{Z}$. Claim 4.5 implies that $N_p \cup P_p \ne \emptyset$. If $N_p \ne \emptyset$ then p satisfies the conditions of case (iv). Otherwise $P_p \ne \emptyset$ and Claim 4.5 implies the existence of an index $h \in P_p$ such that $0 < x_h^* < 1$. But then necessarily $x_h^* = b_h - s^*$ and thus p satisfies the conditions of case (v). □

COROLLARY 4.11. *The problem of optimizing a rational linear function over the set* X^{FM} *(defined on a rational vector* b*) can be solved in polynomial time.*

Proof. Let $\alpha = (h, p, q) \in \mathbb{Q}^1 \times \mathbb{Q}^n \times \mathbb{Q}^n$ and consider the optimization problem

$$\min\{hs + px + qy : (s, x, y) \in X^{FM}\}. \tag{4.6}$$

Observation 1 shows that problem (4.6) is unbounded if and only if $h < 0$ or $p_t + q_t < 0$ or $q_t < 0$ for some $t \in N$. Otherwise there exists an optimal extreme point solution. Let S be the set of all possible values taken by variable s at a vertex of $\mathrm{conv}(X^{FM})$. By Theorem 4.10, $|S| = \mathcal{O}(n^2)$. For each $\bar{s} \in S$, let $V_{\bar{s}}$ be the set of vertices of $\mathrm{conv}(X^{FM})$ such that $s = \bar{s}$ and let $v_{\bar{s}}(\alpha)$ be an optimal solution of the problem

$$\min\{hs + px + qy : (s, x, y) \in V_{\bar{s}}\}.$$

The components of $v_{\bar{s}}(\alpha)$ satisfy $s = \bar{s}$, $y_t = \max\{0, \lceil b_t - \bar{s} \rceil\}$ for $1 \le t \le n$ and

$$x_t^* = \begin{cases} 0 & \text{if } b_t - \bar{s} < 0 \\ b_t - \bar{s} & \text{if } b_t - \bar{s} \ge 0 \text{ and } p_t \ge 0 \\ \lceil b_t - \bar{s} \rceil & \text{if } b_t - \bar{s} \ge 0 \text{ and } p_t < 0 \end{cases}$$

if the value $s = \bar{s}$ corresponds to one of cases (i)–(iv), and similarly for case (v).

Since solving problem (4.6) is equivalent to solving the problem $\min\{\alpha v_{\bar{s}}(\alpha) : \bar{s} \in S\}$, we only need to compute the objective function in $\mathcal{O}(n^2)$ points. This requires $\mathcal{O}(n^3)$ time. □

5. Concluding Remarks. Several other generalizations of the mixing set appear to be interesting, some of which are already being investigated.

A common generalization of the set studied in this paper and the continuous mixing set [6, 9] is the *continuous mixing set with flows*

$$X^{CFM} = \{(s, r, x, y) \in \mathbb{R}_+^1 \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{Z}_+^n : s + r_t + x_t \ge b_t,\ x_t \le y_t,\ 1 \le t \le n\}.$$

Though a compact extended formulation of this set has been found recently [1], the question of finding an inequality description in the original space of variables is still open.

The *mixing-MIR set with divisible capacities*

$$X^{MMIX} = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^n : s + C_t y_t \geq b_t\},$$

where $C_1|C_2|\cdots|C_n$, has been studied by de Farias and Zhao [3]. An interesting question is to give a polyhedral description of $\mathrm{conv}(X^{MMIX})$. The special case when the C_i only take two distinct values has been treated in Van Vyve [8].

Another intriguing question is the complexity status of the problem of optimizing a linear function over the *divisible mixing set*

$$X^{DMIX} = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^{mn}_+ : s + \sum_{j=1}^{m} C_j y_{jt} \geq b_t\},$$

with again $C_1|C_2|\cdots|C_n$. For the case $m = 2$, a compact extended formulation of $\mathrm{conv}(X^{DMIX})$ is given in Conforti and Wolsey [2].

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