



The Module $\mathcal{D}f^s$ for Locally Quasi-homogeneous Free Divisors

FRANCISCO CALDERÓN-MORENO and LUIS NARVÁEZ-MACARRO
*Departamento de Álgebra, Facultad de Matemáticas, Universidad de Sevilla, PO 1160,
41080 Seville, Spain. e-mail: {calderon, narvaez}@algebra.us.es*

(Received: 8 September 2000; accepted: 8 June 2001)

Abstract. We find explicit free resolutions for the \mathcal{D} -modules $\mathcal{D}f^s$ and $\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}$, where f is a reduced equation of a locally quasi-homogeneous free divisor. These results are based on the fact that every locally quasi-homogeneous free divisor is Koszul free, which is also proved in this paper.

Mathematical Subject Classifications (2000). 14F40, 32S20, 32S40.

Key words. de Rham complex, \mathcal{D} -module, free divisor, ideal of linear type, Koszul complex, locally quasi-homogeneous, Spencer complex.

Introduction

In this paper we study the module $\mathcal{D}f^s$, where \mathcal{D} is the ring of germs at $0 \in \mathbb{C}^n$ of linear holomorphic differential operators and f is a reduced local equation of a locally quasi-homogeneous free divisor $D \subset (\mathbb{C}^n, 0)$.

The module $\mathcal{D}f^s$ encodes an enormous amount of geometric information of the singularity $f = 0$, but usually it is hard to work with in an explicit way. We prove the following results (see Corollary 5.8 and Theorem 5.9):

(A) Let $f = 0$ be a reduced local equation of a locally quasi-homogeneous free divisor of \mathbb{C}^n , and let $\{\delta_1, \dots, \delta_{n-1}\}$ be a basis of the module of vector fields vanishing on f . Then

- (1) The δ_i generate the ideal $\text{Ann}_{\mathcal{D}} \mathcal{D}f^s$.
- (2) There exist explicit Koszul–Spencer type free resolutions for the modules $\mathcal{D}f^s$ and $\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}$ built on $\delta_1, \dots, \delta_{n-1}$ and $f, \delta_1, \dots, \delta_{n-1}$, respectively.

Locally quasi-homogeneous free divisors form an important class of divisors with non isolated singularities: normal crossing divisors, the union of reflecting hyperplanes of a complex reflection group, free hyperplane arrangements or the discriminant of stable mappings in Mather’s ‘nice dimensions’ are examples of such divisors.

Let X be a complex analytic manifold. Given a divisor $D \subset X$, let us write $j: U = X \setminus D \hookrightarrow X$ for the corresponding open inclusion and $\Omega^\bullet(*D)$ for the meromorphic de Rham complex with poles along D . In [11], Grothendieck proved that the canonical morphism $\Omega^\bullet(*D) \rightarrow \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism (in the derived category). This result is usually known as (a version of) *Grothendieck's Comparison Theorem*.

In [17], K. Saito introduced the *logarithmic de Rham complex* associated with D , $\Omega_X^\bullet(\log D)$, generalizing the well known case of normal crossing divisors (cf. [8]). In the same paper, K. Saito also introduced the important notion of *free divisor*.

In [7], it is proved that the logarithmic de Rham complex $\Omega_X^\bullet(\log D)$ computes the cohomology of the complement U if D is a locally quasi-homogeneous free divisor (we say that D satisfies the *logarithmic comparison theorem*). In other words, the canonical morphism $\Omega_X^\bullet(\log D) \rightarrow \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism, or using Grothendieck's result, the inclusion $\Omega_X^\bullet(\log D) \hookrightarrow \Omega^\bullet(*D)$ is a quasi-isomorphism. In fact, in [5] it is proved that, in the case of $\dim X = 2$, D is locally quasi-homogeneous if and only if it satisfies the logarithmic comparison theorem.

Since the derived direct image $\mathbf{R}j_*(\mathbb{C}_U)$ is a perverse sheaf (it is the de Rham complex of the holonomic module of meromorphic functions with poles along D [15], II, th. 2.2.4), we deduce that $\Omega_X^\bullet(\log D)$ is perverse for every locally quasi-homogeneous free divisor.

On the other hand, the first author proved the following results [4]:

Let $D \subset X$ be a Koszul free divisor (see Definition 1.6) and \mathcal{J} the left ideal of the ring \mathcal{D}_X of differential operators on X generated by the logarithmic vector fields with respect to D . Then

- (1) The left \mathcal{D}_X -module $\mathcal{D}_X/\mathcal{J}$ is holonomic.
- (2) There is a canonical isomorphism in the derived category

$$\Omega_X^\bullet(\log D) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{J}, \mathcal{O}_X).$$

As a consequence, the logarithmic de Rham complex associated with a Koszul free divisor is a perverse sheaf.

The proof of **(A)** depends strongly on the following result, which has been suggested by the above results (see Theorem 4.3):

(B) Every locally quasi-homogeneous free divisor is Koszul free.

In the first three sections we introduce some material concerning locally quasi-homogeneous free divisors, Koszul free divisors, the notion of linear type ideal and the module $\mathcal{D}f^\circ$.

In the fourth section we include the proof of **(B)** in our previous paper [6].

The fifth section is the main part of this paper and contains the proof of **(A)** and some related results.

In the sixth section we study some examples and we state some problems and conjectures.

The first part of (A) has been proposed (without proof) in [1, page 240] in the particular case of discriminants of versal deformations of simple hypersurface singularities. The normal crossing divisors case has been treated in [10].

1. Locally Quasi-homogeneous and Koszul Free Divisors

1.1. Let X be a n -dimensional complex analytic manifold. We denote by $\pi: T^*X \rightarrow X$ the cotangent bundle, \mathcal{O}_X the sheaf of holomorphic functions on X , \mathcal{D}_X the sheaf of linear differential operators on X (with holomorphic coefficients), $\text{Gr}_{F^\bullet}(\mathcal{D}_X)$ the graded ring associated with the filtration F^\bullet by the order, $\sigma(P)$ the principal symbol of a differential operator P and $\{-, -\}$ the Poisson bracket on \mathcal{O}_{T^*X} or $\text{Gr}_{F^\bullet}(\mathcal{D}_X)$. We will note $\mathcal{O} = \mathcal{O}_{X,p}$, $\mathcal{D} = \mathcal{D}_{X,p}$ and $\text{Gr}_{F^\bullet}(\mathcal{D}) = \text{Gr}_{F^\bullet}(\mathcal{D}_X)_p$ the respective stalks at p , with p a point of X . If $J \subset \mathcal{D}$ is a left ideal, we denote by $\sigma(J)$ the corresponding graded ideal of $\text{Gr}_{F^\bullet}(\mathcal{D})$. Given a divisor $D \subset X$, we denote by $\text{Der}(\log D)$ the \mathcal{O}_X -module of the logarithmic vector fields with respect to D [17]. If f is a local equation of D at p , we denote by $\text{Der}(\log f)$ the stalk at p of $\text{Der}(\log D)$, whose elements are germs at p of vector fields δ such that $\delta(f) \in (f)$.

