Travaux mathématiques, Volume 16 (2005), 315–339,

The modular class of a twisted Poisson structure

by Yvette Kosmann-Schwarzbach

and Camille Laurent-Gengoux

Abstract

We study the geometric and algebraic properties of the twisted Poisson structures on Lie algebroids, leading to a definition of their modular class and to an explicit determination of a representative of the modular class, in particular in the case of a twisted Poisson manifold.

Résumé. Nous étudions des propriétés géométriques et algébriques des structures de Poisson tordues sur les algébroïdes de Lie, permettant une définition de leur classe modulaire et une détermination explicite d'un représentant de celle-ci, en particulier pour le cas d'une variété de Poisson tordue.

1 Introduction

The primary aim of this paper is to extend the definition and properties of the modular class of Poisson manifolds to the case of manifolds with a *twisted Poisson structure*. Moreover we show that the notion of modular class can be extended to the case of Lie algebroids with a twisted Poisson structure.

There are many ways in which the Jacobi identity for a skew-symmetric bracket can be violated. On a manifold with a twisted Poisson structure, the Jacobi identity for the Poisson bracket holds only up to an additional term involving a closed 3-form, called the *background 3-form*. Such structures appeared in string theory. The quantization of an open membrane coupled to a background 3-form was interpreted in the work of Jae-Suk Park [13] as a deformation of the theory without background; the defining condition, equation (4.2) below, appears as the quantum master equation in the Batalin-Vilkovisky quantization of an action functional containing a term, the C-field, which is a closed 3-form. Meanwhile, in [2], Cornalba and Schiappa introduced a star-product deformation which is not only non-commutative but also non-associative; the Jacobi identity for the associated commutator bracket was therefore violated, and that identity took the form that appears in formula (4.4) below. For Klimčik and Strobl [7], equation (4.2) appears as the condition for the constraints for a Lagrangian system to be first-class, i.e., to span a subalgebra of the Poisson algebra, when the action is that of a Poisson σ-model to which is added a term analogous to the term of Wess and Zumino as, e.g., in the Wess-Zumino-Witten model. For this reason, Klimčik et Strobl proposed to call such structures *WZW-Poisson structures* or *WZ-Poisson structures*. "Poisson geometry with a 3-form background" was then studied by Ševera and Weinstein [15] who showed that such a structure is a Dirac structure in a Courant algebroid whose bracket is defined by means of the 3-form. They called it a *Poisson structure with background* or *a twisted Poisson structure*, the term that we have adopted here, in spite of a possible confusion of terminology evoked at the very end of our paper. While any twisted Poisson manifold is locally equivalent to a genuine Poisson manifold, global phenomena make this generalized case interesting.

The *modular vector fields* of Poisson manifolds already figured in Koszul's 1985 article [12], and some of their properties and applications appear in the work of Dufour and Haraki [3], who called them "curl" (rotationnel, in French), and in other papers of the early nineties. Weinstein, in [16], related this notion to the modular automorphism group of von Neumann algebras, gave it the name that has been adopted in the literature, and introduced the notion of modular class.

Given an orientable Poisson manifold, choose a volume form and associate to each smooth function the divergence of its hamiltonian vector field. The map thus obtained is a derivation and is, by definition, the modular vector field. The basic observation is that this vector field is closed in the Poisson cohomology and that its cohomology class, the *modular class*, does not depend on the choice of the volume form. The non-orientable case can be dealt with, replacing volume forms by densities. Further advances, already announced in [16], were made in the article of Evens, Lu and Weinstein [4] where the modular class for a Lie algebroid was defined and where it was shown that the modular class of a Poisson manifold was one half that of its cotangent Lie algebroid. At the same time, Huebschmann developed a powerful algebraic theory in the framework of Lie-Rinehart algebras [5] [6] which recovered the results of [4] when applied to the case where a Lie-Rinehart algebra is the space of sections of a Lie algebroid. Duality properties were proved by these authors and by Xu in [17].

In this article, we shall follow the approach of [8], where the modular vector fields are characterized in terms of the difference of two generating operators of square zero of the Gerstenhaber algebra associated to the given Lie algebroid. After brief preliminary results on operators of order 1 and 2 on graded algebras, we introduce, in Section 3, operators on forms and multivectors and a vector field, defined in terms of a bivector and a 3-form, that are the needed ingredients of the construction of the modular class. This makes sense on an arbitrary vector bundle. In Section 4, we define the Lie algebroids with a twisted Poisson structure, which comprise the cotangent Lie algebroids of twisted Poisson manifolds and the triangular Lie bialgebroids. We show in Section 5 that on a Lie algebroid with a twisted Poisson structure, there exist generators of square zero of the Gerstenhaber algebra of multivectors, defined in terms of the operators of Section 3, and the definition and properties of the modular vector field (Section 6) follow. That vector field is closed in the Lie algebroid cohomology and its class is well-defined (Theorem 6.1). It is the sum of the vector field $X_{\pi,\lambda}$, depending on the bivector π and the volume form λ (or density in the non-orientable case), that appears in the untwisted case, and which is no longer closed in the twisted case, and the vector field, $Y_{\pi,\psi}$, depending on the bivector π and the 3-form ψ defining the twisted Poisson structure. In Section 7, we then recall the construction of Evens, Lu and Weinstein [4] and prove that, as expected, when the Lie algebroid is the tangent bundle of a twisted Poisson manifold, M, the class that we have defined is one half the class, as defined in [4], of the cotangent Lie algebroid T^*M. The examples described in Section 8 show that this is not the case in general, even for Lie algebras considered as Lie algebroids over a point. As another example, we show that the modular classes of the Lie groups equipped with the twisted Poisson structure introduced in [15] vanish. Many of the features of the usual Poisson case can be recovered but new phenomena appear in the case of the Lie algebroids with a twisted Poisson structure.

A full understanding of the relationship between the modular class that we introduce and the modular classes defined in [16] [4] for Lie algebroids, and further justification for the generalization that we propose are provided by the consideration of the relative modular classes [11].

2 Preliminaries: differential operators on graded commutative algebras

By definition, a graded linear operator on a graded commutative algebra $\mathbf{A}$ is of order less than or equal to k if its graded commutator with any $k+1$ left-multiplications by elements of $\mathbf{A}$ vanishes. The graded commutator of graded endomorphims u, of degree $|u|$, and v, of degree $|v|$, of the graded vector space $\mathbf{A}$ is $[u,v] = u \circ v - (-1)^{|u||v|} v \circ u$. Let 1 denote the unit element of $\mathbf{A}$, and let ℓ_a denote left-multiplication by $a \in \mathbf{A}$. For u a graded linear operator on $\mathbf{A}$, and $k = 1$, 2 and 3, we consider the operators $\Phi_u^k : A^{\otimes k} \to A$, defined in [1]. For a, b and c in $\mathbf{A}$,

$$\Phi_u^1(a) = u(a) - u(1)a,$$

$$\Phi_u^2(a)(b) = \Phi_u^1(ab) - \Phi_u^1(a)b - (-1)^{|a||u|} a \Phi_u^1(b),$$

$$\Phi_u^3(a,b)(c) = \Phi_u^2(a)(bc) - \Phi_u^2(a)(b)c - (-1)^{(|a|+|u|)|b|} b\, \Phi_u^2(a)(c).$$

It is easy to prove the following propositions.
- u is of order 0 if and only if $\Phi_u^1 = 0$.
- u is of order $\leqslant 1$ if and only if $\Phi_u^2 = 0$.

In fact, for all $a \in \mathbf{A}, \Phi_u^2(a) = \Phi_{[u,\ell_a]}^1$.

- u is of order $\leqslant 1$ if and only if Φ_u^1 is of order $\leqslant 1$.
- A differential operator u of order $\leqslant 1$ is a derivation if and only if $u(1) = 0$.
- u is of order $\leqslant 2$ if and only if $\Phi_u^3 = 0$.
- u is of order $\leqslant 2$ if and only if $\Phi_u^2(a)$ is a derivation, for all $a \in A$.

In fact, u is of order $\leqslant 2$ if and only if $[u, \ell_a]$ is of order $\leqslant 1$ for all a. This condition is equivalent to $\Phi_u^2(a)$ is of order $\leqslant 1$ for all a, because $\Phi_u^2(a)$ and $[u, \ell_a]$ differ by left-multiplication by $\Phi_u^1(a)$, an operator of order 0. Since $\Phi_u^2(a)(1) = 0$, the operator $\Phi_u^2(a)$ is of order $\leqslant 1$ if and only if it is a derivation.

We remark that the expression $\Phi_u^2(a)(b)$ is skew-symmetric (in the graded sense) in a and b. More precisely,

$$(-1)^{|b|}\Phi_u^2(b)(a) = -(-1)^{(|a|+1)(|b|+1)}(-1)^{|a|}\Phi_u^2(a)(b). \tag{2.1}$$

3 Bivectors and 3-forms

We shall make use of several algebraic constructions which we now describe.

3.1 Conventions

Hereafter A is a vector bundle (later, a Lie algebroid) with base M. By convention, we shall call sections of A vector fields or vectors. More generally, for p and q positive integers, we call sections of $\wedge^p A$, p-vectors and, similarly, we call sections of $\wedge^q(A^*)$, q-forms. The pairing of a p-vector and a p-form is denoted $< \ , \ >$.

Let i_X be the interior product of forms by the vector X, which is a derivation of $\Gamma(\wedge^\bullet A^*)$. More generally, for vectors X_1 and X_2, set

$$i_{X_1 \wedge X_2} = i_{X_1} \circ i_{X_2} \ ,$$

and define inductively the interior product of forms by a multivector. By definition, the interior product of an r-form (r a positive integer) by a section of $\wedge^q A^* \otimes \wedge^p A$ vanishes if $p > r$ and, for $p \leqslant r$, satisfies

$$i_{\xi_1 \wedge \ldots \wedge \xi_q \otimes X_1 \wedge \ldots \wedge X_p}(\alpha_1 \wedge \ldots \wedge \alpha_r) = \xi_1 \wedge \ldots \wedge \xi_q \wedge i_{X_1 \wedge \ldots \wedge X_p}(\alpha_1 \wedge \ldots \wedge \alpha_r) \ .$$

Interior products by elements of $\Gamma(\wedge^q A^* \otimes \wedge^p A)$ are operators of order p on $\Gamma(\wedge^\bullet A^*)$. The interior product of multivectors by multivector-valued forms is similarly defined.

