THE JOURNAL OF SYMBOLIC LOGIC
Volume 57, Number 2, June 1992

THE MODAL LOGIC OF INEQUALITY

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Abstract. We consider some modal languages with a modal operator D whose semantics is based on the relation of inequality. Basic logical properties such as definability, expressive power and completeness are studied. Also, some connections with a number of other recent proposals to extend the standard modal language are pointed at.

§1. Introduction. As is well known, standard (propositional) modal and temporal logic cannot define all the natural assumptions one would like to make on the accessibility relation. One obvious move to try and overcome this lack of expressive power is to extend the languages of modal and temporal logic with new operators. One particular such extension consists in adding an operator D whose semantics is based on the relation of inequality. The proposal to consider the D-operator is due to several people independently, including Koymans [15] and Gargov, Passy and Tinchev [10]. This particular extension of the standard modal language is of interest for a number of reasons. First of all, it shows that some of the most striking deficiencies in expressive power may be removed with relatively simple means. Secondly, several recent proposals to enhance the expressive power of the standard language naturally give rise to considering the D-operator; thus the language with the operators $\diamond$ and D appears as a kind of fixed point amongst the wide range of recently introduced extensions of the standard language. And thirdly, many of the interesting logical phenomena that one encounters in the study of enriched modal languages are illustrated by this particular extension.

Applications of the D-operator can be found in [9], where it has been used in the study of various enriched modal languages, and in [15], where it is applied in the specification of message passing and time-critical systems.

The main subject of this paper is the modal language $\mathscr{L}(\diamond, D)$ whose operators are $\diamond$ and D. The remainder of §1 introduces the basic notions, and examines which of the (anti-) preservation results known from standard modal logic remain valid in the extended formalism. Next, §2 compares the expressive powers of modal languages that contain the D-operator with a number of other modal languages. In

Received December 10, 1990; revised May 16, 1991.

Research supported by the Netherlands Organization for Scientific Research (NWO). This paper is based on my Master's thesis written at the University of Amsterdam, Department of Philosophy, under the supervision of Johan van Benthem; I am grateful to him for his questions and suggestions.

0022-4812/92/5702-0018/$02.90

§3 we present the basic logics in some languages with the D-operator, and we give complete axiomatizations for several special structures; in it we prove that, given the right basic logic in $\mathscr{L}(D)$, all finite extensions thereof are complete; we also discuss analogues of the Sahlqvist theorem for $\mathscr{L}(\Diamond, D)$ and $\mathscr{L}(F, P, D)$. §4 then deals with definability, both of classes of frames and of classes of models.

1.1. Basics. The (multi-) modal languages we consider have an infinite supply of proposition letters ($p, q, r, \ldots$), propositional constants $\bot$, $\top$ and the usual Boolean connectives. Furthermore, they contain one or more unary modal operators. The standard language $\mathscr{L}(\Diamond)$ has operators $\Diamond$ and $\Box$, with $\Diamond$ being regarded as primitive, and $\Box$ being defined as $\neg\Diamond\neg$. (In general, $\mathscr{L}(O_1, \ldots, O_n)$ denotes the (multi-) modal language with operators $O_1, \ldots, O_n$.) We use $\varphi, \psi, \chi, \ldots$ to denote (multi-) modal formulas. The semantic structures are *frames*, i.e. ordered pairs $\langle W, R\rangle$ consisting of a nonempty set W with a binary relation R on W. To save words, we assume that $\mathscr{F}$ denotes the frame $\langle W, R\rangle$. In addition to these frames, structures called *models* will be used, consisting of a frame $\mathscr{F}$ together with a *valuation V on* $\mathscr{F}$ assigning subsets of W to proposition letters. We assume that $\mathscr{M}$ denotes the model $\langle \mathscr{F}, V\rangle$.

$\mathscr{M} \models \varphi[w]$ is defined as usual, the important case being: $\mathscr{M} \models \Diamond\varphi[w]$ iff, for some $v \in W$, Rwv and $\mathscr{M} \models \varphi[v]$. For temporal logic the clause for $\Diamond$ is replaced by two clauses for F and P: $\mathscr{M} \models F\varphi[w]$ iff, for some $v \in W$, Rwv and $\mathscr{M} \models \varphi[v]$; $\mathscr{M} \models P\varphi[w]$ iff, for some $v \in W$, Rvw and $\mathscr{M} \models \varphi[w]$. The semantics of the D-operator is given by $\mathscr{M} \models D\varphi[w]$ iff, for some $v \neq w$, $\mathscr{M} \models \varphi[v]$. From this, notions like $\mathscr{M} \models \varphi$, $\mathscr{F} \models \varphi[w]$, and $\mathscr{F} \models \varphi$ are defined as usual.

G and H are short for $\neg F \neg$ and $\neg P \neg$, respectively. The dual $\neg D \neg$ of D is denoted by $\bar{D}$. Using the D-operator, some useful abbreviations can be defined: $E\varphi := \varphi \vee D\varphi$ (there exists a point at which φ holds); $A\varphi := \varphi \wedge \bar{D}\varphi$ (φ holds at all points); $U\varphi := E(\varphi \wedge \neg D\varphi)$ (φ holds at a unique point).

The fact that some notions are sensitive to the language we are working with is reflected in our notation: e.g. we write $\mathscr{F} \equiv_{\Diamond, D} \mathscr{G}$ to mean that $\mathscr{F}$ and $\mathscr{G}$ validate the same $\varphi \in \mathscr{L}(\Diamond, D)$, and $\mathrm{Th}_{\Diamond, D}(\mathscr{F})$ for the set of formulas in $\mathscr{L}(\Diamond, D)$ that are valid on $\mathscr{F}$.

We will sometimes refer to the first-order languages $\mathscr{L}_0$ and $\mathscr{L}_1$: $\mathscr{L}_0$ has one binary predicate symbol R as well identity; $\mathscr{L}_1$ extends $\mathscr{L}_0$ with unary predicate symbols $P_1, P_2, \ldots, P, Q, \ldots$ corresponding to the proposition letters of the (multi-) modal language. First-order formulas will be denoted by $\alpha, \beta, \gamma, \ldots$. α is called *locally definable* in $\mathscr{L}(O_1, \ldots, O_n)$ if for some $\varphi \in \mathscr{L}(O_1, \ldots, O_n)$, for all $\mathscr{F}$, and all $w \in W$, $\mathscr{F} \models \alpha[w]$ iff $\mathscr{F} \models \varphi[w]$; it is called (*globally*) *definable* in $\mathscr{L}(O_1, \ldots, O_n)$ if for some $\varphi \in \mathscr{L}(O_1, \ldots, O_n)$, for all $\mathscr{F}$, $\mathscr{F} \models \alpha$ iff $\mathscr{F} \models \varphi$.

1.2. (Anti-) preservation and filtrations. It is well known that standard modal formulas are preserved under surjective p-morphisms, disjoint unions and generated subframes:

DEFINITION 1.1. 1. A surjective function f from a frame $\mathscr{F}_1$ to a frame $\mathscr{F}_2$ is called a *p-morphism* if (i) for all $w, v \in W_1$, if R_1wv then $R_2f(w)f(v)$; and (ii) for all $w \in W_1$ and $v \in W_2$, if $R_2f(w)v$ then there is a $u \in W_1$ such that R_1wu and $f(u) = v$.

2. $\mathscr{F}_1$ is called a *generated subframe* of a frame $\mathscr{F}_2$ if (i) $W_1 \subset W_2$; (ii) $R_1 = R_2 \cap (W_2 \times W_2)$; and (iii) for all $w \in W_1$ and $v \in W_2$, if R_2wv then $v \in W_1$.

3. Let $\mathscr{F}_i$ $(i \in I)$ be a collection of disjoint frames. Then the *disjoint union* $\biguplus_{i \in I} \mathscr{F}_i$ is the frame $\langle \bigcup\{W_i : i \in I\}, \bigcup\{R_i : i \in I\}\rangle$.

Here are some examples showing that adding the D-operator to $\mathscr{L}(\Diamond)$ gives an increase in expressive power:

1. $\Diamond p \to Dp$ defines $\forall x \neg Rxx$;
2. $p \vee Dp \to \Diamond p$ defines $R = W^2$;
3. $\Diamond \top \vee D\Diamond \top$ defines $R \neq \varnothing$.

Using the above preservation results, it is easily verified that none of these three conditions is definable in $\mathscr{L}(\Diamond)$. And conversely, the fact that they are definable in $\mathscr{L}(\Diamond, D)$ implies that we no longer have these preservation results in $\mathscr{L}(\Diamond, D)$. Moreover, they can be restored only at the cost of trivializing the constructions concerned.

A fourth important construction in standard modal logic is the following:

DEFINITION 1.2. Let $\mathscr{F}$ be a frame, and $X \subseteq W$. Then $L_R(X) = \{w \in W : \forall v \in W\ (Rwv \to v \in X)\}$. The *ultrafilter extension* $ue(\mathscr{F})$ is the frame $\langle W_{\mathscr{F}}, R_{\mathscr{F}}\rangle$, where $W_{\mathscr{F}}$ is the set of ultrafilters on W, and $R_{\mathscr{F}} U_1 U_2$ holds if, for all $X \subseteq W$, $L_R(X) \in U_1$ implies $X \in U_2$.

Standard modal formulas are *anti-preserved* under ultrafilter extension, i.e. if $ue(\mathscr{F}) \models \varphi$ then $\mathscr{F} \models \varphi$. (Cf. [4, Lemma 2.25].) Perhaps surprisingly, for formulas $\varphi \in \mathscr{L}(\Diamond, D)$ this result still holds good, as one easily deduces from the following result.

PROPOSITION 1.3. *Let V be a valuation on $\mathscr{F}$. Define the valuation $V_{\mathscr{F}}$ on $ue(\mathscr{F})$ by putting $V_{\mathscr{F}}(p) = \{U : V(p) \in U\}$. Then, for all ultrafilters U on W and all formulas $\varphi \in \mathscr{L}(\Diamond, D)$, we have $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \models \varphi[U]$ iff $V(\varphi) \in U$.*

PROOF. We argue by induction on φ. The cases $\varphi \equiv p, \neg\psi, \psi \wedge \chi, \Diamond\psi$ are proved in [4, Lemma 2.25]. The only new case is $\varphi \equiv D\psi$. Suppose $V(D\psi) = \{w : \exists v \neq w\ (v \in V(\psi))\} \in U$. We must find an ultrafilter $U_1 \neq U$ such that $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \models \psi[U_1]$. First assume that U contains a singleton, say, $U = \{X \subseteq W : X \supseteq \{w_0\}\}$. Then $w_0 \in V(D\psi)$, so there exists a $v \neq w_0$ with $v \in V(\psi)$. Since $v \neq w_0$, we must have $\{v\} \notin U$. Let U_1 be the ultrafilter generated by $\{v\}$; then $U \neq U_1$. Furthermore, $v \in V(\psi)$ implies $V(\psi) \in U_1$, and hence $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \models \psi[U_1]$, by the induction hypothesis. It follows that $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \models D\psi[U]$. Next, suppose that U does not contain a singleton. Since $V(D\psi) \in U$, we find some $w_0 \in V(D\psi)$. Let v be a point such that $v \neq w_0$ and $v \in V(\psi)$. Then $\{v\} \notin U$, and we can proceed as in the previous case.