DEFINITION 1.2. A divisor D is Euler-homogeneous at $p \in D$ if there is a local equation h for D around p , and a germ of (logarithmic) vector field δ such that $\delta(h) = h$. A such δ is called a local Euler vector field for f .

The set of points where a divisor is Euler-homogeneous is open.

DEFINITION 1.3 (cf. [7]). A germ of divisor $(D, p) \subset (X, p)$ is quasi-homogeneous if there are local coordinates $x_1, \dots, x_n \in \mathcal{O}_{X,p}$ with respect to which (D, p) has a weighted homogeneous defining equation (with strictly positive weights). A divisor D in a n -dimensional complex manifold X is locally quasi-homogeneous if the germ (D, p) is quasi-homogeneous for each point $p \in D$. A germ of divisor $(D, p) \subset (X, p)$ is locally quasi-homogeneous if the divisor D is locally quasi-homogeneous in a neighborhood of p .

Obviously a locally quasi-homogeneous divisor is Euler-homogeneous at every point.

DEFINITION 1.4. We say that a reduced germ $f \in \mathcal{O}_{X,p}$ is locally quasi-homogeneous if the germ of divisor $(\{f=0\}, p)$ is.

Remark 1.5. A reduced germ $f \in \mathcal{O}_{X,p}$ is locally quasi-homogeneous if and only if for every $q \in \{f=0\}$ near p there are local coordinates $z_1, \dots, z_n \in \mathcal{O}_{X,q}$ and a quasi-homogeneous polynomial $P(t_1, \dots, t_n)$ (with strictly positive weights) such that $f_q = P(z_1, \dots, z_n)$. \square

DEFINITION 1.6 ([17], [4], def. 4.1.1). Let $D \subset X$ be a divisor. We say that D is free at $p \in X$ if $\text{Der}(\log D)_p$ is a free \mathcal{O} -module (of rank n). We say that D is a Koszul free divisor at $p \in X$ if it is free at p and there exists a basis $\{\delta_1, \dots, \delta_n\}$ of $\text{Der}(\log D)_p$ such that the sequence of symbols $\{\sigma(\delta_1), \dots, \sigma(\delta_n)\}$ is regular in $\text{Gr}_{F^*}(\mathcal{D}) = \text{Gr}_{F^*}(\mathcal{D}_X)_p$. If D is a free (resp. Koszul free) divisor at each point of X , we simply say that it is free (resp. Koszul free). We say that a reduced germ $f \in \mathcal{O}_{X,p}$ is free if the divisor $f^{-1}(0)$ is free at p .

Let's remark that a divisor D is automatically Koszul free at every $p \in X \setminus D$.

Remark 1.7. The ideal $I_{D,p} = \text{Gr}_{F^*}(\mathcal{D})\text{Der}(\log D)_p$ is generated by the elements of any basis of $\text{Der}(\log D)_p$. As D is Koszul free at p if and only if $\text{depth}(I_{D,p}, \text{Gr}_{F^*}(\mathcal{D})) = n$ (cf. [14], cor. 16.8), it is clear that the definition of Koszul free divisor does not depend on the election of a particular basis. By the coherence of $\text{Gr}_{F^*}(\mathcal{D}_X)$, if a divisor is Koszul free at a point, then it is Koszul free near that point. \square

We have not found a reference for the following well known proposition (see [14], th. 17.4 for the local case).

PROPOSITION 1.8. *Let $\mathbb{C}\{x\}$ be the ring of convergent power series in the variables $x = x_1, \dots, x_n$ and let G be the graded ring of polynomials in the variables ξ_1, \dots, ξ_t with coefficients in $\mathbb{C}\{x\}$. A sequence $\sigma_1, \dots, \sigma_s$ of homogeneous polynomials in G is regular if and only if the set of zeros $V(I)$ of the ideal I generated by $\sigma_1, \dots, \sigma_s$ has dimension $n + t - s$ in $U \times \mathbb{C}^t$, for some open neighborhood U of 0 (then each irreducible component has dimension $n + t - s$).*

Proof. Let $\mathbb{C}\{x, \xi\}$ be the ring of convergent power series in the variables $x_1, \dots, x_n, \xi_1, \dots, \xi_t$. As the σ_i are homogeneous in G and the ring $\mathbb{C}\{x, \xi\}$ is a flat extension of G , the σ_i are a regular sequence in G if and only if they are a regular sequence in $\mathbb{C}\{x, \xi\}$. But the last condition is equivalent to the equality (*loc. cit.*):

$$\dim_{(0,0)}(V(I)) = \dim(\mathbb{C}\{x, \xi\}/I) = n + t - s.$$

Finally, using the fact that all the σ_i are homogeneous in the variables ξ , the local dimension of $V(I)$ at $(0, 0)$ coincides with its dimension in $U \times \mathbb{C}^t$ for some neighborhood U of 0. \square

COROLLARY 1.9. *Let $D \subset X$ be a free divisor. Let J be the ideal in \mathcal{O}_{T^*X} generated by $\pi^{-1}\text{Der}(\log D)$. Then, D is Koszul free if and only if the set $V(J)$ of zeros of J has dimension n (in this case, each irreducible component of $V(J)$ has dimension n).*

PROPOSITION 1.10. *Let X be a complex manifold of dimension n and let $D \subset X$ be a divisor. Then*

- (1) *Let $X' = X \times \mathbb{C}$ and $D' = D \times \mathbb{C}$. The divisor $D \subset X$ is Koszul free if and only if $D' \subset X'$ is Koszul free.*

- (2) Let Y be another complex manifold of dimension r and let $E \subset Y$ be a divisor. Then: (a) The divisor $(D \times Y) \cup (X \times E)$ is free if $D \subset X$ and $E \subset Y$ are free. (b) The divisor $(D \times Y) \cup (X \times E)$ is Koszul free if $D \subset X$ and $E \subset Y$ are Koszul free.

