THE MOMENT PROBLEM FOR UNIMODAL DISTRIBUTIONS

By N. L. Johnson and C. A. Rogers

University College, London

Summary. Certain inequalities are obtained for the moments of unimodal distributions.

1. Introduction. Let n be one of the numbers 3, 5, 7, $\cdots$, or $+\infty$. Let a real number μ_r be given for each integer r with $1 \leq r < n$. It is known ([1], Chap. III, Sec. 8–12; [2]) that, if there is a (cumulative) distribution function $F(x)$ such that

$$\int_{-\infty}^{\infty} x^r \, dF(x) = \mu_r, \qquad 1 \leq r < n, \tag{1}$$

then

$$\begin{vmatrix} 1 & \mu_1 & \cdots & \mu_s \\ \mu_1 & \mu_2 & \cdots & \mu_{s+1} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \mu_s & \mu_{s+1} & \cdots & \mu_{2s} \end{vmatrix} \geq 0 \tag{2}$$

for all integers s with $2 \leq 2s < n$. Conversely, it is known ([1], loc. cit.; [2]) that if (2) is satisfied with strict inequality for all integers s with $2 \leq 2s < n$, then there is a distribution function $F(x)$ satisfying (1).

We say that a distribution function $F(x)$ is *unimodal*, with mode M, if, for all real numbers $x_1, \cdots, x_4$ satisfying

$$x_1 < x_2 < M < x_3 < x_4, \tag{3}$$

we have

$$\begin{aligned} F(\tfrac{1}{2}\{x_1 + x_2\}) &\leq \tfrac{1}{2}[F(x_1) + F(x_2)], \\ F(\tfrac{1}{2}\{x_3 + x_4\}) &\geq \tfrac{1}{2}[F(x_3) + F(x_4)]. \end{aligned} \tag{4}$$

We prove the following theorem.

Theorem 1. *Let n be one of the numbers 3, 5, 7, $\cdots$, or $+\infty$. Let a real number μ_r be given for each integer r with $1 \leq r < n$. Then there is a unimodal distribution function $F(x)$ with mode zero and with*

$$\int_{-\infty}^{\infty} x^r \, dF(x) = \mu_r, \qquad 1 \leq r < n, \tag{5}$$

if and only if there is a distribution function $G(x)$ such that

$$\int_{-\infty}^{\infty} x^r \, dG(x) = (r+1)\mu_r, \qquad 1 \leq r < n. \tag{6}$$

By use of the special cases $n = 3$ and $n = 5$ of this theorem we obtain the following results.

THEOREM 2. *Let m, M and σ be real numbers, σ being positive. Then there will be a unimodal distribution function with mean m, mode M, and standard deviation σ if and only if*

$$(m - M)^2 \leq 3\sigma^2. \tag{7}$$

THEOREM 3. *Let β_1 and β_2 be real numbers, β_1 being nonnegative. Then there will be a unimodal distribution function with first and second moment-ratios β_1 and β_2, respectively, if and only if*

$$5\beta_2 - 9 \geq \gamma(24\beta_1), \tag{8}$$

where, for all real y, $\gamma(y)$ denotes the largest number x satisfying

$$9x^4 - 2yx^3 - 36yx^2 + 36y^2x + 36y^2 - 6y^3 = 0. \tag{9}$$

It follows from Theorem 3 that a distribution cannot be unimodal if its (β_1, β_2) point falls in the region bounded by the β_1-axis, the limiting line $\beta_2 - \beta_1 - 1 = 0$, and the curve given by

$$5\beta_2 - 9 = \gamma(24\beta_1). \tag{10}$$

This curve meets the β_2-axis at the point (0, 9/5). As β_1 increases, β_2 decreases until the point (27/512, 27/16) is reached. Thereafter β_2 increases with β_1 and the curve is asymptotic to the line

$$60\beta_2 - 64\beta_1 - 81 = 0. \tag{11}$$

The curve is given parametrically by

$$\beta_1 = \frac{108q^4}{(1 - q)(1 + 3q)^3}, \quad 5\beta_2 - 9 = \frac{72q^2(3q - 1)}{(1 - q)(1 + 3q)^2}, \quad (0 \leq q < 1). \tag{12}$$

