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The Modal Status of Antinomies

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What is the modal status of antinomies?[1] Classical modal logic provides no interesting answer to this question because it lets antinomies turn all well-formed formulas (including all modal formulas) into theorems. In the present note we propose two nonclassical modal systems which do not suffer from this defect. Both systems are obtained by supplementing the semantics of Asenjo's and Tamburino's antinomic propositional logic L (see [1], familiarity with which will be assumed in this article) with a very natural-sounding truth condition for modal formulas. The surprising result is that antinomies are in any case both *necessary* and *impossible*: according to the second system we propose, they are both *non-necessary* and *possible* as well. It may be doubted whether these results are in accord with our intuitions. However, it should be remembered that our intuitions were formed during centuries of classical slumber; acquiring the right intuitions in antinomic thinking may simply be a matter of time.

1 The systems

1.1 Language The language is as in [1], p. 19, but add to formation rule 2: if $\mathcal{B}_1$ is a statement form, $\Box\mathcal{B}_1$ is a statement form. Definitions:

$$\neg^*\mathcal{B}_1 =_{df} \mathcal{B}_1 \supset A_1 \;\&\; \neg A_1;\ \Diamond\mathcal{B}_1 =_{df} \neg\Box\neg\mathcal{B}_1.$$

1.2 Semantics An antinomic model is a triple $\langle W, R, V\rangle$, where W is a set (of "possible worlds"), $R \subseteq W \times W$, and V: AT $\times\, W \rightarrow \{0,1,2\}$. (Here AT is the set of atomic statements.) $V(A_i, w) = 0$ or 1, whereas $V(B_i, w) = 2$.

The interpretation function I is defined as follows:

1. $I(A_i, w) = V(A_i, w)$, $I(B_i, w) = V(B_i, w)$.
2. $I(\neg\mathcal{B}_1, w)$, $I(\mathcal{B}_1 \,\$\, \mathcal{B}_2, w)$, where \$ is a truth-functional connective: as given in the tables in [1], p. 18, suitably relativized to the world w.

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$$3.\ I(\Box \mathcal{B}_1, w) = \begin{cases} 0 \text{ if } \forall w'(wRw' \Rightarrow I(\mathcal{B}_1, w') = 0) \\ 1 \text{ if } \exists w'(wRw' \text{ and } I(\mathcal{B}_1, w') = 1) \\ 2 \text{ otherwise.} \end{cases}$$

The motivation for the latter clause is straightforward. Like [1], we read "$I(\mathcal{B}_1, w) = 0(1,2)$" as "$\mathcal{B}_1$ is true and not false (false and not true, true and false) at w". Hence clause 3 is merely another way of stating the familiar and intuitively plausible condition 3′:

$$3'.\ \Box \mathcal{B}_1 \text{ is } \begin{cases} \text{true at } w \text{ (in a model } M\text{) iff } \mathcal{B}_1 \text{ is true at} \\ \quad \text{all } w' \text{ accessible from } w \text{ (in } M\text{)} \\ \\ \text{false at } w \text{ (in } M\text{) iff } \mathcal{B}_1 \text{ is false at some} \\ \quad w' \text{ accessible from } w \text{ (in } M\text{).}^2 \end{cases}$$

We say that $\mathcal{B}_1$ is *valid* in $\langle W, R, V \rangle$ iff for all $w \in W\, I(\mathcal{B}_1, w) \neq 1$ (i.e., iff $\mathcal{B}_1$ is true at all $w \in W$ in the model).

A *serial* antinomic model is an antinomic model satisfying the condition that $\forall w \exists w' wRw'$.

1.3 Axiomatization $\mathcal{A}$-formulas are determined as in [1], pp. 20–21, but add to C2a: if $\mathcal{A}_1$ is an $\mathcal{A}$-formula, $\Box \mathcal{A}_1$ is an $\mathcal{A}$-formula.

The axioms of M are as follows. M1–M13 are the same as L1–L13 ([1], p. 21). To these we add:

M14 $\Box(\mathcal{B}_1 \supset \mathcal{B}_2) \supset (\Box \mathcal{B}_1 \supset \Box \mathcal{B}_2)$.
M15a $\neg \Box \neg^* \mathcal{B}_1 \supset \Diamond \mathcal{B}_1$.[3]
M15b $\Diamond \mathcal{B}_1 \supset \neg \Box \neg^* \mathcal{B}_1$.

The axioms of MD are those of M plus

D $\Box \mathcal{B}_1 \supset \Diamond \mathcal{B}_1$.