Given a bivector π, the vector bundle morphism $\pi^\sharp$ from A^* to A is defined by $< \beta, \pi^\sharp \alpha > = \pi(\alpha, \beta)$, for 1-forms α and β. Thus

$$i_\pi(\alpha \wedge \beta) = -\pi(\alpha, \beta) = < \alpha, \pi^\sharp \beta > \ .$$

Let (e_k) and (ϵ^k), $1 \leqslant k \leqslant N$, where N is the rank of A, be dual local bases of sections of A and A^*, respectively. Then

$$\pi = \frac{1}{2} e_k \wedge \pi^\sharp(\epsilon^k) = -\frac{1}{2}\pi^\sharp(\epsilon^\ell) \wedge e_\ell \ .$$

Here and below, we use the Einstein summation convention. The components, $\pi^{k\ell}$, of π are defined by $\pi = \frac{1}{2}\pi^{k\ell} e_k \wedge e_\ell$, and they satisfy $\pi^\sharp(\epsilon^k) = \pi^{k\ell} e_\ell$.

Let π be a section of $\wedge^2 A$ and ψ a section of $\wedge^3 A^*$. We define $\psi^{(1)}$ by

$$\psi^{(1)}(\xi)(X,Y) = \psi(\pi^\sharp \xi, X, Y) \ ,$$

for $\xi \in \Gamma(A^*)$, X and $Y \in \Gamma A$. Thus $\psi^{(1)}$ is both a vector-valued 2-form and a 2-form-valued vector on A. We further define $\psi^{(2)}$ by

$$\psi^{(2)}(\xi,\eta)(X) = \psi(\pi^\sharp \xi, \pi^\sharp \eta, X) \ ,$$

for ξ and $\eta \in \Gamma(A^*)$, and $X \in \Gamma A$. Thus $\psi^{(2)}$ is both a bivector-valued 1-form and a 1-form-valued bivector on A. In components, setting $\psi = \frac{1}{6}\psi_{k\ell m}\epsilon^k \wedge \epsilon^\ell \wedge \epsilon^m$, we find

$$\psi^{(1)} = \frac{1}{2}\pi^{kp}\psi_{p\ell m}\, \epsilon^\ell \wedge \epsilon^m \otimes e_k \ , \tag{3.1}$$

and

$$\psi^{(2)} = \frac{1}{2}\pi^{kp}\pi^{\ell q}\psi_{pqm}\, \epsilon^m \otimes e_k \wedge e_\ell \ . \tag{3.2}$$

3.2 Operators on forms and multivectors

The following operators are naturally defined on a vector bundle A equipped with a bivector π and a 3-form ψ.

3.2.1 The operator $\underline{\partial}_{\pi,\psi}$

Any form-valued bivector on A acts by interior product on the sections of $\wedge^\bullet A^*$. We define the operator $\underline{\partial}_{\pi,\psi}$ on sections of $\wedge^\bullet A^*$ to be the interior product by the 1-form-valued bivector $\psi^{(2)}$.

Lemma 3.1. *a) The operator $\underline{\partial}_{\pi,\psi}$ is a differential operator of order 2 and of degree -1 on the graded algebra $\Gamma(\wedge^\bullet A^*)$, which vanishes on functions and on 1-forms.*

b) For $q \geqslant 2$, and for all $\alpha_1, \dots, \alpha_q \in \Gamma(A^*)$,

$$\underline{\partial}_{\pi,\psi}(\alpha_1 \wedge \dots \wedge \alpha_q) = \sum_{1 \leqslant k < \ell \leqslant q} (-1)^{k+l}\psi^{(2)}(\alpha_k, \alpha_\ell) \wedge \alpha_1 \wedge \dots \wedge \widehat{\alpha_k} \wedge \dots \wedge \widehat{\alpha}_\ell \wedge \dots \wedge \alpha_q \ . \tag{3.3}$$

(A caret over a factor signifies that the factor is missing.)

Proof. a) We know that, for $p \in \mathbb{N}$, the interior product by a form-valued p-vector is a differential operator of order p, thus part a) follows.

b) Equation (3.3) follows from the definition of the interior product. □

3.2.2 The operator $\underline{d}_{\pi,\psi}$

We now consider $\psi^{(2)}$ as a bivector-valued 1-form on A, we let it act by interior product on the sections of $\wedge^\bullet A$, and we denote this operator by $\underline{d}_{\pi,\psi}$.

Lemma 3.2. *a) The operator $\underline{d}_{\pi,\psi}$ is a derivation of degree $+1$ of the graded algebra $\Gamma(\wedge^\bullet A)$, which vanishes on functions.*

b) For $p \geqslant 1$ and for all $X_1, \ldots, X_p \in \Gamma A$,

$$\underline{d}_{\pi,\psi}(X_1 \wedge \cdots \wedge X_p) = \sum_{k=1}^{p} (-1)^{k+1} \psi^{(2)}(X_k) \wedge X_1 \wedge \cdots \wedge \widehat{X_k} \wedge \cdots \wedge X_p \ . \tag{3.4}$$

Proof. a) Since $\underline{d}_{\pi,\psi}$ is a differential operator of order 1 and it vanishes on functions, it is a derivation.

b) Equation (3.4) follows from the definition of the interior product. □

3.2.3 The operator $\delta_{\pi,\psi}$

We define the operator $\delta_{\pi,\psi}$ on sections of $\wedge^\bullet A^*$ to be the interior product by the 2-form-valued vector $\psi^{(1)}$.

Lemma 3.3. *a) The operator $\delta_{\pi,\psi}$ is a derivation of degree $+1$ of the graded algebra $\Gamma(\wedge^\bullet A^*)$, which vanishes on functions.*

b) For $q \geqslant 1$ and for all $\alpha_1, \ldots, \alpha_q \in \Gamma(A^*)$,

$$\delta_{\pi,\psi}(\alpha_1 \wedge \cdots \wedge \alpha_q) = \sum_{k=1}^{q} (-1)^{k+1} \psi^{(1)}(\alpha_k) \wedge \alpha_1 \wedge \cdots \wedge \widehat{\alpha_k} \wedge \cdots \wedge \alpha_q \ . \tag{3.5}$$

Proof. a) Since $\delta_{\pi,\psi}$ is the interior product by a 2-form-valued vector, it is a derivation.

b) Equation (3.5) follows from the definition of the interior product. □

Remark 3.1. Similarly, an operator of degree -1 on multivectors can be defined as the interior product by the vector-valued 2-form $\psi^{(1)}$. This operator and all three operators defined above are $C^\infty(M)$-linear and can therefore be defined pointwise.

3.3 A vector field

Since $i_\pi\psi$ is a section of A^*, $\pi^\sharp i_\pi \psi$ is a section of A. We set

$$Y_{\pi,\psi} = \pi^\sharp(i_\pi\psi). \tag{3.6}$$

Proposition 3.1. *For any section α of A^*,*

$$< \alpha, Y_{\pi,\psi} >= -i_{\pi^\sharp(\alpha)\wedge\pi}\psi = \frac{1}{2}\,\mathrm{Tr}\Psi_\alpha, \tag{3.7}$$

where $\mathrm{Tr}\Psi_\alpha$ is the trace of the endomorphism of A^* defined by

$$\Psi_\alpha(\beta) = \psi^{(2)}(\alpha, \beta), \tag{3.8}$$

for each section β of A^*.

Proof. From the definition of $Y_{\pi,\psi}$, using the skew-symmetry of $\pi^\sharp$, we obtain

$$< \alpha, Y_{\pi,\psi} >=< \alpha, \pi^\sharp(i_\pi\psi) >= - < i_\pi\psi, \pi^\sharp\alpha >$$

which is indeed equal to $-i_{\pi^\sharp(\alpha)\wedge\pi}\psi$, while the trace of Ψ_α is

$$< \psi^{(2)}(\alpha, \epsilon^k), e_k >= \psi(\pi^\sharp\alpha, \pi^\sharp\epsilon^k, e_k) =< \psi, \pi^\sharp\alpha \wedge \pi^\sharp\epsilon^k \wedge e_k >,$$

where e_k and ϵ^k are dual local bases of sections of A and A^*. The conclusion follows from the relation $\pi = -\frac{1}{2}\,\pi^\sharp\epsilon^k \wedge e_k$. □

Proposition 3.2. *The operators on sections of $\wedge^\bullet A^*$, $\delta_{\pi,\psi}$, $\underline{\partial}_{\pi,\psi}$ and $i_{Y_{\pi,\psi}}$ are related by*

$$[i_\pi, \delta_{\pi,\psi}] = 2\underline{\partial}_{\pi,\psi} - i_{Y_{\pi,\psi}}\ . \tag{3.9}$$

Proof. Since i_π is of order 2 and of degree -2, and $\delta_{\pi,\psi} = i_{\psi^{(1)}}$ is of order 1 and of degree 1, their commutator is of order $\leqslant 2$ and degree -1. Introducing the big bracket as in [14], we know that the term of order 2 in the commutator is the interior product by the big bracket $\{\pi, \psi^{(1)}\}$ of π and $\psi^{(1)}$ (see [9]). A computation shows that $\psi^{(2)} = \frac{1}{2}\{\pi, \psi^{(1)}\}$, therefore the term of order 2 is $2i_{\psi^{(2)}} = 2\underline{\partial}_{\pi,\psi}$. If α is a 1-form, then $[i_\pi, \delta_{\pi,\psi}]\alpha = i_\pi i_{\pi^\sharp\alpha}\psi$, while

$$-i_{Y_{\pi,\psi}}\alpha = - < \pi^\sharp(i_\pi\psi), \alpha >=< i_\pi\psi, \pi^\sharp\alpha >= i_{\pi^\sharp\alpha}i_\pi\psi.$$

Thus equation (3.9) is satisfied for 1-forms. It follows that the term of order 1 in the commutator is $-i_{Y_{\pi,\psi}}$. □

4 Lie algebroids with a twisted Poisson structure

4.1 Lie algebroids

We now assume that the vector bundle A is a Lie algebroid over the base manifold M, with anchor ρ. We recall that ρ is a Lie algebroid morphism from A to TM.