Proof. (1) It is a consequence of [7], Lemma 2.2, (iv) and the fact that $\sigma_1, \dots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X,p}[\xi_1, \dots, \xi_n]$ if and only if $\xi_{n+1}, \sigma_1, \dots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X,(p,i)}[\xi_1, \dots, \xi_n, \xi_{n+1}]$.

- (2) a) It is an immediate consequence of Saito’s criterion (cf. [7], Lemma 2.2, (v)). (b) It is a consequence of a) and Corollary 1.9. □

EXAMPLE 1.11. Examples of Koszul free divisors are:

- (1) Nonsingular divisors.
- (2) Normal crossing divisors.
- (3) Plane curves: If $\dim_{\mathbb{C}} X = 2$, we know that every divisor $D \subset X$ is free [17], cor. 1.7. Let $\{\delta_1, \delta_2\}$ be a basis of $\text{Der}(\log D)_x$. Their symbols $\{\sigma_1, \sigma_2\}$ are obviously linearly independent over \mathcal{O} , and by Saito’s criterion [17], 1.8, they are relatively primes in $\text{Gr}_{F^*}(\mathcal{D} = \mathcal{O}[\xi_1, \xi_2])$. So they form a regular sequence in $\text{Gr}_{F^*}(\mathcal{D})$, and D is Koszul free (see [4], cor. 4.2.2).
- (4) Proposition 1.10 gives a way to obtain Koszul free divisors in any dimension.
- (5) There are irreducible Koszul free divisors in dimensions greater than 2, which are not constructed from divisors in lower dimension [16]: $X = \mathbb{C}^3$ and $D \equiv \{f = 0\}$, with

$$f = 2^8 z^3 - 2^7 x^2 z^2 + 2^4 x^4 z + 2^4 3^2 xy^2 z - 2^2 x^3 y^2 - 3^3 y^4.$$

A basis of $\text{Der}(\log f)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\begin{aligned} \delta_1 &= 6y\partial_x + (8z - 2x^2)\partial_y - xy\partial_z, \\ \delta_2 &= (4x^2 - 48z)\partial_x + 12xy\partial_y + (9y^2 - 16xz)\partial_z, \\ \delta_3 &= 2x\partial_x + 3y\partial_y + 4z\partial_z, \end{aligned}$$

and the sequence $\{\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)\}$ is $\text{Gr}_{F^*}(\mathcal{D})$ -regular.

2. Ideals of Linear Type

DEFINITION 2.1 (cf. [18], §7.2). Let A be a commutative ring, $I \subset A$ an ideal, $\mathcal{R}(I) = \bigoplus_{i=0}^{\infty} I^d t^d \subset A[t]$ the Rees algebra of I and $\text{Sim}(I)$ the symmetric algebra of the A -module I . We say that I is of *linear type* if the canonical (surjective) morphism of graded A -algebras $\text{Sim}(I) \rightarrow \mathcal{R}(I)$ is an isomorphism.

LEMMA 2.2. *Given a commutative ring A and an ideal $I \subset A$ generated by a family of elements $\{a_i\}_{i \in \Lambda}$, the following properties are equivalent:*

- (a) I is of linear type.
- (b) If $\varphi: A[\{X_i\}_{i \in \Lambda}] \rightarrow \mathcal{R}(I)$ is the morphism of graded algebras defined by $\varphi(X_i) = a_i t$, then the kernel of φ is generated by homogeneous elements of degree 1.

Proof. We consider the kernel of the surjective morphism of graded A -algebras $\Phi: A[\{X_i\}_{i \in \Lambda}] \rightarrow \text{Sim}(I)$, defined by $\Phi(X_{i_1} \cdots X_{i_t}) = a_{i_1} \cdots a_{i_t}$. Then $\ker(\Phi) = \ker(\varphi)$ if and only if I is of linear type, $\ker(\Phi)$ is an ideal generated by its homogeneous elements of degree 1, $\ker(\Phi)_1$, and $\ker(\Phi)_1 = \ker(\varphi)_1$. □

The definition and the lemma above sheafify in the obvious way.

The following results concern the case where the ideal I is generated by a regular sequence.

LEMMA 2.3. *Let $\{a_1, \dots, a_m\}$ be an A -sequence. For $p \leq m$, if $\alpha a_1^{s_1} \cdots a_m^{s_m} \in (a_1^{s_1+k_1}, \dots, a_p^{s_p+k_p})$, then $\alpha \in (a_1^{k_1}, \dots, a_p^{k_p})$.*

Proof. For $j = p + 1, \dots, m$, $\{a_1^{s_1+k_1}, \dots, a_p^{s_p+k_p}, a_{p+1}^{s_{p+1}}, \dots, a_j^{s_j}\}$ is a regular A -sequence, and we can prove inductively that

$$\alpha a_1^{s_1} \cdots a_{j-1}^{s_{j-1}} \in (a_1^{s_1+k_1}, \dots, a_p^{s_p+k_p}).$$

For $i = p - 1, \dots, 0$, $\{a_1^{s_1+k_1}, \dots, a_i^{s_i+k_i}, a_{i+1}^{k_{i+1}}, \dots, a_p^{k_p}\}$ is a regular A -sequence, and we inductively prove that

$$\alpha a_1^{s_1} \cdots a_i^{s_i} \in (a_1^{s_1+k_1}, \dots, a_i^{s_i+k_i}, a_{i+1}^{k_{i+1}}, \dots, a_p^{k_p}). \quad \square$$

PROPOSITION 2.4. *Let A be a commutative ring and let $I \subset A$ be an ideal generated by a regular sequence a_1, \dots, a_n . Then, the kernel of the morphism of graded algebras*

$$A[X_1, \dots, X_n] \rightarrow A[t], \quad X_i \mapsto a_i t,$$

is generated by $a_i X_j - a_j X_i$, $1 \leq i < j \leq n$. In particular, I is of linear type.