The rules of both M and MD are modus ponens (R1) and $\mathcal{B}_1 / \Box \mathcal{B}_1$ (R2).

1.4 Soundness and completeness

Theorem *$\mathcal{B}_1$ is valid in the class of all antinomic models (all serial antinomic models) iff $\mathcal{B}_1$ is derivable in M (MD).*

Proof: From right to left: trivial.

From left to right: we define the canonical model $\langle W, R, V \rangle$ for M (MD) as follows:

1. W is the set of all subsets of the language absolutely consistent and complete with respect to M (MD).
2. $R = \{\langle w, w' \rangle \in W \times W$: for all $\mathcal{B}_1$: $w \vdash \Box \mathcal{B}_1 \Rightarrow w' \vdash \mathcal{B}_1\}$. (Here and in the following, $\vdash$ stands for M– (MD–) derivability.)

3. $$V(\mathcal{B}_1, w) = \begin{cases} 0 \text{ if } w \vdash \mathcal{B}_1 \text{ and } w \nvdash \neg\mathcal{B}_1 \\ 1 \text{ if } w \nvdash \mathcal{B}_1 \\ 2 \text{ if } w \vdash \mathcal{B}_1 \text{ and } w \vdash \neg\mathcal{B}_1. \end{cases}$$

Lemma *For all* $\mathcal{B}_1$ *and all* $w \in W$ *in the canonical model*:

$$I(\mathcal{B}_1, w) = \begin{cases} 0 \text{ if } w \vdash \mathcal{B}_1 \text{ and } w \nvdash \neg\mathcal{B}_1 \\ 1 \text{ if } w \nvdash \mathcal{B}_1 \\ 2 \text{ if } w \vdash \mathcal{B}_1 \text{ and } w \vdash \neg\mathcal{B}_1. \end{cases}$$

Proof of lemma: The proof is by induction on the length of $\mathcal{B}_1$. In case $\mathcal{B}_1$ is atomic, the lemma holds by definition. Inductive Hypothesis (I.H.): the lemma holds for $\mathcal{B}_1$, $\mathcal{B}_2$. (i) Then it holds for $\neg\mathcal{B}_1$, $\mathcal{B}_1 \,\$\, \mathcal{B}_2$, where \$ is a truth-functional connective: see [1], proof of Proposition 4.12 (pp. 33–37). (ii) Then it holds for $\Box\mathcal{B}_1$. There are three subcases.

Subcase 1. Suppose $w \vdash \Box\mathcal{B}_1$ and $w \nvdash \neg\Box\mathcal{B}_1$. The first conjunct implies $\forall w'(wRw' \Rightarrow w' \vdash \mathcal{B}_1)$ by definition of R. The second conjunct implies $w \nvdash \neg\Box\neg^*\neg\mathcal{B}_1$ by M15a and R1, hence $w \vdash \Box\neg^*\neg\mathcal{B}_1$ by completeness of w, hence $\forall w'(wRw' \Rightarrow w' \nvdash \neg\mathcal{B}_1)$ by definition of R and absolute consistency of w'. Combination of both consequents and application of I.H. and definition of I yields $I(\Box\mathcal{B}_1, w) = 0$.

Subcase 2. Suppose $w \nvdash \Box\mathcal{B}_1$. Consider $N(w) = \{\mathcal{B}_2 \colon w \vdash \Box\mathcal{B}_2\}$. Suppose $N(w) \cup \{\neg^*\mathcal{B}_1\} \vdash A_1 \,\&\, \neg A_1$. Then $N(w) \vdash \neg^*\neg^*\mathcal{B}_1$ by deduction theorem, hence $N(w) \vdash \mathcal{B}_1$ by antinomic propositional calculus L, which means there are $\mathcal{B}_3, \ldots, \mathcal{B}_n \in N(w)$ such that $\vdash \mathcal{B}_3 \supset (\mathcal{B}_4 \supset \ldots \supset (\mathcal{B}_n \supset \mathcal{B}_1) \ldots)$ by definition of derivability and deduction theorem ($n - 2$ times). R2 yields $\vdash \Box(\mathcal{B}_3 \supset (\mathcal{B}_4 \supset \ldots \supset (\mathcal{B}_n \supset \mathcal{B}_1) \ldots))$, whence $\{\Box\mathcal{B}_3, \ldots, \Box\mathcal{B}_n\} \vdash \Box\mathcal{B}_1$ by M14 and R1 ($n - 2$ times), whence $w \vdash \Box\mathcal{B}_1$ by completeness of w—a contradiction. Hence $N(w) \cup \{\neg^*\mathcal{B}_1\}$ is absolutely consistent. Therefore $\exists w' \in W(wRw'$ and $w' \vdash \neg^*\mathcal{B}_1)$ by Lindenbaum's lemma (compare [1], Lemma 4.11), whence $\exists w'(wRw'$ and $w' \nvdash \mathcal{B}_1)$ by absolute consistency of w', whence $I(\Box\mathcal{B}_1, w) = 1$ by I.H. and definition of I.