We denote the Lie bracket of sections of A and the Gerstenhaber bracket on the graded commutative algebra, $\Gamma(\wedge^\bullet A)$, obtained by extending it as a graded biderivation, by the same symbol, $[\ ,\ ]$. We recall that by definition $[a,.]$ is a derivation of degree $|a|-1$ of $\Gamma(\wedge^\bullet A)$, where $|a|$ is the degree of a. For $a \in \Gamma A$ and $f \in C^\infty(M)$, $[a,f] = \rho(a)\cdot f$ and, for all $a_1,\dots,a_q,b_1,\dots,b_r \in \Gamma A$, $q \geqslant 1$, $r \geqslant 1$,
(4.1)

$$[a_1\wedge\cdots\wedge a_q, b_1\wedge\cdots\wedge b_r]=\sum_{k=1}^{q}\sum_{\ell=1}^{r}(-1)^{k+\ell}[a_k,b_\ell]\wedge a_1\wedge\cdots\wedge\widehat{a_k}\wedge\cdots\wedge a_q\wedge b_1\wedge\cdots\wedge\widehat{b_\ell}\wedge\cdots\wedge b_r.$$

We shall also consider the differential d_A on $\Gamma(\wedge^\bullet A^*)$ which is such that, for $f \in C^\infty(M)$, $d_A f(a) = \rho(a)\cdot f$, for all $a \in \Gamma A$, and for a q-form α, $q \geqslant 1$,

$$\begin{aligned}(d_A\alpha)(a_0,...,a_q) = &\sum_{0\leqslant k\leqslant q}(-1)^k\rho(a_k)\cdot(\alpha(a_0,\dots,\widehat{a_k},\dots,a_q))\\ &+\sum_{0\leqslant k<\ell\leqslant q}(-1)^{k+\ell}\alpha([a_k,a_\ell],a_0,\dots,\widehat{a_k},\dots,\widehat{a_\ell},\dots,a_q)\ ,\end{aligned}$$

for all $a_0,\dots,a_q \in \Gamma A$. The Lie derivation of forms by a section X of A is the operator $\mathcal{L}^A_X = [i_X, d_A]$. When A is TM with the Lie bracket of vector fields, the differential d_A is the de Rham differential of forms, and the Lie derivation coincides with the usual notion.

4.2 Twisted Poisson structures

By definition, (A,π,ψ) is a *Lie algebroid with a twisted Poisson structure* if π is a section of $\wedge^2 A$ and ψ is a d_A-closed section of $\wedge^3 A^*$ such that

$$\frac{1}{2}[\pi,\pi] = (\wedge^3\pi^\sharp)\psi. \tag{4.2}$$

To each function $f \in C^\infty(M)$ is associated the section H_f of A, called the hamiltonian section with hamiltonian f, defined by

$$H_f = \pi^\sharp(d_A f) = -[\pi, f]. \tag{4.3}$$

The bracket of two functions f and g is then defined as

$$\{f,g\} = [H_f, g].$$

This bracket is skew-symmetric and satisfies the following modified Jacobi identity, for all $f,g,h \in C^\infty(M)$,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = \psi(H_f,H_g,H_h)\ . \tag{4.4}$$

This relation is equivalent to

$$H_{\{f,g\}} = [H_f, H_g] + \psi^{(1)}(H_f, H_g).$$

When a twisted Poisson structure is defined on the Lie algebroid TM, the manifold M is called a *twisted Poisson manifold*. The following results were proved by Ševera and Weinstein [15] in the case of $A = TM$ and extend to the case of any Lie algebroid A. See [14] for the case of Lie algebroids.

Theorem 4.1. *Let (A, π, ψ) be a Lie algebroid with a twisted Poisson structure. Then A^* is a Lie algebroid with anchor $\rho \circ \pi^\sharp$, where ρ is the anchor of A, and Lie bracket of sections, α and β, of A^*,*

$$[\alpha, \beta]_{\pi,\psi} = [\alpha, \beta]_\pi + \psi^{(2)}(\alpha, \beta), \tag{4.5}$$

where $[\ ,\ ]_\pi$ is defined by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha - d_A(\pi(\alpha, \beta)). \tag{4.6}$$

The differential of the Lie algebroid A^* is

$$d_{\pi,\psi} = d_\pi + \underline{d}_{\pi,\psi}, \tag{4.7}$$

where, for $X \in \Gamma(\wedge^\bullet A)$, $d_\pi X = [\pi, X]$, and $\underline{d}_{\pi,\psi}$ is defined in 3.2.2.

The map $\pi^\sharp$ satisfies

$$\pi^\sharp[\alpha, \beta]_{\pi,\psi} = [\pi^\sharp\alpha, \pi^\sharp\beta].$$

The case where $\psi = 0$ is that of a Lie algebroid with a Poisson structure, i.e., bivector π such that $[\pi, \pi] = 0$, and the pair (A, A^*) is also called a *triangular Lie bialgebroid*. If moreover $A = TM$, we recover the case of a Poisson manifold.

Remark 4.1. When π is an arbitrary bivector and ψ a 3-form, one can still define a bracket $[\ ,\ ]_{\pi,\psi}$, which does not in general satisfy the Jacobi identity, and a derivation, $d_{\pi,\psi}$, which is not in general of square zero.

Proposition 4.1. *Given a bivector π and a 3-form ψ, the vector field $Y_{\pi,\psi}$ defined by equation (3.6) satisfies*

$$\begin{aligned}&(d_{\pi,\psi} Y_{\pi,\psi})(\alpha, \beta)\\ &= -i_\pi d_A(i_{\pi^\sharp\alpha\wedge\pi^\sharp\beta}\psi) + < (\wedge^3\pi^\sharp)\psi, d_A(\alpha \wedge \beta) > + < d_A\psi, \pi \wedge \pi^\sharp\alpha \wedge \pi^\sharp\beta >,\end{aligned}$$

for all α and $\beta \in \Gamma(A^*)$.

Proof. We shall make use of the fact (see, e.g., [10]) that $\wedge^\bullet\pi^\sharp$ is a chain map, that is to say, for any positive integer q,

$$d_{\pi,\psi} \circ \wedge^q\pi^\sharp = - \wedge^{q+1} \pi^\sharp \circ d_A. \tag{4.8}$$

Thus

$$d_{\pi,\psi} Y_{\pi,\psi} = d_{\pi,\psi} \pi^\sharp (i_\pi \psi) = -(\wedge^2 \pi^\sharp) d_A i_\pi \psi,$$

which implies

$$(d_{\pi,\psi} Y_{\pi,\psi})(\alpha, \beta) = i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta} d_A i_\pi \psi$$
$$= i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta} [d_A, i_\pi]\psi - < i_\pi d_A \psi, \pi^\sharp\alpha \wedge \pi^\sharp\beta > .$$

The first term is

$$i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta} [d_A, i_\pi]\psi = [i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta}, [d_A, i_\pi]]\psi + [d_A, i_\pi] i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta} \psi.$$

The analogue of the Cartan relation for Lie algebroids (see, e.g., [9]) shows that

$$[i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta}, [d_A, i_\pi]] = i_{[\pi^\sharp\alpha \wedge \pi^\sharp\beta, \pi]},$$

where the bracket on the left-hand side is the graded commutator, while the bracket on the right-hand side is the Gerstenhaber bracket of sections of $\wedge^\bullet A$. To conclude the proof, we express $< [\pi^\sharp\alpha \wedge \pi^\sharp\beta, \pi], \psi >$ in terms of $d_A(\alpha \wedge \beta)$. By equation (4.8),

$$[\pi^\sharp\alpha \wedge \pi^\sharp\beta, \pi] = d_\pi(\pi^\sharp\alpha \wedge \pi^\sharp\beta)$$
$$= d_{\pi,\psi}(\pi^\sharp\alpha \wedge \pi^\sharp\beta) - \underline{d}_{\pi,\psi}(\pi^\sharp\alpha \wedge \pi^\sharp\beta) = -(\wedge^3 \pi^\sharp) d_A(\alpha \wedge \beta) - \underline{d}_{\pi,\psi}(\pi^\sharp\alpha \wedge \pi^\sharp\beta).$$

By equation (3.4),

$$< \underline{d}_{\pi,\psi}(\pi^\sharp\alpha \wedge \pi^\sharp\beta), \psi >$$
$$=< (\wedge^2 \pi^\sharp)(i_{\pi^\sharp\alpha}\psi) \wedge \pi^\sharp\beta, \psi > - < (\wedge^2 \pi^\sharp)(i_{\pi^\sharp\beta}\psi) \wedge \pi^\sharp\alpha, \psi >= 0,$$

since the operator $\wedge^2 \pi^\sharp$ is symmetric. Therefore,

$$< [\pi^\sharp\alpha \wedge \pi^\sharp\beta, \pi], \psi >= - < (\wedge^3 \pi^\sharp) d_A(\alpha \wedge \beta), \psi > .$$

The proposition follows. □

Corollary 4.1. *If (A, π, ψ) is a Lie algebroid with a twisted Poisson structure, the $d_{\pi,\psi}$-coboundary of $Y_{\pi,\psi}$ satisfies, for all α and $\beta \in \Gamma(A^*)$,*

$$(4.9) \qquad (d_{\pi,\psi} Y_{\pi,\psi})(\alpha, \beta) = -i_\pi d_A(i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta}\psi) + \frac{1}{2} < [\pi, \pi], d_A(\alpha \wedge \beta) > .$$

We shall make use of this formula in the proof of Theorem 5.2.

