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THE MOMENTS OF THE MAXIMUM OF NORMALLY DISTRIBUTED DEPENDENT VALUES

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Abstract. In this paper the moments of the maximum of a finite number of random values are analyzed. The largest part of analysis is focused on extremes of dependent normal values. For the case of normal distribution, the moments of the maximum of dependent values are expressed through the moments of independent values.

Keywords: dependent random variables, extreme values, moments, normal distribution.

1. Introduction

Suppose $(X_1,...,X_n)$ is an n-dimensional normal vector: $(X_1,...,X_n) \sim N(\mu, V_x)$, where

$$V_X = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_n\rho_{1n} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 & \cdots & \sigma_2\sigma_n\rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_n\sigma_1\rho_{n1} & \sigma_n\sigma_2\rho_{n2} & \cdots & \sigma_n^2 \end{pmatrix}.$$

We form random variable

$$Z_n = \max(X_1,...,X_n).$$

We will focus on calculation of moments of this random variable

$$EZ_n^k = \int_{-\infty}^{\infty} x^m dF_{Z_n}(x).$$

Here $F_X(x) = P(X \le x)$ is a probability distribution function.

There is a number of publications ([1], [2], [3], [4]) for the case of normal vector $(X_1,...,X_n)$ with independent components X_j, $j=\overline{1,n}$. In the case $X_j \sim N(0,1)$, the moments EZ_n and EZ_n^2 could be expressed using only elementary functions up to n = 5 ([5]). For $X_j \sim N(\mu, \sigma_j^2)$, there are formulas ([4]) to calculate EZ_3^{2k-1} based on parameters μ and σ_j. The case of dependent normal values $(X_1, X_2) \sim N(m_1, m_2, \sigma_1^2, \sigma_2^2, \rho)$ is analyzed in [5] and [6]. EZ_2 and EZ_2^2 are presented by distribution parameters. Expressions are proved by applying Z_2 moment generating function. As a consequence, it was shown that EZ_n^2 cannot be expressed only by elementary functions of parameters. This topic is discussed in our earlier paper [7].

2. Statements and Proofs

In this section we will represent the moments of the maximum of dependent normal values by the moments of the maximum of independent normal values. To achieve the result, the inclusion-exclusion principle (sieve method) will be used.

Theorem 1. Suppose that $(X_1,...,X_n) \sim N(0, V_x)$ where

$$V_X = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdots & \cdots & \cdots & \cdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}, \ \rho > 0$$

and $U_j \sim N(0,1)$, $j = \overline{1,n}$.

Then

$$EZ_n^m = \sum_{j=0}^{\left[\frac{m}{2}\right]} \begin{pmatrix} C_k^{2j} (2j-1)!! \rho^j (1-\rho)^{\frac{m}{2}-j} \cdot \\ \cdot E\left(\max(U_1,\ldots,U_n)\right)^{m-2j} \end{pmatrix},$$

$n \geq 1,\ \rho \geq 0.$

Proof. The following is true:

$$X_i \overset{D}{=} Y + \sqrt{1-\rho}\, U_i\,,$$

where $Y \sim N(0,\rho)$ and is independent of U_i, $i = \overline{1,n}$.

Then

$$Z_n \overset{D}{=} Y + \sqrt{1-\rho}\max(U_1,...,U_n)$$

and

$$EZ_n^m = \sum_{k=0}^{m}\left(C_m^k\, EY^k (1-\rho)^{\frac{m-k}{2}} E\left(\max(U_1,\ldots,U_n)\right)^{m-k}\right).$$

Using the fact:

$$EY^k = \begin{cases} 0, \text{ for } k = 2i+1; \\ (k-1)!!\,\rho^{\frac{k}{2}}, \ k = 2i; \end{cases}$$

we conclude the proof of the theorem.

Corollary 1. Suppose that $(X_1,...,X_n) \sim N(\mu,V_x)$ where

$$V_X = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdots & \cdots & \cdots & \cdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}, \ \rho > 0$$

and $U_i \sim N(0,1)$, $i = \overline{1,n}$.

Then

$$EZ_n^s = \sum_{r=0}^{s} C_s^r \mu^r \sigma^{s-r} \cdot$$

$$\cdot \sum_{j=0}^{\left[\frac{s-r}{2}\right]} \begin{pmatrix} C_{s-r}^{2j} (2j-1)!! \rho^j (1-\rho)^{\frac{s-r}{2}-j} \cdot \\ \cdot E\left(\max(U_1,\ldots,U_n)\right)^{s-r-2j} \end{pmatrix},$$

$n \geq 1,\ \rho \geq 0.$

This corollary is proved using Theorem 1 and the fact that

$$EZ_n^s = \sum_{r=0}^{s} C_s^r \mu^r \sigma^{s-r} \cdot$$

$$\cdot E\left(\max(Y_1,\ldots,Y_n)\right)^{s-r}$$

where $(Y_1,...,Y_n) \sim N(0,V_Y)$, $V_Y = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdots & \cdots & \cdots & \cdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}.$

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References

[1] **H. Ruben.** On the moments of order statistics in samples from normal populations. *Biometrika*, 1954, *No*. 41, 200 – 227.

[2] **R.S. Bose, S.S. Gupta** Moments of order statistics from a normal population. *Biometrika*, 1959, *No*. 46, 433 – 440.

[3] **H.A. David.** Order statistics. *Moscow, Nauka*, 1979 (*in Russian*).

[4] **I.N. Kovalenko, A.A. Levitskaja, M.N. Savchik.** Selected problems of combinatorial probability theory. *Kiev*, 1986, 90 – 95 (*in Russian)*.

[5] **M. Cain.** The moment-generating function of the minimum of bivariate normal random variables. T*he American Statistician, Vol.*48, *No*.2, 1994. 124–125.

[6] **A.P. Ker.** On the maximum of bivariate normal random variables. *Extremes, Vol.*4, *No*.2, 1994, 185–190.

[7] **A. Burauskaitė, A. Aksomaitis.** Moments of extremes of normally distributed values. *Lietuvos matematikos rinkinys T*. 43, *Vilnius: IMI*, 2003, 677 – 681.

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