Conversely, assume that $V(D\psi) \notin U$. We have to show that $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \not\models D\psi[U]$. Since $V(D\psi) \notin U$, we have that $X = \{w : \forall v(v \neq w \to v \notin V(\psi))\} \in U$, and hence $X \neq \varnothing$. Let $w_0 \in X$. Clearly, if $w_0 \notin V(\psi)$, then $X = W$ and $V(\psi) = \varnothing$. Consequently, for all ultrafilters $U_1 \neq U$ we have $V(\psi) \notin U_1$. So, by the induction hypothesis, $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \not\models \psi[U_1]$, and hence $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \not\models D\psi[U]$, as required. If, on the other hand, $w_0 \in V(\psi)$, then $X = \{w_0\} = V(\psi)$, and U is generated by X. It follows that, for any ultrafilter $U_1 \neq U$, $X = V(\psi) \notin U_1$. So by the induction hypothesis we have $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \not\models \psi[U_1]$ for such U_1. This implies $\langle ue(\mathscr{F}), V_{\mathscr{F}}\rangle \not\models D\psi[U]$. QED.

COROLLARY 1.4. *For any frame $\mathscr{F}$ and all $\varphi \in \mathscr{L}(\Diamond, D)$, if $ue(\mathscr{F}) \models \varphi$ then $\mathscr{F} \models \varphi$.*

COROLLARY 1.5. *$\exists x\, Rxx$ is not definable in $\mathscr{L}(\Diamond, D)$.*

PROOF. Evidently, $\mathscr{F} = \langle \mathrm{N}, < \rangle \not\models \exists x Rxx$. Elementary reasoning shows that, for any nonprincipal ultrafilter U on N, $R_{\mathscr{F}}UU$. Hence, $ue(\mathscr{F}) \models \exists x Rxx$. Now apply 1.4. QED.

Another important notion from standard modal logic is that of a *filtration*. It has a straightforward adaptation to $\mathscr{L}(\Diamond, D)$:

DEFINITION 1.6. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be models, and let Σ be a set of formulas $\varphi \in \mathscr{L}(\Diamond, D)$ closed under subformulas. A surjective function g: $\mathscr{M}_1 \to \mathscr{M}_2$ is an *extended filtration with respect to* Σ if

1. for all $w, v \in W_1$, if R_1wv then $R_2g(w)g(v)$,
2. for all $w \in W_1$ and all proposition letters p in Σ, $w \in V_1(p)$ iff $g(w) \in V_2(p)$, and
3. for all $w \in W_1$ and all $\triangle\varphi \in \Sigma$, if $\mathscr{M}_2 \models \triangle\varphi[g(w)]$ then $\mathscr{M}_1 \models \triangle\varphi[w]$, where $\triangle \in \{\Diamond, D\}$.

PROPOSITION 1.7. *If g is an extended filtration with respect to Σ from $\mathscr{M}_1$ to $\mathscr{M}_2$, then, for all $w \in W_1$ and all $\varphi \in \Sigma$, $\mathscr{M}_1 \models \varphi[w]$ iff $\mathscr{M}_2 \models \varphi[g(w)]$.*

Recall that the standard example of a filtration in ordinary modal logic is the *modal collapse*: given a model $\mathscr{M}$ and a set Σ that is closed under subformulas, it is defined as the model $\mathscr{M}'$, where, for $g(w) = \{\varphi \in \Sigma: \mathscr{M} \models \varphi[w]\}$ and $W' = g[W]$, $R'g(w)g(v)$ holds iff, for all $\Box\varphi \in \Sigma$, $\Box\varphi \in g(w)$ implies $\varphi \in g(v)$, and $V'(p) = \{g(w): p \in g(w)\}$. To obtain an analogue of the modal collapse for $\mathscr{L}(\Diamond, D)$, take the ordinary modal collapse, and double the points that correspond to more than one point in the original model. A simple inductive proof then shows that corresponding (doubled) points verify the same formulas.

Using the extended collapse one may show in a standard way that formulas $\varphi \in \mathscr{L}(\Diamond, D)$ satisfy the finite model property (and, hence, that the validities in $\mathscr{L}(\Diamond, D)$ form a recursive set): such a $\varphi \in \mathscr{L}(\Diamond, D)$ has a model iff it has a model with at most $2 \cdot 2^n$ worlds, where n is the length of φ. De Smit and van Emde Boas [19] show that for $\varphi \in \mathscr{L}(D)$ one can do considerably better: such a formula has a model iff it has a model with at most $4n$ worlds, where n is the length of φ. Thus, the satisfiability problem for pure D-formulas is NP-complete. The satisfiability for $(\Diamond, D)$-formulas is certainly PSPACE-hard; it is unknown, however, whether this problem is *in* PSPACE.

§2. Some comparisons. In this section we compare modal languages with the D-operator to some languages without it. It is not our aim to give a complete description of all the aspects in which languages of the former kind differ from, or are the same as, languages of the latter kind, but merely to highlight some of the features of the former languages.

2.1. The language $\mathscr{L}(D)$.

PROPOSITION 2.1. *All formulas $\varphi \in \mathscr{L}(D)$ define first-order conditions.*

PROOF. Using the ST-translation as defined in §4.2, such formulas can be translated into equivalent second-order formulas containing only monadic predicate variables. By a result in [1, Chapter IV] these formulas are in turn equivalent to first-order ones. QED.

Proposition 2.1 marks a considerable difference with $\mathscr{L}(\Diamond)$: as is well known, not all $\mathscr{L}(\Diamond)$-formulas correspond to first-order conditions. In the opposite direction, there are also some natural conditions undefinable in $\mathscr{L}(\Diamond)$ that are definable in $\mathscr{L}(D)$. For example, using the preservation of standard modal formulas under generated subframes and disjoint unions, it is easily verified that no finite cardinality is definable in $\mathscr{L}(\Diamond)$; on the other hand, although 2.1 implies that "infinity" is not definable in $\mathscr{L}(D)$, we do have

PROPOSITION 2.2 (KOYMANS). *All finite cardinalities are definable in* $\mathscr{L}(D)$.

PROOF. For $n \in \mathbf{N}$, $|W| \leq n$ is defined by

$$\bigwedge_{1 \leq i \leq n+1} Up_i \rightarrow \bigvee_{1 \leq i < j \leq n+1} E(p_i \wedge p_j),$$

while $|W| > n$ is defined by

$$A\left(\bigvee_{1 \leq i \leq n} p_i\right) \rightarrow E \bigvee_{1 \leq i \leq n} (p_i \wedge Dp_i).$$ QED.

THEOREM 2.3 (Functional Completeness). *On frames $\mathscr{L}(D)$ is equivalent with the language of first-order logic over* $=$.

PROOF. All first-order formulas over identity can be defined as a Boolean combination of formulas expressing the existence of at least a certain number of elements. By 2.2 these formulas are definable in $\mathscr{L}(D)$. The converse follows from 2.1. QED.

2.2. The languages $\mathscr{L}(\Diamond, D)$ and $\mathscr{L}(\Diamond)$. One way to compare the expressive powers of two languages is to examine their ability to discriminate between special (read: well-known) structures. For example, in contrast to $\mathscr{L}(\Diamond)$, $\mathscr{L}(\Diamond, D)$ is able to distinguish $\mathbf{Z}$ from $\mathbf{N}$: $\langle \mathbf{N}, < \rangle \not\equiv_{\Diamond, D} \langle \mathbf{Z}, < \rangle$. This follows from the fact that the existence of a (different) predecessor is expressible in $\mathscr{L}(\Diamond, D)$ by means of the formula $p \wedge \bar{D} \neg p \rightarrow D \Diamond p$.

So $\forall x \exists y (x \neq y \wedge Ryx)$ is an $\mathscr{L}_0$-condition on frames which is definable in $\mathscr{L}(\Diamond, D)$ but not in $\mathscr{L}(\Diamond)$. Other well-known $\mathscr{L}_0$-conditions undefinable in $\mathscr{L}(\Diamond)$ are irreflexivity and anti-symmetry. By the next result, these conditions do have an $\mathscr{L}(\Diamond, D)$-equivalent:

PROPOSITION 2.4. *All Π^1_1-sentences in R, $=$ of the purely universal form*

$$\forall P_1 \cdots \forall P_m \forall x_1 \cdots \forall x_n \mathrm{BOOL}(P_i x_j, Rx_i x_j, x_i = x_j)$$

are definable in $\mathscr{L}(\Diamond, D)$.

PROOF. Let $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ be proposition letters such that each of $p_1, \ldots, p_m$ is different from each of $q_1, \ldots, q_n$. Now take $Uq_1 \wedge \cdots \wedge Uq_n \rightarrow \mathrm{BOOL}(E(q_i \wedge p_j), E(q_i \wedge \Diamond q_j), E(q_i \wedge q_j))$. QED.

It is well known that two finite rooted frames that validate the same formulas $\varphi \in \mathscr{L}(\Diamond)$ are isomorphic. This is improved upon in $\mathscr{L}(\Diamond, D)$:

COROLLARY 2.5. *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are finite frames, then $\mathscr{F}_1 \equiv_{\Diamond, D} \mathscr{F}_2$ iff $\mathscr{F}_1 \cong \mathscr{F}_2$.*

PROOF. Finite frames are isomorphic iff they have the same universal first-order theory. So from 2.4 the result follows. Alternatively, one may give, for each finite frame $\mathscr{F}$, a "characteristic formula" $\chi_{\mathscr{F}}$ such that, for all $\mathscr{G}$, $\mathscr{G} \not\models \neg \chi_{\mathscr{F}}$ iff $G \cong \mathscr{F}$ (cf. §4.1). QED.