5 Generators of the Gerstenhaber algebra of a twisted Poisson structure

By definition, a *generator* of a Gerstenhaber algebra, $(\mathbf{A}, [\ ,\ ]_{\mathbf{A}})$, is an operator u on $\mathbf{A}$ such that

$$[a,b]_{\mathbf{A}} = (-1)^{|a|}(u(ab) - u(a)b - (-1)^{|a|}au(b)), \tag{5.1}$$

for all a and $b \in \mathbf{A}$.

It is clear that any two generators of a Gerstenhaber algebra differ by a derivation of the underlying graded commutative algebra.

Theorem 5.1. *The operator $\partial_\pi + \underline{\partial}_{\pi,\psi}$, where $\partial_\pi = [d_A, i_\pi]$, is a generator of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A^*), [\ ,\ ]_{\pi,\psi})$ associated to the Lie algebroid A with the twisted Poisson structure (π,ψ).*

Proof. By definition of the Lie algebroid bracket of A^*, for 1-forms, α and β,

$$[\alpha,\beta]_{\pi,\psi} = [\alpha,\beta]_\pi + \psi^{(2)}(\alpha,\beta).$$

The differential operator, ∂_π, is of order 2 and of degree -1. Therefore, for each $\alpha \in \Gamma(\wedge^\bullet A^*)$, the map

$$\beta \mapsto (-1)^{|\alpha|}\Phi^2_{\partial_\pi}(\alpha)(\beta) = (-1)^{|\alpha|}(\partial_\pi(\alpha\wedge\beta) - \partial_\pi\alpha\wedge\beta - (-1)^{|\alpha|}\alpha\wedge\partial_\pi\beta)$$

is a derivation of degree $|\alpha|-1$. Since it satisfies the condition of skew-symmetry (2.1), it is enough to show that it coincides with $[\alpha,\beta]_\pi$ when α is of degree 1 and β is of degree 0 or 1. If β is a 0-form, f, then

$$(-1)^{|\alpha|}\Phi^2_{\partial_\pi}(\alpha)(\beta) = (-1)^{|\alpha|}(d_Af\wedge i_\pi\alpha - i_\pi(d_Af\wedge\alpha)) \ .$$

When α is a 1-form, this expression is equal to $i_\pi(d_Af\wedge\alpha) =< d_Af, \pi^\sharp\alpha >$, which is $[\alpha,f]_\pi$ by definition. When α and β are 1-forms,

$$(-1)^{|\alpha|}\Phi^2_{\partial_\pi}(\alpha)(\beta)$$
$$= d_A(\pi(\alpha,\beta)) + i_\pi(d_A\alpha\wedge\beta) - i_\pi(\alpha\wedge d_A\beta) - (i_\pi d_A\alpha)\beta + \alpha(i_\pi d_A\beta) \ ,$$

while

$$[\alpha,\beta]_\pi = i_{\pi^\sharp\alpha}d_A\beta - i_{\pi^\sharp\beta}d_A\alpha + d_A(\pi(\alpha,\beta)) \ ,$$

and both expressions coincide since $i_\pi(d_A\alpha\wedge\beta) - (i_\pi d_A\alpha)\beta = -i_{\pi^\sharp\beta}d_A\alpha$, as can be easily shown.

We now consider the bilinear map $(\alpha,\beta) \mapsto (-1)^{|\alpha|}\Phi^2_{\underline{\partial}_{\pi,\psi}}(\alpha)(\beta)$ which coincides with $\psi^{(2)}$ for α and β of degree 1 and vanishes if α or β is of degree 0. By (2.1), it is skew-symmetric in the graded sense and, for each α, it defines a derivation of degree $|\alpha|-1$, since $\underline{\partial}_{\pi,\psi}$ is a differential operator of order 2. Therefore this bilinear map is the required extension of $\psi^{(2)}$. □

But the square of the generator $\partial_\pi + \underline{\partial}_{\pi,\psi}$ does not vanish in general.

Remark 5.1. When π is an arbitrary bivector and ψ a 3-form, the extension of bracket $[\ ,\ ]_{\pi,\psi}$ as a biderivation on $\Gamma(\wedge^\bullet A^*)$ is not in general a Gerstenhaber algebra bracket. However the proof of the preceding theorem shows that the operator $\partial_\pi + \underline{\partial}_{\pi,\psi}$ is a "generator" of this bracket, in the sense that it satisfies relation (5.1).

Lemma 5.1. *Let ∂ be a generator of the Gerstenhaber algebra $\Gamma(\wedge^\bullet E)$ of a Lie algebroid E, and let U be a section of E^*. Then $\partial + i_U$ is a generator of square zero of $\Gamma(\wedge^\bullet E)$ if and only if $\partial^2 = i_{d_E U}$.*

Proof. The generalization of formula (2.4) in [12] (see, e.g., [8]) implies

$$(\partial + i_U)^2 = \partial^2 + [\partial, i_U] = \partial^2 - i_{d_E U},$$

which proves the claim. □

Theorem 5.2. *The operator $\partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}}$ is a generator of square zero of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A^*), [\ ,\ ]_{\pi,\psi})$ associated to the Lie algebroid A with the twisted Poisson structure (π, ψ).*

Proof. For 1-forms α and β, compute

$$(\partial_\pi + \underline{\partial}_{\pi,\psi})(\alpha \wedge \beta) = -d_A(\pi(\alpha, \beta)) - i_\pi d_A(\alpha \wedge \beta) + i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta}\psi\ .$$

Since $\underline{\partial}_{\pi,\psi}$ vanishes on 1-forms, and since $\partial_\pi d_A(\pi(\alpha, \beta)) = 0$,

$$(\partial_\pi + \underline{\partial}_{\pi,\psi})^2(\alpha \wedge \beta) = i_\pi d_A i_\pi(d_A(\alpha \wedge \beta)) - i_\pi d_A(i_{\pi^\sharp\alpha \wedge \pi^\sharp\beta}\psi)\ .$$

For any closed 3-form τ, $[[i_\pi, d_A], i_\pi]\tau = 2 i_\pi d_A i_\pi \tau$. Since $[[i_\pi, d_A], i_\pi] = i_{[\pi,\pi]}$, where the bracket on the right-hand side is the Gerstenhaber bracket of multivectors, we obtain

$$i_\pi d_A i_\pi \tau = \frac{1}{2}[[i_\pi, d_A], i_\pi]\tau = \frac{1}{2} i_{[\pi,\pi]}\tau.$$

Therefore, in view of Corollary 4.1, when π and ψ satisfy the equations of a twisted Poisson structure,

$$(\partial_\pi + \underline{\partial}_{\pi,\psi})^2(\alpha \wedge \beta) = i_{d_{\pi,\psi} Y_{\pi,\psi}}(\alpha \wedge \beta).$$

The theorem then follows from Theorem 5.1 and the preceding lemma applied to the Lie algebroid $E = A^*$. □

We recall that a Gerstenhaber algebra equipped with a generator of square zero is called a *Batalin-Vilkovisky algebra* or, for short, a *BV-algebra*. We can reformulate the preceding theorem as follows.

If (A, π, ψ) is a Lie algebroid with a twisted Poisson structure, *the algebra $(\Gamma(\wedge^\bullet A^*), [\ ,\ ]_{\pi,\psi})$ is a BV-algebra, with generator $\partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}}$.*

6 Generators and the modular class

For simplicity, we shall assume that the vector bundle A is orientable, i.e., admits a nowhere-vanishing form of top degree, $\lambda \in \Gamma(\wedge^N A^*)$, where N is the rank of A, which we call a volume form. If the bundle is non-orientable, densities must be used instead of volume forms, and the proofs need not be changed.

6.1 A generator of square zero

The N-form λ defines an isomorphism $*_\lambda$ of vector bundles from $\wedge^\bullet A$ to $\wedge^\bullet A^*$ by $*_\lambda V = i_V \lambda$, which induces a map on sections denoted in the same way. If V is a p-vector, then $*_\lambda V$ is an $(N-p)$-form.

Given a twisted Poisson structure on the Lie algebroid A, consider the operator

$$\partial_{\pi,\psi,\lambda} = - *_\lambda d_{\pi,\psi} *_\lambda^{-1},$$

where $d_{\pi,\psi}$ is the differential (4.7).

Proposition 6.1. *The operator $\partial_{\pi,\psi,\lambda}$ is a generator of square zero of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A^*), [\ ,\ ]_{\pi,\psi})$ associated to the Lie algebroid A with the twisted Poisson structure (π, ψ).*

Proof. The square of $\partial_{\pi,\psi,\lambda}$ vanishes since $d_{\pi,\psi}$ is a differential. It is well-known that conjugating the differential of the Lie algebroid by $*_\lambda$ yields the opposite of a generator of the Gerstenhaber bracket. See, e.g., [8]. □

6.2 Properties of sections $X_{\pi,\lambda}$ and $Y_{\pi,\psi}$

In order to generalize the modular vector fields of Poisson manifolds, we shall first consider the section $X_{\pi,\lambda}$ of A such that, for all $\alpha \in \Gamma(A^*)$,

$$< \alpha, X_{\pi,\lambda} > \lambda = \mathcal{L}^A_{\pi^\sharp\alpha}\lambda - (i_\pi d_A \alpha)\lambda \ , \tag{6.1}$$

where $\mathcal{L}^A$ is the Lie derivation. Since the right-hand side of the previous expression is $C^\infty(M)$-linear in α, the section $X_{\pi,\lambda}$ is well-defined.

In particular, $\mathcal{L}^A_{H_f}\lambda =< d_A f, X_{\pi,\lambda} > \lambda$, where, as above, H_f is the hamiltonian section with hamiltonian $f \in C^\infty(M)$. Thus, when $A = TM$, the vector field $X_{\pi,\lambda}$ satisfies, for each $f \in C^\infty(M)$,

$$\mathcal{L}_{H_f}\lambda = (X_{\pi,\lambda} \cdot f)\lambda \ , \tag{6.2}$$

i.e., the function $X_{\pi,\lambda} \cdot f$ is the divergence of H_f with respect to the volume form λ.