Proof. Let g be an homogeneous polynomial of degree m in $A[X_1, \dots, X_n]$ such that $g(a_1, \dots, a_n) = 0$. Let $\text{exp}_g = c X^{e_g}$ be the greatest monomial of g in the inverse lexicographic order, with $e_g = (s_1, \dots, s_t, 0, \dots, 0)$, $s_t \neq 0$. Then

$$g(X_1, \dots, X_n) - \text{exp}_g \in (X_1^{s_1+1}, \dots, X_t^{s_t+1}).$$

By lemma 2.3, $c = \sum_{i=1}^{t-1} \alpha_i a_i \in (a_1, \dots, a_{t-1})$. Then

$$f(X_1, \dots, X_n) = g(X_1, \dots, X_n) - \sum_{i=1}^{t-1} \alpha_i (a_i X_t - a_t X_i) X_1^{s_1} \cdots X_n^{s_n}$$

is an homogeneous polynomial of degree m such that $e_f < e_g$ and

$$f(X_1, \dots, X_n) - g(X_1, \dots, X_n) \in J = (a_i X_j - a_j X_i, 0 < i < j \leq n).$$

In particular, $f(a_1, \dots, a_n) = 0$. Consequently, after a finite number of steps, we will obtain $h(X_1) = c_m X_1^m$, such that $h(X_1) - g(X_1, \dots, X_n) \in J$. So $h(a_1) = c_m a_1^m = 0$, $c_m = 0$ and $g(X_1, \dots, X_n) \in J$. \square

3. The Module $\mathcal{D}f^s$

Let X be a n -dimensional complex analytic manifold, p a point in X and $f \in \mathcal{O} = \mathcal{O}_{X,p}$ a nonzero germ of holomorphic function with $f(p) = 0$. Let D be the (germ of) divisor defined by $f = 0$. The free module of rank one over the ring $\mathcal{O}[f^{-1}, s]$ generated by the symbol f^s has a natural left module structure over the ring $\mathcal{D}[s]$ [2]: the action of a derivation $\delta \in \text{Der}_{\mathbb{C}}(\mathcal{O})$ is given by $\delta(f^s) = \delta(f)sf^{-1}f^s$.

The following lemma is well-known and the proof is straightforward.

LEMMA 3.1. *For every linear differential operator $P \in \mathcal{D}$ of order d , we have*

$$P(f^s) = C_{P,0}f^s + C_{P,1}\binom{s}{1}f^{s-1} + \dots + C_{P,d}\binom{s}{d}f^{s-d}$$

where

$$C_{P,d} = d!\sigma(P)(df) = \{\dots\{\{\sigma(P), f\}, f\} \cdot^d, f\}.$$

Denote by $J_f \subset \mathcal{O}$ the Jacobian ideal associated with f . The surjection

$$\delta \in \text{Der}_{\mathbb{C}}(\mathcal{O}) \mapsto \delta(f) \in J_f$$

and the canonical isomorphism of graded \mathcal{O} -algebras

$$\text{Sim}_{\mathcal{O}}(\text{Der}_{\mathbb{C}}(\mathcal{O})) \simeq \text{Gr}_{F^{\bullet}}(\mathcal{D}) \tag{1}$$

induce a surjective graded morphism of \mathcal{O} -algebras

$$\varphi_f: \text{Gr}_{F^{\bullet}}(\mathcal{D}) \longrightarrow \mathcal{R}(J_f). \tag{2}$$

In coordinates, $\text{Gr}_{F^{\bullet}}(\mathcal{D}) = \mathcal{O}[\xi_1, \dots, \xi_n]$, $\xi_i = \sigma(\partial_i)$ and

$$\varphi_f(\sigma(P)) = \sigma(P)(\partial_1(f)t, \dots, \partial_n(f)t) = \sigma(P)(df)t^d$$

for every differential operator $P \in \mathcal{D}$ of order d .

The homogeneous part of degree 1 of $\ker \varphi_f$ is naturally identified with the \mathcal{O} -module

$$\Theta_f = \{\delta \in \text{Der}_{\mathbb{C}}(\mathcal{O}) \mid \delta(f) = 0\}$$

by means of the canonical isomorphism (1).

Lemma 3.1 implies that $\sigma(\text{Ann}_{\mathcal{D}}f^s) \subset \ker \varphi_f$.

PROPOSITION 3.2. *With the above notations, if J_f is of linear type, then $\sigma(\text{Ann}_{\mathcal{D}}f^s) = \ker \varphi_f$ and the left ideal $\text{Ann}_{\mathcal{D}}f^s$ of \mathcal{D} is generated by Θ_f .*

Proof. By Lemma 2.2, $\ker \varphi_f = \text{Gr}_{F^{\bullet}}(\mathcal{D})\Theta_f \subset \sigma(\text{Ann}_{\mathcal{D}}f^s)$.

The inclusion $\mathcal{D}\Theta_f \subset \text{Ann}_{\mathcal{D}}f^s$ is obvious. Let's prove that $\text{Ann}_{\mathcal{D}}f^s \subset \mathcal{D}\Theta_f$. Clearly, $F^1\text{Ann}_{\mathcal{D}}f^s = \Theta_f$. Suppose $F^{d-1}\text{Ann}_{\mathcal{D}}f^s \subset \mathcal{D}\Theta_f$ and take a differential opera-

tor $P \in F^d \text{Ann}_{\mathcal{D}} f^s \setminus F^{d-1} \text{Ann}_{\mathcal{D}} f^s$. Then, $\sigma(P) \in \ker \varphi_f = \text{Gr}_{F^*}(\mathcal{D})\sigma(\Theta_f)$, and $\sigma(P) = \sum A_i \sigma(\delta_i)$, where $\delta_i \in \Theta_f$ and the A_i are homogeneous of degree $d-1$. Let Q_i be differential operators such that $\sigma(Q_i) = A_i$. We apply the induction hypothesis to $P - \sum_i Q_i \delta_i \in F^{d-1} \text{Ann}_{\mathcal{D}} f^s$ and we conclude the result. \square

PROPOSITION 3.3 (Isolated singularities case, cf. [13], 2.7). *If f has isolated singularity, then $\ker \varphi_f$ is generated by $\partial_i(f)\xi_j - \partial_j(f)\xi_i$, $1 \leq i < j \leq n$. In particular, the left ideal $\text{Ann}_{\mathcal{D}} f^s$ of \mathcal{D} is generated by $\partial_i(f)\partial_j - \partial_j(f)\partial_i$, $1 \leq i < j \leq n$.*

Proof. It is a consequence of Lemma 2.4 and Proposition 3.2. \square

4. Locally Quasi-homogeneous Free Divisors are Koszul Free

PROPOSITION 4.1. *Let U be a connected open set of a complex n -dimensional analytic manifold X and let $\Sigma \subset U$ be a closed analytic set of dimension s . If a sequence $C = \{\sigma_1, \dots, \sigma_{n-s}\}$ of homogeneous polynomials in $\mathcal{O}_X(U)[\xi_1, \dots, \xi_n]$ is regular at every point $q \in U \setminus \Sigma$ (i.e. it is regular in $\mathcal{O}_{X,q}[\xi_1, \dots, \xi_n]$), then it is regular at every point of U .*

