*The Modulus of a Plane Condenser**

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1. Introduction. This paper is a study of the logarithmic capacity and modulus of a plane condenser. In this study a *condenser* is defined as a triple (R, A, B), where R is a connected open set of the extended plane **P**, whose complement $\sim R$ is the union of nonempty disjoint compacta A and B; if A and B are connected, the condenser is called a *ring*. If R is contained in the finite plane **C**, the *capacity* of the condenser (R, A, B) may be defined as

$$\text{cap}\, R = \inf_u \iint_R |\nabla u(x)|^2 \, dx^1 \, dx^2, \tag{1}$$

where x^i denotes the ith coordinate of the point x (no complex multiplication is used in this paper), and the infimum is taken over all continuously differentiable functions u in R with boundary values 0 at A and 1 at B; the capacity of an arbitrary condenser may be defined by means of an auxiliary Möbius transformation. The *modulus* of the condenser is

$$\text{mod}\, R = \frac{2\pi}{\text{cap}\, R},$$

the constant in the numerator being chosen so that the modulus of the annular ring $\{x : r_1 < |x| < r_2\}$ is equal to $\log (r_2/r_1)$.

Our fundamental theorem is a representation for the modulus of a condenser as an "average" of absolute ratios. If x_1, x_2, x_3, x_4 form an ordered quadruple of points in **P**, with $x_2 \neq x_4$, and f is a Möbius transformation such that $f(x_2) = 0$ and $f(x_4) = \infty$, the *absolute ratio* $|x_1, x_2, x_3, x_4|$ is defined by

$$|x_1, x_2, x_3, x_4| = \begin{cases} 1 & \text{if } f(x_1) = f(x_3), \\ |f(x_1)|/|f(x_3)| & \text{if } f(x_1) \neq f(x_3). \end{cases}$$

Then the fundamental theorem is given by the formula

$$\text{mod}\, R = \inf_\sigma \int_B \int_B \int_A \int_A \log |a_1, b_2, b_1, a_2| \, d\sigma(a_1) \, d\sigma(a_2) \, d\sigma(b_1) \, d\sigma(b_2), \tag{2}$$

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where the infimum is taken over all signed measures σ on $A \cup B$ which are unit positive measures on A and unit negative measures on B.

If x_1, x_2, x_3, x_4 are distinct finite points, then

$$|x_1, x_2, x_3, x_4| = \frac{|x_1 - x_2|\,|x_3 - x_4|}{|x_2 - x_3|\,|x_4 - x_1|}. \tag{3}$$

Therefore our representation for mod R may be given in terms of logarithms of Euclidean distances. This makes it possible to obtain simple proofs of the relationships between the modulus of a condenser and the logarithmic capacities of the "plates" A and B.

For the special case in which B is the reflection of A in the unit circle, the representation (2) takes a particularly simple form. In Section 5 we show that our representation then reduces to that given by Tsuji [12] in terms of hyperbolic distances in the unit disc.

In Section 6 we show how the well-known extremal properties of the Grötzsch and Teichmüller rings follow from (2), using simple geometric arguments concerning Euclidean distances.

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2. Preliminary results. If $a \in \mathbf{C}$ and r is any positive number, we denote by $U_r(a)$ the open disc with center a and radius r, and by $S_r(a)$ the circumference at distance r from a. We write $U_r = U_r(0)$ and $S_r = S_r(0)$.

Lebesgue measure in $\mathbf{C}$ may be extended to a Borel measure λ in $\mathbf{P}$, by making the convention $\lambda(\{\infty\}) = 0$. For the remainder of this paper λ will have this meaning, and the word *measure* will signify a signed Borel measure in $\mathbf{P}$. When no range of integration is indicated, an integral is to be taken over the entire support of the associated measure.

The logarithmic capacity [13, p. 55] of a compact set $E \subset \mathbf{C}$ is denoted by $C(E)$, and the Robin constant by $V(E) = \log(1/C(E))$. We say that an arbitrary set $E \subset \mathbf{P}$ has zero logarithmic capacity if every compact subset of $E \cap \mathbf{C}$ has zero logarithmic capacity.

We turn now to condensers. The reader will have already noticed that we sometimes denote the condenser (R, A, B) simply by R, provided there is no danger of confusion. We say that the condenser contains the point x, is bounded, or lies in $\mathbf{C}$ provided the point set R has the property. We say that a Möbius transformation f maps the condenser (R, A, B) onto the condenser (R', A', B') if $f(A) = A'$ and $f(B) = B'$.

In our proofs, it is sometimes convenient to study simple condensers first and then to use a limiting process; the remainder of this section is in preparation for this.

We begin by enlarging the class of competing functions in (1). The arguments of [2, p. 502] show that if $R \subset \mathbf{C}$,

$$(4) \qquad \operatorname{cap} R = \inf_{u \varepsilon \mathcal{A}(R)} \int_R |\nabla u|^2 \, d\lambda,$$

where $\mathcal{A}(R)$ denotes the family of functions which are continuous and absolutely continuous on lines in R, and have boundary values 0 at A and 1 at B. If we agree to write $(R, A, B) \prec (R', A', B')$ provided $A' \subset A$ and $B' \subset B$, it is clear from (4) that cap R is monotonic: if $(R, A, B) \prec (R', A', B')$, then cap $R' \leqq$ cap R.

We say that the condenser (R, A, B) is *exhausted monotonically* by a sequence of condensers $\{(R_n, A_n, B_n)\}$ if $A_n \downarrow A$ and $B_n \downarrow B$.

Lemma 1. *If* $\{(R_n, A_n, B_n)\}$ *forms a monotone exhaustion of* (R, A, B), *then* $\lim_n \operatorname{cap} R_n = \operatorname{cap} R$.

Proof. We may assume that $\infty \varepsilon B$, and hence $\infty \varepsilon B_n$ for all n. From the monotonic character of cap R it is clear that $\underline{\lim} \operatorname{cap} R_n \geqq \operatorname{cap} R$.