Let us call a set T of (multi-) modal formulas (*frame*) *categorical* if, up to isomorphism, there is only one frame validating T; T is called *λ-categorical* if, up to isomorphism, T has only one frame of power λ validating it. (λ-) categoricity is an important notion in first-order logic that is meaningless in standard modal languages: by some elementary manipulations one easily establishes that if $\mathscr{F} \models T$ for some $\mathscr{F}$, where T is a theory in either $\mathscr{L}(\Diamond)$ or $\mathscr{L}(F, P)$, and if I is a set of indices, then for each $i \in I$ there is a frame $\mathscr{F}_i \models T$ such that $\mathscr{F}_i \not\models F_j$ if $i \neq j$. In contrast, for any *finite* frame $\mathscr{F}$ the complete $(\diamond, D)$-theory $\mathrm{Th}_{\diamond,D}(\mathscr{F})$ is easily seen to be categorical by 2.4.

The classical example of an ω-categorical theory in first-order logic is the complete theory of the rationals. By standard techniques one can show that $\mathrm{Th}_{\diamond}(\mathbf{Q})$ is not ω-categorical; but $\mathrm{Th}_{\diamond,D}(\mathbf{Q})$ *is* ω-categorical:

PROPOSITION 2.6. *The complete* $(\diamond, D)$*-theory of* $\mathbf{Q}$ *is* ω*-categorical.*

PROOF. It suffices to give formulas $\varphi \in \mathscr{L}(\diamond, D)$ which are equivalent to the axioms for the theory of dense linear order without endpoints:

$$\begin{array}{ll}
\forall xyz(x < y \wedge y < z \rightarrow x < z), & \diamond\diamond p \rightarrow \diamond p, \\
\forall xy(x < y \wedge y < x \rightarrow x = y), & p \wedge \bar{D}\neg p \rightarrow \Box(\diamond p \rightarrow p), \\
\forall x \neg(x < x), & \diamond p \rightarrow Dp, \\
\forall xy(x = y \vee x < y \vee y < x), & p \rightarrow \diamond q \vee \bar{D}(q \rightarrow \diamond p), \\
\forall xy \exists z(x < y \rightarrow x < z \wedge z < y), & \diamond p \rightarrow \diamond\diamond p, \\
\exists xy(x \neq y), & D\top, \\
\forall x \exists y(x < y), & \diamond\top, \\
\forall x \exists y(y < x), & p \wedge \bar{D}\neg p \rightarrow \diamond p \vee D\diamond p.
\end{array}$$

QED.

The special form of the antecedents of the second and last $(\diamond, D)$-formula in the above list is worth noting. When such an antecedent is true it forces p to act as a so-called *nominal*, i.e., to hold at precisely one point; this then enables us to describe the behavior of $<$ locally, at the unique point at which p holds. (See §2.4 for more on nominals.)

Recall that a *modal sequent* is a pair $\sigma = \langle \Gamma_0, \Theta_0 \rangle$ where Γ_0 and Θ_0 are finite sets of (multi-) modal formulas; $\mathscr{F} \models \sigma$ if, for every V, if $\langle \mathscr{F}, V \rangle \models \Gamma_0$ then there is a $\theta \in \Theta_0$ with $\langle \mathscr{F}, V \rangle \models \theta$. A class K of frames is *sequentially definable* if there is a set L of modal sequents such that $K \doteq \{\mathscr{F} \colon \forall \sigma \in L(\mathscr{F} \models \sigma)\}$. Kapron [14] shows that in $\mathscr{L}(\diamond)$ sequential definability is strictly stronger than ordinary definability. By our remarks in §1.2 and the fact that validity of sequents is preserved under p-morphisms (cf. [14]), it follows that definability in $\mathscr{L}(\diamond, D)$ is still stronger. Furthermore, in $\mathscr{L}(\diamond, D)$ the notions of ordinary and sequential definability coincide; as is pointed out in [13] this is due to the fact that we can define the "universal modality" A in $\mathscr{L}(\diamond, D)$:

PROPOSITION 2.7. *Let K be a class of frames. K is sequentially definable in $\mathscr{L}(\diamond, D)$ iff it is definable in $\mathscr{L}(\diamond, D)$.*

PROOF. One direction is clear. To prove the other one, assume that K is defined by a set L of sequents. For each $\sigma = \langle \{\varphi_0, \dots, \varphi_n\}, \{\psi_0, \dots, \psi_m\} \rangle \in L$ put $\sigma^* := \bigwedge_{0 \le i \le n} A\varphi_i \rightarrow \bigvee_{0 \le i \le m} A\psi_i$. Then K is defined by $\{\sigma^* \colon \sigma \in L\}$. QED.

It should be clear by now that adding the D-operator to $\mathscr{L}(\diamond)$ greatly increases the expressive power. Limitations are easily found, however. As we have seen, $\exists x Rxx$ is still not definable in $\mathscr{L}(\diamond, D)$. And just as with the standard modal language we find that on well-orders a sort of "stabilization of discriminatory power" occurs at a relatively early stage (cf. [5] for a proof of this result for the standard modal language). To prove this we recall that the *clusters* of a transitive frame $\mathscr{F}$ are the equivalence classes of W under the relation $x \sim y$ iff $(Rxy \wedge Ryx) \vee x = y$. Clusters are divided into three kinds: proper, with at least two elements, all reflexive; simple, with one reflexive element; and degenerate, with one irreflexive element.

THEOREM 2.8. *If $\varphi \in \mathscr{L}(\Diamond, D)$, and $\mathscr{F}$ is a well-ordered frame with $\mathscr{F} \not\models \varphi$, then there is a well-ordered frame $\mathscr{G}$ such that $\mathscr{G} < \omega^2$ and $\mathscr{G} \not\models \varphi$.*

PROOF. Suppose that for some V and $w \in W$ we have $\mathscr{M} = \langle \mathscr{F}, V \rangle \models \neg\varphi[w]$. Let Σ^- be the set of subformulas of $\neg\varphi$, and define $\Sigma := \Sigma^- \cup \{\Diamond\psi : D\psi \in \Sigma^-\}$. Let $\mathscr{M}_1$ be the (extended) collapse of $\mathscr{M}$ with respect to Σ. Then $\mathscr{M}_1$ is transitive and linear. Consequently, $\mathscr{M}_1$ may be viewed as a finite linear sequence of clusters.

Next, $\mathscr{M}_1$ will be blown up into a well-ordered model $\mathscr{M}_2$ by replacing each cluster with an appropriate well-order. If $C = \{w\}$ is a degenerate cluster, then C is itself a well-order, and we do nothing. Nondegenerate clusters $\{w_1, \ldots, w_k\}$ are replaced by a copy of ω; the valuation is adapted by verifying p in a newly added n iff $n = i \bmod k$ and $w_i \in V_1(p)$. The resulting model is a well-order, and since $\mathscr{M}_1$ is finite it will have order type $<\omega^2$.

If $w \in W_1$, we write $\bar{w}$ for a point or points corresponding to w in $\mathscr{M}_2$. Then, for all $\psi \in \Sigma$ and $w \in W_1$, $\mathscr{M}_1 \models \psi[w]$ iff $\mathscr{M}_2 \models \psi[\bar{w}]$. This equivalence is proved by induction on ψ. The only nontrivial case is when $\psi \equiv D\chi$ and $\mathscr{M}_2 \models D\chi[\bar{w}]$. In that case one uses the fact that $D\chi \in \Sigma$ implies $\Diamond\chi \in \Sigma$. QED.

From 2.8 and [5, Theorem 5.2] it follows that the well-orders of type $<\omega \cdot k + n$ $(k \leq \omega, n < \omega)$ all have distinct $(\Diamond, D)$-theories, while for $k \geq \omega$ we have $\omega \cdot k + n \equiv_{\Diamond, D} \omega \cdot \omega + n$.

2.3. The languages $\mathscr{L}(\Diamond, D)$ and $\mathscr{L}(F, P)$. On strict linear orders the D-operator becomes definable in $\mathscr{L}(F, P)$: on such frames we have $\mathscr{F} \models (P\varphi \vee F\varphi) \leftrightarrow D\varphi$. In fact, this may be generalized somewhat; call a frame $\mathscr{F}$ *n-connected* $(n > 0)$ if, for any $w, v \in W$ with $w \neq v$, there exists a sequence $w_1, \ldots, w_k$ such that $w_1 = w$, $w_k = v$ and for each j $(1 \leq j < k)$ either Rw_jw_{j+1} or $Rw_{j+1}w_j$. Then, using a suitable translation, one may show that on irreflexive n-connected frames every $(\Diamond, D)$-formula is equivalent to one in $\mathscr{L}(F, P)$. This shows that new results about standard modal languages may be obtained by studying extended ones: for it follows from 2.4 that on the class of irreflexive n-connected frames every purely universal Π^1_1-sentence in R, $=$ is definable in $\mathscr{L}(F, P)$.

By the next result there is no converse to our previous remarks: P is not definable in $\mathscr{L}(\Diamond, D)$, not even on strict linear orders.

THEOREM 2.9. 1. $\langle \mathbf{Q}, < \rangle \not\equiv_{F,P} \langle \mathbf{R}, < \rangle$,

2. $\langle \mathbf{Q}, < \rangle \equiv_{\Diamond, D} \langle \mathbf{R}, < \rangle$.

PROOF. The first part is well known. To prove the second part, assume first that, for some $\varphi \in \mathscr{L}(\Diamond, D)$ and valuation V, $\langle \mathbf{R}, <, V \rangle \not\models \varphi$. Using the ST-translation as defined in §4.2, we find that $\langle \mathbf{R}, <, V \rangle \models \exists x\, ST(\neg\varphi)$. Hence, by the Löwenheim-Skolem theorem, $\langle \mathbf{Q}, <, V' \rangle \models \exists x\, ST(\neg\varphi)$, where $V'(p) = V(p) \upharpoonright \mathbf{Q}$, for all proposition letters p. It follows that $\langle \mathbf{Q}, < \rangle \not\models \varphi$.

Conversely, assume that, for some $\varphi \in \mathscr{L}(\Diamond, D)$ and a valuation V, $\langle \mathbf{Q}, <, V \rangle \not\models \varphi$. Define Σ and $\mathscr{M}_1$ as in the proof of 2.8. Then $\mathscr{M}_1$ is transitive, linear and successive (both to the right and to the left). A model $\mathscr{M}_2$ may then be constructed by replacing each cluster by an ordering of type λ if it is the leftmost cluster, and otherwise, if it is degenerate it and its *nondegenerate* successor (by [18, Lemma 1.1] $\mathscr{M}_1$ does not contain adjacent degenerate clusters) are replaced in one go by an ordering of type $1 + \lambda$; after that, the remaining nondegenerate clusters are also replaced by $1 + \lambda$. The valuation may then be extended to newly added points in such a way that an induction similar to the one in the proof of 2.8 yields $\mathscr{M}_2 \not\models \varphi$. QED.