Lemma 6.1. *The section $X_{\pi,\lambda}$ satisfies*

$$\partial_\pi \lambda = -i_{X_{\pi,\lambda}}\lambda. \tag{6.3}$$

Proof. We shall make use of the fact that for any $\eta \in \Gamma(\wedge^\bullet A^*)$, $\eta \wedge i_\pi \lambda = i_\pi \eta \wedge \lambda$. By the definition of $X_{\pi,\lambda}$,

$$\alpha \wedge i_{X_{\pi,\lambda}} \lambda = d_A(i_{\pi^\sharp \alpha}\lambda) - (i_\pi d_A \alpha)\lambda,$$

for each 1-form α. Since λ is a form of top degree, $i_{\pi^\sharp\alpha}\lambda = \alpha \wedge i_\pi \lambda$. Therefore

$$\alpha \wedge i_{X_{\pi,\lambda}} \lambda = d_A \alpha \wedge i_\pi \lambda - \alpha \wedge d_A i_\pi \lambda - (i_\pi d_A \alpha)\lambda = -\alpha \wedge d_A i_\pi \lambda \ .$$

Thus $i_{X_{\pi,\lambda}}\lambda = -d_A i_\pi \lambda = -\partial_\pi \lambda$. □

We now consider the section $Y_{\pi,\psi}$ defined by (3.6).

Lemma 6.2. *The section $Y_{\pi,\psi}$ satisfies*

$$\underline{\partial}_{\pi,\psi}\lambda = -2\, i_{Y_{\pi,\psi}}\lambda \ . \tag{6.4}$$

Proof. Let $\xi \wedge Q$ be a decomposable form-valued bivector. Then

$$i_{\xi\wedge Q}\lambda = \xi \wedge i_Q \lambda = \varepsilon_\xi *_\lambda Q = *_\lambda\, i_\xi Q = i_{i_\xi Q}\lambda \ ,$$

where ε_ξ is the left exterior product by ξ. Let (e_k) and (ϵ^k), $1 \leqslant k \leqslant N$, be dual local bases of sections of A and A^*, such that $\lambda = \epsilon^1 \wedge \cdots \wedge \epsilon^N$. Let $\sigma = \frac{1}{2}\sigma^{k\ell}_m \epsilon^m \otimes e_k \wedge e_\ell$. Then $i_\sigma \lambda = i_S \lambda$, where $S = \sigma^{k\ell}_k e_\ell$ is twice the trace of σ with respect to the first index. If $\sigma = \psi^{(2)}$, by formulas (3.2) and (3.7), $S = -2Y_{\pi,\psi}$. □

We remark that π and ψ need not satisfy the axioms of a twisted Poisson structure in order for the results of these two lemmas to be valid. Lemma 6.1 expresses the fact that the isomorphism $*_\lambda$ identifies the vector field $X_{\pi,\lambda}$ with the $(N-1)$-form $-\partial_\pi \lambda = -d_A i_\pi \lambda$, a property which is valid for the modular vector fields of Poisson manifolds [8] [16].

6.3 The modular class

Set

$$Z_{\pi,\psi,\lambda} = X_{\pi,\lambda} + Y_{\pi,\psi} \ , \tag{6.5}$$

where $Y_{\pi,\psi}$ is defined by (3.6).

Proposition 6.2. *The section $Z_{\pi,\psi,\lambda}$ satisfies the relation*

$$\partial_{\pi,\psi,\lambda} - (\partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}}) = i_{Z_{\pi,\psi,\lambda}}. \tag{6.6}$$

Proof. Because both $\partial_{\pi,\psi,\lambda}$ and $\partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}}$ are generators of the same Gerstenhaber algebra, they differ by the interior product by a section of A. It is enough to evaluate their difference on the form of top degree λ, and the result follows from the fact that $\partial_{\pi,\psi,\lambda}\,\lambda = 0$, together with Lemmas 6.1 and 6.2. □

By the general properties of [8], we obtain

Theorem 6.1. *The section $Z_{\pi,\psi,\lambda} = X_{\pi,\lambda} + Y_{\pi,\psi}$ of A is a $d_{\pi,\psi}$-cocycle. The cohomology class of $Z_{\pi,\psi,\lambda}$ is independent of the choice of λ.*

Proof. In fact, it follows from Lemma 5.1 that the difference of two generating operators whose squares vanish is the interior product by a 1-cocycle. So the fact that $d_{\pi,\psi}Z_{\pi,\psi,\lambda} = 0$ is a consequence of Theorem 5.2 and of Proposition 6.1. Replacing the form of top degree λ by the form $f\lambda$, where f is a nowhere-vanishing smooth function on the base manifold, adds the coboundary $d_\pi(\ln|f|) = d_{\pi,\psi}(\ln|f|)$ to $X_{\pi,\lambda}$, therefore $Z_{\pi,\psi,f\lambda}$ and $Z_{\pi,\psi,\lambda}$ are cohomologous cocycles. □

Remark 6.1. One can prove directly that $d_{\pi,\psi}Z_{\pi,\psi,\lambda} = 0$, but we have not found a proof simpler than the one given here.

Definition 6.1. The section $Z_{\pi,\psi,\lambda}$ is called a *modular section* (or modular vector field) of (A, π, ψ). The $d_{\pi,\psi}$-cohomology class of $Z_{\pi,\psi,\lambda}$ is called the *modular class* of the Lie algebroid A with the twisted Poisson structure (π, ψ), and A is called *unimodular* if its modular class vanishes. When $A = TM$, the modular class of (TM, π, ψ) is called the *modular class of the twisted Poisson manifold* (M, π, ψ).

The modular classes of twisted Poisson manifolds generalize the modular classes of Poisson manifolds. If $\psi = 0$, then $Z_{\pi,\psi,\lambda} = X_{\pi,\lambda}$, the generator $\partial_{\pi,\psi,\lambda}$ reduces to $\partial_{\pi,\lambda} = -*_\lambda d_\pi *_\lambda^{-1}$, and relation (6.6) reduces to

$$\partial_{\pi,\lambda} - \partial_\pi = i_{X_{\pi,\lambda}} , \tag{6.7}$$

a relation valid for the modular vector fields of a triangular Lie bialgebroid, in particular of a Poisson manifold [8].

6.4 Properties of the modular sections

We now list properties of the modular sections which generalize the properties of the modular vector fields of Poisson manifolds. The modular vector fields of Poisson manifolds and triangular Lie bialgebroids satisfy

$$*_\lambda X_{\pi,\lambda} = -\partial_\pi \lambda ,$$

whereas in the twisted case, equation (6.6) implies

$$*_\lambda Z_{\pi,\psi,\lambda} = -(\partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}})\lambda . \tag{6.8}$$

The modular vector fields of Poisson manifolds and triangular Lie bialgebroids satisfy $*_\lambda X_{\pi,\lambda} = -d_A *_\lambda \pi$, whereas in the twisted case, in view of Proposition 3.2,

$$*_\lambda Z_{\pi,\psi,\lambda} = -(d_A + \delta_{\pi,\psi}) *_\lambda \pi - 3\,\underline{\partial}_{\pi,\psi}\lambda . \tag{6.9}$$

In the untwisted case, the modular vector fields satisfy relation (6.1) for any 1-form α, and therefore

$$< \alpha, X_{\pi,\lambda} > \lambda = \mathcal{L}^A_{\pi^\sharp\alpha}\lambda$$

for any d_A-closed 1-form. In the twisted case, adding relations (6.1) and (3.7), we obtain

$$< \alpha, Z_{\pi,\psi,\lambda} > \lambda = \mathcal{L}^A_{\pi^\sharp\alpha}\lambda - i_\pi((d_A + \delta_{\pi,\psi})\alpha)\lambda \ .$$

We have used the operator $\delta_{\pi,\psi}$ introduced in (3.5) to write the term $i_{\pi^\sharp\alpha}\psi$ as $\delta_{\pi,\psi}\alpha$. The differential d_A of the Lie algebroid A is twisted into the derivation

$$d_{A,\pi,\psi} = d_A + \delta_{\pi,\psi} \tag{6.10}$$

of $\Gamma(\wedge^\bullet A^*)$, which is no longer of square zero, and the modular section $Z_{\pi,\psi,\lambda}$ satisfies

$$< \alpha, Z_{\pi,\psi,\lambda} > \lambda = \mathcal{L}^A_{\pi^\sharp\alpha}\lambda \ , \tag{6.11}$$

for any $d_{A,\pi,\psi}$-closed 1-form α.

6.5 The unimodular case

When (A, π, ψ) is unimodular, i.e., the class of the modular section $Z_{\pi,\psi,\lambda}$ vanishes, the homology and cohomology are isomorphic, the isomorphism being, in fact, defined at the chain level. By definition, the *homology* $H^{\pi,\psi}_\bullet(A)$ of a Lie algebroid with a twisted Poisson structure, (A, π, ψ), is the homology of the complex $(\Gamma(\wedge^\bullet A^*), \partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}})$. The untwisted case, generalizing the Poisson homology of Koszul and Brylinski, was studied by Huebschmann in the framework of Lie-Rinehart algebras [5] [6] and by Xu [17]. The *cohomology* $H^\bullet_{\pi,\psi}(A^*)$ of a Lie algebroid with a twisted Poisson structure is the cohomology of the Lie algebroid A^* defined in Theorem 4.1, i.e., the cohomology of the complex $(\Gamma(\wedge^\bullet A), d_{\pi,\psi})$.

Proposition 6.3. *If (A, π, ψ) is unimodular, for all $k \in \mathbb{N}$, $H^{\pi,\psi}_k(A) \simeq H^{N-k}_{\pi,\psi}(A^*)$, where N is the rank of A.*

This proposition follows from the results of [5] and [6], or [17]. However, to make this paper self-contained, we present a proof.