Now let $\epsilon > 0$ be arbitrary, and select $u \varepsilon \mathcal{A}(R)$ so that $\int_R |\nabla u|^2 \, d\lambda < \operatorname{cap} R + \epsilon$. Fix η, $0 < \eta < \frac{1}{2}$, and define

$$v = \begin{cases} 0 & \text{if } u < \eta, \\ 1 & \text{if } u > 1 - \eta, \\ \dfrac{u - \eta}{1 - 2\eta} & \text{otherwise.} \end{cases}$$

For all large n, $v \varepsilon \mathcal{A}(R_n)$. Hence

$$\operatorname{cap} R_n \leqq \int_R |\nabla v|^2 \, d\lambda \leqq \frac{\operatorname{cap} R + \epsilon}{(1 - 2\eta)^2}.$$

Now let $\eta \to 0$, and then let $\epsilon \to 0$ to obtain $\overline{\lim} \operatorname{cap} R_n \leqq \operatorname{cap} R$. This completes the proof.

We say that a condenser (R, A, B) is *regular* if both A and B have only finitely many components, and none of the components are single points.

Lemma 2. *Every condenser* (R, A, B) *can be exhausted monotonically by regular condensers.*

Proof. We imitate a construction given by Walsh [14]. Again we may assume that $\infty \varepsilon B$.

Let z be a fixed point of R whose coordinates in **C** are both irrational. For each positive integer n, we divide the plane into a lattice $\mathcal{L}_n$ of closed squares by means of the collection of lines

$$x^i = m/2^n, \qquad i = 1, 2; m = 0, \pm 1, \pm 2, \cdots .$$

We now consider all sets E which satisfy the requirements

i) $z \varepsilon E$,
ii) E is an open connected set,
iii) $\bar{E}$ is a union of squares from the lattice $\mathcal{L}_n$,
iv) $\bar{E} \subset R \cap U_n(z)$.

We define R_n to be the union of all such sets E. It is obvious that R_n is an open connected set, $R_n \subset R_{n+1}$, and $R = \cup R_n$.

For all large n, the components of $\sim R_n$ can be grouped into sets A_n and B_n, where $A_n \cap B$ and $B_n \cap A$ are empty. Then $\{(R_n, A_n, B_n)\}$ is the desired exhaustion.

3. The fundamental theorem. If (R, A, B) is any condenser in **P**, we denote by $\mathcal{S}(R)$ the family of signed Borel measures of the form $\sigma = \sigma_A - \sigma_B$, where σ_A is a unit positive measure on A and σ_B is a unit positive measure on B. For any $\sigma \in \mathcal{S}(R)$, we define

$$(5) \qquad I(\sigma) = \int_B \int_B \int_A \int_A \log |a_1, b_2, b_1, a_2| \, d\sigma(a_1) \, d\sigma(a_2) \, d\sigma(b_1) \, d\sigma(b_2).$$

The integral in (5) is taken with respect to the product measure $\sigma \times \sigma \times (-\sigma) \times (-\sigma)$. We discuss now the existence of this integral.

If f is a Möbius transformation which maps the condenser R onto the condenser R', it is clear that any measure $\sigma \in \mathcal{S}(R)$ corresponds in a natural way to a measure $\sigma' \in \mathcal{S}(R')$; we set $\sigma'(f(E)) = \sigma(E)$ for every Borel set E. We see that $I(\sigma)$ exists if and only if $I(\sigma')$ exists, and in this case $I(\sigma) = I(\sigma')$. This means that to settle the existence of $I(\sigma)$ for $\sigma \in \mathcal{S}(R)$, we may assume that $A \cup B$ is bounded. However, in this case it is clear from (3) that $\log |a_1, b_2, b_1, a_2|$ has a finite lower bound for $a_i \in A$ and $b_i \in B$. It follows that $I(\sigma)$ always exists, either as $+\infty$ or a finite number.

To be specific, we see from (3) that if $\sigma \in \mathcal{S}(R)$ and $\infty \notin \operatorname{supp} \sigma$, then

$$(6) \qquad I(\sigma) = \iint \log \frac{1}{|x - y|} \, d\sigma(x) \, d\sigma(y).$$

Logarithmic energy integrals of this kind were studied by M. Riesz [9] and O. Frostman [1]; proofs of the following lemma may be found in these references.

Lemma 3. *If σ is any Borel measure of compact support in* **C** *satisfying $\int d\sigma = 0$, and the energy integral $J(\sigma) = \iint \log (1/|x - y|) \, d\sigma(x) \, d\sigma(y)$ is absolutely convergent, then $J(\sigma) \geqq 0$. Moreover, $J(\sigma) = 0$ if and only if $\sigma = 0$.*

If we combine this result and (6), we see that for $\sigma \in \mathcal{S}(R)$ we have $0 < I(\sigma) \leqq \infty$.

We now define the "transfinite modulus" of the condenser R as

$$(7) \qquad \operatorname{md} R = \inf_{\sigma \in \mathcal{S}(R)} I(\sigma).$$

The fundamental result of this paper, given in Theorem 3, states that $\operatorname{md} R = \operatorname{mod} R$ for any condenser R.

Lemma 4. *If $\operatorname{md} R < \infty$, there exists a measure $\tau \in \mathcal{S}(R)$ such that $I(\tau) = \operatorname{md} R$.*

Proof. By the conformal invariance we may assume that both A and B lie in the disc $U_{1/2}$. We let $\operatorname{md} R = \lim_n I(\sigma_n)$, where $\sigma_n \in \mathcal{S}(R)$. According to

[10, p. 257], we may choose a subsequence of $\{\sigma_n\}$, which we relabel $\{\sigma_n\}$, and a limit measure $\tau \varepsilon \mathcal{S}(R)$, such that $\sigma_n \to \tau$ and $\sigma_n \times \sigma_n \to \tau \times \tau$ in the weak-star topologies.