Subcase 3. Suppose $w \vdash \Box\mathcal{B}_1$ and $w \vdash \neg\Box\mathcal{B}_1$. The first conjunct implies $\forall w'(wRw' \Rightarrow w' \vdash \mathcal{B}_1)$ by definition of R. The second conjunct implies $w \vdash \neg\Box\neg^*\neg\mathcal{B}_1$ by M15b and R1. $\neg\Box\neg^*\neg\mathcal{B}_1$ is an $\mathcal{A}$-formula by C1b and C2a, hence $w \nvdash \Box\neg^*\neg\mathcal{B}_1$ by absolute consistency of w. By the same reasoning as in subcase 2 we have $\exists w'(wRw'$ and $w' \nvdash \neg^*\neg\mathcal{B}_1)$, whence $\exists w'(wRw'$ and $w' \vdash \neg\mathcal{B}_1)$ by completeness of w'. By I.H. we have *not* $\exists w'(wRw'$ and $I(\mathcal{B}_1, w') = 1)$ and *not* $\forall w'(wRw' \Rightarrow I(\mathcal{B}_1, w') = 0)$, whence $I(\Box\mathcal{B}_1, w) = 2$ by definition of I.

This completes the proof of the lemma.

Completeness follows (compare [1], p. 39): Suppose $\nvdash_M \mathcal{B}_1 (\nvdash_{MD} \mathcal{B}_1)$. Then there is a $w \in W$ in the canonical model for M (MD) such that $w \nvdash_M \mathcal{B}_1 (w \nvdash_{MD} \mathcal{B}_1)$ by Lindenbaum's lemma, whence $I(\mathcal{B}_1, w) = 1$ by lemma. The canonical model for M (MD) is an antinomic model (serial antinomic model), hence $\mathcal{B}_1$ is not valid in the class of all antinomic models (all serial antinomic models).

2 *The modal status of antinomies in M and MD*

Observation 1 *If $\mathcal{B}_1$ is an antinomy, the schemas $\Box^k\mathcal{B}_1$ and $\neg\Diamond^k\mathcal{B}_1$ are theorems of M and MD for every $k \geqq 0$ ($k \in \mathbf{N}$).* (*Proof:* by induction on length of formula, using R2.)

Observation 2 *If $\mathcal{B}_1$ is an antinomy, every instance of the schema $\Sigma\mathcal{B}_1$, where Σ is any* (!) *sequence of occurrences of $\neg$, $\Box$, and $\Diamond$, is a theorem of MD.* (*Proof:* by induction on length of formula, using R1, R2, and D.)

The latter observation does not imply, however, that *every* statement concerning the modal status of antinomies is provable in MD. In fact, there are infinitely many of such statements which are unprovable; for example, each statement of the form $\neg{*}\Sigma B_1$, where Σ is any sequence of occurrences of $\neg$, $\Box$, and $\Diamond$, is invalid in the class of all serial models and therefore neither provable in M nor in MD. Hence:

Observation 3 *M and MD are absolutely consistent.*

NOTES

1. An *antinomy* is, syntactically speaking, a provable statement whose negation is also provable; semantically, it is a statement that is both true and false at all possible worlds in all models.

2. Using the definition of $\Diamond$, the corresponding condition for $\Diamond\mathcal{B}_1$ turns out to be: $\Diamond\mathcal{B}_1$ is true at w (in M) iff $\mathcal{B}_1$ is true at some w' such that wRw' (in M); $\Diamond\mathcal{B}_1$ is false at w (in M) iff $\mathcal{B}_1$ is false at all w' such that wRw' (in M). Similar truth conditions (for tensed instead of modalized formulas) are to be found in [2], Section 3.2.

3. M15a is interderivable with $\neg\Box(\mathcal{B}_1 \supset \mathcal{B}_2) \supset \Diamond\mathcal{B}_1$.

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