2.4. The language $\mathscr{L}(\diamond, D)$ and some other enriched languages. In [6] a simple method of incorporating reference into modal logic is presented by introducing a new sort of atomic symbols—*nominals*—into the modal language. These new symbols combine with other symbols of the language in the usual way to form formulas. Their only nonstandard feature is that they are true at *exactly one point* in a model. Let $\mathscr{L}_n(\diamond)$ denote the language $\mathscr{L}(\diamond)$ with nominals added to it. From [6] we know that $\mathscr{L}_n(\diamond)$ is much more expressive than $\mathscr{L}(\diamond)$: important classes of frames undefinable in $\mathscr{L}(\diamond)$ become definable in $\mathscr{L}_n(\diamond)$. But it turns out that $\mathscr{L}(\diamond, D)$ is even more expressive than $\mathscr{L}_n(\diamond)$. To see this, let $n_0, n_1, n_2, \ldots$ range over nominals, let $p_0, p_1, p_2, \ldots$ denote the proposition letters in $\mathscr{L}_n(\diamond)$ and $\mathscr{L}(\diamond, D)$, and define $\tau\colon \mathscr{L}_n(\diamond) \to \mathscr{L}(\diamond, D)$ by putting $\tau(p_i) = p_{2i}$ and $\tau(n_i) = p_{2i+1}$, and by letting τ commute with the connectives and operators. Given a formula $\varphi \in \mathscr{L}_n(\diamond)$, let $n_1, \ldots, n_k$ be the nominals occurring in φ, and define $(\varphi)^* \in \mathscr{L}(\diamond, D)$ to be $U\tau(n_1) \wedge \cdots \wedge U\tau(n_k) \to \tau(\varphi)$.

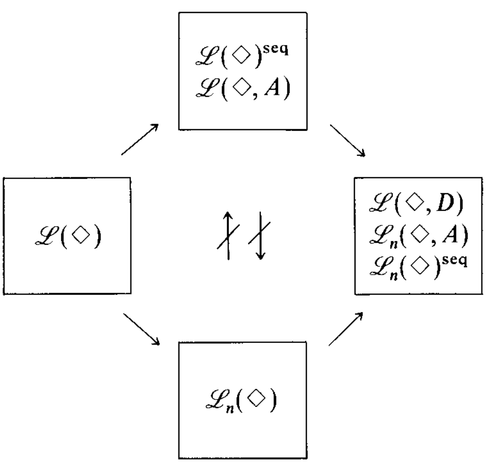
PROPOSITION 2.10. *Every class of frames that is definable in $\mathscr{L}_n(\diamond)$ is definable in $\mathscr{L}(\diamond, D)$, but not conversely.*

PROOF. The first part follows from the observation that, for any formula $\varphi \in \mathscr{L}_n(\diamond)$ and any model $\langle W, R, V\rangle$, $\langle W, R, V\rangle \models \varphi[w]$ iff $\langle W, R, V^*\rangle \models \varphi^*[w]$, where $V^*(p) = V(\tau^{-1}(p))$. The second part follows from 2.2 and the fact that 1 is the only cardinality definable in $\mathscr{L}_n(\diamond)$ (cf. [6]). QED.

In [6] and [9] the extension $\mathscr{L}_n(\diamond, A)$ of $\mathscr{L}_n(\diamond)$ is studied—here A is the operator defined in §1.1, whose semantics is given by $\mathscr{M} \models A\varphi[w]$ iff, for all $v \in W$, $\mathscr{M} \models \varphi[v]$; it is sometimes called the *shifter* (in [6]), or the *universal modality* (in [9]). By the above observations $\mathscr{L}_n(\diamond, A)$ is no more expressive than $\mathscr{L}(\diamond, D)$. Moreover, by a nice result in [9] the converse holds as well:

THEOREM 2.11 (Gargov and Goranko). *A class of frames is definable in $\mathscr{L}_n(\diamond, A)$ iff it is definable in $\mathscr{L}(\diamond, D)$.*

Combining results from this section and earlier ones together with results from [9] and [13], we arrive at the following picture:



(Here, $\mathscr{L}(\Diamond)^{\text{seq}}$ is $\mathscr{L}(\Diamond)$ with sequential definability; each box contains languages that are equivalent with respect to definability of frames, and arrows point to more expressive languages.)

§3. Axiomatics. We first study logics in the language $\mathscr{L}(D)$; we show that, given the right basic logic in $\mathscr{L}(D)$, all its finite extensions are complete. After that we consider the basic logics in $\mathscr{L}(\Diamond, D)$ and $\mathscr{L}(F, P, D)$, as well as some logics axiomatizing familiar classes of frames, or special structures. Finally, we will discuss Sahlqvist theorems for $\mathscr{L}(\Diamond, D)$ and $\mathscr{L}(F, P, D)$.

3.1. The logics in $\mathscr{L}(D)$.

DEFINITION 3.1. DL^- is propositional logic plus the following schemata:

(A1) $\bar{D}(p \to q) \to (\bar{D}p \to \bar{D}q)$,

(A2) $p \to \bar{D}Dp$ (symmetry),

(A3) $DDp \to (p \vee Dp)$ (pseudotransitivity).

As rules of inference it has modus ponens, substitution, and a "necessitation rule" for $\bar{D}$:

$$\vdash \varphi \Rightarrow \vdash \bar{D}\varphi.$$

THEOREM 3.2 (Koymans). *Let $\Sigma \cup \{\varphi\} \subseteq \mathscr{L}(D)$. Then $\Sigma \vdash_{DL^-} \varphi$ iff $\Sigma \models \varphi$.*

PROOF. Soundness is immediate. To prove completeness, assume $\Sigma \nvdash_{DL^-} \varphi$, and let $\Delta \supseteq \Sigma \cup \{\neg\varphi\}$ be a maximal DL^--consistent set. Consider $W_\Delta :=$ $\{\Gamma\colon \exists n(R_D)^n\Delta\Gamma\}$, where Γ ranges over maximal DL^--consistent sets and R_D is the canonical relation defined as follows: $R_D\Gamma_1\Gamma_2$, iff, for all $D\psi \in \Gamma_1$, $\psi \in \Gamma_2$. Then

$$\forall xy(R_Dxy \to R_Dyx) \quad \text{and} \quad \forall xyz(R_Dzy \wedge R_Dyz \to R_Dxz \vee x = z).$$

If there are any R_D-reflexive points, let c be such a point; replace it with *two* points c_1, c_2, and adapt R_D by putting $R_Dc_1c_2$, and conversely, and by putting R_Dc_iw (R_Dwc_i) if R_Dcw (R_Dwc) $(i = 1, 2)$. In the resulting structure R_D is real inequality, and φ is refuted somewhere. QED.

Hence, one may be inclined to think that DL^- is the basic logic in $\mathscr{L}(D)$, just as K is the basic logic in $\mathscr{L}(\Diamond)$. However, DL^- is, so to speak, not as stable as K: in $\mathscr{L}(\Diamond)$ incompleteness phenomena occur only with more exotic extensions of K (cf. [2]); in contrast, here is a very simple incomplete extension of DL^-:

EXAMPLE 3.3. Consider the system $DL^- + (\varphi \to D\varphi)$. Then $DL^- + (\varphi \to D\varphi)$ $\models \bot$, since no frame validates $DL^- + (\varphi \to D\varphi)$. On the other hand, $DL^- +$ $(\varphi \to D\varphi) \nvdash \bot$. To see this, recall that a *general frame* is a triple $\mathfrak{F} = \langle W, R, \mathscr{W}\rangle$, where $\mathscr{W} \subseteq P(W)$ contains $\varnothing$, and is closed under the Boolean operations as well as the operator L_R (cf. 1.2); valuations on a general frame should take their values inside $\mathscr{W}$. Now let $\mathfrak{F} = \langle W, R, \mathscr{W}\rangle$, where $\mathscr{F} = \langle\{0,1\}, \varnothing\rangle$ and $\mathscr{W} = \{\varnothing, \{0,1\}\}$ (so D is interpreted using the relation $R = \varnothing$). Then $\mathfrak{F} \models DL^- + (\varphi \to D\varphi)$. Therefore, $DL^- + (\varphi \to D\varphi)$ is incomplete.

To avoid incompleteness phenomena such as those sketched above, we follow some suggestions by Yde Venema and Valentin Goranko, and add the following rule of inference to DL^-:

(IR) If, for all proposition letters p not occurring in φ, $\vdash p \wedge \bar{D}\neg p \to \varphi$, then $\vdash \varphi$.

The idea of using special kinds of derivation rules to obtain completeness results originates with Dov Gabbay, who used an irreflexivity rule to axiomatize the set of $\Diamond$-formulas valid on irreflexive frames (cf. [8]). A rule analogous to (IR) has been used to obtain completeness results for axiom systems in languages with nominals (cf. [6], [8], and [10]).

Let DL denote DL^- plus the rule (IR). Note that, given the substitution rule, (IR) is in fact equivalent to a finitary rule: if, for *some* proposition letter p not occurring in φ, $\vdash p \wedge \bar{D}\neg p \to \varphi$, then $\vdash \varphi$. Our next aim is to prove that, in terms of general consequence, DL has no effects over DL^-. To this end it suffices to show that DL precisely axiomatizes the basic logic in $\mathscr{L}(D)$.

Let $L \supseteq DL$ be a logic. A set of formulas Γ is called *L-nice* if Γ is maximal L-consistent, and if, for some proposition letter p, $p \wedge \bar{D}\neg p \in \Gamma$.

PROPOSITION 3.4. *Let $L \supseteq DL$. Every L-consistent set can be extended to an L-nice set.*

PROOF. This is a combination of the usual Lindenbaum construction plus applications of the rule (IR). Note that we may have to add new proposition letters to our language. (Cf. [10] or [21] for details.) QED.

An *i-canonical general frame* for a logic L extending DL in $\mathscr{L}(D)$ is a tuple $\mathfrak{F}_L = \langle W, R_D, \mathscr{W} \rangle_L$ such that W is a set of L-nice sets R_D-generated by a single L-nice set, i.e., $W = \{\Gamma\colon \exists n \geq 0\, (R_D)^n \Sigma\Gamma\}$ for some L-nice Σ; $R_D = \{\langle \Sigma, \Gamma \rangle\colon$ for all $\bar{D}\varphi \in \Sigma$, $\varphi \in \Gamma\}$; and $\mathscr{W} = \{X \subseteq W\colon \exists\varphi \in \mathscr{L}(D)\, \forall\Delta \in W(\varphi \in \Delta \leftrightarrow \Delta \in X)\}$. Finally, $\mathscr{F}_L$ is an *i-canonical frame* if it is the full frame underlying some i-canonical general frame.