2. Proof of Theorem 1. We first suppose that $F(x)$ is a unimodal distribution function satisfying the conditions (5), and having mode zero. Then the conditions (4) are satisfied for all $x_1, \cdots, x_4$ satisfying (3) with $M = 0$. Thus $F(x)$ is a nondecreasing function which is convex for $x < 0$ and concave for $x > 0$. It follows that the one-sided differential coefficients $F'_-(x)$, $F'_+(x)$ exist for all nonzero values of x, and are equal except possibly for an enumerable number of values of x (see [3]). We define a function $f(x)$ by the equations

$$\begin{aligned} f(0) &= 0; \\ f(x) &= F'_+(x), \qquad x \neq 0. \end{aligned} \tag{13}$$

Then $f(x)$ is a nonnegative function which is nondecreasing for $x < 0$ and which is nonincreasing for $x > 0$. Further, if I is any closed bounded interval not containing the point $x = 0$, the incremental ratio $\{F(y) - F(x)\}/(y - x)$ is bounded for all distinct points x and y of I (see [3], pp. 91–96). Hence $F(x)$ is absolutely continuous in I and thus is an indefinite integral of $f(x)$ in I. Thus, if we write

$$j(x) = \begin{cases} F(-0), & x < 0, \\ F(0), & x = 0, \\ F(+0), & x > 0, \end{cases} \tag{14}$$

we have

$$F(x) = \int_0^x f(\xi)\, d\xi + j(x) \tag{15}$$

for all x.

From the monotonicity properties of $f(x)$ we have

$$|x|^{r+1} f(x) \le 2^{r+1} \left| \int_{x/2}^{x} \xi^r f(\xi)\, d\xi \right| = 2^{r+1} \left| \int_{x/2}^{x} \xi^r\, dF(\xi) \right|,$$

so that by the convergence of the integrals in (5)

$$x^{r+1} f(x) \to 0 \quad \text{as} \quad x \to \pm\infty \tag{16}$$

for all $r < n$.

Now consider the function $G(x)$ defined by

$$G(x) = F(x) - xf(x). \tag{17}$$

From (16), $G(-x) \to 0$ and $G(x) \to 1$ as $x \to +\infty$. Also from (15),

$$G(x) = \int_0^x \{f(\xi) - f(x)\}\, d\xi + j(x), \tag{18}$$

so that $G(x)$ is a nondecreasing function of x. Hence $G(x)$ is a distribution function.

Let r be an integer with $1 \le r < n$ and let X and Y be positive numbers. Then

$$\begin{aligned} \int_{-X}^{Y} x^r\, dG(x) &= \int_{-X}^{Y} x^r\, dF(x) - \int_{-X}^{Y} x^r\, d\{xf(x)\} \\ &= \int_{-X}^{Y} x^r\, dF(x) - [x^{r+1} f(x)]_{-X}^{Y} + r \int_{-X}^{Y} x^r f(x)\, dx \\ &= (r+1) \int_{-X}^{Y} x^r\, dF(x) - Y^{r+1} f(Y) + (-X)^{r+1} f(-X), \end{aligned} \tag{19}$$

the integration by parts being justified since $xf(x)$ is of bounded variation by (17), and the final step being justified by (15). Hence, using (16) and (5),

$$\int_{-\infty}^{\infty} x^r dG(x) = (r+1) \int_{-\infty}^{\infty} x^r\, dF(x) = (r+1)\mu_r \tag{20}$$

for all r with $1 \le r < n$. This proves the second assertion of the theorem.

We note that, since $F(x)/x$ is absolutely continuous in every closed bounded interval not including the point $x = 0$, it follows by (17) that the function $F(x)$ is given in terms of $G(x)$ by

$$(21) \qquad F(x) = \begin{cases} -x \int_{-\infty}^{x} \xi^{-2} G(\xi)\, d\xi, & x < 0, \\ G(0), & x = 0, \\ x \int_{x}^{\infty} \xi^{-2} G(\xi)\, d\xi, & x > 0, \end{cases}$$

or, on integration by parts, by

$$(22) \qquad F(x) = \begin{cases} \int_{-\infty}^{x} (1 - x\xi^{-1})\, dG(\xi), & x < 0, \\ G(0), & x = 0, \\ 1 - \int_{x}^{\infty} (1 - x\xi^{-1})\, dG(\xi), & x > 0. \end{cases}$$

We now prove the first assertion of the theorem. Suppose that there exists a distribution function $G(x)$ with

$$(23) \qquad \int_{-\infty}^{\infty} x^r\, dG(x) = (r + 1)\mu_r, \qquad 1 \leq r < n.$$