Now $\log (1/|x - y|)$ is continuous for $x \varepsilon A$ and $y \varepsilon B$, so

$$\lim \int_B \int_A \log \frac{1}{|x - y|} \, d\sigma_n(x) \, d\sigma_n(y) = \int_B \int_A \log \frac{1}{|x - y|} \, d\tau(x) \, d\tau(y).$$

As in [13, p. 55], we now have

$$\begin{aligned} \text{md}\, R &= \lim_n \left(\int_A \int_A + \int_B \int_B + \int_A \int_B + \int_B \int_A \right) \log \frac{1}{|x - y|} \, d\sigma_n(x) \, d\sigma_n(y) \\ &\geqq \left(\int_A \int_A + \int_B \int_B + \int_A \int_B + \int_B \int_A \right) \log \frac{1}{|x - y|} \, d\tau(x) \, d\tau(y) \\ &= I(\tau) \geqq \text{md}\, R, \end{aligned}$$

and so $I(\tau) = \text{md}\, R$.

Lemma 5. *If $\{R_n\}$ forms a monotone exhaustion of the condenser R, then* $\lim_n \text{md}\, R_n = \text{md}\, R$.

Proof. Again we may assume that A and B lie inside the disc $U_{1/2}$. Since $R_n \prec R$, we know $\text{md}\, R_n \leqq \text{md}\, R$, and therefore $\overline{\lim}\, \text{md}\, R_n \leqq \text{md}\, R$.

To prove the complementary inequality, it is necessary to consider separately the cases $\text{md}\, R < \infty$ and $\text{md}\, R = \infty$; we will only consider the first case, since the second is similar.

Since $\text{md}\, R < \infty$, we have $\text{md}\, R_n < \infty$ for each n, and we may select measures $\tau_n \varepsilon \mathcal{S}(R_n)$ such that $\text{md}\, R_n = \iint \log (1/|x - y|) \, d\tau_n(x) \, d\tau_n(y)$. Arguing as in Lemma 4, we find a subsequence $\{\tau_n\}$ and a limit distribution τ such that $\tau_n \to \tau$ and $\tau_n \times \tau_n \to \tau \times \tau$ weakly. It follows that $\underline{\lim}\, \text{md}\, R_n = \underline{\lim}\, I(\tau_n) \geqq I(\tau) \geqq \text{md}\, R$, and the proof is complete.

We now investigate the potential function which arises from the mass distribution τ. In studying potentials we again use a formulation which is invariant under Möbius transformations.

For any $\sigma \varepsilon \mathcal{S}(R)$ and $z \varepsilon \mathbf{P}$ we define the potential of σ with respect to z as the function

$$u_{\sigma,z}(x) = \int_B \int_A \log |x, a, z, b| \, d\sigma(a) \, d\sigma(b). \tag{8}$$

The right-hand side of (8) is understood to mean the negative of the integral of $\log |x, a, z, b|$ with respect to the product measure $\sigma \times (-\sigma)$. It will be shown in Theorem 1 that this integral always exists, if $\sigma = \tau$.

The next two lemmas follow easily from the definition of the absolute ratio.

Lemma 6. *Let (R, A, B) be any condenser in $\mathbf{P}$, and let $z \varepsilon \mathbf{P}$. Suppose that $\sigma \varepsilon \mathcal{S}(R)$ has no point masses, and that $u_{\sigma,z}(x)$ exists and is finite in a set E containing* *the point y. Then* $u_{\sigma,y}(x)$ *exists and is finite for every point in E, and*

$$u_{\sigma,z}(x) - u_{\sigma,z}(y) = u_{\sigma,y}(x), \quad \text{if} \quad x \in E.$$

Lemma 7. Let $\sigma \in \mathcal{S}(R)$ *and* $\infty \notin \operatorname{supp} \sigma$. *Then* $u_{\sigma,\infty}(\infty) = 0$; *for* $x \in \mathbf{C}$, $u(x) \equiv u_{\sigma,\infty}(x)$ *exists (as* $+\infty$, $-\infty$, *or a finite number) and*

$$u(x) = \int \log \frac{1}{|x - y|} \, d\sigma(y).$$

In order to establish the following results, we must use some celebrated techniques of potential theory. In Lemma 8 we present an analogue of Maria's maximum principle [7], and in Theorem 1 we employ the method of balayage, as presented in [1].

Lemma 8. Let (R, A, B) be a condenser in **P**. *Suppose* $\sigma \in \mathcal{S}(R)$ *and* $\infty \in R$. *Suppose* $u = u_{\sigma,\infty}$ *has the finite bounds*

$$u(x) \leqq V_A \quad \text{if} \quad x \in A \cap \operatorname{supp} \sigma,$$

and

$$u(x) \geqq V_B \quad \text{if} \quad x \in B \cap \operatorname{supp} \sigma.$$

Then $V_A \geqq 0$, $V_B \leqq 0$, *and*

$$V_B \leqq u(x) \leqq V_A \quad \text{for all} \quad x \in \mathbf{P}.$$

Proof. We set $A_\sigma = A \cap \operatorname{supp} \sigma$ and $B_\sigma = B \cap \operatorname{supp} \sigma$, and observe that according to the preceding lemma, u is superharmonic in $\sim B_\sigma$ and subharmonic in $\sim A_\sigma$. Therefore, to prove the bounds on $u(x)$ it suffices to show that $\overline{\lim}\, u(x) \leqq V_A$ as $x \in \sim A_\sigma$ approaches ∂A_σ, and $\underline{\lim}\, u(x) \geqq V_B$ as $x \in \sim B_\sigma$ approaches ∂B_σ. It will follow by setting $x = \infty$ that $V_B \leqq 0 \leqq V_A$.