Proof. Let $p \in \Sigma$ and let $\pi: U \times \mathbb{C}^n \rightarrow U$ be the projection. By Proposition 1.8, we have to prove that the ideal $I = (\sigma_1, \dots, \sigma_{n-s})$ defines an analytic set $V = V(I) \subset U \times \mathbb{C}^n$ of dimension $n+s$. By hypothesis, we know that C is regular on $U \setminus \Sigma$, and so (*loc. cit.*) the dimension of (every irreducible component of) $V \cap \pi^{-1}(U \setminus \Sigma)$ is $n+s$. Now, let W be an irreducible component of V . It has, at least, dimension $n+s$. If W is contained in $\pi^{-1}(\Sigma) = \Sigma \times \mathbb{C}^n$, then it must be equal to $\pi^{-1}(\Sigma)$. If not, $\dim W = \dim(W \cap \pi^{-1}(U \setminus \Sigma)) \leq \dim(V \cap \pi^{-1}(U \setminus \Sigma)) = n+s$. So, we conclude that W has dimension $n+s$. \square

COROLLARY 4.2. *Let D be a free divisor in some analytic manifold X and let $\Sigma \subset D$ a discrete set of points. If D is Koszul free at every point $x \in D \setminus \Sigma$, then D is Koszul free (at every point of X).*

THEOREM 4.3. *Every locally quasi-homogeneous free divisor is Koszul free.*

Proof. We proceed by induction on the dimension t of the ambient manifold X . For $t=1$, the theorem is trivial and for $t=2$, the theorem is directly proved in example 1.11, 3. Now, we suppose that the result is true for $t < n$, and let D be a locally quasi-homogeneous free divisor of a complex analytic manifold X of dimension n . Let $p \in D$ and let $\{\delta_1, \dots, \delta_n\}$ be a basis of the logarithmic derivations of D at p .

Thanks to [7], prop. 2.4 and Lemma 2.2, (iv), there is an open neighborhood U of p such that for each $q \in U \cap D$, with $q \neq p$, the germ of pair (X, D, q) is isomorphic to a product $(\mathbb{C}^{n-1} \times \mathbb{C}, D' \times \mathbb{C}, (0, 0))$, where D' is a locally quasi-homogeneous free divisor. Induction hypothesis implies that D' is a Koszul free divisor at 0. Then, by Proposition 1.10.1, D is a Koszul free divisor at q too. We have then proved that D is a Koszul free divisor in $U \setminus \{p\}$. We conclude by using Corollary 4.2. \square

COROLLARY 4.4. *Every free divisor that is locally quasi-homogeneous at the complement of a discrete set is Koszul free.*

In particular, the last corollary gives rise a new proof of the fact that every divisor in dimension 2 is Koszul free (cf. 1.11, 3)).

5. The Module $\mathcal{D}f^s$ for Locally Quasi-Homogeneous Free Divisors

5.1. In this section, $f \in \mathcal{O} = \mathcal{O}_{X,p}$ will be a reduced locally quasi-homogeneous free germ 1.4, 1.6. That means that $D = \{f = 0\}$ is a locally quasi-homogeneous free divisor near p .

We will also assume that

- (1) The equation f and its Euler vector field E are globally defined on X .
- (2) $E(q) \neq 0$ for every $q \in X \setminus \{p\}$.
- (3) $\text{Der}(\log D)$ is \mathcal{O}_X -free (of rank $n = \dim X$).

In order to proceed inductively on the dimension of the ambient variety when working with such f 's, we quote the following direct consequence of [9], Lemmas 1.3, 1.5 (see also [7], prop. 2.4).

PROPOSITION 5.2. *Let $f \in \mathcal{O}_{X,p}$ a reduced locally quasi-homogeneous free germ and let D be the divisor $f = 0$. For $q \in D \setminus \{p\}$ close to p , there are local coordinates $z_1, \dots, z_n \in \mathcal{O}_{X,q}$ centered at q and a quasi-homogeneous polynomial $G'(t_1, \dots, t_{n-1})$ in $n - 1$ variables which is also a locally quasi-homogeneous free germ in $\mathcal{O}_{\mathbb{C}^{n-1},0}$ and such that $f_q = G'(z_1, \dots, z_{n-1})$.*

We call $\tilde{\Theta}_f$ the \mathcal{O}_X -sub-module (and Lie algebra) of $\text{Der}(\log D)$ whose sections are vector fields annihilating f . Denote by $\mathcal{J}_f \subset \mathcal{O}_X$ the jacobian ideal sheaf associated with f . The stalk of $\tilde{\Theta}_f$ (resp. of \mathcal{J}_f) at p is then Θ_f (resp. J_f).

As in (2), we have a surjective graded morphism of \mathcal{O}_X -algebras

$$\Phi_f: \text{Gr}_{F^\bullet}(\mathcal{D}_X) \longrightarrow \mathcal{R}(\mathcal{J}_f),$$

whose stalk at p is φ_f .

We have

$$\text{Der}(\log D) = \tilde{\Theta}_f \oplus (\mathcal{O}_X E), \quad \text{Der}(\log f) = \Theta_f \oplus (\mathcal{O} E), \tag{3}$$

and $\tilde{\Theta}_f, \Theta_f$ are free of rank $n - 1$.

PROPOSITION 5.3. *The Koszul complex associated with $\tilde{\Theta}_f \subset \text{Der}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n}) = \text{Gr}_{F^\bullet}^1(\mathcal{D}_X) \subset \text{Gr}_{F^\bullet}(\mathcal{D}_X)$:*

$$0 \rightarrow \mathrm{Gr}_{F^\bullet}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \tilde{\Theta}_f \xrightarrow{d_{-n+1}} \cdots \xrightarrow{d_{-2}} \mathrm{Gr}_{F^\bullet}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^1 \tilde{\Theta}_f \xrightarrow{d_{-1}} \mathrm{Gr}_{F^\bullet}(\mathcal{D}_X),$$

$$d_{-k}(F \otimes (\sigma_1 \wedge \cdots \wedge \sigma_k)) = \sum_{i=1}^k (-1)^{i-1} P\sigma_i \otimes (\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \cdots \wedge \sigma_k),$$

is exact.