Let $F(x)$ be the corresponding function defined by the equation (22). Then it is clear that $F(x)$ is a nondecreasing function of x and that $F(-x) \to 0$ and $F(x) \to 1$ as $x \to +\infty$. Hence $F(x)$ is a distribution function. Also, if a, h, b, k are any real numbers with

$$a - h < a < a + h < 0 < b - k < b < b + k,$$

then

$$(24) \qquad \begin{aligned} &F(a + h) - 2F(a) + F(a - h) \\ &\quad = \int_{a-h}^{a} \xi^{-1}(a - h - \xi)\, dG(\xi) + \int_{a}^{a+h} \xi^{-1}(\xi - a - h)\, dG(\xi) \geq 0, \end{aligned}$$

and a similar argument shows that

$$(25) \qquad F(b + k) - 2F(b) + F(b - k) \leq 0.$$

Hence the conditions (4) are satisfied whenever $x_1, \cdots, x_4$ satisfy (3) with $M = 0$. Thus $F(x)$ is a unimodal distribution function with mode at $x = 0$.

Now, if $x < 0$,

$$(26) \qquad \begin{aligned} F(x) &= \int_{-\infty}^{x} (1 - x\xi^{-1})\, dG(\xi) \\ &= \int_{-\infty}^{x} \left\{ \int_{\xi}^{x} (-\xi^{-1})\, d\eta \right\} dG(\xi) \\ &= \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\eta} (-\xi^{-1})\, dG(\xi) \right\} d\eta; \end{aligned}$$

and similarly, if $x > 0$,

$$F(x) = 1 - \int_x^\infty \left\{ \int_\eta^\infty \xi^{-1}\, dG(\xi) \right\} d\eta. \tag{27}$$

Consequently

$$\begin{aligned}\int_{-\infty}^{\infty} x^r\, dF(x) &= \int_{-\infty}^{0} x^r \left\{ \int_{-\infty}^{x} (-\xi^{-1})\, dG(\xi) \right\} dx + \int_0^\infty x^r \left\{ \int_x^\infty \xi^{-1}\, dG(\xi) \right\} dx \\ &= \int_{-\infty}^{0} (-\xi^{-1}) \left\{ \int_\xi^0 x^r\, dx \right\} dG(\xi) + \int_0^\infty \xi^{-1} \left\{ \int_0^\xi x^r\, dx \right\} dG(\xi) \\ &= (r+1)^{-1} \int_{-\infty}^{\infty} \xi^r\, dG(\xi) = \mu_r,\end{aligned}$$

for all r with $1 \le r < n$. Thus $F(x)$ is a unimodal distribution with mode zero, satisfying (5). This completes the proof of the theorem.

By combining Theorem 1 with the results quoted in Section 1 we obtain the following

COROLLARY. *If there is a unimodal distribution function $F(x)$ with mode zero and with*

$$\int_{-\infty}^{\infty} x^r\, dF(x) = \mu_r, \qquad 1 \le r < n, \tag{28}$$

the condition

$$\begin{vmatrix} 1 & 2\mu_1 & \cdots & (s+1)\mu_s \\ 2\mu_1 & 3\mu_2 & \cdots & (s+2)\mu_{s+1} \\ \vdots & \vdots & & \\ (s+1)\mu_s & (s+2)\mu_{s+1} & \cdots & (2s+1)\mu_{2s} \end{vmatrix} \ge 0 \tag{29}$$

is satisfied for each integer s with $2 \le 2s < n$. Conversely, if (29) is satisfied with strict inequality for all integers s with $2 \le 2s < n$, then there is a unimodal distribution function $F(x)$ having mode zero and satisfying (28).

3. Proof of Theorem 2. First suppose that there is a unimodal distribution function with mean m, mode M and standard deviation σ. Then there is a unimodal distribution function $F(x)$ having mode zero and satisfying (5) with

$$n = 3, \qquad \mu_1 = m - M, \qquad \mu_2 = \sigma^2 + (m - M)^2. \tag{30}$$

So by the case $n = 3$ of the Corollary to Theorem 1 we have

$$\begin{vmatrix} 1 & 2(m-M) \\ 2(m-M) & 3\sigma^2 + 3(m-M)^2 \end{vmatrix} \ge 0,$$

that is,

$$(m - M)^2 \le 3\sigma^2.$$

Now suppose that m, M and σ are real numbers, with $\sigma > 0$, satisfying (7). To prove the existence of a unimodal distribution with mean m, mode M and standard deviation σ, it clearly suffices to prove the existence of a unimodal distribution function $F(x)$ having mode zero and satisfying (5) when the condition (30) is satisfied. Hence, by Theorem 1 it suffices to prove the existence of a distribution function $G(x)$ with $2(m - M)$ and $3\sigma^2 + 3(m - M)^2$ for its first and second moments, respectively. Since

$$3\sigma^2 + 3(m - M)^2 \geq \{2(m - M)\}^2$$

by (7), the existence of such a function $G(x)$ is clear. This proves the theorem.