Fix $a_0 \in \partial A_\sigma$. Since $u(x) \leqq V_A$ for $x \in A_\sigma$, it is clear that in A_σ, σ has no point masses; hence for any $\epsilon > 0$ we can choose $\delta > 0$ so small that $\sigma(A_\delta) < \epsilon$, where $A_\delta = A_\sigma \cap \overline{U_\delta(a_0)}$. Let $x \in U_\delta(a_0) - A_\sigma$. Select $a_1 \in A_\delta$ so that $|x - a_1| \leqq |x - a|$ for any $a \in A_\delta$. Then $|a_1 - a| \leqq |x - a_1| + |x - a| \leqq 2\,|x - a|$ for any $a \in A_\delta$. Thus for x sufficiently near a_0 we have

$$\begin{aligned}
&\int_{A_\delta} \log \frac{1}{|x - a|} \, d\sigma(a) \\
&\qquad \leqq \sigma(A_\delta) \log 2 + \int_{A_\delta} \log \frac{1}{|a_1 - a|} \, d\sigma(a) \\
&\qquad \leqq \epsilon \log 2 + \int_{\mathbf{C}} \log \frac{1}{|a_1 - a|} \, d\sigma(a) - \int_{\mathbf{C}-A_\delta} \log \frac{1}{|a_1 - a|} \, d\sigma(a) \\
&\qquad \leqq \epsilon \log 2 + V_A - \left[\int_{\mathbf{C}-A_\delta} \log \frac{1}{|x - a|} \, d\sigma(a) - \epsilon \right]
\end{aligned}$$

or $u(x) \leqq \epsilon \log 2 + \epsilon + V_A$. Since ϵ was arbitrary, we have $\overline{\lim}\, u(x) \leqq V_A$ as $x \in \sim A_\sigma$ approaches $a_0 \in \partial A_\sigma$.

A similar proof shows that $\underline{\lim}\ u(x) \geqq V_B$ as $x \in \sim B_\sigma$ approaches ∂B_σ, so the lemma is established.

Theorem 1. *Let (R, A, B) be any condenser in* $\mathbf{P}$ *for which* $\text{md } R = I(\tau) < \infty$. *Let $u = u_{\tau,\infty}$ be the equilibrium potential, given by (8). Then $u(x)$ exists and is finite for all $x \in \mathbf{P}$. In fact, there exist finite constants $V_A \geqq 0$ and $V_B \leqq 0$ such that*

i) $V_A - V_B = \text{md } R$,

ii) $V_B \leqq u(x) \leqq V_A$ *for all* $x \in \mathbf{P}$,

iii) $u(x) = V_A$ *in A, except for a set of zero capacity,*

iv) $u(x) = V_B$ *in B, except for a set of zero capacity.*

Moreover, u is harmonic everywhere in R.

Proof. First we consider the case $A \cup B \subset U_{1/2}$. We define $V_A = \int_A u\, d\tau$ and $V_B = -\int_B u\, d\tau$. Since $V_A > -\infty$, $V_B < \infty$, and $V_A - V_B = \text{md } R < \infty$, it follows that V_A and V_B are finite.

We now show that $u(x) \geqq V_A$ on A, except for a set of zero capacity. For this we set $E = \{x \in A : u(x) < V_A\}$ and suppose the logarithmic capacity $C(E) > 0$. Now if $E_n = \{x \in A : u(x) \leqq V_A - 1/n\}$, then E_n is closed, and $E = \cup E_n$; thus $C(E_n) > 0$ for all large n. Otherwise stated, there exists an $\epsilon > 0$ such that $F = \{x \in A : u(x) \leqq V_A - 2\epsilon\}$ is closed and $C(F) > 0$.

However, we have $V_A = \int_A u\, d\tau$, so there exists $a_0 \in A \cap \text{supp } \tau$ such that $u(a_0) > V_A - \epsilon$. Since u is lower semicontinuous in A, $u(x) > V_A - \epsilon$ in a neighborhood N of a_0; and clearly we can arrange the choice of N so that $d(N, B \cup F) > 0$. We set $\tau(N) = m > 0$.

Since $C(F) > 0$, we can find a positive measure ψ with support on F such that $I(\psi) \neq \infty$ and $\psi(F) = m > 0$. Now we define a measure σ with support in $A \cup B$ as follows: $\sigma = \psi$ on F, $\sigma = -\tau$ on N, and $\sigma = 0$ elsewhere. It is clear that $I(\sigma) < \infty$. For $0 < h < 1$, $\tau + h\sigma \in \mathcal{S}(R)$, so

$$
\begin{aligned}
0 \leqq I(\tau + h\sigma) - I(\tau) &= 2h \int u\, d\sigma + h^2 I(\sigma) \\
&< 2h[(V_A - 2\epsilon)m - (V_A - \epsilon)m] + h^2 I(\sigma) \\
&= -h[2m\epsilon - hI(\sigma)].
\end{aligned}
$$

Since this last expression is negative for small h, we have obtained a contradiction. Therefore $u(x) \geqq V_A$ on A, except for a set of zero capacity.

A similar proof shows that $u(x) \leqq V_B$ on B, except for a set of zero capacity.

We next show that $u(x) \leqq V_A$ for $x \in A \cap \text{supp } \tau$. If in fact $u(a_1) > V_A$, where $a_1 \in A \cap \text{supp } \tau$, then by the lower semicontinuity of u in A there exist a number $\epsilon > 0$ and a neighborhood N_1 of a_1 such that $u(x) > V_A + \epsilon$ in N_1; again we may require that $d(N_1, B) > 0$. Now $\tau(N_1) > 0$, so

$$
V_A = \int_{A \cap N_1} u\, d\tau + \int_{A - N_1} u\, d\tau \geqq (V_A + \epsilon)\tau(N_1) + V_A(1 - \tau(N_1)) > V_A,
$$

which is impossible. The proof that $u(x) \geqq V_B$ in $B \cap \text{supp } \tau$ is similar.

It now follows from our form of Maria's maximum principle that u satisfies $V_A \leqq u(x) \leqq V_B$ everywhere in $\mathbf{P}$, $V_A \geqq 0$, and $V_B \leqq 0$. Hence the theorem has been proved in the case $\infty \in R$.