Proof. We need to prove that some (or any) basis $\{\delta_1, \dots, \delta_{n-1}\}$ of $\tilde{\Theta}_f$ form a regular sequence in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_X)$, but such a basis can be augmented to a basis $\{\delta_1, \dots, \delta_{n-1}, E\}$ of $\mathrm{Der}(\log D)$, that we know by Theorem 4.3 to form a regular sequence in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_X)$. \square

PROPOSITION 5.4. *With the hypothesis of 5.1, if the augmented graded complex of $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_X)$ -modules*

$$0 \rightarrow \mathrm{Gr}_{F^\bullet}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \tilde{\Theta}_f \xrightarrow{d_{-n+1}} \cdots \xrightarrow{d_{-1}} \mathrm{Gr}_{F^\bullet}(\mathcal{D}_X) \xrightarrow{\Phi_f} \mathcal{R}(\mathcal{J}_f) \rightarrow 0 \tag{4}$$

is exact on $X - \{p\}$, then it is exact everywhere.

Proof. We know that Φ_f is surjective. By Proposition 5.3, the only thing to prove is $\ker \Phi_f = \mathrm{Im} d_{-1}$. We can proceed separately on each homogeneous component:

$$0 \rightarrow \mathrm{Gr}_{F^\bullet}^{m-n+1}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \tilde{\Theta}_f \xrightarrow{d_{-n+1}^m} \cdots \xrightarrow{d_{-1}^m} \mathrm{Gr}_{F^\bullet}^m(\mathcal{D}_X) \xrightarrow{\Phi_f^m} \mathcal{J}_f^m \rightarrow 0.$$

Let's consider the coherent \mathcal{O}_X -module $\mathcal{F} = \mathrm{Im} d_{-1}^m$ and the short sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathrm{Gr}_{F^\bullet}^m(\mathcal{D}_X) \xrightarrow{\Phi_f^m} \mathcal{J}_f^m \rightarrow 0. \tag{5}$$

By Proposition 5.3 and the fact that the cohomology with support $H_p^i(\mathcal{O}_X)$ vanishes for $i \neq n$, we deduce that $H_p^i(\mathcal{F}) = 0$ for $i = 0, 1$ and $H_p^0(\mathcal{J}_f^m) = 0$. These properties and the exactness of (5) on $X - \{p\}$ imply the proposition (cf. [12], (8.14)). \square

The following lemma is clear.

LEMMA 5.5. *Let $g \in \mathcal{O}_{n-1} = \mathbb{C}\{y_1, \dots, y_{n-1}\}$ and call $f = g$, but as an element in $\mathcal{O}_n = \mathbb{C}\{y_1, \dots, y_n\}$. Then:*

- (1) $\ker \varphi_f$ is generated by $\ker \varphi_g$ and $\sigma(\partial_{y_n})$.
- (2) Θ_f is generated by Θ_g and ∂_{y_n} .

THEOREM 5.6. *Let $f \in \mathcal{O} = \mathcal{O}_{X,p}$ be a reduced locally quasi-homogeneous free germ. Then the graded complex of $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ -modules*

$$0 \rightarrow \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \otimes_{\mathcal{O}} \bigwedge^{n-1} \Theta_f \xrightarrow{\varepsilon_{-n+1}} \cdots \xrightarrow{\varepsilon_{-1}} \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \xrightarrow{\varphi_f} \mathcal{R}(\mathcal{J}_f) \rightarrow 0$$

is exact. In particular, the kernel of the morphism

$$\mathrm{Gr}_{f^s}(\mathcal{D}) \xrightarrow{\varphi_f} \mathcal{R}(J_f)$$

is the ideal generated by Θ_f and then the jacobian ideal J_f is of linear type.

Proof. By the exactness of (5.3), the only thing to prove is that $\ker \varphi_f$ is generated by $\sigma(\Theta_f)$. We will use induction on $n = \dim X$. If $n = 2$, we apply Proposition 3.3. We suppose that the result is true if the ambient variety has dimension $n - 1$. By Proposition 5.4, we need to prove the exactness of the complex (4) on $U \setminus \{x\}$, for some open neighborhood U of x , or equivalently, that $\ker \Phi_f$ is generated by $\sigma(\Theta_f)$ at every $q \in U \setminus \{x\}$. The result is then a consequence of Proposition 5.2, Lemma 5.5 and the induction hypothesis. \square

DEFINITION 5.7. The Spencer complex^{*} for $\tilde{\Theta}_f$ is the complex of free left \mathcal{D}_X -modules given by:

$$0 \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \tilde{\Theta}_f \xrightarrow{\varepsilon_{-n+1}} \dots \xrightarrow{\varepsilon_{-2}} \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^1 \tilde{\Theta}_f \xrightarrow{\varepsilon_{-1}} \mathcal{D}_X,$$

$$\varepsilon_{-1}(P \otimes \delta) = P\delta;$$

$$\varepsilon_{-k}(P \otimes (\delta_1 \wedge \dots \wedge \delta_k))$$

$$= \sum_{i=1}^k (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \dots \wedge \hat{\delta}_i \wedge \dots \wedge \delta_k) +$$

$$+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \dots \wedge \hat{\delta}_i \wedge \dots \wedge \hat{\delta}_j \wedge \dots \wedge \delta_k).$$

In a similar way we define the Spencer complex for Θ_f , which is the stalk at p of the Spencer complex for $\tilde{\Theta}_f$.

Both Spencer complexes can be augmented by considering the obvious maps $\mathcal{D}_X \rightarrow \mathcal{D}_X f^s, \mathcal{D} \rightarrow \mathcal{D}f^s$.

COROLLARY 5.8. *With the hypothesis of 5.1, we have*

- (a) *The Spencer complex for Θ_f is a resolution of $\mathcal{D}f^s$. In particular, the left ideal $\mathrm{Ann}_{\mathcal{D}f^s}$ is generated by Θ_f .*
- (b) *The left ideal $\mathrm{Ann}_{\mathcal{D}[s]f^s}$ is generated by Θ_f and $E - s$.*
- (c) *The left ideal $\mathrm{Ann}_{\mathcal{D}} \eta$, where η is the class of f^s in the quotient $\mathcal{D}[s]f^s / \mathcal{D}[s]f^{s+1}$, is generated by Θ_f and f .*

Proof. For (a) we proceed as in [4], prop. 4.1.3 by using Proposition 3.2 and Theorem 5.6. Property (b) follows easily from (a), and property (c) follows from (a) and (b). \square

^{*}It should be noticed that such complex was originally used by Chevalley and Eilenberg in the setting of the cohomology of Lie algebras (cf. [19], 7.7).