Proof. Let λ be a nowhere-vanishing form of top degree, and compare $\partial_{\pi,\psi,f\lambda}$ and $\partial_{\pi,\psi,\lambda}$, where f is a nowhere-vanishing function on M. Then, by the graded Leibniz identity, for any multivector X and any function g,

$$d_\pi(gX) = [\pi, g] \wedge X + g\ d_\pi X = -H_g \wedge X + g\ d_\pi X.$$

Since $\underline{d}_{\pi,\psi}$ is $C^\infty(M)$-linear, this relation implies

$$d_{\pi,\psi}(gX) = -H_g \wedge X + g\ d_{\pi,\psi}X.$$

For any p-form α,

$$\partial_{\pi,\psi,f\lambda}(\alpha) = - *_{f\lambda} d_{\pi,\psi} *_{f\lambda}^{-1} \alpha = - *_{\lambda} d_{\pi,\psi} *_{\lambda}^{-1} \alpha + f *_{\lambda} (H_{\frac{1}{f}} \wedge *_{\lambda}^{-1}\alpha)$$

$$= \partial_{\pi,\psi,\lambda}\alpha + f i_{H_{\frac{1}{f}}} \alpha = \partial_{\pi,\psi,\lambda}\alpha + i_{H_{\ln|f|}}\alpha.$$

We have used the fact that $i_X \circ *_\lambda = *_\lambda \circ \varepsilon_X$, where ε_X is the left exterior product by X, which implies that $*_\lambda(X \wedge *_\lambda^{-1}\alpha) = i_X\alpha$.

If the modular class vanishes, there exists $g \in C^\infty(M)$ such that $Z_{\pi,\psi,\lambda} = H_g$. Set $f = e^{-g}$. Then

$$\partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}} = \partial_{\pi,\psi,\lambda} - i_{Z_{\pi,\psi,\lambda}} = \partial_{\pi,\psi,\lambda} - i_{H_g} = \partial_{\pi,\psi,f\lambda} \ .$$

In other words, the map $V \mapsto *_{f\lambda}V$ is a chain map from $(\Gamma(\wedge^\bullet A), d_{\pi,\psi})$ to $(\Gamma(\wedge^\bullet A^*), \partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}})$. □

7 Comparison with the ELW-modular class

As stated in Theorem 4.1, when (A, π, ψ) is a Lie algebroid with a twisted Poisson structure, the dual vector bundle A^* is a Lie algebroid with anchor $\rho \circ \pi^\sharp$ and bracket $[\ ,\ ]_{\pi,\psi}$. Since in [6] and [4], general notions for Lie algebroids were defined, a comparison is in order. In [4], Evens, Lu and Weinstein defined the characteristic class of a Lie algebroid with a representation in a line bundle and the modular class of a Lie algebroid E – which we shall call the ELW-modular class of E. We shall compare what we have called the modular class of (A, π, ψ) to the ELW-modular class of the Lie algebroid $(A^*, \rho \circ \pi^\sharp, [\ ,\ ]_{\pi,\psi})$, and conclude that in the case of $A = TM$, the first is one half the second, a result that is not valid in the case of a Lie algebroid in general.

7.1 The characteristic class

We recall the construction of [4]. Let E be a Lie algebroid over a manifold M, with anchor ρ and Lie bracket of sections $[\ ,\ ]_E$, and let D be a representation of E on a line bundle L over M, $x \in \Gamma E \mapsto D_x \in \mathrm{End}(\Gamma L)$. By definition, the map D is $C^\infty(M)$-linear and $D_x(f\mu) = f D_x\mu + (\rho(x) \cdot f)\,\mu$, for all $x \in \Gamma E$, $\mu \in \Gamma L$ and $f \in C^\infty(M)$, and $D_{[x,y]_E} = [D_x, D_y]$, for all x and y in ΓE. The *characteristic class of E associated to the representation D on L* is the class of the d_E-cocycle $\theta_s \in \Gamma(E^*)$ defined by

$$D_x s =< \theta_s, x > s,$$

where s is a nowhere-vanishing section of L. If L is not trivial, the class of L is defined as one half that of its square.

Assume that ∂ is a generating operator of the Gerstenhaber bracket, $[\ ,\ ]_E$, of $\Gamma(\wedge^\bullet E)$. Set

$$D^\partial_x(\mu) = [x,\mu]_E - (\partial x)\mu = -x \wedge \partial\mu \ , \tag{7.1}$$

for $x \in \Gamma E$ and $\mu \in \Gamma(\wedge^N E)$. Then D^∂ is a representation of E on $\wedge^N E$, and the associated characteristic class is the class of the cocycle $\xi \in \Gamma(E^*)$ such that

$$< \xi, x > \mu = -x \wedge \partial\mu \ .$$

If, in particular, (A,π,ψ) is a Lie algebroid with a twisted Poisson structure, we can consider the Lie algebroid $E = A^*$, with Lie bracket of sections $[\ ,\ ]_{\pi,\psi}$, and generator $\partial = \partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}}$. If λ is a nowhere-vanishing section of $\Gamma(\wedge^N A^*)$ (or a density in the non-orientable case), then, by equation (6.8), $\partial\lambda = -i_{Z_{\pi,\psi,\lambda}}\lambda$. The associated characteristic class is the class of the $d_{\pi,\psi}$-cocycle $\theta \in \Gamma A$ such that

$$< \alpha, \theta > \lambda = \alpha \wedge i_{Z_{\pi,\psi,\lambda}}\lambda =< \alpha, Z_{\pi,\psi,\lambda} > \lambda \ ,$$

for all $\alpha \in \Gamma(A^*)$. Therefore,

Proposition 7.1. *The characteristic class of A^* associated to the representation (7.1), $\alpha \mapsto D^\partial_\alpha$, of A^* on $\wedge^N(A^*)$, where $\partial = \partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}}$, coincides with the modular class of the Lie algebroid A with the twisted Poisson structure (π,ψ).*

7.2 The ELW-modular class of A^*

In [4], the modular class of a Lie algebroid E is defined as follows. Let $L^E = \wedge^N E \otimes \wedge^n T^* M$, where n is the dimension of M. Define a representation D^E of E on L^E by

$$D^E_x(\omega \otimes \mu) = [x,\omega]_E \otimes \mu + \omega \otimes \mathcal{L}_{\rho(x)}\mu, \tag{7.2}$$

for $x \in \Gamma E$ and $\omega \otimes \mu \in \Gamma(L^E)$. Here $\mathcal{L}$ is the Lie derivation of forms on M by vector fields.

Definition 7.1. The *ELW-modular class* of the Lie algebroid E is the characteristic class of E associated to the representation D^E of E on L^E.

Proposition 7.2. *The modular class of a twisted Poisson manifold is equal to one half the ELW-modular class of its cotangent bundle Lie algebroid.*

Proof. Let λ be a volume form on M, i.e., a nowhere-vanishing section of $\wedge^n T^* M$. By definition, the ELW-modular class of the Lie algebroid T^*M is the class of the vector field U such that

$$< \alpha, U > \lambda \otimes \lambda = [\alpha, \lambda]_{\pi,\psi} \otimes \lambda + \lambda \otimes \mathcal{L}_{\pi^\sharp\alpha}\lambda \ , \tag{7.3}$$

for all 1-forms α. For any generator ∂ of the Gerstenhaber bracket $[\ ,\ ]_{\pi,\psi}$,

$$[\alpha, \lambda]_{\pi,\psi} = (\partial\alpha)\lambda - \alpha \wedge \partial\lambda \ .$$

If $\partial = \partial_\pi + \underline{\partial}_{\pi,\psi} + i_{Y_{\pi,\psi}}$, then $\partial\alpha =< \alpha, Y_{\pi,\psi} > -i_\pi d\alpha$, while, by equation (6.8), $\partial\lambda = -i_{Z_{\pi,\psi,\lambda}}\lambda$, and therefore

$$[\alpha, \lambda]_{\pi,\psi} = (< \alpha, Z_{\pi,\psi,\lambda} + Y_{\pi,\psi} > -i_\pi d\alpha)\lambda \ .$$

Since by definition $X_{\pi,\psi}$ satisfies $\mathcal{L}_{\pi^\sharp\alpha}\lambda = (< \alpha, X_{\pi,\psi} > +i_\pi d\alpha)\lambda$, the vector field U is such that

$$< \alpha, U > \lambda \otimes \lambda = 2 < \alpha, Z_{\pi,\psi,\lambda} > \lambda \otimes \lambda \ .$$

Therefore $U = 2Z_{\pi,\psi,\lambda}$. □

In the Poisson case, $\partial = \partial_\pi$ and $\partial_\pi\lambda = -i_{X_{\pi,\lambda}}\lambda$, and we obtain the relation $< \alpha, U > \lambda \otimes \lambda = 2 < \alpha, X_{\pi,\lambda} > \lambda \otimes \lambda$, which gives a new proof of the result of [4] stating that the modular class of a Poisson manifold, defined as the class of $X_{\pi,\lambda}$, characterized by an equation such as (6.2), is equal to one half the ELW-modular class of the Lie algebroid $(T^*M, \pi^\sharp, [\ ,\ ]_\pi)$.

The simple relationship between the modular class of $A = TM$ and the ELW-modular class of $A^* = T^*M$ does not in general hold for a Lie algebroid with a twisted Poisson structure. In particular, in Section 8, we shall show that the two classes may be different for a Lie algebra considered as a Lie algebroid over a point, even in the usual, untwisted case.

7.3 Modular class and gauge transformations

Assume that we are given, on the cotangent bundle T^*M of a manifold M, two Lie algebroid structures, denoted by $(T^*M, \rho, [\, ,\,])$ and $(T^*M, \rho', [\, ,\,]')$ respectively, together with a Lie algebroid isomorphism, $\sigma : T^*M \to T^*M$, over the identity of the base M. According to formula (7.2), each Lie algebroid acts on $L^{T^*M} = \wedge^n T^*M \otimes \wedge^n T^*M$, where n is the dimension of M, and we denote by D and D' these representations. Define $\tau : L^{T^*M} \to L^{T^*M}$ by $\omega \mapsto \det(\sigma)\,\omega$, for all $\omega \in \Gamma(L^{T^*M})$.

Lemma 7.1. *The isomorphism τ is an intertwining operator for the representations D and $D' \circ \sigma$, i.e.,*

$$D'_{\sigma(\alpha)}\tau(\omega) = \tau(D_\alpha\omega), \tag{7.4}$$

for all $\alpha \in \Gamma(T^*M)$, $\omega \in \Gamma(L^{T^*M})$.