If (R, A, B) is an arbitrary condenser with md $R < \infty$, and $z \in R$ is arbitrary, it follows from what we have already proved that $u_{\tau,z}$ exists everywhere and is harmonic on R; moreover, there exist constants $V_{A,z} \geqq 0$ and $V_{B,z} \leqq 0$ such that $V_{A,z} - V_{B,z} = \text{md } R$ and $V_{B,z} \leqq u_{\tau,z}(x) \leqq V_{A,z}$, with respective equality on the sets B and A, except for sets of zero logarithmic capacity.

It follows from Lemma 6 that $u = u_{\tau,\infty}$ exists everywhere and is harmonic in R; moreover,

$$V_{B,z} - u_{\tau,z}(\infty) \leqq u(x) \leqq V_{A,z} - u_{\tau,z}(\infty),$$

with respective equality on the sets B and A, except for sets of zero logarithmic capacity. Finally, if we set $V_A = V_{A,z} - u_{\tau,z}(\infty)$ and $V_B = V_{B,z} - u_{\tau,z}(\infty)$, it is clear that $V_A \geqq 0$, $V_B \leqq 0$, and $V_A - V_B = \text{md } R$, and the proof is complete.

With the aid of this theorem, we may show that the measure τ of Lemma 4 is unique. In fact, if md $R = I(\tau) = I(\tau') < \infty$, and $A \cup B \subset U_{1/2}$, then $J(\tau - \tau')$ is an absolutely convergent integral with value $I(\tau) + I(\tau') - 2 \int u_{\tau,\infty}\, d\tau' = 0$. It follows from Lemma 3 that $\tau = \tau'$.

Theorem 2. *Let (R, A, B) be a condenser in* $\mathbf{P}$ *with* md $R = I(\tau) < \infty$, *satisfying any one of the following conditions*:

i) *the Dirichlet problem is solvable in R,*
ii) *R is a ring,*
iii) *R is regular.*

Then $u = u_{\tau,\infty}$ is continuous everywhere and

$$u(x) = V_A \quad \text{if} \quad x \in A, \qquad u(x) = V_B \quad \text{if} \quad x \in B. \tag{9}$$

Proof. Since u is known to be finite everywhere, it follows from Lemma 6 that we may prove the therorem under the additional assumption that $\infty \in R$.

We begin by showing that whenever (9) holds, u is continuous everywhere. We know that u is continuous in R. If $a \in A$, then u is lower semicontinuous at a, so by Theorem 1 we have $\overline{\lim}\, u(x) \leqq V_A = u(a) \leqq \underline{\lim}\, u(x)$ as $x \to a$. Thus u is continuous at points of A, and a similar argument shows that u is continuous at points of B.

Now we prove that (9) holds under any of the hypotheses i), ii), iii).

If i) holds, there exists a harmonic function v in R with boundary values V_A at A and V_B at B. Now $v - u$ is bounded and harmonic in R; from the last theorem we see that $v - u$ has zero boundary values, except for a set of zero logarithmic capacity on ∂R. It follows from a well-known theorem [13, p. 77] that $v - u \equiv 0$ in R.

To prove (9), we let $a_0 \in A$. According to the preceding paragraph, for each

$\epsilon > 0$ we can choose $\delta > 0$ so that $u(x) > V_A - \epsilon$ for $x \in U_\delta(a_0) \cap R$; according to Theorem 1, the same inequality holds for $x \in U = U_\delta(a_0)$, except for a set of Lebesgue measure zero. Since u is superharmonic at a_0, $u(a_0) \geqq \lambda(U)^{-1} \int_U u\, d\lambda \geqq V_A - \epsilon$. Since ϵ was arbitrary, we see that $u(a_0) \geqq V_A$, and therefore $u(a_0) = V_A$. A similar proof shows that $u(b_0) = V_B$ for $b_0 \in B$.

If ii) or iii) holds, it is known that the Dirichlet problem for R is solvable [13, p. 7], and hence we can apply part i). This completes the proof.

Even if the condenser does not satisfy any of the conditions of this theorem, the superharmonic and subharmonic inequalities show that u is continuous at all *interior* points of $A \cup B$. We can in fact show that $\overline{\operatorname{supp} \tau} \subset \partial(A \cup B)$. Suppose, for example, that $a \in A \cap \operatorname{supp} \tau$, where $\bar{U} = \overline{U_r(a)} \subset \operatorname{int} A$. If μ is the uniform distribution of mass 1 on the circumference $S_r(a)$, and v is the logarithmic potential of μ, then

$$V_A = \int u\, d\mu = \int v\, d\tau \leqq V_A - \int_{\bar{U}} \log \frac{1}{|x - a|}\, d\tau(x) + \log \frac{1}{r}\, \tau(\bar{U}) < V_A,$$

a contradiction.

Theorem 3 (*Fundamental theorem*). *If* (R, A, B) *is any condenser in* **P**, $\operatorname{mod} R = \operatorname{md} R$.

Proof. By conformal invariance, we may assume that $\infty \in B$. Let $\{(R_n, A_n, B_n)\}$ be the exhaustion of R described in Lemma 2. Let $\operatorname{md} R_n = I(\tau_n)$; according to the last theorem, the potential function $u_n(x) = \int \log (1/|x - y|)\, d\tau_n(y)$ has constant boundary values $V_{A,n}$ and $V_{B,n}$ on the sets A_n and B_n. Then the function $v_n(x) = (\operatorname{md} R_n)^{-1}(u_n(x) - V_{B,n})$ has boundary values 0 and 1 on ∂R_n, and is harmonic in R_n. If the level curves $v_n = c_0$ and $v_n = c_1$ bound the region $R_{n'}$, and if the constants c_0, c_1 are near 0, 1, then

$$\begin{aligned}\int_{R_n'} |\nabla v_n|^2\, d\lambda &= \int_{\partial R_n'} v_n \frac{\partial v_n}{\partial \nu}\, ds \\ &= \frac{(c_1 - c_0)}{\operatorname{md} R_n} \int_{v_n = c_1} \frac{\partial u_n}{\partial \nu}\, ds \\ &= \frac{(c_1 - c_0)}{\operatorname{md} R_n} \int_{A_n} (2\pi)\, d\tau_n \\ &= \frac{(c_1 - c_0)2\pi}{\operatorname{md} R_n}.\end{aligned}$$

If we let $c_0 \to 0$ and $c_1 \to 1$, and apply the monotone convergence theorem, we get $\operatorname{mod} R_n = \operatorname{md} R_n$. Finally, we conclude from Lemmas 1 and 5 that $\operatorname{mod} R = \operatorname{md} R$.