PROPOSITION 3.5. *Let $\mathfrak{F}_L$ be an i-canonical general frame for some logic L extending DL in $\mathscr{L}(D)$. Then R_D is real inequality.*

PROOF. By definition R_D holds between any two different elements of W. To see that it holds only between different elements of W, let $\Delta \in W$. Since Δ is L-nice, we $p \wedge \bar{D}\neg p \in \Delta$, for some proposition letter p. Hence, $\neg R_D \Delta\Delta$. QED.

THEOREM 3.6. *Let $\Sigma \cup \{\varphi\} \subseteq \mathscr{L}(D)$. Then $\Sigma \vdash_{DL} \varphi$ iff $\Sigma \models \varphi$.*

PROOF. Soundness is immediate. To prove completeness, assume that $\Sigma \nvdash_{DL} \varphi$. By 3.4, $\Sigma \cup \{\neg\varphi\}$ can be extended to a nice set Σ'. Consider an i-canonical frame $\mathscr{F} = \langle W, R_D \rangle$ such that $\Sigma' \in W$. Let V be the *canonical* valuation, i.e., $V(p) = \{\Gamma \in W\colon p \in \Gamma\}$. Then $\langle W, R_D, V \rangle \models \neg\varphi[\Sigma']$. QED.

It follows from 3.2 and 3.6 that the rule (IR) is superfluous in the basic logic. However, it does yield new consequences in extensions of DL: $DL + (\varphi \to D\varphi)$ is inconsistent, and thus complete. (To see that it is inconsistent, note first that, for any proposition letter p, $DL + (\varphi \to D\varphi) \vdash (p \wedge \bar{D}\neg p \to \bot)$; hence, by the rule (IR), $DL + (\varphi \to D\varphi) \vdash \bot$.) Better still, if a *simple extension* of a logic L is an extension of L with only finitely many axiom schemas, then, with the rule (IR) added to our basic logic in $\mathscr{L}(D)$, we can prove the completeness of *every* simple extension of the basic logic in $\mathscr{L}(D)$. To do so we need a number of lemmas and definitions, the first of which will be given next.

DEFINITION 3.7. Let $\mathfrak{F}_L = \langle W, R_D, \mathscr{W} \rangle_L$ be an i-canonical general frame for a logic $L \supseteq DL$. A set $X \subseteq W$ is *definable in* $\mathfrak{F}_L$ if $X \in \mathscr{W}$. A valuation V is *definable in* $\mathfrak{F}_L$ if, for every $\varphi \in \mathscr{L}(D)$, $V(\varphi)$ is definable.

PROPOSITION 3.8. *Let $\mathfrak{F}_L = \langle W, R_D, \mathscr{W} \rangle_L$ be an i-canonical general frame for a logic $L \supseteq DL$. Let $X \subseteq W$ be finite or cofinite. Then X is definable in $\mathfrak{F}_L$.*

Proposition 3.9. *Let $\mathfrak{F}_L = \langle W, R_D, \mathcal{W} \rangle_L$ be an i-canonical general frame for a logic $L \supseteq DL$. Let V be a valuation. If, for all proposition letters p, $V(p)$ is either finite or cofinite, then V is definable in $\mathfrak{F}_L$.*

Proof. This follows from 3.8 and the fact that, for any φ, $V(D\varphi)$ is either $\varnothing$, W, or the complement of a singleton. QED.

Our strategy for proving the completeness of every simple extension of DL is as follows. Instead of trying to push validities on an i-canonical general frame down to its underlying full frame (as is done in e.g. the original proof of Sahlqvist's theorem; cf. [17]), we will try and lift refutations on an i-canonical frame up to the i-canonical general frame it is derived from. Our main tool in doing so is a version of the Ehrenfeucht-Fraïssé theorem for monadic first order logic over identity. (For full details and a proof of this result we refer the reader to [22, §1.7].)

For the time being we will work in a fixed finite language having $p_0, \ldots, p_{k-1}$ as its only proposition letters. The monadic first order language into which this restricted modal language translates via the ST-translation is denoted $\mathcal{L}^k$; so $\mathcal{L}^k$ only has k unary predicate letters $P_0, \ldots, P_{k-1}$. Let $\mathcal{M} = \langle W, P_0, \ldots, P_{k-1} \rangle$ be an $\mathcal{L}^k$-model. If $X \subseteq W$, then $X^0 = X$ and $X^1 = W \backslash X$. For $s \in 2^k$ the *s-slot* is

$$W_s^{\mathcal{M}} = P_0^{s(0)} \cap \cdots \cap P_{k-1}^{s(k-1)}.$$

An s-slot is *P_i-positive* (*P_i-negative*) if $s(i) = 0$ ($s(i) = 1$).

Let $\mathcal{M} = \langle W, P_0, \ldots, P_{k-1} \rangle$ and $\mathcal{M}' = \langle W', P'_0, \ldots, P'_{k-1} \rangle$ be two $\mathcal{L}^k$-models. We write $\mathcal{M} \equiv_n \mathcal{M}'$ if $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same $\mathcal{L}^k$-sentences of quantifier rank at most n. For two sets X, Y we write $X \approx_n Y$ iff $|X| = |Y| < n$ or $|X|, |Y| \geq n$; by extension we put $\mathcal{M} \approx_n \mathcal{M}'$ iff, for all $s \in 2^k$, $W_s^{\mathcal{M}} \approx_n W_s^{\mathcal{M}'}$.

Theorem 3.10. *For any two $\mathcal{L}^k$-models $\mathcal{M}$ and $\mathcal{M}'$ we have $\mathcal{M} \equiv_n \mathcal{M}'$ iff $\mathcal{M} \approx_n \mathcal{M}'$.*

Lemma 3.11. *Let $\varphi \in \mathcal{L}(D)$. Let W be any nonempty set. If, for some valuation V, $\langle W, V \rangle \not\models \varphi$, then there is a valuation V' with $\langle W, V' \rangle \not\models \varphi$ and such that $V'(p)$ is either finite or cofinite for all proposition letters p.*

Proof. Let $\mathcal{M} = \langle W, V \rangle$. Let n be the quantifier rank of $\forall x\, ST(\varphi)$. Assume $p_0, \ldots, p_{k-1}$ are the proposition letters occurring in φ. For proposition letters q not occurring in φ, we may assume that $V(q) = \varnothing$. Write $P_0, \ldots, P_{k-1}$ for $V(p_0), \ldots, V(p_{k-1})$. We will give a procedure for turning an infinite and coinfinite extension of one of the proposition letters $p_0, \ldots, p_{k-1}$ into a finite or cofinite one. If needed, we can repeat this procedure for the remaining proposition letters to establish the lemma.

Assume that P_0 is infinite and coinfinite. Now, P_0 is the union of all P_0-positive s-slots, and P_0^c is the union of all P_0-negative s-slots ($s \in 2^k$). Thus, there must be $t, r \in 2^k$ with t P_0-positive and r P_0-negative such that both $W_t^{\mathcal{M}}$ and $W_r^{\mathcal{M}}$ are infinite. Choose n elements $w_0, \ldots, w_{n-1} \in W_t^{\mathcal{M}}$. We define a valuation V' as follows. Informally, the idea is to put all elements of $W_t^{\mathcal{M}} \backslash \{w_0, \ldots, w_{n-1}\}$ into $W_r^{\mathcal{M}}$. Formally,

- $V'(p_0) = (V(p_0) \backslash W_t^{\mathcal{M}}) \cup \{w_0, \ldots, w_{n-1}\}$;

while for p_i with $i > 0$,

- if t and r are both P_i-positive or both P_i-negative, then $V'(p_i) = V(p_i)$,

• if t is P_i-positive and r is P_i-negative, then $V'(p_i) = (V(p_i) \backslash W_t^{\mathscr{M}}) \cup \{w_0, \ldots, w_{n-1}\}$, and

• if t is P_i-negative and r is P_i-positive, then $V'(p_i) = V(p_i) \cup (W_t^{\mathscr{M}} \backslash \{w_0, \ldots, w_{n-1}\})$.

Note that if $V(p_i)$ $(i \neq 0)$ was already finite or cofinite, then so is $V'(p_i)$. Let $\mathscr{M}' = \langle W, V' \rangle$. Then $\mathscr{M} \approx_n \mathscr{M}'$, as is easily checked. Thus, by 3.10, $\mathscr{M} \equiv_n \mathscr{M}'$, and so $\mathscr{M}' \not\models \varphi$.

If $V'(p_0)$ is (still) neither finite nor cofinite, we repeat the above procedure until all P_0-positive slots are finite. An upper bound for the number of times we may have to do this is 2^{k-1}. QED.

THEOREM 3.12. *Let $\varphi \in \mathscr{L}(D)$. Then $DL + \varphi$ is complete.*

PROOF. Let L denote $DL + \varphi$. Assume $\nvdash_L \varphi$. Then for some i-canonical general frame $\mathfrak{F}_L = \langle W, R_D, \mathscr{W} \rangle_L$ we have $\mathfrak{F}_L \not\models \psi$. To prove the theorem we show that $\mathscr{F}_L = \langle W, R_D \rangle \models L$, and $\mathscr{F}_L \not\models \psi$. The latter is immediate from $\mathfrak{F}_L \not\models \psi$. To prove the former, assume that $\mathscr{F}_L \not\models \varphi$. So, for some valuation V, $\langle \mathscr{F}_L, V \rangle \not\models \varphi$. By 3.11 there is a valuation V' such that $\langle \mathscr{F}, V' \rangle \not\models \varphi$, and such that $V'(p)$ is finite or cofinite for all proposition letters p. But then, by 3.9, V' is definable in $\mathfrak{F}_L$. Hence $\mathfrak{F}_L \not\models \varphi$, a contradiction. QED.

3.2. Logics in $\mathscr{L}(\Diamond, D)$ and $\mathscr{L}(F, P, D)$. The basic logic DL_m in $\mathscr{L}(\Diamond, D)$ is $DL + K + (\Diamond p \rightarrow p \vee Dp)$; its rules of inference are those of DL plus those of K. The basic logic DL_t in $\mathscr{L}(F, P, D)$ is $DL + K_t + (Fp \rightarrow p \vee Dp)$; its rules of inference are those of DL plus those of K_t.