4. Proof of Theorem 3. We first state without proof an elementary algebraic lemma.

LEMMA. *Let β_1, β_2 be real numbers with $\beta_1 \geq 0$. Suppose that there is a real number δ satisfying*

$$3 - \delta^2 > 0, \tag{31}$$

$$(3 - \delta^2)(5\beta_2 - 9 - 4\delta\sqrt{\beta_1}) \geq 16\beta_1; \tag{32}$$

then

$$5\beta_2 - 9 \geq \gamma(24\beta_1), \tag{33}$$

where $\gamma(y)$ is the function defined in the statement of Theorem 3. Conversely, if (33) *is satisfied with strict inequality, then there is a real δ satisfying* (31) *and satisfying* (32) *with strict inequality.*

Now suppose that there is a unimodal distribution function with first and second moment-ratios β_1 and β_2 respectively. Then there is a unimodal distribution function $H(x)$ with the numbers $0, 1, \sqrt{\beta_1}, \beta_2$ as its first four moments. Let δ be the mode of $H(x)$. Then the function $F(x) = H(x + \delta)$ is a unimodal distribution function with mode zero and moments

$$\mu_1 = -\delta, \quad \mu_2 = 1 + \delta^2, \quad \mu_3 = \sqrt{\beta_1} - 3\delta - \delta^3,$$
$$\mu_4 = \beta_2 - 4\delta\sqrt{\beta_1} + 6\delta^2 + \delta^4. \tag{34}$$

So these numbers $\mu_1, \cdots, \mu_4$ satisfy the condition (29) for $s = 1$ and for $s = 2$. These conditions reduce to the inequalities

$$3 - \delta^2 \geq 0, \tag{35}$$

$$(3 - \delta^2)(5\beta_2 - 9 - 4\delta\sqrt{\beta_1}) \geq 16\beta_1. \tag{36}$$

When $3 - \delta^2 > 0$ it follows from the Lemma that the condition (8) is satisfied. When $\delta = \pm\sqrt{3}$ we note that the function $G(x)$ defined in the proof of Theorem 1 has first and second moments $2\mu_1 = \mp 2\sqrt{3}$, $3\mu_2 = 12$, and so has standard deviation zero. Thus the rth moment of $G(x)$ is $(r + 1)\mu_r = (\mp\sqrt{3})^r$. This implies that $\beta_1 = 0$, $\beta_2 = 9/5$, so that the inequality (8) is satisfied in this case.

Now suppose that (8) is satisfied. First consider the case when (8) is satisfied with strict inequality. Then by the Lemma there is a number δ satisfying the conditions (35) and (36) with strict inequality. Hence the numbers $\mu_1, \cdots, \mu_4$ defined by (34) satisfy the condition (29) with strict inequality for $s = 1$ and for $s = 2$. So by the Corollary to Theorem 1 there is a unimodal distribution function with moments $\mu_1, \cdots, \mu_4$ satisfying (34). This unimodal distribution function has β_1 and β_2 for its first and second moment-ratios.

We have finally to consider the case when (8) is satisfied with equality. Since the function $108\, q^4(1 - q)^{-1}(1 + 3q)^{-3}$ increases from the value 0 and tends to $+\infty$ as q increases from 0 and tends to 1, we can choose a number q with $0 \leq q < 1$ such that

$$\beta_1 = 108q^4(1 - q)^{-1}(1 + 3q)^{-3}. \tag{37}$$

Then, since (8) is satisfied with equality, we have

$$5\beta_2 - 9 = 72q^2(3q - 1)(1 - q)^{-1}(1 + 3q)^{-2}. \tag{38}$$

It is not difficult to verify that the distribution function $F(x)$ defined by

$$F(x) = \begin{cases} 0, & x < 0, \\ q + (1 - q)x, & 0 \leq x \leq 1, \\ 1, & x > 1, \end{cases} \tag{39}$$

has β_1 and β_2 for its first and second moment-ratios. This completes the proof of the theorem.

REFERENCES

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