Let's call $\Xi_f = \Theta_f \oplus (\mathcal{O}f)$ (resp. $\tilde{\Xi}_f = \tilde{\Theta}_f \oplus (\mathcal{O}_X f)$), which is a free sub- \mathcal{O} -module (respectively, sub- \mathcal{O}_X -module) and a Lie subalgebra of \mathcal{D} (resp. of \mathcal{D}_X). It can be also canonically embedded in $\text{Gr}_{F^\bullet}(\mathcal{D})$ (resp. $\text{Gr}_{F^\bullet}(\mathcal{D}_X)$) equipped with the Poisson bracket $\{-, -\}$. As in 5.3 and 5.7, we define the Koszul complex associated with $\Xi_f \subset \text{Gr}_{F^\bullet}(\mathcal{D})$ (resp. $\tilde{\Xi}_f \subset \text{Gr}_{F^\bullet}(\mathcal{D}_X)$) and the Spencer complex associated with $\Xi_f \subset \mathcal{D}$ (resp. $\tilde{\Xi}_f \subset \mathcal{D}_X$). The Koszul (resp. Spencer) complex associated with $\Xi_f \subset \text{Gr}_{F^\bullet}(\mathcal{D})$ (resp. with $\Xi_f \subset \mathcal{D}$) is obviously the stalk at p of the Koszul (resp. of the Spencer) complex associated with $\tilde{\Xi}_f \subset \text{Gr}_{F^\bullet}(\mathcal{D}_X)$ (resp. with $\tilde{\Xi}_f \subset \mathcal{D}_X$).

THEOREM 5.9. *With the hypothesis of 5.1, the following properties hold:*

- (1) *The Koszul complex associated with $\Xi_f \subset \text{Gr}_{F^\bullet}(\mathcal{D})$ is exact.*
- (2) *The Spencer complex associated with $\Xi_f \subset \mathcal{D}$ is a free resolution of $\mathcal{D}[s]f^s / \mathcal{D}[s]f^{s+1}$.*

Proof. For the first property, call \mathbf{K} the Koszul complex associated with $\tilde{\Xi}_f \subset \text{Gr}_{F^\bullet}(\mathcal{D}_X)$. The Koszul complex associated with $\Xi_f \subset \text{Gr}_{F^\bullet}(\mathcal{D})$ is the stalk at p of \mathbf{K} .

We proceed by induction on the dimension of the ambient variety. If that dimension is 1, $\Xi_f = \mathcal{O}f$, and the Koszul complex associated with f is clearly exact. Suppose the result true if the dimension of the ambient variety is $< n$.

Now, suppose $\dim X = n$.

Let $\delta_1, \dots, \delta_{n-1}$ be a basis of $\tilde{\Theta}_f$ in some small enough neighborhood U of p . According to proposition 4.1, we need to prove that \mathbf{K} is exact on $U \setminus \{p\}$.

For every $q \in U$ with $f(q) \neq 0$, the germ of f at q is an unit and by Proposition 5.3, the complex \mathbf{K} is exact at q .

Let q be a point in $D = \{f = 0\}$, $q \neq p$. By Proposition 5.2, there are local coordinates $z_1, \dots, z_n \in \mathcal{O}_{X,q}$ and a quasi-homogeneous polynomial $G'(t_1, \dots, t_{n-1}) \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ in $n-1$ variables which is also a locally quasi-homogeneous free germ in $\mathcal{O}_{\mathbb{C}^{n-1},0}$ and such that $f_q = G'(z_1, \dots, z_n)$.

Let $G(t_1, \dots, t_n) \in \mathcal{O}_{\mathbb{C}^n,0}$ be the same polynomial as $G'(t_1, \dots, t_{n-1})$ but considered in n variables. The exactness of \mathbf{K}_q is then equivalent to the exactness of the Koszul complex associated with $\Xi_G \subset \text{Gr}_{F^\bullet} \mathcal{D}_{\mathbb{C}^n,0}$.

Let's write $\mathcal{O}_m = \mathbb{C}\{t_1, \dots, t_m\}$ and call ζ'_i the principal symbol of $\partial/\partial t_i$.

Let

$$\{\delta'_1, \dots, \delta'_{n-2}\} \subset \bigoplus_{i=1}^{n-1} \mathcal{O}_{n-1} \frac{\partial}{\partial t_i}$$

be a basis of $\Theta_{G'}$. A basis of Θ_G is then

$$\left\{ \delta'_1, \dots, \delta'_{n-2}, \frac{\partial}{\partial t_n} \right\} \subset \bigoplus_{i=1}^n \mathcal{O}_n \frac{\partial}{\partial t_i}.$$

Call σ'_i the principal symbol of δ'_i , $i = 1, \dots, n-2$.

By induction hypothesis we know that the Koszul complex associated with $\Xi_{G'} \subset \text{Gr}_{F^*} \mathcal{D}_{\mathbb{C}^{n-1},0} = \mathcal{O}_{n-1}[\zeta'_1, \dots, \zeta'_{n-1}]$ is exact or, equivalently, that $\sigma'_1, \dots, \sigma'_{n-2}, G'$ is a regular sequence in $\mathcal{O}_{n-1}[\zeta'_1, \dots, \zeta'_{n-1}]$. That implies that $\sigma'_1, \dots, \sigma'_{n-2}, \zeta'_n, G = G'$ is a regular sequence in $\mathcal{O}_n[\zeta'_1, \dots, \zeta'_n]$, i.e. that the Koszul complex associated with $\Xi_G \subset \text{Gr}_{F^*} \mathcal{D}_{\mathbb{C}^n,0}$ is exact, and the result is proved.

For the second property, we filter the Spencer complex associated with $\Xi_f \subset \mathcal{D}$ as in [10], prop. 2.3.4:

$$\text{deg}(\Theta_f) = 1, \quad \text{deg}(f) = 0.$$

Its graded complex coincides with the Koszul complex associated with $\Xi_f \subset \text{Gr}_{F^*}(\mathcal{D})$, and then the Spencer complex is exact. To conclude, we use Corollary 5.8, (c). \square

Remark 5.10. From Corollary 5.8(b), and following the proof of Theorem 5.9, we can also prove that under the hypothesis of 5.1, the following results hold:

- (a) The Spencer complex over $\mathcal{D}[s]$ associated with $\Theta_f \oplus (\mathcal{O}(E - s))$ is a $\mathcal{D}[s]$ -free resolution of $\mathcal{D}[s]f^s$.
- (b) The Spencer complex over \mathcal{D} associated with $\Theta_f \oplus (\mathcal{O}(E + k))$ is a \mathcal{D} -free resolution of $\mathcal{D}f^{-k} = \mathcal{O}[f^{-1}]$ for any integer $k \gg 0$.