Proof. For any sections ω_1 and ω_2 of $\wedge^n T^*M$, $\tau(\omega_1 \otimes \omega_2) = (\wedge^n\sigma)(\omega_1) \otimes \omega_2$. Hence

$$\begin{aligned} D'_{\sigma(\alpha)}\tau(\omega_1 \otimes \omega_2) &= [\sigma(\alpha), (\wedge^n\sigma)(\omega_1)]' \otimes \omega_2 + (\wedge^n\sigma)(\omega_1) \otimes \mathcal{L}_{\rho'(\sigma(\alpha))}\omega_2 \\ &= (\wedge^n\sigma)([\alpha, \omega_1]) \otimes \omega_2 + (\wedge^n\sigma)(\omega_1) \otimes \mathcal{L}_{\rho(\alpha)}\omega_2 = \det(\sigma)D_\alpha(\omega_1 \otimes \omega_2), \end{aligned}$$

since σ is an isomorphism of Lie algebroids. □

Choose a volume form λ on T^*M. Let U (resp., U') be the representative of the ELW-modular class of the Lie algebroid $(T^*M, \rho, [\,,\,])$ (resp., $(T^*M, \rho', [\,,\,]'))$ with volume form λ (resp. $\mu = \sqrt{|\det(\sigma)|}\lambda$).

Proposition 7.3. *The vector fields U and U' are related by $U = {}^t\sigma\, U'$, where ${}^t\sigma$ is the transpose of σ.*

Proof. For some locally constant function $\epsilon \in \{-1, +1\}$, $\tau(\lambda \otimes \lambda) = \epsilon\mu \otimes \mu$. Therefore, by equation (7.4),

$$\epsilon D'_{\sigma(\alpha)}\mu \otimes \mu = \tau(D_\alpha \lambda \otimes \lambda).$$

By the definition of the modular vector field given by equation (7.3),

$$< \sigma(\alpha), U' > \epsilon\mu \otimes \mu =< \alpha, U > \tau(\lambda \otimes \lambda).$$

Hence $< \sigma(\alpha), U' >=< \alpha, U >$, and the result follows. □

A twisted Poisson structure on a given manifold, M, can be modified by a *gauge transformation*. Assume that B is a 2-form on M such that for all $m \in M$, the linear automorphism of $T^*_m M$, $\sigma_B : \alpha \mapsto \alpha + i_{\pi^\sharp\alpha}B$, is invertible. Define a bivector π' by

$$(\pi')^\sharp = \pi^\sharp(\mathrm{Id} + B^\flat \circ \pi^\sharp)^{-1} ,$$

where $B^\flat : TM \to T^*M$ is defined by $B^\flat(X) = i_X B$. Then $(\pi', \psi - dB)$ is a twisted Poisson structure that is said to be obtained by a *gauge transformation* from (π, ψ) [15].

Proposition 7.4. *A vector field X on M is in the modular class of (TM, π, ψ) if and only if $X + \pi^\sharp i_X B$ is in the modular class of $(TM, \pi', \psi - dB)$.*

Proof. According to [15], the map σ_B is a Lie algebroid isomorphism from T^*M equipped with the Lie algebroid structure associated to (π, ψ) to T^*M equipped with the Lie algebroid structure associated to $(\pi', \psi - dB)$. It follows from Propositions 7.2 and 7.3 that its transpose, $X \mapsto X + \pi^\sharp i_X B$, maps modular class to modular class. □

8 Examples

Example 1. The case where $(\wedge^3\pi^\sharp)\psi = 0$. Whenever (M, π) is a Poisson manifold and ψ is a closed 3-form satisfying $(\wedge^3\pi^\sharp)\psi = 0$, then (M, π, ψ) is a twisted Poisson manifold. One can prove that the assumption $(\wedge^3\pi^\sharp)\psi = 0$ implies $Y_{\pi,\psi} = 0$. Hence, for any volume form λ, the modular vector field of (M, π, ψ) is equal to $X_{\pi,\lambda}$, which is the modular vector field of the Poisson manifold (M, π). In this case therefore, the modular class of the twisted Poisson manifold (M, π, ψ) is equal to the modular class of the Poisson manifold (M, π). This conclusion holds more generally in the case of a Lie algebroid with a twisted Poisson structure of this type.

Example 2. The case where $\pi^\sharp$ is invertible. Let (π, ψ) be a twisted Poisson structure on A such that $\pi \in \Gamma(\wedge^2 A)$ is of maximal rank at each point. Let ω be the non-degenerate 2-form such that $\omega^\flat = (\pi^\sharp)^{-1}$. One can show that the modular vector field $Z_{\pi,\psi,\lambda}$, where λ is the volume form $\omega^{\frac{N}{2}}$, vanishes. The proof rests on the properties $d_A\omega = \psi$ and $\frac{N}{2}\omega^{\frac{N}{2}-1} = i_\pi \omega^{\frac{N}{2}}$. Therefore the Lie algebroids with a twisted Poisson structure whose bivector is of maximal rank are unimodular, a result which extends the fact that symplectic manifolds are unimodular.

Example 3. Twisted Poisson structures on Lie groups. Consider the example of a twisted Poisson structure defined on a dense open subset of a Lie group [15]. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and assume that $\mathfrak{g}$ is equipped with an invariant non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$. For any $X \in \mathfrak{g}$ (resp., $\alpha \in \mathfrak{g}^*$), we denote by $\overline{X} \in \mathfrak{g}^*$ (resp., $\underline{\alpha} \in \mathfrak{g}$) the image of X (resp., α) under the isomorphism of $\mathfrak{g}$ to $\mathfrak{g}^*$ (resp., $\mathfrak{g}^*$ to $\mathfrak{g}$) induced by $\langle\ ,\ \rangle$. For any $X \in \wedge^\bullet \mathfrak{g}$, let X^R and X^L be the corresponding right- and left-invariant multivector fields. Dually, for any $\alpha \in \wedge^\bullet \mathfrak{g}^*$, we denote by α^R and α^L the corresponding right- and left-invariant forms. Then, for any $\alpha \in \mathfrak{g}^*$, $d\alpha^L = (d_\mathfrak{g}\alpha)^L$, while $d\alpha^R = -(d_\mathfrak{g}\alpha)^R$, where $d_\mathfrak{g}$ is the Chevalley-Eilenberg differential of $\mathfrak{g}$ and d is the de Rham differential.

The canonical 3-form ψ on $\mathfrak{g}$, defined by $\psi(X, Y, Z) = \frac{1}{2}\langle X, [Y, Z]\rangle$, for X, Y and $Z \in \mathfrak{g}$, satisfies $\psi^R = \psi^L$. The corresponding bi-invariant form on G is called the Cartan 3-form, and we shall denote it by the same symbol. It satisfies, for any $X \in \mathfrak{g}$,

$$i_{X^L}\psi = -\frac{1}{2}(d_\mathfrak{g}\overline{X})^L \qquad \text{and} \qquad i_{X^R}\psi = -\frac{1}{2}(d_\mathfrak{g}\overline{X})^R. \tag{8.1}$$

Let G_0 be the dense open set of elements $g \in G$ such that -1 is not an eigenvalue of Ad_g. Equivalently, G_0 is the subset of elements $g \in G$ such that the linear map $\alpha \mapsto \alpha^R(g) + \alpha^L(g)$ is an isomorphism from $\mathfrak{g}^*$ to $T^*_g G$. A twisted Poisson structure on G_0 is given by ψ and the bivector π defined by

$$\pi^\sharp(\alpha^L + \alpha^R) = 2(\underline{\alpha}^L - \underline{\alpha}^R)\ ,$$

Proposition 8.1. *The twisted Poisson manifold (G_0, π, ψ) is unimodular.*

Proof. We first evaluate $Y_{\pi,\psi}$ on the 1-forms $\alpha^R + \alpha^L$, obtaining

$$i_{Y_{\pi,\psi}}(\alpha^L + \alpha^R) = -i_{\pi \wedge \pi^\sharp(\alpha^L+\alpha^R)}\psi = -2i_{\pi\wedge(\underline{\alpha}^L - \underline{\alpha}^R)}\psi.$$

By equation (8.1), we obtain

$$i_{Y_{\pi,\psi}}(\alpha^L + \alpha^R) = i_\pi((d_\mathfrak{g}\alpha)^L - (d_\mathfrak{g}\alpha)^R). \tag{8.2}$$

We can choose a volume form λ which is both left- and right-invariant. By definition,

$$(i_{X_{\pi,\lambda}}(\alpha^L + \alpha^R))\,\lambda = \mathcal{L}_{\pi^\sharp(\alpha^L+\alpha^R)}\lambda - (i_\pi d(\alpha^L + \alpha^R))\,\lambda\ .$$

Since λ is left- and right-invariant, $\mathcal{L}_{\pi^\sharp(\alpha^L+\alpha^R)}\lambda = 2(\mathcal{L}_{\underline{\alpha}^L}\lambda - \mathcal{L}_{\underline{\alpha}^R}\lambda) = 0$, thus

$$i_{X_{\pi,\lambda}}(\alpha^L + \alpha^R) = -i_\pi((d_{\mathfrak{g}}\alpha)^L - (d_{\mathfrak{g}}\alpha)^R). \tag{8.3}$$

Equations (8.2) and (8.3) imply that $Z_{\pi,\psi,\lambda} = 0$. □

Example 4. Lie algebras with a triangular r-matrix. Let $\mathfrak{g}$ be a real Lie algebra of dimension N, and let $r \in \wedge^2\mathfrak{g}$ such that $[r,r] = 0$, this bracket being the algebraic Schouten bracket on $\wedge^\bullet\mathfrak{g}$. Such an r is called a *triangular r-matrix.* Then $\mathfrak{g}^*$ is a Lie algebra with bracket $[\alpha,\beta]_r = \mathrm{ad}^*_{r^\sharp\alpha}\beta - \mathrm{ad}^*_{r^\sharp\beta}\alpha$. Let λ be a non-zero element in $\wedge^N\mathfrak{g}^*$. The class that we have just defined is the class of the element $X_{r,\lambda}$ of $\mathfrak{g}$ such that

$$< \alpha, X_{r,\lambda} > \lambda = (\partial_r\alpha)\lambda + d_{\mathfrak{g}}(i_{r^\sharp\alpha}\lambda),$$

or, equivalently, by equation (6.3),

$$< \alpha, X_{r,\lambda} > \lambda = -\alpha \wedge \partial_r\lambda,$$

for $\alpha \in \mathfrak{g}^*$. On the other hand, the ELW-modular class of $\mathfrak{g}^*$ considered as a Lie algebroid over a point is the class, in the Lie algebra cohomology of $\mathfrak{g}$, of the element $\widetilde{X}_{r,\lambda}$ of $\mathfrak{g}$ such that

$$< \alpha, \widetilde{X}_{r,\lambda} > \lambda = [\alpha, \lambda]_r.$$

Since ∂_r generates the bracket $[\ ,\ ]_r$, $[\alpha,\lambda]_r = (\partial_r\alpha)\lambda - \alpha \wedge \partial_r\lambda$. Thus the class of $X_{r,\lambda}$ is one half that of $\widetilde{X}_{r,\lambda}$ if and only if $d_{\mathfrak{g}}(i_{r^\sharp\alpha}\lambda)$ vanishes for all $\alpha \in \mathfrak{g}^*$.