An *i-canonical general frame* for a logic L extending DL_m in $\mathscr{L}(\Diamond, D)$ is a tuple $\mathfrak{F}_L = \langle W, R_D, R_\Diamond, \mathscr{W} \rangle_L$, where W and R_D are as in §3.1, while $\mathscr{W} = \{X \subseteq W$: $\exists \varphi \in \mathscr{L}(\Diamond, D) \forall \Delta \in W(\varphi \in \Delta \leftrightarrow \Delta \in X)\}$, and $R_\Diamond = \{\langle \Sigma, \Gamma \rangle$: for all $\Box \psi \in \Sigma, \psi \in \Gamma\}$. As before, an *i-canonical frame* is a full frame underlying an i-canonical general frame. Analogous definitions may be given for extensions of DL_t, where the canonical relations are denoted R_F, R_P, and R_D.

THEOREM 3.13. 1. *Let $\Sigma \cup \{\varphi\} \subseteq \mathscr{L}(\Diamond, D)$. Then $\Sigma \vdash_{DL_m} \varphi$ iff $\Sigma \models \varphi$.*

2. *Let $\Sigma \cup \{\varphi\} \subseteq \mathscr{L}(F, P, D)$. Then $\Sigma \vdash_{DL_t} \varphi$ iff $\Sigma \models \varphi$.*

PROOF. Similar to the proof of 3.6. Note that by the additional axiom $\Diamond p \rightarrow p \vee Dp$, any set W of maximal DL_m-consistent sets that is closed under R_D is also closed under $R_\Diamond$. Analogous remarks hold for DL_t and the canonical relations R_F and R_P. QED.

Next we present axioms in $\mathscr{L}(\Diamond, D)$ for some special structures and familiar classes of frames. Here is a list of axioms together with the corresponding conditions on frames:

(A4)	$\Diamond\Diamond p \rightarrow \Diamond p$	transitivity,
(A5)	$p \rightarrow \Diamond p$	reflexivity,
(A6)	$p \wedge \bar{D}\neg p \rightarrow \Box(\Diamond p \rightarrow p)$	anti-symmetry,
(A7)	$\Diamond p \rightarrow Dp$	irreflexivity,
(A8)	$p \rightarrow \Diamond q \vee \bar{D}(q \rightarrow \Diamond p)$	linearity,

(A9)	$\Diamond\top$	successiveness to the right,
(A10)	$p \wedge \bar{D}\neg p \to \Diamond p \vee D\Diamond p$	successiveness to the left,
(A11)	$\Box(\Box p \to p) \to (\Diamond\Box p \to p)$	discreteness,
(A12)	$\Diamond p \to \Diamond\Diamond p$	denseness.

THEOREM 3.14. 1. DL_m + (A4)–(A6) *is complete with respect to partial orders.*
2. DL_m + (A4) + (A7) *is complete with respect to strict partial orders.*
3. DL_m + (A4) + (A8) *is complete with respect to linear orders.*
4. DL_m + (A4) + (A7) + (A8) *is complete with respect to strict linear orders.*

PROOF. Assume that $\Sigma \nvdash \varphi$ in DL_m + (A4)–(A6). As in the proof of 3.6 we can construct an i-canonical frame containing an element that extends $\Sigma \cup \{\neg\varphi\}$. Using the characteristic axioms, it is routine to check that $\mathscr{F}$ is a partial order. (Note that to be able to apply (A6) we need to know that every point (i.e. maximal consistent set) in the i-canonical frame contains a "unique" proposition letter.) Also, one easily verifies that φ is refuted under the canonical valuation. Cases 2, 3, and 4 of the theorem may be proved in a similar way. QED.

THEOREM 3.15. 1. DL_m + (A4) + (A7)–(A9) + (A11) *axiomatizes* $\mathrm{Th}_{\Diamond,D}(\mathbf{N})$.
2. DL_m + (A4) + (A7)–(A11) *axiomatizes* $\mathrm{Th}_{\Diamond,D}(\mathbf{Z})$.
3. DL_m + (A4) + (A7)–(A9) + (A12) *axiomatizes* $\mathrm{Th}_{\Diamond,D}(\mathbf{Q})$ ($=\mathrm{Th}_{\Diamond,D}(\mathbf{R})$ *by* 2.9).

PROOF. To prove 1, 2 and 3, start by constructing an i-canonical frame as in the proof of 3.6. In the case of 3 the resulting structure will be isomorphic to $\langle \mathbf{Q}, < \rangle$. In the case of 1 or 2 one may apply an appropriate version of the techniques of [18] to turn the frame into a frame based on **N** or **Z**. QED.

What about decidability of the above logics? Using extended filtrations (cf. 1.7), one easily establishes that both DL_m + (A4)–(A6) and DL_m + (A4) + (A8) have the *finite frame property* (f.f.p.); from this their decidability follows in a standard way.

As for DL_m + (A4) + (A7), note that it does not have the f.f.p.: any frame $\mathscr{F}$ with $\mathscr{F} \models DL_m$ + (A4) + (A7) and $\mathscr{F} \not\models \neg\Box\Diamond\top$ must be infinite. However, DL_m + (A4) + (A7) does have the finite model property (f.m.p.)—thus showing that Segerberg's theorem (which says that the f.f.p. and the f.m.p. are equivalent in $\mathscr{L}(\Diamond)$) fails in $\mathscr{L}(\Diamond, D)$. In fact, DL_m + (A4) + (A7) may be shown to be complete with respect to the class of finite models $\mathscr{M} = \langle \mathscr{F}, V \rangle$ which satisfy $\mathscr{F} \models DL_m$ + (A4), and, for any $\varphi \in \mathscr{L}(\Diamond, D)$, if $\{w\colon Rww\} \cap V(\varphi) \neq \varnothing$, then $|V(\varphi)| \geq 2$. Soundness is immediate. The easy proof of the completeness is too lengthy to be included here, so we only mention some steps in it. By 3.14 there is a model $\mathscr{M} = \langle \mathscr{F}, V \rangle$ with $\mathscr{F} \models DL_m$ + (A4) + (A7) and $\mathscr{M} \not\models \varphi[w]$, for some $w \in \mathscr{M}$. Let $\Sigma \ni \neg\varphi$ be some finite set of formulas that is closed under subformulas, and that satisfies $\Diamond\psi \in \Sigma \Rightarrow D\psi \in \Sigma$. We define a nonstandard model $\mathscr{M}'$ as follows; let g, W', R', and V' be as in our remarks following 1.7; define R_D by $R_D g(v) g(u)$ iff, for all $\bar{D}\psi \in g(v)$, $\psi \in g(u)$. Then, using R_D as the interpretation of D, $\mathscr{M}' \not\models \varphi$, and, moreover, R' is transitive, $R' \subseteq R_D$, R_D holds between any two different points, and $\mathscr{M}'$ is finite. Next, one may use the "doubling-points" technique of 3.2 to obtain a model $\mathscr{M}'' \not\models \varphi$ in which R_D is real inequality, and which satisfies all our requirements.

Using the fact that DL_m + (A4) + (A7) has the f.m.p., one may establish the decidability of this logic. The decidability of DL_m + (A4) + (A7) + (A8) and of

$\mathrm{Th}_{\Diamond,D}(\mathbf{Q})$ may be proved in a similar fashion. To obtain decidability results for $\mathrm{Th}_{\Diamond,D}(\mathbf{N})$ and $\mathrm{Th}_{\Diamond,D}(\mathbf{Z})$ one may apply Rabin-Gabbay techniques (cf. [6, Chapter 5] for a similar move in $\mathscr{L}_n(F,P)$).

We end this subsection with a general question due to Gargov and Goranko [9]. Let L extend K in $\mathscr{L}(\Diamond)$ with schemas $\{\varphi_i: i \in I\}$. The *minimal extension* of L in $\mathscr{L}(\Diamond, D)$ is DL_m plus the schemas φ_i read as schemas over $\mathscr{L}(\Diamond, D)$. The *transfer problem* is the following: if L has property P, does its minimal extension have P? Here we will consider only one of the many obvious properties one may study in this context: incompleteness.

Let L be a logic in $\mathscr{L}(\Diamond)$. To show that if L is incomplete, then so is its minimal extension L' in $\mathscr{L}(\Diamond, D)$, it suffices to show that L' is conservative over L. To this end, assume $L \nvdash \varphi$. Then, by the completeness of K, we find a model $\mathscr{M}$, and a $w \in \mathscr{M}$, such that $\mathscr{M} \models L^* \cup \{\neg\varphi\}[w]$, where L^* is the set of all $\mathscr{L}(\Diamond)$-instances of the axioms of L. Now, obviously, $\mathscr{M} \models DL_m$, and also $\mathscr{M} \models L^{**}[w]$, where L^{**} is the set of $\mathscr{L}(\Diamond, D)$-instances of the axioms of L (this is because, for any set $V(\varphi)$, $V(D\varphi)$ is either $\varnothing$, $V(\top)$, or the complement of $V(\varphi)$). But then $L' \nvdash \varphi$.

(As an aside, new and fairly simple incomplete logics occur as well: let X be $DL_m + (\Diamond\varphi \to D\varphi) + (\Diamond\Diamond\varphi \to \Diamond\varphi) + (\Box\Diamond\varphi \to \Diamond\Box\varphi)$. Then $X \models \bot$ since $\Diamond\varphi \to D\varphi$ defines irreflexivity of R, while, given $\Diamond\Diamond\varphi \to \Diamond\varphi$, $\Box\Diamond\varphi \to \Diamond\Box\varphi$ defines $\forall x \exists y(Rxy \to \forall z(Ryz \to z = y))$. However, by a routine argument involving general frames, $X \nvdash \bot$.)

3.3. On Sahlqvist theorems for $\mathscr{L}(\Diamond, D)$ and $\mathscr{L}(F,P,D)$. We start with some preliminary remarks. A formula in $\mathscr{L}(\Diamond, D)$ is called a *Sahlqvist formula* if it is a conjunction of formulas of the form $\bar{\mathbf{t}}_1 \cdots \bar{\mathbf{t}}_n(\varphi \to \psi)$, where $\bar{\mathbf{t}}_i \in \{\Box, \bar{D}\}$, ψ is positive (in the usual syntactic sense), and φ is a so-called Sahlqvist antecedent, i.e., it is built up from formulas, $\bar{\mathbf{t}}_1 \cdots \bar{\mathbf{t}}_m(\neg)p$ (with $\bar{\mathbf{t}}_i \in \{\Box, \bar{D}\}$) using only $\vee$, $\wedge$, $\Diamond$ and D. (Sahlqvist forms in $\mathscr{L}(\Diamond)$ or $\mathscr{L}(F,P,D)$ are defined similarly.) A formula is called a *weak Sahlqvist form* if it is a Sahlqvist form in which the Sahlqvist antecedents do not contain any formulas of the form $\bar{\mathbf{t}}_1 \cdots \Box \cdots \bar{\mathbf{t}}_m p$ (with $\bar{\mathbf{t}}_i$ either empty or as before).