6. Examples and Questions

We know several (related) kind of free divisors:

[LQH]

Locally quasi-homogeneous (Definition 1.3).

[EH]

Euler homogeneous (Definition 1.2).

[LCT]

Free divisors satisfying the logarithmic comparison theorem.

[KF]

Koszul free (Definition 1.6).

[P]

Free divisors such that the complex $\Omega_X^\bullet(\log D)$ is a perverse sheaf.

We have then the following implications: [LQH] \Rightarrow [EH] (obvious), [LQH] \Rightarrow [LCT] by [7], th. 1.1, [LCT] \Rightarrow [P], by [15], II, th. 2.2.4) [KF] \Rightarrow [P] by [4] th. 4.2.1, [LQH] \Rightarrow [KF] by Theorem 4.3.

EXAMPLE 6.1 (Free divisors in dimension 2). We recall Theorem 3.9 from [5]: Let X be a complex analytic manifold of dimension 2 and $D \subset X$ a divisor. The following conditions are equivalent:

- (1) D is Euler homogeneous.
- (2) D is locally quasi-homogeneous.
- (3) The logarithmic comparison theorem holds for D .

Consequently, in dimension 2 we have:

$$[\text{LQH}] \Leftrightarrow [\text{EH}] \Leftrightarrow [\text{LCT}]$$

and $[\text{KF}]$ (cf. 1.11, 3) and $[\text{P}]$ ([4]) always hold. In particular,

$$[\text{KF}] \not\Leftrightarrow [\text{LQH}], [\text{EH}], [\text{LCT}].$$

Examples of plane curves not satisfying logarithmic comparison theorem are, for instance, the curves of the family (cf. [5]):

$$x^q + y^q + xy^{p-1} = 0, \quad p \geq q + 1 \geq 5.$$

EXAMPLE 6.2 (An example in dimension 3). Let's consider $X = \mathbb{C}^3$ and $D = \{f = 0\}$, with $f = xy(x+y)(x+yz)$ [4]. A basis of $\text{Der}(\log D)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\delta_1 = xy\partial_x + y^2\partial_y - 4(x+yz)\partial_z,$$

$$\delta_2 = x(x+3y)\partial_x - y(3x+y)\partial_y + 4x(z-1)\partial_z,$$

$$\delta_3 = x\partial_x + y\partial_y$$

the determinant of the coefficients matrix being $-16f$ and

$$\delta_1(f) = 0, \quad \delta_2(f) = 0, \quad \delta_3(f) = 4f.$$

In particular, D is Euler homogeneous ($E = (1/4)\delta_3$) and we know [5] that it satisfies the logarithmic comparison theorem. Let $I \subset \mathcal{O}_{T^*X}$ be the ideal generated by the symbols $\{\sigma_1, \sigma_2, \sigma_3\}$ of the basis of $\text{Der}(\log D)$. By corollary 1.9, D is not Koszul free, because the dimension of $V(I)$ at $((0, 0, \lambda), 0) \in T^*X$ is 4, and neither is D locally quasi-homogeneous. So

$$[\text{LCT}] \not\Leftrightarrow [\text{KF}], [\text{LQH}], \quad [\text{EH}] \not\Leftrightarrow [\text{KF}], [\text{LQH}].$$

Finally, for the only missing relation, we quote the following conjecture from [5]:

CONJECTURE 6.3. *If the logarithmic comparison theorem holds for D , then D is Euler homogeneous.*

EXAMPLE 6.4. Let's see in the example 6.2 that the left ideal $\text{Ann}_D(f^s)$ is not generated by Θ_f and then, J_f is not an ideal of linear type.

Here, we set $X = \mathbb{C}^3$, $p = (0, 0, 0)$ and $E = (1/4)\delta_3$. The \mathcal{O} -modules Θ_f and $\text{Der}(\log f) = \Theta_f \oplus \mathcal{O} \cdot E$ are generated by $\{\delta_1, \delta_2\}$ and $\{\delta_1, \delta_2, E\}$, respectively. The symbols $\sigma_1 = \sigma(\delta_1)$, $\sigma_2 = \sigma(\delta_2)$ form a $\text{Gr}_{F^*}(D)$ -regular sequence (the proof is

analogous to Example 1.11, 3)). Then, as in the proof of [4], prop. 4.1.2, we have $\sigma(\mathcal{D}\Theta_f) = \text{Gr}_{F^*}(\mathcal{D})\sigma(\Theta_f)$. For

$$P = 2y^2\partial_x\partial_y - 2y^2\partial_y^2 - (2xz + 6yz)\partial_x\partial_z + 10yz\partial_y\partial_z + 8z(1-z)\partial_z^2 + \\ + (2x - 4y)\partial_y\partial_z - x\partial_x - y\partial_y - 8z\partial_z + 4\partial_z$$

and $R = \mathbb{C}[x, y, z]$, $S = R[\xi_1, \xi_2, \xi_3]$, $\mathfrak{m} = R(x, y, z)$ we check that

- (1) $P \in \text{Ann}_{\mathcal{D}_X}(f^s)$,
- (2) $(S(\sigma_1, \sigma_2) : \sigma(P)) = S(x, y)$, and then $(S(\sigma_1, \sigma_2) : \sigma(P)) \cap R = R(x, y)$.

So, $\sigma(P) \notin R_{\mathfrak{m}}[\xi_1, \xi_2, \xi_3]\sigma(\Theta_f)$ and, by faithful flatness,

$$\sigma(P) \notin \mathcal{O}[\xi_1, \xi_2, \xi_3]\sigma(\Theta_f) = \text{Gr}_{F^*}(\mathcal{D})\sigma(\Theta_f).$$

We conclude that $P \notin \mathcal{D}\Theta_f$.

PROBLEM 6.5. We do not know whether a free divisor defined by a quasi-homogeneous polynomial (with strictly positive weights) is locally quasi-homogeneous.

PROBLEM 6.6. We do not know any example of a free divisor $D \subset X$ whose logarithmic de Rham complex $\Omega_X^*(\log D)$ is not perverse.

Acknowledgement

Both authors are supported by PB97-0723.

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