Example 4.1. Triangular r-matrix on the non-abelian 2-dimensional Lie algebra. In this simple example, one class vanishes, while the other does not. Let $\mathfrak{g}$ be the non-abelian 2-dimensional Lie algebra with basis (e_1, e_2) such that $[e_1, e_2] = e_1$, with the triangular r-matrix $\pi = r = e_1 \wedge e_2$, and $\psi = 0$. For the dual basis (ϵ^1, ϵ^2), $[\epsilon^1, \epsilon^2]_r = -\epsilon^2$, the dual Lie algebra is in fact isomorphic to $\mathfrak{g}$. Let $\lambda = \epsilon^1 \wedge \epsilon^2$. The modular vector $X_{r,\lambda}$ vanishes since $i_{X_{r,\lambda}}\lambda = -d_{\mathfrak{g}} i_r\lambda$ and $d_{\mathfrak{g}}$ vanishes on $\wedge^0\mathfrak{g}$. On the other hand, the ELW-modular class of $\mathfrak{g}$ is the class of $\widetilde{X}_{r,\lambda} = -e_1$, and this element is non-zero, therefore its class does not vanish.

Remark. The ELW-modular class of $\mathfrak{g}^*$ is equal to the class of the infinitesimal modular character of $\mathfrak{g}^*$, which is the element of $\mathfrak{g}$, $\alpha \mapsto \mathrm{Tr}(\mathrm{ad}_\alpha)$, and it also coincides with the modular class of the linear Poisson manifold $\mathfrak{g}$, dual to the Lie algebra $\mathfrak{g}^*$. In the example above, alternatively, we can show that the constant vector field $\widetilde{X}_{r,\lambda} = -e_1$ is not globally hamiltonian with respect to the linear

Poisson structure π^r on $\mathfrak{g}$, the dual of the Lie algebra $(\mathfrak{g}^*, [\ ,\ ]_r)$. If (x_1, x_2) are the coordinates on $\mathfrak{g}$ with respect to the chosen basis, and $x \in \mathfrak{g}$, then $\pi^r(x) = -x_2 e_1 \wedge e_2$. Since, for a function $u \in C^\infty(\mathfrak{g})$, $d_{\pi^r} u = [\pi^r, u] = x_2(\frac{\partial u}{\partial x_2} e_1 - \frac{\partial u}{\partial x_1} e_2)$, the condition $-e_1 = d_{\pi^r} u$ is clearly not realizable along the axis $x_2 = 0$.

Example 4.2. Triangular r-matrix on $\mathfrak{sl}(2,\mathbb{R})$. On $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, define $\pi = r = X_+ \wedge H$, where X_+, X_- and H denote the usual basis of $\mathfrak{sl}_2(\mathbb{R})$ such that $[H, X_+] = 2X_+$, $[H, X_-] = -2X_-$ and $[X_+, X_-] = H$. The relation $[r, r] = 0$ is satisfied, hence (r, ψ) with $\psi = 0$ is a twisted Poisson structure on $\mathfrak{sl}(2,\mathbb{R})$. In view of equation (6.8), in the particular case where $\psi = 0$, for any non-zero $\lambda \in \wedge^3 \mathfrak{g}^*$, the modular vector $Z_{r,\lambda}$ is defined by $*_\lambda Z_{r,\lambda} = -d_{\mathfrak{g}} i_r \lambda$, We choose $\lambda = H^* \wedge X_+^* \wedge X_-^*$, where X_+^*, X_-^*, H^* is the dual basis, and we compute

$$*_\lambda Z_{r,\lambda} = -d_{\mathfrak{g}} i_r \lambda = -d_{\mathfrak{g}} X_-^* = -2H^* \wedge X_-^* = i_{2X_+} \lambda.$$

Hence the modular vector is $Z_{r,\lambda} = 2X_+$ and the modular class is not trivial.

It is straightforward to check that $d_{\mathfrak{g}} i_{r^\sharp \alpha} \lambda$ vanishes for any $\alpha \in \mathfrak{sl}(2,\mathbb{R})^*$. Therefore, according to the above statement concerning the case of triangular r-matrices, the ELW-modular class of the Lie algebra $\mathfrak{g}^*$ is equal to twice the modular class. This conclusion can be verified by computing the infinitesimal modular character of the Lie algebra $\mathfrak{g}^*$.

Example 5. A twisted r-matrix. Let $\mathfrak{g}$ be the Lie algebra of the group of affine transformations of $\mathbb{R}^2$. Denote by u_1, u_2 a basis of $\mathbb{R}^2$ (that we identify with the abelian subalgebra of translations) and $e_{i,j}$, $i, j \in \{1, 2\}$, the basis of $\mathfrak{gl}(2,\mathbb{R})$ given by $e_{i,j}(u_k) = [e_{i,j}, u_k] = \delta_{jk} u_i$. The dual basis is denoted by $(e_{1,1}^*, e_{1,2}^*, e_{2,1}^*, e_{2,2}^*, u_1^*, u_2^*)$.

Define $\pi = r = e_{1,1} \wedge e_{2,2} + u_1 \wedge u_2$ and $\psi = -(e_{1,1}^* + e_{2,2}^*) \wedge u_1^* \wedge u_2^*$. It is easy to check that ψ is closed, and the following computation shows that (r, ψ) is a twisted Poisson structure on $\mathfrak{g}$,

$$\frac{1}{2}[r, r] = [e_{1,1} \wedge e_{2,2}, u_1 \wedge u_2] = (e_{1,1} - e_{2,2}) \wedge u_1 \wedge u_2 = -(\wedge^3 r^\sharp)((e_{1,1}^* + e_{2,2}^*) \wedge u_1^* \wedge u_2^*).$$

To compute the modular field, we first evaluate

$$Y_{\pi,\psi} = r^\sharp(e_{1,1}^* + e_{2,2}^*) = e_{2,2} - e_{1,1}.$$

Then we set $\lambda = e_{2,2}^* \wedge e_{1,1}^* \wedge e_{1,2}^* \wedge e_{2,1}^* \wedge u_2^* \wedge u_1^*$, and we obtain

$$X_{\pi,\lambda} = - *_\lambda^{-1} d_{\mathfrak{g}} i_r \lambda = - *_\lambda^{-1} d_{\mathfrak{g}}(e_{1,2}^* \wedge e_{2,1}^* \wedge u_2^* \wedge u_1^* + e_{2,2}^* \wedge e_{1,1}^* \wedge e_{1,2}^* \wedge e_{2,1}^*)$$
$$= - *_\lambda^{-1} (e_{1,2}^* \wedge e_{2,1}^* \wedge d_{\mathfrak{g}}(u_2^* \wedge u_1^*)) = e_{2,2} - e_{1,1}.$$

Hence the modular class of $(\mathfrak{g}, r, \psi)$ is $2(e_{2,2} - e_{1,1})$.

Remark. Whereas to a Lie algebroid with a Poisson structure, (A, π), is associated a Lie bialgebroid structure on (A, A^*), to a Lie algebroid with a twisted Poisson structure, (A, π, ψ), is associated a *quasi-Lie bialgebroid* structure on (A, A^*).

This was proved by Roytenberg in [14]. The Lie algebroid structure on A^* is that described in Theorem 4.1, while the bracket on A is $[\ ,\ ]_A + \psi^{(1)}$, and the associated derivation of $\Gamma(\wedge^\bullet A^*)$ is the operator $d_{A,\pi,\psi} = d_A + \delta_{\pi,\psi}$ introduced in (6.10). If, in particular, A is a Lie algebra $\mathfrak{g}$, and $r \in \wedge^2\mathfrak{g}$ and a $d_{\mathfrak{g}}$-cocycle $\psi \in \wedge^3\mathfrak{g}^*$ satisfy the twisted Poisson condition, $\frac{1}{2}[r,r] = (\wedge^3 r^\sharp)\psi$, then the pair $(\mathfrak{g}, \mathfrak{g}^*)$ becomes a *quasi-Lie bialgebra*. To avoid confusion, we stress that the term "twisted" is used here to denote an r-matrix which is not triangular but satisfies a non-linear condition involving a 3-cocycle, while in the theory of Lie quasi-bialgebras, a"twist" or "twisting" is the modification of an r-matrix by the addition of an element $t \in \wedge^2\mathfrak{g}$. The case of "twisted r-matrices" in the former sense deserves to be further explored.

Acknowledgments We are grateful to Alan Weinstein for his very valuable comment on the non-orientable case, and we thank the referee for his remarks.

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Yvette Kosmann-Schwarzbach
Centre de Mathématiques Laurent Schwartz, U.M.R. du CNRS 7640
École Polytechnique
F-91128 Palaiseau
yks@math.polytechnique.fr

Camille Laurent-Gengoux
Département de Mathématiques
Université de Poitiers
F-86962 Futuroscope Cedex
Camille.Laurent@math.univ-poitiers.fr