From the fundamental theorem we can easily obtain precise relationships between the modulus of a condenser and the logarithmic Robin constants of its "plates." For example, we have the following form of the theorem on the "reduced modulus" [5, p. 3].

Theorem 4. *If $A \subset \mathbf{C}$ is compact and $\sim A$ is connected,*

$$\lim_{t\to\infty} [\mathrm{mod}\,(U_t - A) - \log t] = V(A).$$

Proof. Let $A \subset U_r$. We conclude immediately from Theorem 3 that for $t > r$,

$$\mathrm{mod}\,(U_t - A) \geqq V(A) + \log(1/t) + 2\log(t - r),$$
$$\mathrm{mod}\,(U_t - A) \leqq V(A) + \log(1/t) + 2\log(t + r),$$

since it is known [13, p. 84] that $V(S_t) = \log(1/t)$. We conclude that

$$V(A) + 2\log(t - r)/t \leqq \mathrm{mod}\,(U_t - A) - \log t \leqq V(A) + 2\log(t + r)/t,$$

and the theorem follows immediately.

From Theorem 4 we may deduce an upper bound for mod R which is simple and useful: for any condenser of the form $(R, A, \sim U_1)$,

$$\mathrm{mod}\, R \leqq V(A). \tag{10}$$

In fact,

$$\begin{aligned} V(A) &= \lim_{t\to\infty} [\mathrm{mod}\,(U_t - A) - \log t] \\ &\geqq \lim_{t\to\infty} [\mathrm{mod}\,(U_t - U_1) + \mathrm{mod}\,(U_1 - A) - \log t] \\ &= \mathrm{mod}\, R. \end{aligned}$$

4. The discrete form of the fundamental theorem. It is possible to give a discrete formulation for the transfinite modulus of a condenser (R, A, B), by using the techniques of Pólya and Szegö [8]. To avoid degenerate cases, suppose in this section that A and B are infinite sets. Among all choices of points $a_1, \cdots, a_n \in A$ and $b_1, \cdots, b_n \in B$ let

$$W_n = \inf \binom{n}{2}^{-1} \sum_{1\leq i<j\leq n} \log |a_i, b_j, b_i, a_j|. \tag{11}$$

It is not hard to see that the sequence $\{W_n\}$ is monotonic. To prove this we select points $a_1, \cdots, a_n \in A$ and $b_1, \cdots, b_n \in B$ with

$$\binom{n}{2} W_n = \sum_{i<j} \log |a_i, b_j, b_i, a_j|.$$

Then for each fixed integer l, $1 \leqq l \leqq n$,

$$\binom{n}{2} W_n \geqq \sum_{\substack{1\leq j\leq n \\ j\neq l}} \log |a_l, b_j, b_l, a_j| + \binom{n-1}{2} W_{n-1}.$$

If we add these inequalities for $1 \leqq l \leqq n$, we get

$$n\binom{n}{2}W_n \geqq 2\binom{n}{2}W_n + n\binom{n-1}{2}W_{n-1},$$

which reduces to $W_n \geqq W_{n-1}$.

It follows that the sequence $\{W_n\}$ has a limit $W(R)$, which is either a finite number or $+\infty$.

Theorem 5. We have mod $R = W(R)$.

Proof. By the invariance under Möbius transformations, we may assume that A and B are bounded.

1) $W(R) \leqq$ md R. Since this inequality is obvious if md $R = \infty$, we may assume that md $R < \infty$. Now for any points $a_1, \cdots, a_n \varepsilon A$ and $b_1, \cdots, b_n \varepsilon B$ we have

$$\binom{n}{2}W_n \leqq \sum_{i<j} \log \frac{1}{|a_i - a_j|} + \sum_{i<j} \log \frac{1}{|b_i - b_j|} - \sum_{i \neq j} \log \frac{1}{|a_i - b_j|}.$$

The sense of the inequality is not changed if we integrate with respect to any unit positive measures μ, ν. In particular, if md $R = I(\tau)$, $\mu = \tau \mid A$, and $\nu = (-\tau) \mid B$,

$$\int \cdots \int \binom{n}{2} W_n \, d\mu(a_1) \cdots d\mu(a_n) \, d\nu(b_1) \cdots d\nu(b_n)$$
$$\leqq \sum_{i<j} \int \cdots \int \log \frac{1}{|a_i - a_j|} \, d\mu(a_1) \cdots d\mu(a_n) \, d\nu(b_1) \cdots d\nu(b_n) + \cdots$$

or

$$\binom{n}{2}W_n \leqq \sum_{i<j} \int_A \int_A \log \frac{1}{|a_1 - a_2|} \, d\mu(a_1) \, d\mu(a_2)$$
$$+ \sum_{i<j} \int_B \int_B \log \frac{1}{|b_1 - b_2|} \, d\nu(b_1) \, d\nu(b_2) - \sum_{i \neq j} \int_B \int_A \log \frac{1}{|a - b|} \, d\mu(a) \, d\nu(b).$$

Therefore $W_n \leqq I(\tau) =$ md R, which proves 1).

2) md $R \geqq W(R)$. From the construction given in the proof of Lemma 2, it is clear that we can find a monotone exhaustion $\{(R_m, A_m, B_m)\}$ of (R, A, B) with the following properties: for each m, A_m and B_m are closed domains, and $d(\partial R_m, A \cup B) > 0$. We set $\delta = d(A_1, B_1)$, and for each m, select $n = n(m)$ so that $(\pi n)^{-\frac{1}{2}} < d(\partial R_m, A \cup B)$.