The Sahlqvist theorem for $\mathscr{L}(\Diamond)$ says that for a Sahlqvist form $\varphi \in \mathscr{L}(\Diamond)$, we have that φ corresponds to a first-order condition C_φ on frames, and that $K + \varphi$ is complete with respect to the class of frames satisfying C_φ (cf. [17]). Our next aim is to give a proof of the correspondence half of a Sahlqvist theorem for $\mathscr{L}(\Diamond, D)$. We need the following notation. For the remainder of this section we use T ($T_0, T_1, \ldots$) as a binary relation symbol to stand for either identity, R, or inequality. The set operators M_T and L_T are defined by $M_T(S) = \{w: \exists v(wTv \wedge v \in S)\}$ and $L_T(S) = (M_T(S^c))^c$. T may be associated with (modal) operators $\mathbf{t}$ and $\bar{\mathbf{t}}$ in the following way. If T is the identity, both $\mathbf{t}$ and $\bar{\mathbf{t}}$ are the identity function; if $T = R$, then $\mathbf{t} = \Diamond$ and $\bar{\mathbf{t}} = \Box$; if T is inequality, then $\mathbf{t} = D$ and $\bar{\mathbf{t}} = \bar{D}$.

To each modal formula φ we associate a set operator F^φ as follows. Let $P_1, \ldots, P_k$ be sets, and let $\vec{P}$ abbreviate $P_1, \ldots, P_k$. Then $F^{p_i}(\vec{P}) = P_i$ $(1 \le i \le k)$, while, for other nonmodal φ, F^φ is the obvious Boolean set operation. Also, $F^{\Diamond\varphi}(S_1, \ldots, S_n) = M_R(F^\varphi(S_1, \ldots, S_n))$, and $F^{D\varphi}(S_1, \ldots, S_n) = M_{\neq}(F^\varphi(S_1, \ldots, S_n))$. The functions $F^{\Box\varphi}$ and $F^{\bar{D}\varphi}$ are defined dually.

LEMMA 3.16. *For any sets X and Y, $X \subseteq L_{T_1} \cdots L_{T_n}(Y)$ iff $M_{\check{T}_n} \cdots M_{\check{T}_1}(X) \subseteq Y$.*

THEOREM 3.17. *Let φ be a Sahlqvist formula in $\mathscr{L}(\Diamond, D)$. Then φ corresponds to a first order condition on frames, effectively obtainable from φ.*

PROOF. Similar to the proof of [4, Theorem 9.10] or [17, Theorem 8]. Assume φ has the form $\bar{\mathbf{t}}_1 \cdots \bar{\mathbf{t}}_m(\psi \to \chi)$ with $\bar{\mathbf{t}}_i \in \{\Box \downarrow \bar{D}\}$ (the more general case is a straightforward generalization). Let $p_1, \ldots, p_k$ be all the proposition letters occurring in φ. Having $\langle W, R\rangle \models \varphi$ means having $\forall P, v, u(vT_1 \circ \cdots \circ T_m u \wedge u \in F^{\psi}(P) \to u \in F^{\chi}(\vec{P}))$, where $T_1, \ldots, T_m$ are the relations corresponding to $\bar{\mathbf{t}}_1, \ldots, \bar{\mathbf{t}}_m$ respectively. Using such equivalences as

$$\forall \cdots (\Phi \wedge x \in F^{\varphi_1 \vee \varphi_2}(\vec{P}) \to \Psi) \leftrightarrow \bigwedge_{i=1,2} \forall \cdots (\Phi \wedge x \in F^{\varphi_i}(\vec{P}) \to \Psi),$$

$$\forall \cdots (\Phi \wedge x \in F^{\mathbf{t}\varphi}(\vec{P}) \to \Psi) \leftrightarrow \forall \cdots \forall y(\Phi \wedge xTy \wedge y \in F^{\varphi}(\vec{P}) \to \Psi),$$

$$\forall \cdots (\Phi \wedge x \in F^{\bar{\mathbf{t}}_1 \cdots \bar{\mathbf{t}}_n \neg p}(\vec{P}) \to \Psi) \leftrightarrow \forall \cdots (\Phi \to \Psi \vee x \in F^{\mathbf{t}_1 \cdots \mathbf{t}_n p}(\vec{P})),$$

this formula may be rewritten as a conjunction of formulas of the form

$$(1) \qquad \forall \vec{P}\vec{x}\vec{u}\left(\Phi \wedge \bigwedge_{j=1}^{k} \bigwedge_{i=1}^{m_j} x_{ij} \in L_{T_{n_{ij}}}(\cdots(L_{T_{1_{ij}}}(P_j))\cdots) \to \bigvee_{j=1}^{h} u_j \in F^{\chi_j}(\vec{P})\right),$$

where Φ is a quantifier-free formula in $\mathscr{L}_0$ ordering its variables in a certain way (each variable occurs to the right of an R or $\neq$ only once), and where all χ_j's are monotone. By 3.16 we have $\bigwedge_{i=1}^{m_j} x_{ij} \in L_{T_{n_{ij}}} \cdots L_{T_{1_{ij}}}(P_j)$ iff $\bigcup_{i=1}^{m_j} M_{\breve{T}_{1_{ij}}} \cdots M_{\breve{T}_{n_{ij}}}(\{x_{ij}\}) \subseteq P_j$. Thus by universal instantiation (1) implies the first-order formula

$$(2) \quad \forall \vec{x}\vec{u}\left(\Phi \to \bigvee_{j=1}^{h} u_j \in F^{\chi_j}\left(\bigcup_{i=1}^{m_1} M_{\breve{T}_{1_{i1}}} \cdots M_{\breve{T}_{n_{i1}}}(\{x_{i1}\}), \ldots, \bigcup_{i=1}^{m_k} M_{\breve{T}_{1_{ik}}} \cdots M_{\breve{T}_{n_{ik}}}(\{x_{ik}\})\right)\right).$$

But, conversely, by the monotonicity of the functions F^{χ_j} (2) implies (1), and we are done. QED.

What about the completeness half of a Sahlqvist theorem for $\mathscr{L}(\Diamond, D)$? An earlier version of this paper did contain a proof for the completeness half. However, Yde Venema found a serious mistake in it; he subsequently proved a full Sahlqvist theorem for $\mathscr{L}(F, P, D)$. Unfortunately, his proof has no adaptation to the Sahlqvist fragment of $\mathscr{L}(\Diamond, D)$, for it relies heavily upon the fact that if a set X is definable (in the sense of Definition 3.7, but with some obvious changes) in an i-canonical general frame for a logic $L \subseteq \mathscr{L}(F, P, D)$, then so is the cone $\{y\colon xR_F y$ for some $x \in X\}$. In general, such cones need not be definable in general frames for logics in $\mathscr{L}(\Diamond, D)$ (cf. [21]). For a special subclass of Sahlqvist forms we do have the following result.

THEOREM 3.18. *Let φ be a weak Sahlqvist form in $\mathscr{L}(\Diamond, D)$. Then φ corresponds to a first-order condition C_φ effectively obtainable from φ, and $DL_m + \varphi$ is complete with respect to the class of frames satisfying C_φ.*

PROOF. The correspondence half is a subcase of 3.17. For a proof of the completeness half we refer the reader to [21]. QED.

Although the class of weak Sahlqvist forms is strictly smaller than the class of all Sahlqvist forms, it is still a large one, which contains $\mathscr{L}(\Diamond, D)$-equivalents of many important first-order conditions on binary relations. For example, by inspecting the proof of 2.4 one can see that it contains equivalents of all Horn-like first-order sentences of the form $\forall \vec{x}(\alpha \to \beta)$, where α and β are positive quantifier-free $\mathscr{L}_0$-formulas.

§4. Definability. We first make a remark or two about definability of classes of frames. After that we give a characterization of the $\mathscr{L}_0$-formulas that are equivalent to a $\diamond$, D-formula on models, and apply this result to obtain a model-theoretic characterization of the definable classes of models.

4.1. Definability of classes of frames. The study of definability of classes of frames in $\mathscr{L}(\diamond, D)$ in the spirit of [11] has been undertaken in [9] and [12]. For the sake of completeness we repeat the main definability result from the latter papers.

A *general* ultraproduct of frames $\mathscr{F}_i$ is an ultraproduct of the full general frames $\langle \mathscr{F}_i, 2^{W_i} \rangle$. (Cf. [4].)

DEFINITION 4.1. $\mathscr{F}'$ is a *collapse* of the general frame $\mathfrak{F} = \langle \mathscr{F}, \mathscr{W} \rangle$ if $\mathscr{F}'$ is a subframe of $\mathscr{F}$ and if there exists a subframe $\mathfrak{G}$ of $\mathfrak{F}$ such that $(\mathscr{F}')^+ \cong (\mathfrak{G})^+$ and, for each $x \in W'$, $\{y\colon Rxy\} \subseteq [R'(x)]_{\mathfrak{G}^+}$, where $[X]_{\mathfrak{G}^+}$ is the least element of $(\mathfrak{G})^+$ containing X, and $(\cdot)^+$ is the mapping defined in [4, Chapter 4] that takes (general) frames to modal algebras.

THEOREM 4.2 (Gargov and Goranko). *A class of frames is definable in $\mathscr{L}(\diamond, D)$ iff it is closed under isomorphisms and collapses of general ultraproducts of frames.*

Gargov and Goranko arrived at 4.2 by using an appropriate kind of modal algebras. For an important special case a purely modal proof may be given:

PROPOSITION 4.3. *A class K of finite frames is definable in $\mathscr{L}(\diamond, D)$ iff it is closed under isomorphisms.*

PROOF. Let $\mathscr{F}$ be a finite frame with $W = \{w_1, \ldots, w_n\}$, and $\mathscr{F} \models \mathrm{Th}_{\diamond,D}(K)$. Assume $p_1, \ldots, p_n$ are different proposition letters. Define $\chi_{\mathscr{F}}$ by

$$\bigwedge_{1 \leq i \leq n} Ep_i \wedge A\Big(\bigvee_{1 \leq i \leq n} (p_i \wedge \neg Dp_i)\Big) \wedge A\Big(\bigwedge_{1 \leq i \neq j \leq n} (p_i \rightarrow \neg p_j)\Big)$$
$$\wedge\, A\Big(\bigwedge_{1 \leq i, j \leq n} (p_i \rightarrow Op_j)\Big),$$

where $O \equiv \diamond$ if Rw_iw_j holds, and $O \equiv \neg\diamond$ otherwise. Then, for any frame $\mathscr{G}$, there is a valuation V with $\langle \mathscr{G}, V \rangle \not\models \neg\chi_{\mathscr{F}}$ iff $\mathscr{G} \cong \mathscr{F}$. In particular $\mathscr{F} \not\models \neg\chi_{\mathscr{F}}$. Hence $\neg\chi_F \notin \mathrm{Th}_{\diamond,D}(K)$. Thus, for some $\mathscr{G} \in K$, $\mathscr{G} \not\models \neg\chi_{\mathscr{F}}$. So $\mathscr{F} \in K$. QED.