Now let $a_1', \cdots, a_n' \varepsilon A$ and $b_1', \cdots, b_n' \varepsilon B$ be points at which the infimum in (11) is assumed, and set

$$W_n' = \binom{n}{2}^{-1}\left[\sum_{i<j} \log \frac{1}{|a_i' - a_j'|} + \sum_{i<j} \log \frac{1}{|b_i' - b_j'|} - \sum_{i,j} \log \frac{1}{|a_i' - b_j'|}\right]. \tag{12}$$

Since $\lim_n (W_n' - W_n) = 0$, we have $\lim_n W_n' = W(R)$. We let A_i', B_j' be discs of radius $\epsilon = (\pi n)^{-\frac{1}{2}}$ centered about a_i', b_j', and define the density function

$\rho(x) = \sum \rho_i(x)$, where

$$\rho_i(x) = \begin{cases} +1 & \text{if} \quad x \in A_i', \\ -1 & \text{if} \quad x \in B_i', \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\sigma_n \equiv \rho\lambda \in \mathcal{S}(R_m)$.

As in the earlier proofs, it is now necessary to write $I(\sigma_n)$ as the sum of four double integrals, and to estimate each double integral separately. To this end, we note first that $\log (1/|x - y|)$ is superharmonic in x, for fixed y. Thus

$$\int_{A_i'} \log \frac{1}{|x-y|}\, d\lambda(x) \leqq \frac{1}{n} \log \frac{1}{|y - a_i'|}$$

and hence

$$\sum_{i,j} \int_{A_j'} \int_{A_i'} \log \frac{1}{|x-y|}\, d\lambda(x)\, d\lambda(y) \leqq \frac{1}{n} \sum_{i,j} \int_{A_j'} \log \frac{1}{|y - a_i'|}\, d\lambda(y).$$

The integrals in the summation on the right may be estimated by a similar argument, if $i \neq j$, since $\log (1/|y - a_i'|)$ is superharmonic. And by direct computation, each of the n integrals occurring for $i = j$ has the value $\pi\epsilon^2/2 - \pi\epsilon^2 \log \epsilon = O((\log n)/n)$. Therefore

$$(13) \qquad \int_{A_m} \int_{A_m} \log \frac{1}{|x-y|}\, d\sigma_n(x)\, d\sigma_n(y) \leqq \frac{1}{n^2} \sum_{i \neq j} \log \frac{1}{|a_i' - a_j'|} + O\left(\frac{\log n}{n}\right).$$

Similarly,

$$(14) \qquad \int_{B_m} \int_{B_m} \log \frac{1}{|x-y|}\, d\sigma_n(x)\, d\sigma_n(y) \leqq \frac{1}{n^2} \sum_{i \neq j} \log \frac{1}{|b_i' - b_j'|} + O\left(\frac{\log n}{n}\right).$$

Next we observe that for $x \in B_j'$ and $y \in A_m$ we have

$$1/|x - y| \geqq (1/|y - b_j'|)(\delta/(\delta + \epsilon)).$$

Hence

$$\sum_{i,j} \int_{A_i'} \int_{B_j'} \log \frac{1}{|x-y|}\, d\lambda(x)\, d\lambda(y) \geqq \frac{1}{n} \sum_{i,j} \int_{A_i'} \left[\log \frac{1}{|y - b_j'|} + \log \frac{\delta}{\delta + \epsilon}\right] d\lambda(y).$$

The integrals in the summation on the right may be estimated by a similar argument, since $1/|y - b_j'| \geqq (1/|a_i' - b_j'|)(\delta/(\delta + \epsilon))$ for $y \in A_i'$. Therefore

$$(15) \quad -\int_{A_m} \int_{B_m} \log \frac{1}{|x-y|}\, d\sigma_n(x)\, d\sigma_n(y) \geqq \frac{1}{n^2} \sum_{i,j} \log \frac{1}{|a_i' - b_j'|} + 2 \log \frac{\delta}{\delta + \epsilon}.$$

Adding estimates (13), (14) and (15), we obtain

$$\begin{aligned}\operatorname{md} R_m &\leqq I(\sigma_n)\\ &\leqq \frac{2}{n^2}\binom{n}{2}W_n' - 4\log\frac{\delta}{\delta+\epsilon} + O\left(\frac{\log n}{n}\right).\end{aligned}$$

Now as $m \to \infty$, clearly $n \to \infty$ and $\epsilon \to 0$, so $\operatorname{md} R = \lim \operatorname{md} R_m \leqq W(R)$.

5. Symmetric condensers. We now consider condensers of the form (R, A, B), where A lies in the upper half-plane D, and $B = \tilde{A}$ is the reflection of A in ∂D; condensers of this form are called *symmetric* condensers.

If R is a symmetric condenser for which $\operatorname{md} R = I(\tau) < \infty$, we will prove that the measure τ is likewise symmetric; that is, if E is any Borel set, and $\tilde{E}$ is the reflection of E in ∂D, then $\tau(E) = -\tau(\tilde{E})$. From this we show that Tsuji's representation for the modulus of a ring in terms of Green's functions is a special case of our fundamental theorem.

Lemma 9. *Let (R, A, B) be a symmetric condenser for which* $\operatorname{md} R = I(\tau) < \infty$. *Then the measure τ is symmetric.*

Proof. We define a new measure $\tilde{\tau} \in \mathcal{S}(R)$ as follows: if E is any Borel set and $\tilde{E}$ is the reflection of E in the real axis, let $\tilde{\tau}(E) = -\tau(\tilde{E})$. Since the condenser R is symmetric, we have

$$\begin{aligned} I(\tilde{\tau}) &= \iint \log\frac{1}{|x-y|}\, d\tau(\bar{x})\, d\tau(\bar{y})\\ &= \iint \log\frac{1}{|\bar{x}-\bar{y}|}\, d\tau(x)\, d\tau(y)\\ &= I(\tau).\end{aligned}$$

It follows from the uniqueness of τ that $\tau = \tilde{\tau}$, and therefore τ is symmetric.

This result shows that in the minimum problem $I(\tau) = \min \{I(\sigma) : \sigma \in \mathcal{S}(R)\}$ we may restrict ourselves to those $\sigma \in \mathcal{S}(R)$ which are symmetric.

We now recall that the Green's function for the upper half-plane D is

$$G(x, y) = \log\frac{|x-\bar{y}|}{|x-y|}.$$

If (R, A, B) is a symmetric condenser, and $\sigma \in \mathcal{S}(R)$ is a symmetric measure, then

$$I(\sigma) = 2\int_A\int_A G(x, y)\, d\sigma(x)\, d\sigma(y), \tag{16}$$

and for $x \in D$,

$$u_{\sigma,\infty}(x) = \int_A G(x, y)\, d\sigma(y). \tag{17}$$

Theorem 6 (*Tsuji* [12]). *Let (R, A, B) be any ring in* P *with* $\operatorname{mod} R < \infty$. *Let $G(x, y)$ denote the Green's function for $\sim B$, and let $\mathfrak{M}$ be the family of unit* *positive measures μ with support in A. Then*

$$\inf_{\mu \in \mathfrak{M}} \int_A \int_A G(x, y)\, d\mu(x)\, d\mu(y) = \operatorname{mod} R,$$

and there exists a unique $\nu \in \mathfrak{M}$ for which the infimum is attained. The potential $w(x) = \int_A G(x, y)\, d\nu(y)$ is continuous everywhere in $\sim B$, and has the constant value mod R *on A.*

Proof. If $\sim B$ is the upper half-plane, the theorem follows from applying the results of Section 3 to the symmetric condenser $(R^+, A, \tilde{A})$, using formulas (16) and (17) for $I(\sigma)$ and $u_{\sigma,\infty}$. For arbitrary rings, the theorem is an immediate consequence of this special case and the Riemann mapping theorem.

6. The Grötzsch and Teichmüller rings. We now apply our fundamental theorem to prove the extremal properties for the Grötzsch and Teichmüller rings. Several proofs of these results are known, but the ones we give here have a particular geometric appeal.

For $0 < p < 1$ the Grötzsch ring $R_G(p)$ consists of the unit disc minus the segment $\{(x^1, x^2) : 0 \leqq x^1 \leqq p, x^2 = 0\}$. For $q > 0$ the Teichmüller ring $R_T(q)$ consists of the finite plane minus the segment $\{(x^1, x^2) : -1 \leqq x^1 \leqq 0, x^2 = 0\}$ and the ray $\{(x^1, x^2) : q \leqq x^1 < \infty, x^2 = 0\}$.

Theorem 7 (Teichmüller [11]). *Let (R, A, B) be a ring for which $(-1, 0)$, $(0, 0) \in A$ and $z, \infty \in B$. Then*

$$\operatorname{mod} R \leqq \operatorname{mod} R_T(|z|).$$

Proof. Let $\epsilon > 0$ be arbitrary, and select distinct finite points $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ in the complement of $R_T(|z|)$ so that

$$\binom{n}{2}^{-1}\left[\sum_{i<j} \log \frac{1}{|a_i - a_j|} + \sum_{i<j} \log \frac{1}{|b_i - b_j|} - \sum_{i \neq j} \log \frac{1}{|a_i - b_j|}\right] < \operatorname{mod} R_T(|z|) + \epsilon.$$

Let A intersect the circle $S_{|a_i|}$ in a_i' and let B intersect $S_{|b_i|}$ in b_i'. Since points lying on two concentric circles assume their greatest and smallest distances when they lie on the same diameter, we have $|a_i' - a_j'| > |a_i - a_j|$, $|b_i' - b_j'| > |b_i - b_j|$ and $|a_i' - b_j'| < |a_i - b_j|$. Therefore

$$\begin{aligned}\operatorname{mod} R &\leqq \binom{n}{2}^{-1}\left[\sum \log \frac{1}{|a_i' - a_j'|} + \sum \log \frac{1}{|b_i' - b_j'|} - \sum \log \frac{1}{|a_i' - b_j'|}\right] \\ &\leqq \binom{n}{2}^{-1}\left[\sum \log \frac{1}{|a_i - a_j|} + \sum \log \frac{1}{|b_i - b_j|} - \sum \log \frac{1}{|a_i - b_j|}\right] \\ &\leqq \operatorname{mod} R_T(|z|) + \epsilon.\end{aligned}$$

The theorem follows if we let $\epsilon \to 0$.

The extremal property for the Grötzsch ring can be deduced from the corresponding property for the Teichmüller ring, by making the following observation. Let x, y, z be points in **C**, with y lying on the line segment from x to z. Let (R_1, A_1, B_1) be a ring for which $x, y \varepsilon A_1$ and $z, \infty \varepsilon B_1$; and let (R_2, A_2, B_2) be the ring for which A_2 is the line segment from x to y and B_2 is the parallel ray from z to ∞. Then mod $R_1 \leqq$ mod R_2.

Theorem 8 (*Grötzsch* [3]). *Let* (R, A, B) *be a ring for which* A *contains the points* z *and* $(0, 0)$, *and* $B = \sim U_1$. *Then*

$$\operatorname{mod} R \leqq \operatorname{mod} R_G(|z|).$$

Proof. We consider the ring (R^+, A, A^*), where A^* is the reflection of A in the unit circumference S_1. We define the ring $R_G^+(|z|)$ in a similar way. By the above remark,

$$\operatorname{mod} R = (\operatorname{mod} R^+)/2 \leqq (\operatorname{mod} R_G^+(|z|))/2 = \operatorname{mod} R_G(|z|),$$

which completes the proof.

Often this result is combined with an upper bound for the modulus of the Grötzsch ring. Bounds of this sort may be deduced from the general estimate (10). If (10) is applied directly to the Grötzsch ring, we obtain mod $R_G(p) \leqq \log(4/p)$, the usual estimate [5, p. 7; 6, p. 64], and slightly better upper bounds may be obtained by applying (10) to other rings which are conformally equivalent to $R_G(p)$.

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