4.2. Definability of classes of models. Standard modal formulas, when interpreted on models, are equivalent to a special kind of first-order formulas. Adding the D-operator does not change this.

DEFINITION 4.4. Let x be a fixed variable. The *standard translation* $ST(\varphi)$ of a formula $\varphi \in \mathscr{L}(\diamond, D)$ is defined as follows: it commutes with the Boolean connectives, and $ST(p) = Px$, $ST(\diamond\psi) = \exists y(Rxy \wedge ST(\psi)[x := y])$, and $ST(D\psi) = \exists y(x \neq y \wedge ST(\psi)[x := y])$, where y is a variable not occurring in $ST(\psi)$.

Since the equivalences $\mathscr{M} \models \varphi[w]$ iff $\mathscr{M} \models ST(\varphi)[w]$ and $\mathscr{M} \models \varphi$ iff $\mathscr{M} \models \forall x\, ST(\varphi)$ hold, well-known facts about $\mathscr{L}_1$ become applicable for $\mathscr{L}(\diamond, D)$. $\mathscr{L}_1$-formulas of the form $ST(\varphi)$ for some $\varphi \in \mathscr{L}(\diamond, D)$ can be described independently in the following way:

DEFINITION 4.5. The set of *MD-formulas* is the least set X of $\mathscr{L}_1$-formulas such that $Px \in X$, for unary predicate symbols P and all variables x; if $\alpha \in X$ then $\neg\alpha \in X$; if $\alpha, \beta \in X$ have the same free variable, then $\alpha \wedge \beta \in X$; and if $\alpha \in X$, x, y are distinct variables, and y is α's free variable, then $\exists y(Rxy \wedge \alpha)$, $\exists y(x \neq y \wedge \alpha) \in X$.

The semantic characterization of MD-formulas we give generalizes a corresponding result for $\mathscr{L}(\Diamond)$ in [4]. However, whereas the proof given there uses an elementary chain construction, the proof we present uses saturated models. Clearly, the characterization will also be a characterization of the (translations of the) $(\Diamond, D)$-formulas in $\mathscr{L}_1$.

DEFINITION 4.6. A binary relation Z is called a *p-relation* between two models $\mathscr{M}_1$ and $\mathscr{M}_2$ if the following four conditions hold:

1. If Zwv, then w and v verify the same proposition letters.
2. If Zwv, $w' \in W_1$ and R_1ww', then $Zw'v'$ for some $v' \in W_2$ with R_2vv'; if Zwv, $v' \in W_2$ and R_2vv', then $Zw'v'$ for some $w' \in W_1$ with R_1ww'.
3. If Zwv, $w' \in W_1$ and $w \neq w'$, then $Zw'v'$ for some $v' \in W_2$ with $v \neq v'$; if Zwv, $v' \in W_2$ and $v \neq v'$, then $Zw'v'$ for some $w' \in W_1$ with $w \neq w'$.
4. $\mathrm{dom}(Z) = W_1$ and $\mathrm{ran}(Z) = W_2$.

An $\mathscr{L}_1$-formula $\alpha(x_1, \ldots, x_n)$ is *invariant for p-relations* if, for all models $\mathscr{M}_1$ and $\mathscr{M}_2$, all p-relations Z between $\mathscr{M}_1$ and $\mathscr{M}_2$, and all $w_1, \ldots, w_n \in W_1$ and $w'_1, \ldots, w'_n \in W_2$ such that $Zw_1w'_1, \ldots, Zw_nw'_n$, we have $\mathscr{M}_1 \models \alpha[w_1, \ldots, w_n]$ iff $\mathscr{M}_2 \models \alpha[w'_1, \ldots, w'_n]$.

THEOREM 4.7. *An $\mathscr{L}_1$-formula containing exactly one free variable x is equivalent to an MD-formula iff it is invariant for p-relations.*

PROOF. A simple induction proves that every MD-formula is invariant for p-relations.

Conversely, assume that the $\mathscr{L}_1$-formula α has this property, and suppose x is α's free variable. Define $MD(\alpha) := \{\beta\colon \beta \text{ is an MD-formula}, \alpha \models \beta, FV(\beta) \subseteq \{x\}\}$. We will prove that $MD(\alpha) \models \alpha$. Then, by compactness, there is a $\beta \in MD(\alpha)$ with $\models \alpha \leftrightarrow \beta$. Assume $\mathscr{M} \models MD(\alpha)[w]$; we have to show that $\mathscr{M} \models \alpha[w]$. Introduce a new constant $\underline{w}$ to stand for the object w, and define $\mathscr{L}^* = \mathscr{L}_1 \cup \{\underline{w}\}$. Expand $\mathscr{M}$ to an $\mathscr{L}^*$-model $\mathscr{M}^*$ by interpreting $\underline{w}$ as w. In the remainder of this proof we use the following notation: if $\beta \in \mathscr{L}_1$ then $\beta^* \equiv \beta[x := \underline{w}]$; and if T is a set of $\mathscr{L}_1$-formulas then $T^* := \{\beta^*\colon \beta \in T\}$.

Let $T := \{\beta\colon \mathscr{M} \models \beta[w], \beta \text{ is an MD-formula}, FV(\beta) \subseteq \{x\}\}$. By compactness we find an $\mathscr{L}^*$-model $\mathscr{N}^*$ with $\mathscr{N}^* \models T^* \cup \{\alpha^*\}$. By [7, Theorem 6.1.1] there are ω-saturated elementary extensions $\mathscr{M}_1^* =: \langle W_1, R_1, w_1, V_1\rangle \succ \mathscr{M}^*$ and $\mathscr{N}_1^* =: \langle W_2, R_2, w_2, V_2\rangle \succ \mathscr{N}^*$ such that both w_1 and w_2 realize T, and such that $\mathscr{N}_1^* \models \alpha^*$.

Define a relation $Z \subseteq W_1 \times W_2$ between (the $\mathscr{L}_1$-reducts of) $\mathscr{M}_1^*$ and $\mathscr{N}_1^*$ by putting Zwv iff, for all $\varphi \in \mathscr{L}(\Diamond, D)$,

$$\langle W_1, R_1, V_1\rangle \models \varphi[w] \quad \text{iff} \quad \langle W_2, R_2, V_2\rangle \models \varphi[v].$$

We verify that Z is in fact a p-relation by checking the conditions of 4.6. Condition 1 is trivial. We only check half of condition 2: assume that R_1ww' and Zwv, with $w, w' \in W_1$ and $v \in W_2$. We have to prove $\exists v' \in W_2(R_2vv' \wedge Zwv')$. Define $\Psi := \{\varphi \in \mathscr{L}(\Diamond, D)\colon \mathscr{M}_1^* \models \varphi[w']\}$. Then $ST(\Psi) \cup \{R\underline{v}y\}$ is finitely satisfiable in $(\mathscr{N}_1^*, v)$. Hence, by saturation, $(\mathscr{N}_1^*, v) \models ST(\Psi) \cup \{R\underline{v}y\}[v']$, for some $v' \in W_2$. But then we have $Zw'v'$. Condition 3 is similar to condition 2, and condition 4 is immediate from condition 3 and the fact that Zw_1w_2.

Finally, by invariance for p-relations, $\mathscr{N}_1^* \models \alpha^*$ yields $\mathscr{M}_1^* \models \alpha^*$. Since $\mathscr{M}^* \prec \mathscr{M}_1^*$ it follows that $\mathscr{M}^* \models \alpha^*$, and so $\mathscr{M} \models \alpha[w]$. QED.

Next we apply 4.7 to obtain a definability result for classes of models. To this end we find it convenient to take frames $\langle \mathscr{F}, w\rangle$ with a distinguished world w (as in Kripke's original publications) as the basic notion of frame. Similarly, the basic notion of model is taken to be $\langle \mathscr{F}, w, V\rangle$.

THEOREM 4.8. *Let M be a class of models. Then $M = \{\mathscr{M}\ (=\langle W, R, w, V\rangle)$: $\mathscr{M} \models \varphi[w]\}$ for some $\varphi \in \mathscr{L}(\Diamond, D)$ iff M is closed under p-relations and ultraproducts, while its complement is closed under ultraproducts.*

PROOF. Introduce a new constant $\underline{w}$ to stand for the object w, and define $\mathscr{L}^* := \mathscr{L}_1 \cup \{\underline{w}\}$. As before we write β^* for $\beta[x := \underline{w}]$.

If $M = \{\mathscr{M}\ (=\langle W, R, w, V\rangle)$: $\mathscr{M} \models \varphi[w]\}$ for some $\varphi \in \mathscr{L}(\Diamond, D)$, then M is closed under p-relations and ultraproducts. The complement of M is defined by $\{\neg ST(\varphi)^*\}$, hence closed under ultraproducts.

For the other direction, suppose that $\mathscr{M}$ and its complement satisfy the stated conditions. Since $\mathscr{M}$ is closed under p-relations, it and its complement are closed under isomorphisms. So by [7, Corollary 6.1.16] there is an $\mathscr{L}^*$-sentence α^* such that, for all $\mathscr{L}^*$-models $\mathscr{M}$, $\mathscr{M} \in M$ iff $\mathscr{M} \models \alpha^*$. From the fact that M is closed under p-relations one easily derives that α is closed under p-relations between "ordinary" models. Therefore, by 4.7 α is equivalent to an MD-formula with the same free variable. Hence α is equivalent to $ST(\varphi)$ for some formula $\varphi \in \mathscr{L}(\Diamond, D)$. QED.

REMARK 4.9. In [16] Piet Rodenburg uses a proof similar to the one we gave for 4.7 to characterize the definable classes of models of intuitionistic propositional logic. A reading of this characterization led to 4.8.

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